BULL. AUSTRAL. MATH. SOC. 94B25, 05C12, 05A17
VOL. 57 (1998) [367-376]

PERFECT CODES ON THE TOWERS OF HANOI GRAPH

CHI-KWONG LI AND INGRID NELSON

We characterise all the perfect k-error correcting codes that can be defined on the graph associated with the Towers of Hanoi puzzle. In particular, a short proof for the existence of 1-error correcting code on such a graph is given.

1. INTRODUCTION

In the study of recurrence relations, one common example is the following combinatorial game known as the *Towers of Hanoi puzzle.*

Initially, there are 3 pegs and n circular disks of increasing size on one peg with the largest disk on the bottom. These disks are to be transferred one at a time onto another of the pegs with the provision that one is never allowed to place a larger disk on top of a smaller one. The problem is to determine the number of moves necessary for the transfer.

For convenience, we call the three pegs P_0, P_1, and P_2, and label the disk as $D_1, \ldots, D_n$, where D_1 has the smallest radius. Define a *legal configuration* of the disks on the three pegs to be an arrangement of the disks on the pegs so that no larger disk is on the top of a smaller one. Then one easily checks that there is a one-one correspondence between all legal configurations with the space $\mathbb{Z}_3^n$ of ternary sequences of length n, such that a given $\mathbf{x} = x_1 \cdots x_n \in \mathbb{Z}_3^n$ corresponds to the configuration with D_i lying on P_j if $x_i = j$. For example, 101 corresponds to the configuration that D_1 and D_3 lie on P_1, and D_2 lies on P_0 (see Figure 1).

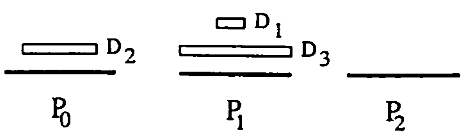

Figure 1

We shall call the legal configuration corresponding to $\mathbf{x} \in \mathbb{Z}_3^n$ the $\mathbf{x}$-configuration. The sequences with all entries equal to the same $i \in \{0, 1, 2\}$ are called the *perfect states* corresponding to the configurations with all disks lying on the same peg.

One can construct a graph with all $\mathbb{Z}_3^n$ as the vertex set, where two vertices $\mathbf{x}$ and $\mathbf{y}$ are connected by an edge if there is a legal move in the Towers of Hanoi puzzle that

Received 26th August, 1997

transforms the x-configuration to the y-configuration. This graph is called the Towers of Hanoi graph, denoted by H_n, and first appeared in [6]. We depict H_1 and H_2 in Figure 2.

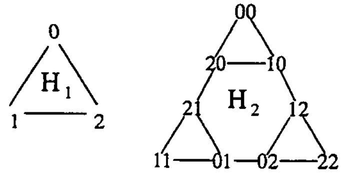

Figure 2

The Towers of Hanoi puzzle, its graph, and their generalisations have generated much interesting research (for example, see [5]). In fact, the graph H_n can be constructed from H_{n-1} by the following algorithm:

Step 1. Let $\widetilde{H}_{n-1}$ be the mirror image of H_{n-1} about a vertical line passing through the top perfect state.

Step 2. Construct $H_{n-1}^{(i)}$ by appending i to the end of each vertex of $\widetilde{H}_{n-1}$ to form a sequence of length n for $i = 0, 1, 2$.

Step 3. Put $H_{n-1}^{(0)}$ in the top, rotate $H_{n-1}^{(1)}$ by 120 degrees clockwise and put it in the left bottom corner, rotate $H_{n-1}^{(2)}$ by 120 degrees counterclockwise and put it in the right bottom corner.

Step 4. Connect the vertex $0\cdots01$ in $H_{n-1}^{(1)}$ with the vertex $0\cdots02$ in $H_{n-1}^{(2)}$, connect the vertex $1\cdots10$ in $H_{n-1}^{(0)}$ with the vertex $1\cdots12$ in $H_{n-1}^{(2)}$, connect the vertex $2\cdots20$ in $H_{n-1}^{(0)}$ with the vertex $2\cdots21$ in $H_{n-1}^{(1)}$.

One easily sees that this algorithm will generate all $\mathbf{x} \in \mathbb{Z}_3^n$ as vertices, and all the legal moves in the Towers of Hanoi puzzle as edges. We give the graphical representation of the situation in Figure 3.

