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Ekedahl–Oort Strata for Good Reductions of Shimura Varieties of Hodge Type

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Abstract. For a Shimura variety of Hodge type with hyperspecial level structure at a prime p, Vasiu and Kisin constructed a smooth integral model (namely the integral canonical model) uniquely determined by a certain extension property. We define and study the Ekedahl–Oort stratifications on the special fibers of those integral canonical models when $p > 2$. This generalizes Ekedahl–Oort stratifications defined and studied by Oort on moduli spaces of principally polarized abelian varieties and those defined and studied by Moonen, Wedhorn, and Viehmann on good reductions of Shimura varieties of PEL type. We show that the Ekedahl–Oort strata are parameterized by certain elements w in the Weyl group of the reductive group in the Shimura datum. We prove that the stratum corresponding to w is smooth of dimension $l(w)$ (*i.e.*, the length of w) if it is non-empty. We also determine the closure of each stratum.

1 Introduction

Ekedahl–Oort strata were first defined and studied by Ekedahl and Oort for Siegel modular varieties in late 1990's in [19]. Let g, n be integers such that $g > 0$ and $n > 2$, and let $\mathscr{A}_{g,n}$ be the moduli scheme of principally polarized abelian schemes over $\mathbb{F}_p$-schemes with a symplectic level n structure. Then $\mathscr{A}_{g,n}$ is smooth over $\mathbb{F}_p$. Let $\mathcal{A}$ be the universal abelian scheme over $\mathscr{A}_{g,n}$. For a field k of characteristic $p > 0$, a k-point s of $\mathscr{A}_{g,n}$ gives a principally polarized abelian variety $(\mathcal{A}_s, \psi)$ over k. The polarization $\psi: \mathcal{A}_s \to \mathcal{A}_s^\vee$ induces an isomorphism $\mathcal{A}_s[p] \simeq \mathcal{A}_s^\vee[p]$ that will still be denoted by ψ.

Let C be the set of isomorphism classes of self-dual BT-1s of height $2g$ over $\overline{\mathbb{F}_p}$. For a class $c \in C$, we fix a self dual BT-1 (H_c, ψ_c) in this class. Let $\mathscr{A}_{g,n}^c$ be the set of points s in $\mathscr{A}_{g,n} \otimes \overline{\mathbb{F}_p}$ such that there exists an algebraically closed field $\overline{k}$ and embeddings of $k(s)$ and $\overline{\mathbb{F}_p}$, such that the pairs $(\mathcal{A}_s[p], \psi) \otimes \overline{k}$ and $(H_c, \psi_c) \otimes \overline{k}$ are isomorphic. The subset $\mathscr{A}_{g,n}^c$ is called an Ekedahl–Oort stratum.

Oort proved in [19] that C is of cardinality 2^g, and each $\mathscr{A}_{g,n}^c$ is non-empty and locally closed in $\mathscr{A}_{g,n} \otimes \overline{\mathbb{F}_p}$. Moreover, he proved that each stratum is quasi-affine, and gave a dimension formula. Ekedahl and van der Geer then computed the cycle classes of Ekedahl–Oort strata in [3].

By studying Ekedahl–Oort stratification, Oort re-proved a theorem by Faltings and Chai that $\mathscr{A}_{g,n}$ is geometrically irreducible. However, his proof does not make use of characteristic zero arguments and the irreducibility of the moduli space of characteristic zero is actually a corollary of this theorem.

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The theory of Ekedahl–Oort strata has been generalized by works of Moonen [15, 16], Wedhorn [26], Moonen–Wedhorn [17] and Viehmann–Wedhorn [24] to Shimura varieties of PEL type, and works of Vasiu [23] to Shimura varieties of Hodge type. We remark that these papers use different methods: [15, 16] use canonical filtrations on Dieudonné modules attached to BT-1s, [26] uses moduli of BT-ns, [17] and [24] use F-zips, while [23] uses truncated F-crystals.

Pink, Wedhorn, and Ziegler developed systematically technical tools to study Ekedahl–Oort strata for Shimura varieties in [20, 21]. Results in [20] were already used in [24] to obtain very explicit results on Ekedahl–Oort strata for PEL Shimura varieties (*e.g.*, see [24, Theorem 7.1] for a combinatorial description for closure of a stratum).

In this paper, we establish and study Ekedahl–Oort strata for Shimura varieties of Hodge type using [21]. The advantage of this method is that we could work schematically and get explicit statements. Now we explain the main results of this paper.

Let (G, X) be a Shimura datum of Hodge type with good reduction at $p > 2$. We will always assume here and in the main body of the paper that $p > 2$ unless otherwise mentioned. Let $\overline{G}$ be the reduction of G. Let $\mathrm{Sh}_K(G, X)$ be the Shimura variety with K small enough and hyperspecial at p. Let $\mathscr{S}$ be the integral canonical model constructed by Vasiu and Kisin, and let $\mathscr{S}_0$ be its special fiber. The main results of this paper are as follows.

