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EXTENDING SURJECTIVE MAPS PRESERVING THE NORM OF SYMMETRIC KUBO-ANDO MEANS

EMMANUEL CHETCUTI AND CURT HEALEY

ABSTRACT. In [4], the authors addressed the question of whether surjective maps preserving the norm of a symmetric Kubo-Ando mean can be extended to Jordan $*$-isomorphisms. The question was affirmatively answered for surjective maps between the positive definite cones of unital C^*-algebras for certain specific classes of symmetric Kubo-Ando means. Here, we give a comprehensive answer to this question for surjective maps between the positive definite cones of AW^*-algebras preserving the norm of any symmetric Kubo-Ando mean.

1. INTRODUCTION

Recently, in [4], considerable attention was given to the problem of characterizing those maps between the positive definite cones of unital C^*-algebras that preserve the norm of a given Kubo-Ando mean. We recall that a binary operation σ on the positive definite cone $\mathscr{B}(H)^{++}$ of the algebra $\mathscr{B}(H)$ of bounded operators on the Hilbert space H, is called a *Kubo-Ando connection* if it satisfies the following properties:

(i) If $A \leq C$ and $B \leq D$, then $A\sigma B \leq C\sigma D$.
(ii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$.
(iii) If $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n\sigma B_n \downarrow A\sigma B$ (We write $A_n \downarrow A$ when (A_n) is monotonic decreasing and SOT-convergent to A).

A *Kubo-Ando mean* is a Kubo-Ando connection with the normalization condition $I\sigma I = I$. The most fundamental connections are:

- the *sum* $(A, B) \mapsto A + B$,
- the *parallel sum* $(A, B) \mapsto A : B = (A^{-1} + B^{-1})^{-1}$,
- the *geometric mean*

$$(A, B) \mapsto A\sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}.$$

The domain of definition can easily be extended from $\mathscr{B}(H)^{++}$ to the positive semi-definite cone $\mathscr{B}(H)^{+}$. For details, refer to the introduction section in [3].

In [7, Theorem 3.2], it is shown that there is an affine order isomorphism from the class of Kubo-Ando connections onto the class of operator monotone functions via the map $f(xI) = I\sigma(xI)$ for $x > 0$. Moreover, it is also shown that $f(A) = I\sigma A$ for every $A \in \mathscr{B}(H)^{+}$, which implies that

$$A\sigma B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}, \quad \forall\, A \in \mathscr{B}(H)^{++}, B \in \mathscr{B}(H)^{+}.$$

The function f is called the *representing function* of σ. We further recall that if σ is a Kubo-Ando connection with representing function f, then the representing function of the 'reversed' Kubo-Ando connection $(A, B) \mapsto B\sigma A$ is the transpose f°, defined by $f^\circ(x) := xf(x^{-1})$. The Kubo-Ando connection is said to be symmetric if it coincides with its reverse; that is, a Kubo-Ando connection is symmetric if and only if the representing function f satisfies $f = f^\circ$ as shown in [7, Corollary

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4.2]. The Kubo-Ando means are precisely the Kubo-Ando connections whose representing function satisfy the normalizing condition $f(1) = 1$.

Since every finite Borel measure on $[0, \infty]$ is regular (i.e a Radon measure), we have that operator monotone functions correspond to positive finite Borel measures on $[0, \infty]$ by Löwner's Theorem (see [5]): To every operator monotone function f corresponds a unique positive and finite Borel measure m on $[0, \infty]$ such that

$$f(x) = \int_{[0,\infty]} \frac{x(1+t)}{x+t}\,\mathrm{d}m(t) \;=\; m(\{0\}) \,+\, x\,m(\{\infty\}) \,+\, \int_{(0,\infty)} \frac{1+t}{t}(t:x)\,\mathrm{d}m(t) \quad (x>0). \tag{1}$$

It is easy to see that $f(0+) = m(\{0\})$, $f^{\circ}(0+) = m(\{\infty\})$. Finally, by [7, Theorem 3.4], there is an affine isomorphism from the class of positive finite Borel measures on $[0, \infty]$ onto the class of Kubo-Ando connections. This is given by the formula

$$A\sigma B = \alpha A + \beta B + \int_{(0,\infty)} \frac{1+t}{t}(tA : B)\,\mathrm{d}m(t) \quad A, B \in \mathscr{B}(H)^+ \tag{2}$$

where $\alpha = m(\{0\})$ and $\beta = m(\{\infty\})$. In the case of a symmetric Kubo-Ando connection, $\alpha = \beta$. For further details on the provenance of the integral representation (2), the reader is referred to [7, Theorem 3.2].