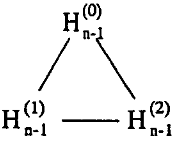

Figure 3

Define the *distance* $d(\mathbf{x}, \mathbf{y})$ between two vertices $\mathbf{x}$ and $\mathbf{y}$ to be the length of the shortest path joining the two vertices. Clearly, $d(\mathbf{x}, \mathbf{y})$ corresponds to the minimum number of legal moves needed in the Towers of Hanoi puzzle to transform the x-configuration to the y-configuration. For example (for example, see [2]), the distance between 2 perfect states in H_n equals $2^n - 1$, which is the maximum distance between any two vertices in H_n. The distance function d defines a metric on $\mathbb{Z}_3^n$, and for any nonnegative integer k, one may define the *radius-k ball* centred at $\mathbf{x} \in \mathbb{Z}_3^n$ to be the set

$$B(\mathbf{x}, k) = \{\mathbf{y} \in \mathbb{Z}_3^n : d(\mathbf{x}, \mathbf{y}) \leqslant k\}.$$

In the study of coding theory (see [4] for general background), one would like to

https://doi.org/10.1017/S0004972700031774 Published online by Cambridge University Press

partition $\mathbb{Z}_3^n$ as a disjoint union of $B(\mathbf{x}_1,k),\ldots,B(\mathbf{x}_M,k)$ for a suitable choice of $\mathbf{C}_n(k)=\{\mathbf{x}_1,\ldots,\mathbf{x}_M\}\subseteq\mathbb{Z}_3^n$. If such a partition exists, the set $\mathbf{C}_n(k)$ will be called a *perfect k-error correcting* ternary code on H_n.

The purpose of this note is to determine those (n,k) pairs for which a perfect k-error correcting code exists on H_n, and characterise $\mathbf{C}_n(k)$ if it exists. In particular, we shall give a short proof for the existence of a perfect 1-error correcting code on H_n (see [**2**] for the original proof).

It is worthwhile to point out that the existence of the k-error correcting codes on $\mathbb{Z}_3^n$ depends heavily on the structure of the radius k-balls. If a different metric is used, then the structure of the radius k-balls will be completely changed, and hence the corresponding coding theory problem will change accordingly. For instance, if the widely used Hamming metric $\widetilde{d}(\mathbf{x},\mathbf{y})=$ the number of no zero entries in the vector $(\mathbf{x}-\mathbf{y})$, then a perfect 1-error ternary correcting code rarely exists (see [**4**]).

Note also that other coding theory problems have been studied using the Towers of Hanoi graph [**3**].

2. Results and Proofs

We begin with a short proof of the fact that perfect 1-error correcting codes always exists on H_n. See [**2**] for the original proof.

Theorem 2.1. *Let n be a positive integer. Then the collection $\mathbf{C}_n(1)$ of ternary sequences of length n with an even number of terms equal to 1 and an even number of terms equal to 2 is a perfect 1-error correcting code on H_n. Moreover, we have*

$$\left|\mathbf{C}_n(1)\right|=\begin{cases}(3^n+3)/4 & \text{if } n \text{ is even,}\\ (3^n+1)/4 & \text{if } n \text{ is odd.}\end{cases}$$

Proof: We need to show that every $\mathbf{y}\in\mathbb{Z}_3^n$ lies in one and only one $B(\mathbf{x},1)$ with $\mathbf{x}\in\mathbf{C}_n(1)$. Denote by n_i the number of terms in $\mathbf{y}$ that equal i for $i=0,1,2$.

First, suppose $\mathbf{y}\in\mathbf{C}_n(1)$, that is, n_1 and n_2 are even. Then $\mathbf{y}\in B(\mathbf{y},1)$, and it is clear that $\mathbf{y}$ cannot be transformed to another $\mathbf{x}\in\mathbf{C}_n(1)$ by just one legal move. Hence, $\mathbf{y}$ lies in $B(\mathbf{y},1)$ but not in $B(\mathbf{x},1)$ for any other $\mathbf{x}\in\mathbf{C}_n(1)$. Next, suppose $\mathbf{y}\notin\mathbf{C}_n(1)$. Then either

(i) both n_1 and n_2 are odd, or

(ii) exactly one of n_1 or n_2 is odd.

If (i) holds, then one can consider the $\mathbf{y}$-configuration and transfer the smallest disk on P_1 and P_2 from one peg to the other peg. Clearly, the resulting configuration corresponds to a ternary sequence $\mathbf{x}\in\mathbf{C}_n(1)$, and the proposed move is the only single legal move on the $\mathbf{y}$-configuration that will lead to a configuration corresponding to a sequence in $\mathbf{C}_n(1)$. Thus $\mathbf{y}\in B(\mathbf{x},1)$, but is not in any other unit ball $B(\mathbf{z},1)$ with $\mathbf{z}\in\mathbf{C}_n(1)$.

https://doi.org/10.1017/S0004972700031774 Published online by Cambridge University Press

Suppose (ii) holds, and suppose n_i is odd with $i \in \{1,2\}$. Then one can consider the $\mathbf{y}$-configuration and transfer the smallest disk on P_0 and P_i from one peg to the other peg. Clearly, the resulting configuration corresponds to a ternary sequence $\mathbf{x} \in \mathbf{C}_n(1)$, and the proposed move is the only single legal move on the $\mathbf{y}$-configuration that will lead to a configuration corresponding to a ternary sequence in $\mathbf{C}_n(1)$. Thus $\mathbf{y} \in B(\mathbf{x},1)$, but is not in any other unit ball $B(\mathbf{z},1)$ with $\mathbf{z} \in \mathbf{C}_n(1)$.