(a) Fixing a symplectic embedding, we construct a $\overline{G}$-zip of type μ over $\mathscr{S}_0$. See Definition 2.2.1 for the definition, and Theorem 3.4.1 for this result. This $\overline{G}$-zip induces a morphism $\zeta\colon \mathscr{S}_0 \to \overline{G}\text{-}\mathsf{Zip}^{\mu}_{\kappa}$, where $\overline{G}\text{-}\mathsf{Zip}^{\mu}_{\kappa}$ is the stack of $\overline{G}$-zips of type μ (see [21] or our §2.2).

(b) (Theorem 4.1.2) The morphism ζ is smooth.

(c) Inverse images of $\overline{\mathbb{F}_p}$-points of $\overline{G}\text{-}\mathsf{Zip}^{\mu}_{\kappa}$ are Ekedahl–Oort strata, so they are locally closed in $\mathscr{S}_0 \otimes \overline{\mathbb{F}_p}$. Moreover, all the possible strata are given by a certain subset of the Weyl group of $\overline{G}$.

(d) (Proposition 4.1.4) There is a dimension formula for each stratum assuming that it is non-empty. There is also a description of Zariski closure of a stratum. There is a unique stratum that is open dense in $\mathscr{S}_0 \otimes \mathbb{F}_p$. This stratum is called the ordinary stratum. There is at most one zero dimensional stratum in $\mathscr{S}_0 \otimes \mathbb{F}_p$, which is called the superspecial stratum.

We remark that our results are compatible with main results in Vasiu's [23]. For example, his Basic Theorem D(d) in 12.2 asserts that the number of strata is at most $[W_G : W_P]$, which is the same as our Proposition 4.1.4. We also remark that Vasiu's method works when $p = 2$, but our method, based on Kisin's [7], has restrictions when $p = 2$. In fact, Kisin assumes [7, 2.3.4] in his construction of integral models and integral automorphic sections, and as a result, we have to impose that condition to follow his constructions.

There are recent preprints closely related to this paper. D. Wortmann proves in [29] that the μ-ordinary locus coincides with the ordinary Ekedahl–Oort stratum, and hence open dense. This is a generalization of the fact that the ordinary Newton stratum coincides with the ordinary Ekedahl–Oort stratum on Siegel modular varieties. The author proves in [30] that Ekedahl–Oort stratifications are independent of choices of symplectic embeddings. There are also works of Koskivirta–Wedhorn [9] and Goldring–Koskivirta [4] on Hasse invariants on Shimura varieties of Hodge type.

2 F-zips and G-zips

2.1 F-zips

In this section, we will follow [17, 21] to introduce F-zips. Let S be a scheme, and let M be a locally free O_S-module of finite rank. By a descending (resp. ascending) filtration $C^\bullet$ (resp. $D_\bullet$) on M, we always mean a separating and exhaustive filtration such that $C^{i+1}(M)$ is a locally direct summand of $C^i(M)$ (resp. $D_i(M)$ is a locally direct summand of $D_{i+1}(M)$).

Let $\mathsf{LF}(S)$ be the category of locally free O_S-modules of finite rank, and let $\mathsf{FilLF}^\bullet(S)$ be the category of locally free O_S-modules of finite rank with descending filtration. For two objects $(M, C^\bullet(M))$ and $(N, C^\bullet(N))$ in $\mathsf{FilLF}^\bullet(S)$, a morphism $f\colon(M, C^\bullet(M)) \to (N, C^\bullet(N))$ is a homomorphism of O_S-modules such that $f(C^i(M)) \subseteq C^i(N)$. We also denote by $\mathsf{FilLF}_\bullet(S)$ the category of locally free O_S-modules of finite rank with ascending filtration. For two objects $(M, C^\bullet)$ and $(M', C'^\bullet)$ in $\mathsf{FilLF}^\bullet(S)$, their tensor product is defined to be $(M \otimes M', T^\bullet)$ with $T^i = \sum_j C^j \otimes C'^{i-j}$. Similarly for $\mathsf{FilLF}_\bullet(S)$. For an object $(M, C^\bullet)$ in $\mathsf{FilLF}^\bullet(S)$, one defines its dual to be

$$(M, C^\bullet)^\vee = \big({}^\vee M := M^\vee, {}^\vee C^i := (M/C^{1-i})^\vee\big);$$

and for an object $(M, D_\bullet)$ in $\mathsf{FilLF}_\bullet(S)$, one defines its dual to be

$$(M, D_\bullet)^\vee = \big({}^\vee M := M^\vee, {}^\vee D_i := (M/D_{-1-i})^\vee\big).$$

It is clear from the convention that $(M, C^\bullet)^\vee = ({}^\vee M, {}^\vee C^\bullet) = (M^\vee, {}^\vee C^\bullet)$, and similar with $D_\bullet$.

If S is over $\mathbb{F}_p$, we will denote by $\sigma\colon S \to S$ the morphism that is the identity on the topological space and p-th power on the sheaf of functions. For an S-scheme T, we will write $T^{(p)}$ for the pull back of T via σ. For a quasi-coherent O_S-module M, $M^{(p)}$ means the pull back of M via σ. For a σ-linear map $\varphi\colon M \to M$, we will denote its linearization by $\varphi^{\mathrm{lin}}\colon M^{(p)} \to M$.