After the exposition on general Kubo-Ando means, we can now define the property under study for symmetric Kubo-Ando means.

Definition 1. Let σ be a Kubo-Ando mean and let $\mathscr{A}$ and $\mathscr{B}$ be unital C^*-subalgebras of $\mathscr{B}(H)$. A surjective map ϕ between the positive definite cones of $\mathscr{A}$ and $\mathscr{B}$ is said to preserve the norm of σ if

$$\|A\sigma B\| = \|\phi(A)\sigma\phi(B)\|\,, \quad \forall\, A, B \in \mathscr{A}^{++}.$$

A natural question to ask is whether a surjective map $\phi : \mathscr{A}^{++} \to \mathscr{B}^{++}$ preserving the norm of a symmetric Kubo-Ando mean is an order isomorphism. By an *order isomorphism*, we mean a map ϕ such that $A \le B \iff \phi(A) \le \phi(B)$ where $A, B \in \mathscr{A}^{++}$. Let $A, B \in \mathscr{A}^{++}$. Then by [3, Theorem 6],

$$\begin{aligned} A \le B &\iff \|A\sigma X\| \le \|B\sigma X\|\,, \quad \forall\, X \in \mathscr{A}^{++} \\ &\iff \|\phi(A)\sigma\phi(X)\| \le \|\phi(B)\sigma\phi(X)\|\,, \quad \forall\, X \in \mathscr{A}^{++} \\ &\iff \phi(A) \le \phi(B). \end{aligned} \tag{3}$$

Moreover, ϕ is norm preserving. This can be seen by recalling [7, Theorem 3.3] and noting that

$$\|A\| = \|A\sigma A\| = \|\phi(A)\sigma\phi(A)\| = \|\phi(A)\|\,, \quad \forall\, A \in \mathscr{A}^{++}. \tag{4}$$

The question we now tackle is whether a surjective map preserving the norm of a symmetric Kubo-Ando mean extends to a Jordan $*$-isomorphism. Let us recall that a *Jordan $*$-isomorphism* is a bijective linear map $J : \mathscr{A} \to \mathscr{B}$ such that $J(A^*) = J(A)^*$ and $J(AB+BA) = J(A)J(B)+J(B)J(A)$ for $A, B \in \mathscr{A}$. The problem under study has been stated explicitly in the open problem section of [9], and we reformulate it here to its most general form.

Problem 1. Do surjective maps between positive cones of unital C^*-subalgebras of $\mathscr{B}(H)$ that preserve the norm of a symmetric Kubo-Ando mean extend to Jordan $*$-isomorphisms?

The above problem has been solved for the arithmetic mean [4, Theorem 2.4] and the geometric mean [2, Theorem 1], but for the harmonic mean, it has been solved in the case of AW^*-algebras [4, Theorem 2.16]. Our aim is to provide a complete answer for to the above problem for general symmetric means in the setting of AW^*-subalgebras of $\mathscr{B}(H)$.

2. Preliminary Considerations

Let us first recall [4, Lemma 2.3] and provide the proof here for completeness' sake.

Lemma 2. *Let $\mathscr{A}, \mathscr{B}$ be unital C^*-algebras and $\phi : \mathscr{A}^{++} \to \mathscr{B}^{++}$ a surjective norm preserving order isomorphism. Then $\phi(tI) = tI$ for $t > 0$.*

Proof. Given that $\|\phi(tI)\| = \|tI\| = t$, it follows that $\phi(tI) \leq tI$. Furthermore, there exists $A \in \mathscr{A}^{++}$ such that $\phi(A) = tI$. Since $\|A\| = \|\phi(A)\| = t$, we have that $A \leq tI$. Hence, we have $tI = \phi(A) \leq \phi(tI)$. This proves the lemma. □

Moreover, by [4, Lemma 2.8], $\psi_\epsilon(A) = \phi(A + \epsilon I) - \epsilon I$ is a surjective norm preserving order isomorphism between the positive semi-definite cones of $\mathscr{A}$ and $\mathscr{B}$ for some $\epsilon > 0$ such that $\psi_\epsilon(tI) = tI$ for all $t > 0$. Let us provide the proof here for completeness' sake.