Note that if $\mathbf{x} \in \mathbb{Z}_3^n$ is a perfect state, then applying one legal move to the $\mathbf{x}$-configuration may lead to two possible outcomes depending on where the smallest disk is transferred. Thus $B(\mathbf{x},1)$ has three elements. If $\mathbf{x} \in \mathbb{Z}_3^n$ is not a perfect state, then there are 3 possible outcomes with a legal move, namely, one may transfer the smallest disk to one of the two other pegs, or one may transfer a disk between the two pegs not containing the smallest disk. In such case, $B(\mathbf{x},1)$ has four elements. Now, the collection of $B(\mathbf{x},1)$ with $\mathbf{x} \in \mathbf{C}_n(1)$ form a partition of $\mathbb{Z}_3^n$. If n is even, then all 3 perfect states belong to $\mathbf{C}_n(1)$. Hence 3 of the $B(\mathbf{x},1)$ will have 3 elements, and the rest will have 4 elements. Thus

$$3^n = |\mathbb{Z}_3^n| = \sum_{\mathbf{x}\in\mathbf{C}_n(1)} \left|B(\mathbf{x},1)\right| = 3 \times 3 + 4 \times \Big(|\mathbf{C}_n(1)| - 3\Big),$$

and thus $\left|\mathbf{C}_n(1)\right| = (3^n + 3)/4$. If n is odd, there is only one perfect state in $\mathbf{C}_n(1)$. By a similar argument, one sees that $\left|\mathbf{C}_n(1)\right| = (3^n + 1)/4$. ∎

We have several remarks in connection with the above theorem. Details and proofs can be found in [**2**].

1. The above proof actually suggests an easy decoding algorithm for $\mathbf{C}_n(1)$.
2. It is not difficult to show that one of the perfect states must be a codeword for any 1-error correcting code on H_n. If one assumes that the top corner vertex $0 \cdots 0$ is a codeword, then the code must be $\mathbf{C}_n(1)$.
3. By the general theory of coding theorem (for example, [**4**, Theorem 1.9]), we see that $d(\mathbf{x},\mathbf{y}) \geqslant 3$ for any $\mathbf{x},\mathbf{y} \in \mathbf{C}_n(1)$. This fact can also be proved independently using arguments similar to those in our proof of the theorem.

Next, we turn to the case when $k > 1$. We have the following result.

THEOREM 2.2. *Suppose $n, k > 1$ are integers. There exists a perfect k-error correcting code $\mathbf{C}_n(k)$ on H_n if and only if*

(a) $k \geqslant 2^{n-2} \cdot 3$, *or*

(b) $k = 2^{n-1} - 1$.

Furthermore, if (a) *holds, then a perfect k-error correcting code must consist of a single vertex $\mathbf{c}$ of H_n such that $B(\mathbf{c},k)$ contains all perfect states; if* (b) *holds, the only perfect k-error correcting code is the set of the three perfect states.*

As we shall see in the following proofs, it is rather easy to check that if n and k satisfy the condition (a) or (b), then there are perfect k-error-correcting codes $\mathbf{C}_n(k)$ as described in the theorem. The non-trivial part is the necessity part of the theorem, that is, the non-existence of any other perfect codes for $k > 1$.

If $n > 1$, we always assume that H_n can be decomposed into $H_{n-1}^{(i)}$ for $i = 0, 1, 2$, as described in the introduction.

We first establish several lemmas concerning the vertices of H_n.