Definition 2.1.1 Let S be an $\mathbb{F}_p$-scheme. By an F-zip over S, we mean a tuple $\underline{M} = (M,\ C^\bullet,\ D_\bullet,\ \varphi_\bullet)$ such that

(i) M is an object in $\mathsf{LF}(S)$, *i.e.*, M is a locally free sheaf of finite rank on S;
(ii) $(M, C^\bullet)$ is an object in $\mathsf{FilLF}^\bullet(S)$, *i.e.*, $C^\bullet$ is a descending filtration on M;
(iii) $(M, D_\bullet)$ is an object in $\mathsf{FilLF}_\bullet(S)$, *i.e.*, $D_\bullet$ is an ascending filtration on M;
(iv) $\varphi_i\colon C^i/C^{i+1} \to D_i/D_{i-1}$ is a σ-linear map whose linearization

$$\varphi_i^{\mathrm{lin}}\colon (C^i/C^{i+1})^{(p)} \longrightarrow D_i/D_{i-1}$$

is an isomorphism.

By a morphism of F-zips

$$\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet) \longrightarrow \underline{M'} = (M', C'^\bullet, D'_\bullet, \varphi'_\bullet),$$

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we mean a morphism of $\mathcal{O}_S$-modules $f: M \to N$, such that for all $i \in \mathbb{Z}$, $f(C^i) \subseteq C'^i$, $f(D_i) \subseteq D'_i$, and f induces a commutative diagram

$$\begin{tikzcd} C^i/C^{i+1} \arrow[r, "\varphi_i"] \arrow[d, "f"'] & D_i/D_{i-1} \arrow[d, "f"] \\ C'^i/C'^{i+1} \arrow[r, "\varphi'_i"] & D'_i/D'_{i-1}. \end{tikzcd}$$

Remark 2.1.2 Let S be a locally Noetherian $\mathbb{F}_p$-scheme, and X be an abelian scheme or a K3 surface over S; then $\mathrm{H}^i_{\mathrm{dR}}(X/S)$ has a natural F-zip structure. See [27, 1.6, 1.7, and 1.11] for more details and examples.

Example 2.1.3 ([21, Example 6.6]) The Tate F-zips of weight d is

$$\mathbf{1}(d) := (\mathcal{O}_S, C^\bullet, D_\bullet, \varphi_\bullet),$$

where

$$C^i = \begin{cases} \mathcal{O}_S & \text{for } i \le d, \\ 0 & \text{for } i > d, \end{cases} \qquad D_i = \begin{cases} 0 & \text{for } i < d, \\ \mathcal{O}_S & \text{for } i \ge d, \end{cases}$$

and φ_d is the Frobenius.

One can talk about tensor products and duals in the category of F-zips.

Definition 2.1.4 ([21, Definition 6.4]) Let $\underline{M}$, $\underline{N}$ be two F-zips over S; then their tensor product is the F-zip $\underline{M} \otimes \underline{N}$, consisting of the tensor product $M \otimes N$ with induced filtrations $C^\bullet$ and $D_\bullet$ on $M \otimes N$, and induced σ-linear maps

$$\begin{tikzcd} \mathrm{gr}^i_C(M \otimes N) \arrow[d, "\cong"] & & \mathrm{gr}^D_i(M \otimes N) \\ \bigoplus_j \mathrm{gr}^j_C(M) \otimes \mathrm{gr}^{i-j}_C(N) \arrow[rr, "\oplus_j \varphi_j \otimes \varphi_{i-j}"] & & \bigoplus_j \mathrm{gr}^D_j(M) \otimes \mathrm{gr}^D_{i-j}(N), \arrow[u, "\cong"'] \end{tikzcd}$$

whose linearization are isomorphisms.

Definition 2.1.5 ([21, Definition 6.5]) The dual of an F-zip $\underline{M}$ is the F-zip $\underline{M}^\vee$ consisting of the dual sheaf of $\mathcal{O}_S$-modules $M^\vee$ with the dual descending filtration of $C^\bullet$ and dual ascending filtration of $D_\bullet$, and σ-linear maps whose linearization are isomorphisms

$$\big(\mathrm{gr}^i_C(M^\vee)\big)^{(p)} = \big((\mathrm{gr}^{-i}_C M)^\vee\big)^{(p)} \xrightarrow{((\varphi^{\mathrm{lin}}_{-i}))^{-1\vee}} (\mathrm{gr}^D_{-i} M)^\vee \cong \mathrm{gr}^D_i(M^\vee).$$

For the Tate F-zips introduced in Example 2.1.3, we have natural isomorphisms $\mathbf{1}(d) \otimes \mathbf{1}(d') \cong \mathbf{1}(d + d')$ and $\mathbf{1}(d)^\vee \cong \mathbf{1}(-d)$. The d-th Tate twist of an F-zip $\underline{M}$ is defined as $\underline{M}(d) := \underline{M} \otimes \mathbf{1}(d)$, and there is a natural isomorphism $\underline{M}(0) \cong \underline{M}$.

Definition 2.1.6 A morphism between two objects in $\mathsf{LF}(S)$ is said to be *admissible* if the image of the morphism is a locally direct summand. A morphism $f\colon (M, C^\bullet) \to (M', C'^\bullet)$ in $\mathsf{FilLF}^\bullet(S)$ (resp. $f\colon (M, D_\bullet) \to (M', D'_\bullet)$ in $\mathsf{FilLF}_\bullet(S)$) is called *admissible* if for all i, $f(C^i)$ (resp. $f(D_i)$) is equal to $f(M) \cap C'^i$ (resp. $f(M) \cap D'_i$) and is a locally direct summand of M'. A morphism between two F-zips $\underline{M} \to \underline{M'}$ in F-$\mathsf{Zip}(S)$ is called *admissible* if it is admissible with respect to the two filtrations.