Lemma 3. *Let $\mathscr{A}, \mathscr{B}$ be unital C^*-algebras and $\phi : \mathscr{A}^{++} \to \mathscr{B}^{++}$ a surjective norm preserving order isomorphism. Define $\psi_\epsilon : \mathscr{A}^{+} \to \mathscr{B}^{+}$ by $\psi_\epsilon(A) = \phi(A + \epsilon I) - \epsilon I$ where $\epsilon > 0$, then ψ_ϵ is a surjective norm preserving order isomorphism such that $\psi_\epsilon(tI) = tI$ for all $t > 0$.*

Proof. Let $A, B \in \mathscr{A}^{+}$, and $\epsilon > 0$. Then

$$\begin{aligned} A \leq B &\iff A + \epsilon I \leq B + \epsilon I \\ &\iff \phi(A + \epsilon I) \leq \phi(B + \epsilon I) \\ &\iff \phi(A + \epsilon I) - \epsilon I \leq \phi(B + \epsilon I) - \epsilon I \\ &\iff \psi_\epsilon(A) \leq \psi_\epsilon(B) \end{aligned}$$

which implies that ψ_ϵ is an order isomorphism. We next show that ψ_ϵ is surjective. Let $B \in \mathscr{B}^{+}$. Then there is some $A \in \mathscr{A}^{++}$ such that $\phi(A) = B + \epsilon I$. Since $\phi(A) \geq \epsilon I = \phi(\epsilon I)$, we have $A \geq \epsilon I$, which allows us to conclude that $\psi_\epsilon(A - \epsilon I) = \phi(A) - \epsilon I = B$. Therefore, ψ_ϵ is surjective. Since $\|\psi_\epsilon(A)\| = \|\phi(A + \epsilon I) - \epsilon I\| = \|\phi(A + \epsilon I)\| - \epsilon = \|A + \epsilon I\| - \epsilon = \|A\|$, we have that ψ_ϵ is norm preserving. Finally, $\psi_\epsilon(tI) = \phi((t + \epsilon)I) - \epsilon I = (t + \epsilon)I - \epsilon I = tI$.

□

Before recalling [10, Lemma 3.1], let us provide some definitions. For a unital C^*-algebra $\mathscr{A}$ we define $E_t(\mathscr{A}) := \{0 \leq A \leq tI\}$, where t is a positive real number. We recall that $E_1(\mathscr{A})$ is called the effect algebra associated with $\mathscr{A}$. Let

$$\Delta_t(\mathscr{A}) := \{(a, b) \in E_t(\mathscr{A}) \times E_t(\mathscr{A}) : a \wedge b = 0\}$$

where $a \wedge b$ denotes the infimum of a, b in $\mathscr{A}^{+}$. Endow $\Delta_t(\mathscr{A})$ with the order relation $(a_1, a_2) \leq (b_1, b_2) \iff a_1 \leq b_1$ and $b_2 \leq a_2$, where $(a_1, a_2), (b_1, b_2) \in \Delta_t(\mathscr{A})$. Finally, denote the set of projections of $\mathscr{A}$ by $P(\mathscr{A})$. We recall that for any non-empty family of projections in $\mathscr{A}$, if the infimum in $P(\mathscr{A})$ exists, then this will be also the infimum of the family, taken in $\mathscr{A}^{+}$.

Lemma 4. *Let $\mathscr{A}$ be a unital C^*-algebra. Then the following conditions are equivalent:*

(1) *P is a projection.*
(2) *The pair (P, Q) is maximal in $\Delta_1(\mathscr{A})$ for some $Q \in E_1(\mathscr{A})$.*

Proof. (1) ⇒ (2). Suppose that P is a projection. Consider the pair $(P, I - P)$. Let $A \in \mathscr{A}^{+}$ such that $A \leq P$ and $A \leq I - P$. Then $(I - P)A(I - P) \leq (I - P)P(I - P) = 0$. Similarly, $PAP = 0$ which implies $A = (I - P)AP + PA(I - P)$, but since $(I - P)A = A$, and $PA = A$, we have $A = 2A$, so $A = 0$.

Suppose $(P, I - P) \leq (A, B)$ where $(A, B) \in \Delta_1(\mathscr{A})$. Since $P \leq A \leq I$, we have $AP = P$. Thus, $(I - P)AP = 0$ and $PA(I - P) = 0$. Since $A = P + (I - P)A(I - P)$, we have $(I - P)A(I - P) \leq A$ and $(I - P)A(I - P) \leq (I - P) \leq B$ which implies that $(I - P)A(I - P) = 0$. Therefore, $A = P$, and, similarly, it can be concluded that $B = I - P$.

