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Erik Walsberg, ***Metric Geometry in a Tame Setting***, University of California, Los Angeles, 2015. Supervised by Matthias Aschenbrenner. MSC: Primary 03C64. Keywords: o-minimal structures, metric spaces, tame geometry.

Abstract

The thesis is about the topology and geometry of metric spaces definable in an o-minimal expansion $\mathcal{R}$ of an ordered field $(R, <, +, \cdot)$. A definable metric space is a pair (X, d) consisting of a definable set $X \subseteq R^k$ and a definable $(R, +, <)$-valued metric. If $X \subseteq R^k$ is definable and e is the restriction of the usual euclidean metric on R^k to X then (X, e) is a definable metric space, in this way the geometry of definable sets may be considered as a special case of the geometry of definable metric spaces. Examples of definable metric spaces whose geometry is unlike that of any definable set are given by the hyperbolic plane ($\mathbb{R}_{\exp}$-definable) and certain subriemannian spaces ($\mathbb{R}_{\text{an}}$-definable). The main theorem of the thesis is the following: *Let (X, d) be a definable metric space. Then one of the following holds*:

1. *There is an infinite definable $A \subseteq X$ such that (A, d) is discrete.*
2. *There is a definable set $Z \subseteq R^l$, for some l, such that (X, d) is definably homeomorphic to Z equipped with its induced euclidean topology.*

If $(R, <, +, \cdot)$ is the ordered field of real numbers, then a definable set A is infinite if and only if it is uncountable. As a separable metric space cannot contain an uncountable discrete subset the theorem above shows that a separable metric space definable in an o-minimal expansion of the real field is definably homeomorphic to a definable set equipped with its induced euclidean topology. This reduces the topology of separable definable metric spaces in o-minimal expansions of the real field to the topology of definable sets. Perhaps surprisingly, there are interesting examples of nonseparable metric spaces definable in $(\mathbb{R}, <, +, \cdot)$, geometric realizations of Cayley graphs of "definable group actions".

Later in the thesis, the theory of imaginaries in real closed valued fields is used to prove the following: *If $\mathcal{X}$ is an $(\mathbb{R}, <, +, \cdot)$-definable family of compact metric spaces then the collection of Gromov–Hausdorff limits of sequences of elements of $\mathcal{X}$ forms an $(\mathbb{R}, <, +, \cdot)$-definable family of metric spaces.* This theorem is an analogue of a result proven by van den Dries on Hausdorff limits of definable families of sets. Its proof gives a connection between the model theory of valued fields and the geometry of definable metric spaces.

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Anton Freund, ***Type-two well-ordering principles, admissible sets, and Π^1_1-comprehension***, University of Leeds, UK, 2018. Supervised by Michael Rathjen. MSC: 03B30, 03D60, 03F05. Keywords: well-ordering principles, admissible sets, Π^1_1-comprehension, dilators, beta-proofs, Bachmann-Howard ordinal, primitive recursive set theory, slow consistency, proof length, Paris-Harrington principle.

Abstract

This thesis introduces a well-ordering principle of type two, which we call the Bachmann-Howard principle. The main result states that the Bachmann-Howard principle is equivalent to the existence of admissible sets and thus to Π^1_1-comprehension. This solves a conjecture of Rathjen and Montalbán. The equivalence is interesting because it relates "concrete" notions from ordinal analysis to "abstract" notions from reverse mathematics and set theory.

A type-one well-ordering principle is a map T which transforms each well-order X into another well-order $T[X]$. If T is particularly uniform then it is called a dilator (due to Girard). Our Bachmann-Howard principle transforms each dilator T into a well-order $\mathrm{BH}(T)$.