With admissible morphisms, tensor products, and duals defined as above, the categories $\mathsf{LF}(S)$, $\mathsf{FilLF}^\bullet(S)$, $\mathsf{FilLF}_\bullet(S)$ become O_S-linear exact rigid tensor categories (see [21, 4.A, 4.C, 4.D]). The admissible morphisms, tensor products, duals, and the Tate object $\mathbf{1}(0)$ make F-$\mathsf{Zip}(S)$ an $\mathbb{F}_p$-linear exact rigid tensor category (see [21, 6]). The natural forgetful functors

$$F\text{-}\mathsf{Zip}(S) \to \mathsf{LF}(S), \quad F\text{-}\mathsf{Zip}(S) \to \mathsf{FilLF}^\bullet(S), \quad F\text{-}\mathsf{Zip}(S) \to \mathsf{FilLF}_\bullet(S)$$

are exact functors.

Remark 2.1.7 For a morphism in $\mathsf{LF}(S)$, $\mathsf{FilLF}^\bullet(S)$, $\mathsf{FilLF}_\bullet(S)$, or F-$\mathsf{Zip}(S)$, the property of being admissible is local for the fpqc topology (see [21, Lemmas 4.2 and 6.8]).

2.2 G-zips

We will introduce G-zips following [21, Section 3]. Note that the authors of [21] work with reductive groups over a general finite field $\mathbb{F}_q$ containing $\mathbb{F}_p$, and q-Frobenius. But we do not need the most general version of G-zips, as our reductive groups are connected and defined over $\mathbb{F}_p$.

Let G be a connected reductive group over $\mathbb{F}_p$, let k be a finite extension of $\mathbb{F}_p$, and let $\chi\colon \mathbb{G}_{m,k} \to G_k$ be a cocharacter over k. Let P_+ (resp. P_-) be the parabolic subgroup of G_k such that its Lie algebra is the sum of spaces with non-negative weights (resp. non-positive weights) in $\mathrm{Lie}(G_k)$ under $\mathrm{Ad} \circ \chi$. Let U_+ (resp. U_-) be the unipotent radical of P_+ (resp. P_-), and let L be the common Levi subgroup of P_+ and P_-. Note that L is also the centralizer of χ.

Definition 2.2.1 Let S be a scheme over k. A G-zip of type χ over S is a tuple $\underline{I} = (I, I_+, I_-, \iota)$ consisting of a right G_k-torsor I over S, a right P_+-torsor $I_+ \subseteq I$ (*i.e.*, the inclusion $I_+ \subseteq I$ is such that it is compatible for the P_+-action on I_+ and the G_κ-action on I), a right $P_-^{(p)}$-torsor $I_- \subseteq I$ (similarly as for $I_+ \subseteq I$), and an isomorphism of $L^{(p)}$-torsors $\iota\colon I_+^{(p)}/U_+^{(p)} \to I_-/U_-^{(p)}$.

A morphism $(I, I_+, I_-, \iota) \to (I', I'_+, I'_-, \iota')$ of G-zips of type χ over S consists of equivariant morphisms $I \to I'$ and $I_\pm \to I'_\pm$ that are compatible with inclusions and the isomorphisms ι and ι'.

Here by a torsor over S of an fpqc group scheme G/S, we mean an fpqc scheme X/S with a G-action $\rho\colon X \times_S G \to X$ such that the morphism $X \times G \to X \times_S X$, $(x, g) \to (x, x \cdot g)$ is an isomorphism.

The category of G-zips of type χ over S will be denoted by G-$\mathsf{Zip}_k^\chi(S)$. With the evident notation of pull back, the G-$\mathsf{Zip}_k^\chi(S)$ form a fibered category over the category

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of schemes over k, denoted by $G\text{-Zip}_k^\chi$. Noting that morphisms in $G\text{-Zip}_k^\chi(S)$ are isomorphisms, $G\text{-Zip}_k^\chi$ is a category fibered in groupoids.

Theorem 2.2.2 *The fibered category $G\text{-Zip}_k^\chi$ is a smooth algebraic stack of dimension 0 over k.*

Proof This is [21, Corollary 3.12]. ∎

2.2.1 Some Technical Constructions about G-zips

We need more information about the structure of $G\text{-Zip}_k^\chi$. First, we need to introduce some standard G-zips as in [21].

Construction 2.2.3 ([21, Construction 3.4]) Let S/k be a scheme. For a section $g \in G(S)$, one associates a G-zip of type χ over S as follows. Let $I_g = S \times_k G_k$ and $I_{g,+} = S \times_k P_+ \subseteq I_g$ be the trivial torsors. Then $I_g^{(p)} \cong S \times_k G_k = I_g$ canonically, and we define $I_{g,-} \subseteq I_g$ as the image of $S \times_k P_-^{(p)} \subseteq S \times_k G_k$ under left multiplication by g. Then left multiplication by g induces an isomorphism of $L^{(p)}$-torsors

$$\iota_g\colon I_{g,+}^{(p)}/U_+^{(p)} = S\times_k P_+^{(p)}/U_+^{(p)} \cong S\times_k P_-^{(p)}/U_-^{(p)} \xrightarrow{\sim} g(S\times_k P_-^{(p)})/U_-^{(p)} = I_{g,-}/U_-^{(p)}.$$

We thus obtain a G-zip of type χ over S, denoted by $\underline{I}_g$.