LEMMA 2.3. *Let $\mathbf{w}$ be a vertex in H_n, and let d_0, d_1, d_2 be the distance between $\mathbf{w}$ and the top, left bottom, and right bottom perfect states of H_n, respectively. Then*

(a) *d_0, d_1, d_2 cannot have the same (even or odd) parity.*

(b) *$d_i = d_j$ for some $i \neq j$ if and only if $d_{3-i-j} = 0$, that is, $\mathbf{w}$ is a perfect state.*

PROOF: The proof can be done by induction. If $n = 1$, the conclusions (a) and (b) clearly hold. Suppose $n \geqslant 2$, and the conclusions hold for H_{n-1}. Let $\mathbf{w}$ be a vertex of H_n. We may assume that $\mathbf{w}$ lies in the subgraph $H_{n-1}^{(0)}$ of H_n by a suitable 120 degrees rotation of H_n. Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are the top, left bottom, and right bottom perfect states of H_n, and let $\tilde{\mathbf{y}}, \tilde{\mathbf{z}}$ be the left and right bottom perfect states of $H_{n-1}^{(0)}$ (see Figure 4). Then $d_0 = d(\mathbf{w}, \mathbf{x})$,

$$\begin{aligned} d_1 &= d(\mathbf{w}, \mathbf{y}) = d(\mathbf{w}, \tilde{\mathbf{y}}) + d(\tilde{\mathbf{y}}, \mathbf{y}) = d(\mathbf{w}, \tilde{\mathbf{y}}) + 2^{n-1}, \\ d_2 &= d(\mathbf{w}, \mathbf{z}) = d(\mathbf{w}, \tilde{\mathbf{z}}) + d(\tilde{\mathbf{z}}, \mathbf{z}) = d(\mathbf{w}, \tilde{\mathbf{z}}) + 2^{n-1}. \end{aligned}$$

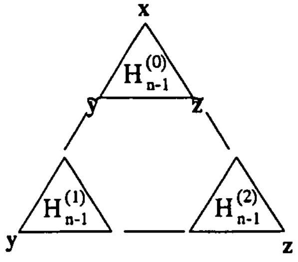

Figure 4

One can then apply the induction assumption on $d(\mathbf{w}, \mathbf{x})$, $d(\mathbf{w}, \tilde{\mathbf{y}})$ and $d(\mathbf{w}, \tilde{\mathbf{z}})$ to get the conclusions on d_0, d_1 and d_2. ∎

LEMMA 2.4. *Let n, k be positive integers. Suppose $\mathbf{w}$ is a vertex in H_n such that $B(\mathbf{w}, k)$ does not contain any of the three perfect states. Then there is a subgraph R of H_n which is isomorphic to H_1 such that $B(\mathbf{w}, k)$ contains exactly one vertex of R.*

PROOF: Consider subgraphs in H_n of the form H_m so that the vertex set V of the subgraph satisfies $\mathbf{w} \in V \subseteq B(\mathbf{w}, k)$. Let S be such a subgraph with the maximum number of vertices, that is, largest possible m, and let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be the perfect states of S. Since $B(\mathbf{w}, k)$ does not contain any perfect state of H_n, we see that each of $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ is connected to some vertices that are not in S (see Figure 5). By Lemma 2.3 (a), we

https://doi.org/10.1017/S0004972700031774 Published online by Cambridge University Press

see that $n_0 = k - d(\mathbf{w}, \mathbf{x})$, $n_1 = k - d(\mathbf{w}, \mathbf{y})$ and $n_2 = k - d(\mathbf{w}, \mathbf{z})$ cannot all have the same parity. In particular, we may assume that n_0 is odd. (Otherwise, apply 120 degrees rotations to H_n to make n_0 odd.) Let $\mathbf{v}$ be a vertex in H_n satisfying $d(\mathbf{w}, \mathbf{v}) = k$ and $d(\mathbf{x}, \mathbf{v}) = n_0$, and let R be the subgraph of H_n, which is isomorphic to H_1 and has $\mathbf{v}$ as a vertex. If one follows a path in H_n from $\mathbf{w}$ to $\mathbf{v}$, one sees that the two other vertices of the subgraph R will have a distance $k+1$ from $\mathbf{w}$. Hence $B(\mathbf{w}, k)$ contains only the vertex $\mathbf{v}$ of R. □

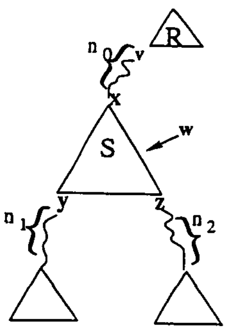

Figure 5

We are now ready to prove our theorem. While the theorem is stated in terms of the conditions on n and k, we divide the proof into different cases according to the size of $\mathbf{C}_n(k)$. In particular, we shall show that $\left|\mathbf{C}_n(k)\right|$ can only be 1 or 3. We begin with the case when $\left|\mathbf{C}_n(k)\right| = 1$.

LEMMA 2.5. *Suppose $n, k > 1$ are positive integers. Then there exists a perfect k-error correcting code on H_n consisting of one codeword if and only if $k \geqslant 2^{n-2} \cdot 3$.*

PROOF: Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be the top, left and right bottom perfect states of H_n, and let $\tilde{\mathbf{y}}, \tilde{\mathbf{z}}$ be the left and right bottom perfect states of $H_{n-1}^{(0)}$ (see Figure 4).