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(2) $\Rightarrow$ (1). Suppose (P,Q) is maximal in $\Delta_1(\mathscr{A})$. The function $f:[0,1]\to[0,1]$ defined by $f(0):=0$ and $f(t)=2(1+t^{-1})^{-1}$ is strictly increasing and surjective. In particular, f is continuous. The function $A\to f(A)$ is an order isomorphism of $E_1(\mathscr{A})$ onto itself. Therefore, $(f(P),f(Q))\in\Delta_1(\mathscr{A})$. Since $t\le f(t)$ for every $t\in[0,1]$, we have $(P,Q)\le(f(P),f(Q))$. This implies that $(P,Q)=(f(P),f(Q))$. By the Spectral Mapping Theorem, $\mathrm{sp}(P)\subseteq\{0,1\}$, so P is a projection. □

Remark 1. *Let $\mathscr{A},\mathscr{B}$ be unital C^*-algebras and let $\psi:\mathscr{A}^+\to\mathscr{B}^+$ be a surjective order isomorphism such that $\psi(tI)=tI$. Then (A,B) is maximal in $\Delta_t(\mathscr{A})$ iff $(\psi(A),\psi(B))$ is maximal in $\Delta_t(\mathscr{B})$. In particular, $P\in P(\mathscr{A})$ iff $\psi(P)\in P(\mathscr{B})$ by Lemma 4.*

Lemma 5. *Let $\mathscr{A},\mathscr{B}$ be unital C^*-algebras. If $\psi:\mathscr{A}^+\to\mathscr{B}^+$ is a surjective order isomorphism such that $\psi(tI)=tI$ for all $t>0$, then for any projection P in $\mathscr{A}$, $\psi(tP)=t\psi(P)$ for all $t>0$.*

Proof. We claim that (P_1,P_2) is maximal in $\Delta_1(\mathscr{A})$ if and only if (tP_1,tP_2) is maximal in $\Delta_t(\mathscr{A})$. Let (P_1,P_2) be maximal in $\Delta_1(\mathscr{A})$. That $P_1\wedge P_2=0 \iff tP_1\wedge tP_2=0$ is clear. Suppose $(A,B)\in\Delta_t(\mathscr{A})$ such that $(tP_1,tP_2)\le(A,B)$, then $(P_1,P_2)\le(t^{-1}A,t^{-1}B)$ and $(t^{-1}A,t^{-1}B)\in\Delta_1(\mathscr{A})$ which by maximality of (P_1,P_2) implies that $(P_1,P_2)=(t^{-1}A,t^{-1}B)$, so $(tP_1,tP_2)=(A,B)$. Similar arguments are used to prove the reverse implication.

Let $t>0$. Since $(P,(I-P))$ is maximal in $\Delta_1(\mathscr{A})$, we have that $(tP,t(I-P))$ is maximal in $\Delta_t(\mathscr{A})$ which implies that $(\psi(tP),\psi(t(I-P)))$ is maximal in $\Delta_t(\mathscr{B})$ by Remark 1. Subsequently, $(t^{-1}\psi(tP),t^{-1}\psi(t(I-P)))$ is maximal in $\Delta_1(\mathscr{B})$, so $(t^{-1}\psi(tP),t^{-1}\psi(t(I-P)))\in P(\mathscr{B})\times P(\mathscr{B})$.

If $t>1$, then $(\psi(P),\psi(I-P))\le(\psi(tP),\psi(t(I-P)))$. Since $\mathrm{rng}(\psi(P))\subseteq\mathrm{rng}(\psi(tP))=\mathrm{rng}(t^{-1}\psi(tP))$ and $\mathrm{rng}(\psi(I-P))\subseteq\mathrm{rng}(t^{-1}\psi(t(I-P)))$, we can conclude that $(\psi(P),\psi(I-P))\le(t^{-1}\psi(tP),t^{-1}\psi(t(I-P)))$, so by maximality $\psi(P)=t^{-1}\psi(tP)$.