Lemma 2.2.4 *Any G-zip of type χ over S is étale locally of the form $\underline{I}_g$.*

Proof This is [21, Lemma 3.5]. ∎

Now we will explain how to write $G\text{-Zip}_k^\chi$ in terms of a quotient of an algebraic variety by the action of a linear algebraic group following [21, Section 3].

Denote by $\mathrm{Frob}_p\colon L \to L^{(p)}$ the relative Frobenius of L, and by $E_{G,\chi}$ the fiber product

$$\begin{tikzcd} E_{G,\chi} \arrow[rr] \arrow[d] & & P_-^{(p)} \arrow[d] \\ P_+ \arrow[r] & L \arrow[r, "\mathrm{Frob}_p"] & L^{(p)}. \end{tikzcd}$$

Then we have

(2.1) $$E_{G,\chi}(S) = \left\{ (p_+ := lu_+,\ p_- := l^{(p)}u_-) : l \in L(S), u_+ \in U_+(S), u_- \in U_-^{(p)}(S) \right\}.$$

It acts on G_k from the left-hand side as follows. For $(p_+, p_-) \in E_{G,\chi}(S)$ and $g \in G_k(S)$, $(p_+, p_-)\cdot g := p_+ g p_-^{-1}$.

To relate $G\text{-Zip}_k^\chi$ to the quotient stack $[E_{G,\chi}\backslash G_k]$, we need the following constructions in [21]. First, for any two sections $g, g' \in G_k(S)$, there is a natural bijection between the set

$$\mathrm{Transp}_{E_{G,\chi}(S)}(g, g') := \{(p_+, p_-) \in E_{G,\chi}(S) \mid p_+ g p_-^{-1} = g'\}$$

and the set of morphisms of G-zips $\underline{I}_g \to \underline{I}_{g'}$ (see [21, Lemma 3.10]). So we define a category $\mathcal{X}$ fibered in groupoids over the category of k-schemes as follows. For any scheme S/k, let $\mathcal{X}(S)$ be the small category whose underlying set is $G(S)$, and for any two elements $g, g' \in G(S)$, the set of morphisms is the set $\mathrm{Transp}_{E_{G,\chi}(S)}(g, g')$.

Theorem 2.2.5 *There is a fully faithful morphism $\mathcal{X} \to G\text{-}\mathsf{Zip}_k^{\chi}$ given by sending $g \in \mathcal{X}(S) = G(S)$ to $\underline{I}_g$. It induces an isomorphism $[E_{G,\chi}\backslash G_k] \to G\text{-}\mathsf{Zip}_k^{\chi}$.*

Proof This is [21, Proposition 3.11]. ∎

3 Integral Canonical Models and G-zips

3.1 Construction of Integral Canonical Models

Integral canonical models are constructed by Vasiu in [22] and Kisin in [7]. We will first follow [13] to introduce Shimura varieties, and then follow [7] to introduce integral canonical models.

Definition 3.1.1 Let G be a connected reductive group over $\mathbb{Q}$. We will write $\mathbb{S}$ for the Deligne torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. Let $h: \mathbb{S} \to G_{\mathbb{R}}$ be a homomorphism of algebraic groups and let X be the $G(\mathbb{R})$-conjugacy class of h. Then the pair (G, X) is called a *Shimura datum* if the following conditions are satisfied:

(i) $\mathrm{Ad} \circ h$ induces a Hodge structure of type $(-1,1) + (0,0) + (1,-1)$ on $\mathrm{Lie}(G_{\mathbb{R}})$;
(ii) the conjugation action of $h(i)$ on $G_{\mathbb{R}}^{\mathrm{ad}}$ gives a Cartan involution.
(iii) G^{ad} has no simple factor over $\mathbb{Q}$ onto which h has trivial projection.

Let (G, X) be a Shimura datum, and K be a compact open subgroup of $G(\mathbb{A}_f)$ that is small enough. The complex manifold $\mathrm{Sh}_K(G,X)_{\mathbb{C}} = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K)$ has a unique structure of a complex quasi-projective variety by results of Baily–Borel. The Shimura datum (G, X) gives the $G(\mathbb{R})$-orbit X of the real manifold $\mathbf{Hom}_{\mathbb{R}}(\mathbb{S}, G_{\mathbb{R}})(\mathbb{R})$. For $x \in X$ with corresponding homomorphism $h_x: \mathbb{S} \to G_{\mathbb{R}}$, we have a cocharacter

$$\varpi_x: \mathbb{G}_{m,\mathbb{C}} \xrightarrow{\mathrm{id} \times 0} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{\cong} \mathbb{S}_{\mathbb{C}} \xrightarrow{h_{x,\mathbb{C}}} G_{\mathbb{C}}.$$

The $G(\mathbb{C})$-orbit of ϖ_x in $\mathbf{Hom}_{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}}, G_{\mathbb{C}})$ depends only on X and is defined over a finite extension $E/\mathbb{Q}$, called the reflex field of (G, X). By results of Deligne, Milne, Borovoi, Shih, and others, $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ has a canonical model $\mathrm{Sh}_K(G, X)$ over E. We refer the reader to [13, Chapter 12] and [14, Chapter 2, 2.17] for more details.