If $k \geqslant 2^{n-2} \cdot 3$, then one can choose $\mathbf{c}$ so that $d(\mathbf{c}, \tilde{\mathbf{y}}) = 2^{n-2}$ and $d(\mathbf{c}, \tilde{\mathbf{z}}) = 2^{n-2} - 1$. One easily checks that all vertices of H_n lie in $B(\mathbf{c}, k)$.

Conversely, suppose there is a perfect k-error correcting code consisting of the single codeword $\mathbf{c}$. Then all perfect states belong to $B(\mathbf{c}, k)$. To show that $k \geqslant 2^{n-2} \cdot 3$, suppose $\mathbf{c}$ lies in $H_{n-1}^{(0)}$, otherwise apply a suitable 120 degrees rotation to H_n. Then

$$2^{n-1} - 1 = d(\tilde{\mathbf{z}}, \tilde{\mathbf{y}}) \leqslant d(\tilde{\mathbf{z}}, \mathbf{c}) + d(\mathbf{c}, \tilde{\mathbf{y}}).$$

It follows that

$$\begin{aligned} k &= \max\left\{d(\mathbf{c}, \mathbf{z}), d(\mathbf{c}, \mathbf{y})\right\} \\ &= \max\left\{d(\mathbf{c}, \tilde{\mathbf{z}}) + d(\mathbf{z}, \tilde{\mathbf{z}}), d(\mathbf{c}, \tilde{\mathbf{y}}) + d(\mathbf{y}, \tilde{\mathbf{y}})\right\} \\ &= \max\left\{d(\mathbf{c}, \tilde{\mathbf{z}}), d(\mathbf{c}, \tilde{\mathbf{y}})\right\} + 2^{n-1} \\ &\geqslant [2^{n-1} \cdot 3 - 1]/2. \end{aligned}$$

Since k is an integer, we have $k \geqslant 2^{n-2} \cdot 3$. ∎

Next, we consider the case when $1 < \left|\mathbf{C}_n(k)\right| \leqslant 3$.

LEMMA 2.6. *Suppose $n, k > 1$ are positive integers. The following conditions are equivalent.*

(a) *There exists a perfect k-error correcting code $\mathbf{C}_n(k)$ on H_n with $1 < \left|\mathbf{C}_n(k)\right| \leqslant 3$.*

(b) *There is a unique perfect k-error correcting code on H_n consisting of the 3 perfect states.*

(c) $k = 2^{n-1} - 1$.

PROOF: It is clear that (b) implies (a). If (c) holds, then the perfect states will form a perfect k-error correcting code, and thus (a) is true. It remains to prove that both (b) and (c) follow from (a).

Now, suppose (a) holds. Let $\mathbf{x}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}$ be the top, left and right bottom perfect states of $H_{n-1}^{(0)}$ (see Figure 4).

We first show that it is impossible to have $\left|\mathbf{C}_n(k)\right| = 2$. Suppose there is such a perfect code, and $\mathbf{C}_n(k)$ consists of $\mathbf{c}_1$ and $\mathbf{c}_2$. We may assume that they are not vertices of $H_{n-1}^{(0)}$ by a suitable 120 degrees rotation of H_n. Then $\mathbf{x}$ lies in one of two radius-k balls centred at the codewords. Without loss of generality, we may assume that $\mathbf{x}$ lies in $B(\mathbf{c}_1, k)$ and $\mathbf{c}_1$ is a vertex of $H_{n-1}^{(1)}$. Then $k \geqslant d(\mathbf{c}_1, \mathbf{x}) > d(\tilde{\mathbf{y}}, \mathbf{x})$, or in other words, $k \geqslant 2^{n-1}$ Clearly, $\mathbf{c}_2$ cannot lie in $H_{n-1}^{(1)}$; otherwise, $\tilde{\mathbf{y}} \in B(\mathbf{c}_1, k) \cap B(\mathbf{c}_2, k)$ by the above condition on k. Also, $\mathbf{c}_2$ cannot lie in $H_{n-1}^{(2)}$, otherwise, $\tilde{\mathbf{z}} \in B(\mathbf{c}_1, k) \cap B(\mathbf{c}_2, k)$. Thus $\mathbf{C}_n(k)$ cannot have exactly 2 elements.