If $t<1$, then $(\psi(tP),\psi(t(I-P)))\le(\psi(P),\psi(I-P))$. Since $\mathrm{rng}(t^{-1}\psi(tP))=\mathrm{rng}(\psi(tP))\subseteq\mathrm{rng}(\psi(P))$ and $\mathrm{rng}(t^{-1}\psi(t(I-P)))\subseteq\mathrm{rng}(\psi(I-P))$, we have $(t^{-1}\psi(tP),t^{-1}\psi(t(I-P))\le(\psi(P),\psi(I-P))$, so by maximality $\psi(P)=t^{-1}\psi(tP)$. □

Let $\mathscr{A}$ be a unital C^*-algebra, and let $0<\epsilon_1<\epsilon_2$. Consider maps $\psi_{\epsilon_1},\psi_{\epsilon_2}$ of the type stated in Lemma 3. Let ϕ also denote the map as specified in Lemma 3. Then

$$\phi(tP+\epsilon_1 I)\le\phi(tP+\epsilon_2 I),\quad \forall\, P\in P(\mathscr{A}),\, t>0.$$

By Lemma 5,

$$t\psi_{\epsilon_1}(P)+\epsilon_1 I\le t\psi_{\epsilon_2}(P)+\epsilon_2 I,\quad \forall\, P\in P(\mathscr{A}),\, t>0.$$

As $t\to\infty$,

$$\psi_{\epsilon_1}(P)\le\psi_{\epsilon_2}(P),\quad \forall\, P\in P(\mathscr{A}). \tag{5}$$

Moreover, we claim that $\psi_{\epsilon_1}(P)=\psi_{\epsilon_2}(P)$. Let $Q=\psi_{\epsilon_2}(P)-\psi_{\epsilon_1}(P)$. Then Q is a projection which is orthogonal to $\psi_{\epsilon_1}(P)$. Furthermore, $\psi_{\epsilon_2}^{-1}(Q)=R\le P$, which implies that $\psi_{\epsilon_1}(R)\le\psi_{\epsilon_1}(P)$. By Remark 1, R is a projection, so (5) implies $\psi_{\epsilon_1}(R)\le\psi_{\epsilon_2}(R)=Q$ which implies that $\psi_{\epsilon_1}(R)=0$. Since $\psi_\epsilon(0)=0$ for any $\epsilon>0$, we have $Q=0$, and the claim is proved.

Let $P\in P(\mathscr{A})$. Then

$$\left\|A\sigma\big(P+\epsilon I\big)\right\|=\left\|\phi(A)\sigma\big(\psi_\epsilon(P)+\epsilon I\big)\right\|,\quad \forall\,\epsilon>0.$$

Since $\psi_\epsilon(P)=S$ for all $\epsilon>0$, we can use [3, Remark 1 (i)] to conclude that as $\epsilon\to0$, we have

$$\|A\sigma P\|=\|\phi(A)\sigma S\|.$$

This implies that the above can be written as

$$\|A\sigma P\|=\|\phi(A)\sigma\psi_\epsilon(P)\|,\quad \forall\,\epsilon>0. \tag{6}$$

Furthermore, if $A=T+\delta I$ for some $T\in P(\mathscr{A})$ and $\delta>0$, then

$$\left\|\big(T+\delta I\big)\sigma P\right\|=\left\|\big(\psi_\epsilon(T)+\delta I\big)\sigma\psi_\epsilon(P)\right\|,\quad \forall\,\epsilon>0. \tag{7}$$

3. Results

Let $\mathscr{A},\mathscr{B}$ be AW^*-subalgebras of $\mathscr{B}(H)$, and $\phi:\mathscr{A}^{++}\to\mathscr{B}^{++}$ be a surjective map which preserves the norm of a symmetric Kubo-Ando mean σ. We shall show that there exists a Jordan $*$-isomorphism $J:\mathscr{A}\to\mathscr{B}$ which extends ϕ, i.e. $\phi(A)=J(A)$ holds for all $A\in\mathscr{A}^{++}$. If f is the representing function of σ, then the proof shall be split into the cases when $f(0+)=0$ and $f(0+)>0$. The proof for when $f(0+)=0$ requires the characterization of surjective positive homogenous order isomorphisms. By ϕ being positive homogenous, we mean that $\phi(tA)=t\phi(A)$ for $A\in\mathscr{A}^{++}$ and $t>0$.

Theorem 6. [8, Theorem 13] *Let $\mathscr{A}$, $\mathscr{B}$ be unital C^*-algebras. The map $\phi:\mathscr{A}^{++}\to\mathscr{B}^{++}$ is a surjective positive homogenous order isomorphism if and only if it is of the form*

$$\phi(A)=CJ(A)C,\quad \forall\, A\in\mathscr{A}^{++} \tag{8}$$

where $C\in\mathscr{B}^{++}$ and $J:\mathscr{A}\to\mathscr{B}$ is a Jordan $*$-isomorphism.