Let $p \geq 3$ be a prime, and let $G_{\mathbb{Z}_p}$ be a reductive group over $\mathbb{Z}_p$ whose generic fiber is $G_{\mathbb{Q}_p}$. Let $K = K_p K^p$ with $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$, and K^p be an open compact subgroup of $G(\mathbb{A}_f^p)$ that is small enough. Let v be a prime of O_E over (p); then v is unramified over p. We write $O_{E,(v)}$ for the localization of O_E at v. Assume that the Shimura datum (G, X) is of Hodge type, *i.e.*, there is an embedding of Shimura data $(G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), X')$. Then by [7, Lemmas 2.3.1 and 2.3.2], for the chosen $G_{\mathbb{Z}_p}$, there exists a lattice $V_{\mathbb{Z}} \subseteq V$, such that ψ restricts to a pairing $V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$ and $G_{\mathbb{Z}_{(p)}}$, the closure of G in $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ with $G_{\mathbb{Z}_p} = G_{\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$, is reductive. Moreover, by [7, Proposition 1.3.2], there is a tensor $s \in V_{\mathbb{Z}_{(p)}}^{\otimes}$ defining $G_{\mathbb{Z}_{(p)}} \subseteq \mathrm{GL}(V_{\mathbb{Z}_{(p)}})$, *i.e.*, for any $\mathbb{Z}_{(p)}$-algebra

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R, we have

$$G_{\mathbb{Z}_{(p)}}(R) = \left\{ g \in \mathrm{GL}(V_{\mathbb{Z}_{(p)}})(R) \mid g(s \otimes 1) = s \otimes 1 \right\}.$$

Here $V_{\mathbb{Z}_{(p)}} := V_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)}$, and $V^{\otimes}_{\mathbb{Z}_{(p)}}$ is a finite free $\mathbb{Z}_{(p)}$-module that is obtained from $V_{\mathbb{Z}_{(p)}}$ by using the operations of taking duals, tensor products, symmetric powers, exterior powers, and direct sums finitely many times.

Let $K'_p \subseteq \mathrm{GSp}(V_{\mathbb{Q}_p}, \psi)$ be the stabilizer of $V_{\mathbb{Z}_p} := V_{\mathbb{Z}} \otimes \mathbb{Z}_p$. Then by [7, Lemma 2.1.2], we can choose $K' = K'_p K'^p$ such that K'^p contains K^p and K' leaves $V_{\widehat{\mathbb{Z}}}$ stable, making the finite morphism

$$\mathrm{Sh}_K(G, X) \longrightarrow \mathrm{Sh}_{K'}(\mathrm{GSp}(V, \psi), X')_E$$

a closed embedding.

Let $d = |V^{\vee}_{\mathbb{Z}} / V_{\mathbb{Z}}|$, and $g = \dim(V)/2$. Then $\mathrm{Sh}_{K'}(\mathrm{GSp}(V, \psi), X')$ is closed in the generic fiber of $\mathscr{A}_{g,d,K'} \otimes O_{E,(v)}$, where $\mathscr{A}_{g,d,K'}$ is the fine moduli scheme of g-dimensional abelian schemes over $\mathbb{Z}_{(p)}$-schemes equipped with a degree d polarization and a level K'-structure (see [18, Theorem 7.9]). Let $\mathscr{S}_K(G, K)^-$ be the Zariski closure of $\mathrm{Sh}_K(G, X)$ in $\mathscr{A}_{g,d,K'} \otimes O_{E,(v)}$ with the reduced induced scheme structure, and let $\mathscr{S}_K(G, X)$ be the normalization of $\mathscr{S}_K(G, K)^-$. Let $\mathcal{A}$ be the universal abelian scheme on $\mathscr{A}_{g,d,K'}$. Then

$$\mathcal{V} = \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}|_{\mathrm{Sh}_K(G,X)} / \mathrm{Sh}_K(G, X)) \quad (\text{resp. } \mathcal{V}^{\circ} = \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}|_{\mathscr{S}_K(G,X)} / \mathscr{S}_K(G, X)))$$

is a vector bundle on $\mathrm{Sh}_K(G, X)$ (resp. $\mathscr{S}_K(G, X)$). By the construction of [7, Section 2.2], the tensor $s \in V^{\otimes}_{\mathbb{Z}_{(p)}}$ gives a section s_{dR} of $\mathcal{V}^{\otimes}$, which is horizontal with respect to the Gauss–Manin connection.

Here we collect some of the main results in [7].

Theorem 3.1.2

(i) *The scheme $\mathscr{S}_K(G, X)$ is smooth over $O_{E,(v)}$, and*

$$\mathscr{S}_{K_p}(G, X) := \varprojlim_{K^p} \mathscr{S}_{K_p K^p}(G, X)$$

is an inverse system with finite étale transition maps, whose generic fiber is $G(\mathbb{A}^p_f)$-equivariantly isomorphic to $\mathrm{Sh}_{K_p}(G, X) := \varprojlim_{K^p} \mathrm{Sh}_{K_p K^p}(G, X)$.