Next, suppose $\mathbf{C}_n(k)$ consists of 3 elements: $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2$. Notice that each of $H_{n-1}^{(i)}$ for $i = 0, 1, 2$, has a codeword; that is, $\mathbf{c}_i$ is a vertex of $H_{n-1}^{(i)}$. If this is not the case, one can use the arguments in the preceding paragraph to show that two of the $B(\mathbf{c}_i, k)$ will have non-empty intersection. If $B(\mathbf{c}_i, k)$ only contains those vertices in $H_{n-1}^{(i)}$ for $i = 0, 1, 2$, then we have $d(\mathbf{c}_0, \tilde{\mathbf{y}}) = d(\mathbf{c}_0, \tilde{\mathbf{z}})$. By Lemma 2.3 (b), we see that $\mathbf{c}_0 = \mathbf{x}$ and $k = 2^{n-1} - 1$. Similarly, one can show that $\mathbf{c}_1 = \mathbf{y}$ and $\mathbf{c}_2 = \mathbf{z}$. Hence condition (b) and (c) of the lemma hold.

In the following, we show that any other construction of $\mathbf{C}_n(k)$ is impossible. Suppose it is not the case, and suppose one of the $B(\mathbf{c}_i, k)$ contains some vertices in $H_{n-1}^{(j)}$ for some $i \neq j$. We may assume that $(i, j) = (0, 2)$ by applying some suitable rotation and reflection about the vertical line passing through $\mathbf{x}$. Furthermore, decompose $H_{n-1}^{(2)}$ into $H_{n-2}^{(0)}$, $H_{n-2}^{(1)}$, and $H_{n-2}^{(2)}$ as shown in Figure 6.

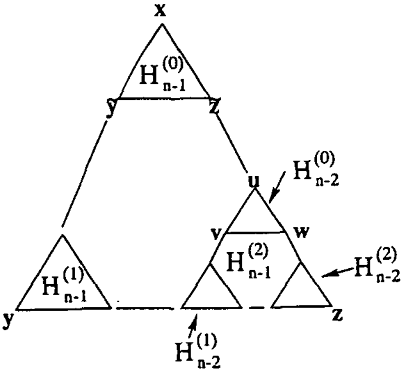

Figure 6

Suppose $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are the top, left and right bottom perfect states of $H_{n-2}^{(0)}$. Then $\mathbf{u} \in B(\mathbf{c}_0, k)$ by our assumption. Note that

$$d(\mathbf{c}_0, \mathbf{v}) = d(\mathbf{c}_0, \tilde{\mathbf{z}}) + d(\tilde{\mathbf{z}}, \mathbf{v}) = d(\mathbf{c}_0, \tilde{\mathbf{z}}) + d(\tilde{\mathbf{z}}, \mathbf{w}) = d(\mathbf{c}_0, \mathbf{w}).$$

Thus either

(i) $\mathbf{v}, \mathbf{w} \in B(\mathbf{c}_0, k)$, or

(ii) $\{\mathbf{v}, \mathbf{w}\} \cap B(\mathbf{c}_0, k) = \emptyset$.

We first show that (i) cannot happen. If $\mathbf{v}, \mathbf{w} \in B(\mathbf{c}_0, k)$, then $k \geqslant d(\mathbf{c}_0, \mathbf{w}) \geqslant d(\tilde{\mathbf{z}}, \mathbf{w}) = 2^{n-2}$, and we must have $\mathbf{c}_2$ lying in $H_{n-2}^{(1)}$ or $H_{n-2}^{(2)}$. If $\mathbf{c}_2$ is a vertex of $H_{n-2}^{(1)}$, then $\mathbf{v} \in B(\mathbf{c}_2, k)$ as $k \geqslant 2^{n-2}$. If $\mathbf{c}_2$ is a vertex of $H_{n-2}^{(2)}$, then $\mathbf{w} \in B(\mathbf{c}_2, k)$ as $k \geqslant 2^{n-2}$. In both cases, we have $B(\mathbf{c}_0, k) \cap B(\mathbf{c}_2, k) \neq \emptyset$, which is a contradiction.

Next, we show that (ii) is also impossible. Note that by an argument similar to that in the preceding paragraph, one can show that even if $B(\mathbf{c}_1, k)$ contains some vertices in $H_{n-1}^{(2)}$, it cannot include all the vertices in $H_{n-2}^{(1)}$. Thus $B(\mathbf{c}_1, k)$ cannot contain $\mathbf{v}$ or $\mathbf{w}$, and so we must have $\mathbf{v}, \mathbf{w} \in B(\mathbf{c}_2, k)$. If $\mathbf{c}_2$ is a vertex in $H_{n-2}^{(0)}$, then $k < 2^{n-2}$. Otherwise, we have $d(\mathbf{c}_2, \mathbf{u}) \leqslant 2^{n-2} \leqslant k$ and thus $\mathbf{u} \in B(\mathbf{c}_2, k) \cap B(\mathbf{c}_0, k)$. But then $k < 2^{n-2} \leqslant d(\mathbf{c}_i, \mathbf{z})$ for any i, contradicting the fact that $\mathbb{Z}_3^n = \bigcup_{i=0}^{2} B(\mathbf{c}_i, k)$. Thus, $\mathbf{c}_2$ must be in either $H_{n-2}^{(1)}$ or $H_{n-2}^{(2)}$. Now consider the vertex $\tilde{\mathbf{w}}$ on the shortest path from $\mathbf{u}$ to $\mathbf{w}$ such that $d(\mathbf{c}_0, \tilde{\mathbf{w}}) = k + 1$. Clearly, we have