It is clear in this theorem that if $\phi(I)=I$ then $C=I$. We further recall the following characterisation of Kubo-Ando connections with representing function f satisfying $f(0+)=0$.

Proposition 7. [9, Lemma 2] *Let $f:(0,\infty)\to(0,\infty)$ be a non-trivial (i.e. not affine) operator monotone function satisfying $f(0+)=0$ and let σ denote the Kubo-Ando connection associated to f. For $A\in\mathscr{B}(H)^{++}$ and non-zero projection $P\in\mathscr{B}(H)$*

$$\|A\sigma P\|=f^\circ\left(\frac{1}{\max\{\lambda\geq 0:\lambda P\leq PA^{-1}P\}}\right).$$

Since the geometric mean is a symmetric mean with representation function f such that $f(t)=t^{1/2}$, we have

$$\|A\#P\|^2=\frac{1}{\max\{\lambda\geq 0:\lambda P\leq PA^{-1}P\}}. \tag{9}$$

Therefore,

$$\|A\sigma P\|=f^\circ(\|A\#P\|^2). \tag{10}$$

We shall also require the following result.

Proposition 8. [2, Lemma 11] *Let $\mathscr{A}$ be an AW^*-algebra and σ a symmetric Kubo-Ando connection with corresponding representation function $f$such that $f(0+)=0$. For $A,B\in\mathscr{A}^{++}$, we have*

$$A\leq B\iff \|A\sigma P\|\leq\|B\sigma P\|,\quad \forall\, P\in P(AW^*(I,A^{-1}-B^{-1})).$$

Proof. Let $T=A^{-1}-B^{-1}$ and consider $AW^*(I,T)$. Let us first recall that any commutative AW^*-algebra is algebra $*$-isomorphic to some $C(X)$ where X is compact, Hausdorff, and extremally disconnected. By an extremally disconnected set we mean a set such that the closure of every open set is open. [1, Theorem 1 Section 7].

Therefore, let f_T be the corresponding function of T in $C(X)$. If T is not positive then the spectrum $\sigma(T)$ contains some negative number, so there is some ϵ such that $\sigma(T)\cap]-\infty,-\epsilon[\neq\emptyset$. Consider the projection P_ϵ associated with $\overline{f_T^{-1}(-\infty,-\epsilon)}$. Since $f_T(\overline{f_T^{-1}(-\infty,-\epsilon)})\subseteq\overline{f_T(f_T^{-1}(-\infty,-\epsilon))}\subseteq(-\infty,-\epsilon]$, we have $P_\epsilon TP_\epsilon\leq-\epsilon P_\epsilon$, so $P_\epsilon A^{-1}P_\epsilon+\epsilon P_\epsilon\leq P_\epsilon B^{-1}P_\epsilon$. Therefore, for $\lambda>0$

$$\lambda P_\epsilon\leq P_\epsilon A^{-1}P_\epsilon\implies(\lambda+\epsilon)P_\epsilon\leq P_\epsilon B^{-1}P_\epsilon.$$

Furthermore, since $\|A\#P\|\neq 0$ for any $P\in P(\mathscr{A})$, by (9), (10), and the injectivity of f, it can be concluded that $\|B\sigma P_\epsilon\|<\|A\sigma P_\epsilon\|$ which is a contradiction. □

For $f(0+)>0$, we shall require the characterisation of order isomorphisms which are norm preserving and orthogonality preserving in both directions. A map $\psi:\mathscr{A}^+\to\mathscr{B}^+$ is said to be orthogonality preserving in both directions when $AB=0\iff\psi(A)\psi(B)=0$ for $A,B\in\mathscr{A}^+$.

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Lemma 9. [4, Lemma 2.3] *Let $\mathscr{A},\mathscr{B}$ be C^*-algebras such that at least one is unital, and let $\psi:\mathscr{A}^+\to\mathscr{B}^+$ be a surjective order isomorphism such that ψ is norm preserving and orthogonality preserving in both directions. Then ψ extends to a Jordan $*$-isomorphism $J:\mathscr{A}\to\mathscr{B}$.*

Before proceeding with the proof, let us recall [3, Lemma 1].