(ii) *The scheme $\mathscr{S}_{K_p}(G, X)$ satisfies the a certain extension property.*
Namely, for any regular and formally smooth $O_{E,(v)}$-scheme X, any morphism $X \otimes E \to \mathscr{S}_{K_p}(G, X)$ extends uniquely to a morphism $X \to \mathscr{S}_{K_p}(G, X)$.

(iii) *The section s_{dR} extends to a section of $\mathcal{V}^{\circ\otimes}$, which will still be denoted by s_{dR}. For any closed point $x \in \mathscr{S}_K(G, X) \otimes \mathbb{F}_p$ and any lifting $\widetilde{x} \in \mathscr{S}_K(G, X)(W(k(x)))$, we have*

 (a) *the scheme $\mathbf{Isom}_{W(k(x))}((V^{\vee}_{\mathbb{Z}_p} \otimes W(k(x)), s \otimes 1), (\mathcal{V}^{\circ}_{\widetilde{x}}, s_{\mathrm{dR},\widetilde{x}}))$ is a trivial right $G_{\mathbb{Z}_p} \otimes W(k(x))$-torsor;*

 (b) *for any $t \in \mathbf{Isom}_{W(k(x))}((V^{\vee}_{\mathbb{Z}_p} \otimes W(k(x)), s \otimes 1), (\mathcal{V}^{\circ}_{\widetilde{x}}, s_{\mathrm{dR},\widetilde{x}}))(W(k(x)))$, $G_{\mathbb{Z}_p} \otimes W(k(x))$ acts faithfully on $\mathcal{V}^{\circ}_{\widetilde{x}}$ via $g(v) := tgt^{-1}(v)$, for all $v \in \mathcal{V}^{\circ}_{\widetilde{x}}$. The Hodge filtration on $\mathcal{V}^{\circ}_{\widetilde{x}}$ is induced by a cocharacter of $G_{\mathbb{Z}_p} \otimes W(k(x))$.*

Proof (i) and (ii) are [7, Theorem 2.3.8(1) and (2)]. The first sentence of (iii) is [7, Corollary 2.3.9].

The proof of (iii)(a) is hidden inside [7]. We write F for $q.f.(W(k(x)))$ and $\widetilde{x}_F$ for the F-point of $\mathscr{S}_K(G,X)$ given by the composition

$$\operatorname{Spec}(F) \hookrightarrow \operatorname{Spec}(W(k(x))) \xrightarrow{\widetilde{x}} \mathscr{S}_K(G,X).$$

Let $\mathcal{A}$ be the universal abelian scheme as before. Then there is an isomorphism $V_{\mathbb{Z}_p}^\vee \to \mathrm{H}^1_{\text{ét}}(\mathcal{A}_{\widetilde{x}_F}, \mathbb{Z}_p)$ taking s to $s_{\text{ét},\widetilde{x}_F}$. Here the right-hand side is the p-adic étale cohomology of the abelian variety $\mathcal{A}_{\widetilde{x}_F}$ over F.

Then by [7, Corollary 1.4.3 (3)], there is an isomorphism

$$\mathrm{H}^1_{\text{ét}}(\mathcal{A}_{\widetilde{x}_F}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k(x)) \longrightarrow \mathbb{D}\big(\mathcal{A}_x[p^\infty]\big)\big(W(k(x))\big)$$

taking $s_{\text{ét},\widetilde{x}_F} \otimes 1$ to a Frobenius-invariant tensor s_0. Here we write $\mathbb{D}(-)$ for the Dieudonné functor as in [7]. We remark that this is just an isomorphism, which is highly non-canonical. But by the construction in [7, Corollary 2.3.9], s_0 gives $s_{\mathrm{dR},\widetilde{x}}$ by the canonical identification

$$\mathbb{D}\big(\mathcal{A}_x[p^\infty]\big)\big(W(k(x))\big) \cong \mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{A}_{\widetilde{x}}/W(k(x))\big).$$

This proves (iii)(a).

For (iii)(b), see the proof of [7, Corollary 1.4.3 (4)]. Note that Kisin actually proves the Hodge filtration on $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_{\widetilde{x}})$ is a $G_{\mathbb{Z}_p}$-filtration, but in his statement, he only states it for the special fiber. ∎

3.2 Construction of the G-zip at a Point

In this and the following subsections, we show how to get a $\overline{G}$-zips over $\mathscr{S}_0$ using $\overline{\mathcal{V}^\circ}$, where $\overline{G}$ is the special fiber of $G_{\mathbb{Z}_p}$ considered in the previous subsection. We use 'G-zip' in the title here (and also that of 3.2.6, 3.3), as we want to keep the notations in titles simple and coherent.

We will first say something about cocharacters inducing the Hodge filtrations, as they are crucial data in the definition of $\overline{G}$-zips, and will also be used in 2.4 to get the torsor I over $\mathscr{S}_0$.