$$k = d(\mathbf{c}_2, \tilde{\mathbf{w}}) = d(\mathbf{c}_2, \mathbf{w}) + d(\mathbf{w}, \tilde{\mathbf{w}}).$$

Similarly, if $\tilde{\mathbf{v}}$ is the vertex on the shortest path from $\mathbf{u}$ to $\mathbf{v}$ such that $d(\mathbf{c}_0, \tilde{\mathbf{v}}) = k + 1$, then

$$k = d(\mathbf{c}_2, \tilde{\mathbf{v}}) = d(\mathbf{c}_2, \mathbf{v}) + d(\mathbf{v}, \tilde{\mathbf{v}}).$$

Evidently, $d(\mathbf{u}, \tilde{\mathbf{w}}) = d(\mathbf{u}, \tilde{\mathbf{v}}) = (k+1) - d(\mathbf{c}_0, \mathbf{u})$. It follows that $d(\mathbf{w}, \tilde{\mathbf{w}}) = d(\mathbf{v}, \tilde{\mathbf{v}})$ and $d(\mathbf{c}_2, \mathbf{v}) = d(\mathbf{c}_2, \mathbf{w})$. However, this is impossible because $d(\mathbf{c}_2, \mathbf{v}) \leqslant 2^{n-2} < d(\mathbf{c}_2, \mathbf{w})$ if $\mathbf{c}_2 \in H_{n-2}^{(1)}$, and $d(\mathbf{c}_2, \mathbf{w}) \leqslant 2^{n-2} < d(\mathbf{c}_2, \mathbf{v})$ if $\mathbf{c}_2 \in H_{n-2}^{(2)}$. ∎

Finally, we consider the case when $3 < \left|\mathbf{C}_n(k)\right|$.

Lemma 2.7. *Suppose $n, k > 1$ are positive integers. Then there is no perfect k-error correcting code $\mathbf{C}_n(k)$ on H_n with $3 < \left|\mathbf{C}_n(k)\right|$.*

Proof: Suppose there is a perfect k-error correcting code $\mathbf{C}_n(k)$ on H_n with $3 < \left|\mathbf{C}_n(k)\right|$. Then there is a codeword $\mathbf{c}$ such that $B(\mathbf{c}, k)$ does not contain any of the three perfect states of H_n. By Lemma 2.4, there exists a subgraph U of H_n with vertices $\mathbf{u}_0, \mathbf{u}_1$ and $\mathbf{u}_2$ isomorphic to H_1 such that $B(\mathbf{c}, k) \cap U = \mathbf{u}_0$. Furthermore, we assume that $\mathbf{u}_1$ is connected to a subgraph V with vertices $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$, which is also isomorphic to H_1. Similarly, $\mathbf{u}_2$ is connected to a subgraph W with vertices $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$, which is also isomorphic to H_1. We depict the situation in Figure 7.

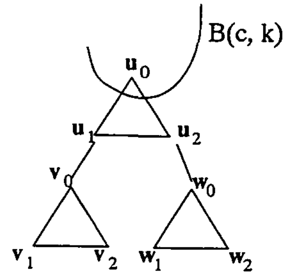

Figure 7

Let $\mathbf{d}$ be a codeword such that $\mathbf{u}_1 \in B(\mathbf{d}, k)$. Since $k \geqslant 2$, we can move at least 2 steps from $\mathbf{u}_1$ along the shortest path from $\mathbf{u}_1$ to $\mathbf{d}$ in H_n. Clearly, $d(\mathbf{d}, \mathbf{u}_1) = k$ and hence $d(\mathbf{d}, \mathbf{v}_0) = k - 1$ so that $B(\mathbf{c}, k) \cap B(\mathbf{d}, k) = \emptyset$. As a result, $d(\mathbf{d}, \mathbf{v}_i) = k - 2$ for $i = 1$ or 2. Similarly, let $\mathbf{e}$ be a codeword such that $\mathbf{u}_2 \in B(\mathbf{e}, k)$. One then sees that $d(\mathbf{e}, \mathbf{w}_0) = k - 1$, and $d(\mathbf{e}, \mathbf{w}_i) = k - 2$ for $i = 1$ or 2.