Lemma 10. *Let $f:(0,\infty)\to(0,\infty)$ be an operator monotone function and let m denote the positive and finite Borel measure associated to f via (1). For every Borel subset Δ of $[0,\infty]$ satisfying $m(\Delta)>0$, the function f_Δ defined on $(0,\infty)$ by*

$$f_\Delta : x\mapsto \int_\Delta \frac{x(1+t)}{x+t}\,\mathrm{d}m(t)$$

is operator monotone. In particular, if $m((0,\infty))\neq 0$, the function h defined by

$$h(x):=\int_{(0,\infty)}\frac{x(1+t)}{x+t}\,\mathrm{d}m(t)=f(x)-f(0+)-f^\circ(0+)x \quad (x>0)$$

is operator monotone. If f is symmetric, then so is h.

We are now in a position to prove the main result of this paper.

Theorem 11. *Let $\mathscr{A},\mathscr{B}$ be AW^*-subalgebras of $\mathscr{B}(H)$. A surjective map $\phi:\mathscr{A}^{++}\to\mathscr{B}^{++}$ preserves the norm of a symmetric Kubo-Ando mean σ if and only if there is a Jordan $*$-isomorphism $J:\mathscr{A}\to\mathscr{B}$ which extends ϕ, i.e. $\phi(A)=J(A)$ holds for all $A\in\mathscr{A}^{++}$.*

Proof. Sufficiency is trivial. The proof shall be split into two cases depending on the behaviour of the representation function f at 0.

Case 1: $f(0+)=0$. Let $A\in\mathscr{A}^{++}$, $P\in P(\mathscr{A})$, and $\epsilon>0$. Then by (6)

$$\|A\sigma P\|=\|\phi(A)\sigma\psi_\epsilon(P)\|\,.$$

By (10) and the fact that f is injective,

$$\|A\#P\|=\|\phi(A)\#\psi_\epsilon(P)\|\,. \tag{11}$$

Let $t>0$,

$$\|tA\#P\|=t^{1/2}\,\|A\#P\|=t^{1/2}\,\|\phi(A)\#\psi_\epsilon(P)\|=\|t\phi(A)\#\psi_\epsilon(P)\|\,. \tag{12}$$

By (11) and (12),

$$\|\phi(tA)\#\psi_\epsilon(P)\|=\|t\phi(A)\#\psi_\epsilon(P)\|\,.$$

By Proposition 8, we can then conclude that $t\phi(A)=\phi(tA)$ which proves that ϕ is positive homogenous. Therefore, by Theorem 6, we conclude that ϕ extends to a Jordan $*$-isomorphism.

Case 2: $f(0+)>0$. Let $P,Q\in P(\mathscr{A})$ be two orthogonal projections, and consider $(Q+\delta I)\sigma P$ where $\delta>0$. By (7), for maps of the type in Lemma 3 the following equation is obtained

$$\big\|\big(Q+\delta I\big)\sigma P\big\|=\big\|\big(\psi_\epsilon(Q)+\delta I\big)\sigma\psi_\epsilon(P)\big\|\,,\quad \forall\,\epsilon>0. \tag{13}$$

We shall show that ψ_ϵ is orthogonality preserving in both directions. If $m((0,\infty))\neq 0$, by Lemma 10 we can denote by σ_h the symmetric Kubo-Ando connection corresponding to $h(x)=f(x)-\alpha-\alpha x$ where $\alpha=f(0+)$. Let τ denote the Kubo-Ando connection on $\mathscr{B}(H)^+\times\mathscr{B}(H)^+$ defined by the equation

$$A\tau B=\int_{(0,\infty)}\frac{1+t}{t}(tA:B)\,\mathrm{d}m(t).$$

Since

$$h(xI)=\int_{(0,\infty)}\frac{x(1+t)}{(x+t)}I\,\mathrm{d}m(t)=\int_{(0,\infty)}\frac{1+t}{t}(tI:xI)\,\mathrm{d}m(t)=I\tau(xI),$$

we have $\sigma_h=\tau$ by [7, Theorem 3.2]. Therefore, by (2) we can decompose $(Q+\delta I)\sigma P$ in the following way

$$(Q+\delta I)\sigma P=\alpha(Q+\delta I+P)+(Q+\delta I)\sigma_h P.$$

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Since $P(Q+\delta I)^{-1}P = P\big((1+\delta)^{-1}Q+\delta^{-1}(I-Q)\big)P = \delta^{-1}P$, by Proposition 7 it can be concluded that

$$\left\|\big(Q+\delta I\big)\sigma_h P\right\| = h(\delta) \quad \text{since} \quad \max\{\lambda \geq 0 : \lambda P \leq P(Q+\delta I)^{-1}P\} = \delta^{-1}.$$