We consider 2 cases. First, suppose $\mathbf{v}_2$ and $\mathbf{w}_1$ are adjacent. Since $d(\mathbf{v}_1, \mathbf{w}_1) = 2$ and $d(\mathbf{v}_2, \mathbf{w}_1) = 1$, we see that

$$d(\mathbf{w}_1, \mathbf{d}) \leqslant \min\left\{d(\mathbf{w}_1, \mathbf{v}_i) + d(\mathbf{v}_i, \mathbf{d}) : i = 1, 2\right\} \leqslant k.$$

Thus $\mathbf{w}_1 \in B(\mathbf{d}, k) \cap B(\mathbf{e}, k)$, which is a contradiction unless $\mathbf{d} = \mathbf{e}$. However, this is impossible as shown in the following. If $k = 2$, then $\mathbf{d}$ must be chosen from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2\}$. But none of the choices will lead to $d(\mathbf{d}, \mathbf{v}_0) = k - 1 = d(\mathbf{d}, \mathbf{w}_0)$. Thus we may assume that $k > 2$, and $d(\mathbf{d}, \mathbf{v}_1) = k - 2 = d(\mathbf{d}, \mathbf{w}_2)$. But then $\mathbf{d}$ must lie in some subgraph R isomorphic to H_r so that the vertex set of R is contained in $B(\mathbf{d}, k - 3)$. If one moves k steps along a path from $\mathbf{d}$ to the subgraph U, either one can reach exactly one vertex of U or all the three vertices of U (see Proof of Lemma 2.4). Thus, it is impossible to have $\mathbf{u}_1, \mathbf{u}_2 \in B(\mathbf{d}, k)$.

Next, suppose $\mathbf{v}_2$ and $\mathbf{w}_1$ are not adjacent. Then U must be lying at the bottom of a certain subgraph S of H_n which is isomorphic to some H_m with $m > 1$, and either

(i) both V and W are in the same H_m, or

https://doi.org/10.1017/S0004972700031774 Published online by Cambridge University Press

(ii) only one of V or W is also in H_m.

In both cases, there will be a subgraph $\tilde{S}$ of S isomorphic to H_2 containing U and one of V or W, say W (see Figure 8). Since both $\mathbf{u}_0$ and $\mathbf{w}_2$ have a distance 2 from the top perfect state of $\tilde{S}$, which has a distance $k-2$ from $\mathbf{c}$, we see that $\mathbf{w}_2 \in B(\mathbf{c}, k)$. However, $d(\mathbf{e}, \mathbf{w}_0) = k - 1$ implies that $\mathbf{w}_2$ also belongs to $B(\mathbf{e}, k)$, which is a contradiction. ∎

One can easily combine Lemmas 2.5 – 2.7 to get the conclusion of Theorem 2.2.

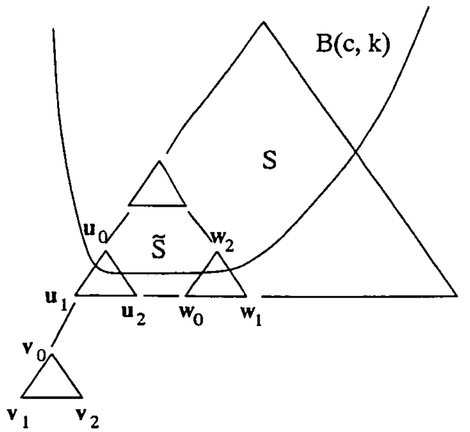

Figure 8

References

[1] R.A. Brualdi, *Introductory combinatorics* (North-Holland, New York, 1992).

[2] P. Cull and I. Nelson, 'Error-correcting codes on the Towers of Hanoi graphs', (preprint).

[3] M.C. Er, 'The cyclic towers of Hanoi and pseudo ternary codes', *J. Inform. Optim. Sci.* **6** (1986), 271–277.

[4] R. Hill, *A first course in coding theory* (Oxford University Press, Oxford, 1986).

[5] A.M. Hinz, 'The Tower of Hanoi', *Enseign. Math.* **35** (1989), 289–321.

[6] R.S. Scorer, P.M. Grundy and C.A.B. Smith, 'Some binary games', *Math. Gaz.* **280** (1944), 96–103.

Department of Mathematics
The College of William and Mary
Williamsburg, VA 23187
United States of America
e-mail: ckli@math.wm.edu

Lockheed Martin Federal Systems
6304 Spine Road
Boulder CO 80302
United States of America
e-mail: ingrid.nelson@lmco.com