Therefore,

$$\begin{aligned} \|(Q+\delta I)\sigma P\| &= \|\alpha(Q+\delta I+P)+(Q+\delta I)\sigma_h P\| \\ &\leq \|\alpha(Q+\delta I+P)+\|(Q+\delta I)\sigma_h P\|\, I\| \\ &\leq \|\alpha(Q+\delta I+P)+h(\delta)I\|\,. \end{aligned} \tag{14}$$

Furthermore,

$$\begin{aligned} \left\|\big(\psi_\epsilon(Q)+\delta I\big)\sigma\psi_\epsilon(P)\right\| &= \left\|\alpha\big(\psi_\epsilon(Q)+\delta I+\psi_\epsilon(P)\big)+(\psi_\epsilon(Q)+\delta I)\sigma_h\psi_\epsilon(P)\right\| \\ &\geq \|\alpha(\psi_\epsilon(Q)+\psi_\epsilon(P))\| \end{aligned} \tag{15}$$

where the above follows because $(\psi_\epsilon(Q)+\delta I)\sigma_h\psi_\epsilon(P)$ is a positive operator. Therefore, by using the inequalities (14) and (15) in equation (13),

$$\|\alpha(Q+\delta I+P)+h(\delta)I\| \geq \|\alpha(\psi_\epsilon(Q)+\psi_\epsilon(P))\|\,.$$

By letting $\delta \to 0$ and using the fact that $h(0+)=0$, it can be concluded that

$$1 \geq \|Q+P\| \geq \|\psi_\epsilon(Q)+\psi_\epsilon(P)\|\,. \tag{16}$$

Since by Remark 1 $\psi_\epsilon(Q)$ and $\psi_\epsilon(P)$ are projections, we have that they are orthogonal to each other. Moreover, if $m((0,\infty))=0$, then σ is the arithmetic mean and it is clear that (16) holds.

Let $A, B \in \mathscr{A}^+$ be such that $AB=0$ and $\max\{\|A\|, \|B\|\}=t$. Furthermore, denote by P_A and P_B the range projections corresponding to A and B. The range projections are elements of $P(\mathscr{A})$ by [6, Theorem 7]. Since $\mathrm{img}(B) \subseteq \ker(A)$, we can conclude that $AP_B=0$; similarly, $P_AP_B=0$. Since these are orthogonal, we have $\psi_\epsilon(P_A)\psi_\epsilon(P_B)=0$. Furthermore, $A \leq tP_A$, so $\psi_\epsilon(A) \leq t\psi_\epsilon(P_A)$ by Lemma 5. Thus, $\mathrm{rng}(\psi_\epsilon(A)) \subseteq \mathrm{rng}(\psi_\epsilon(P_A)) \subseteq \ker(\psi_\epsilon(P_B)) \subseteq \ker(\psi_\epsilon(B))$, so $\psi_\epsilon(A)\psi_\epsilon(B)=0$. By Remark 1, similar arguments can be used to show that if $\psi_\epsilon(A)\psi_\epsilon(B)=0$, then $AB=0$. Thus, by Lemma 9, ψ_ϵ extends to a Jordan $*$-isomorphism.

Let $A \in \mathscr{A}^+$, and $\epsilon_2 > \epsilon_1 > 0$. Then

$$\begin{aligned} \psi_{\epsilon_2}(A) &= \phi\big((A+(\epsilon_2-\epsilon_1)I)+\epsilon_1 I\big)-\epsilon_2 I \\ &= \psi_{\epsilon_1}\big((A+(\epsilon_2-\epsilon_1)I)\big)+\epsilon_1 I-\epsilon_2 I \\ &= \psi_{\epsilon_1}(A) \end{aligned}$$

where we used the fact that ψ_{ϵ_1} is linear and unital. Therefore, the family of maps $\{\psi_\epsilon\}_{\epsilon>0}$ is just one map ψ which extends to a Jordan $*$-isomorphism. Let $A \in \mathscr{A}^{++}$ be such that $A \geq \epsilon I$. Then

$$\psi(A) = \psi(A-\epsilon I)+\psi(\epsilon I) = \phi(A),$$

so ϕ extends to a Jordan $*$-isomorphism. □

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Emmanuel Chetcuti, Department of Mathematics, Faculty of Science, University of Malta, Msida MSD 2080 Malta
Email address: emanuel.chetcuti@um.edu.mt

Curt Healey, Department of Mathematics, Faculty of Science, University of Malta, Msida MSD 2080 Malta
Email address: curt.c.healey.13@um.edu.mt

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