3.2.1 Basics about Cocharacters

Proposition 3.2.1 *Let G be a reductive group over a scheme S and* $\mathbf{Hom}(\mathbb{G}_m, G)$ *be the fpqc-sheaf of cocharacters denoted by Z. Then we have the following:*

(i) *Z is represented by a smooth and separated scheme over S;*
(ii) *the fpqc-quotient of Z by the adjoint action of G is represented by a disjoint union of connected finite étale S-schemes;*
(iii) *assume that G has a maximal torus T over S. Let $X_*(T)$ be the scheme of cocharacters, and let W be the Weyl group scheme with respect to T. Then*
   (a) *$T \subseteq G$ induces an isomorphism of fpqc-sheaves $W\backslash X_*(T) \cong G\backslash Z$;*
   (b) *if $S = \operatorname{Spec} R$ with R a henselian local ring with residue field k such that G_k is quasi-split, then the natural map $X_*(T)(R) \to (W\backslash X_*(T))(R)$ is surjective.*

Proof The first statement follows from [2, Corollary 4.2, Chapter XI]. For (ii), one can work with open affines of S. But by [2, Corollary 3.20, Chapter XIV], maximal tori exist Zariski locally. We can assume that S is affine such that there exists a maximal torus $T \subseteq G$. Note that $X_*(T)$ is étale and locally finite over S, and that W is a finite étale group scheme (see [2, 3.1, Chapter XXII]).

Maximal tori are fpqc locally G-conjugate, so the inclusion $T \subseteq G$ induces an isomorphism of fpqc-sheaves $W\backslash X_*(T) \cong G\backslash Z$. To prove (ii), it suffices to prove that $W\backslash X_*(T)$ is represented by an étale and locally finite scheme over S. To see that $W\backslash X_*(T)$ is representable, note that $W \times X_*(T)$ and $X_*(T) \times X_*(T)$ are both étale over S, so the morphism

$$\alpha\colon W \times X_*(T) \longrightarrow X_*(T) \times X_*(T), \quad (w, v) \longmapsto (v, w \cdot v)$$

is also étale, and hence has open image. But it is also closed, since there is a finite étale cover S' of S, such that W and $X_*(T)$ become constant, and the image of α is just copies of S', which is closed in $(X_*(T) \times X_*(T))_{S'}$. Let R be the image of α, then its projections to $X_*(T)$ induced by projections of $X_*(T) \times X_*(T)$ to its factors are both finite. So by [8, Chapter 1, Propositions 5.14 and 5.16 b)], $W\backslash X_*(T)$ is represented by an étale separated scheme over S.

To see that the quotient is locally finite, one still works over S'. For an S'-point of $X_*(T)_{S'}$, its orbit under $W(S')$ is just copies of S'. Let $X' \subseteq X_*(T)_{S'}$ be an open and closed subscheme such that it contains precisely one copy of S' in each $W(S')$-orbit. Then $X' \cong (W\backslash X_*(T))_{S'}$, and hence $W\backslash X_*(T)$ is locally finite.

To finish the proof of the proposition, we only need to prove (iii)(b). If $R = k$ is a field, then the statement follows from [10, Lemma 1.1.3]. We remark that although it is stated for fields containing $\mathbb{Q}$ there, its proof works for general fields. But for a henselian local ring R, noting that both $X_*(T)$ and $W\backslash X_*(T)$ are étale, we have

$$X_*(T)(R) = X_*(T)(k) \quad \text{and} \quad (W\backslash X_*(T))(R) = (W\backslash X_*(T))(k),$$

and hence (iii)(b). ■

3.2.2 A Cocharacter Defined Over $W(\kappa)$

Now we come back to notations introduced after Definition 3.1.1. Let $\kappa = O_E/v$, and $Z = \mathbf{Hom}(\mathbb{G}_m, G_{\mathbb{Z}_{(p)}})$. Let $T \subseteq G_{\mathbb{Z}_{(p)}}$ be a maximal torus, and W_T be the Weyl group scheme, then by the above proposition, $G_{\mathbb{Z}_{(p)}}\backslash Z \cong W_T\backslash X_*(T)$ is a union of connected finite étale $\mathbb{Z}_{(p)}$-schemes. As explained at the beginning of Section 3.1, the Shimura datum gives a $G_{\mathbb{C}}$-orbit $[\varpi_x]$ of $Z_{\mathbb{C}}$ which is defined over E, and hence a connected component $C \cong \operatorname{Spec} O_{E,(p)}$ of $G_{\mathbb{Z}_{(p)}}\backslash Z$. Noting that $G_{\mathbb{Z}_{(p)}} \otimes \mathbb{F}_p$ is quasi-split, by (iii)(b) of the previous proposition, the κ-point of C induced by $O_{E,(p)} \to O_E/v = \kappa$ comes from a κ-point of $X_*(T)$, which lifts to a $W(\kappa)$-point of $X_*(T)$. The cocharacter corresponding to this point is such that for any embedding $W(\kappa) \to \mathbb{C}$, its image in $Z_{\mathbb{C}}$ lies in $[\varpi_x]$. As by our construction, its image in $G_{W(\kappa)}\backslash Z_{W(\kappa)}$ lies in $C_{W(\kappa)} = O_{E,v}$.

3.2.3 An Easy Lemma

To get started, we need one preparation, namely the next lemma. It is probably well known, but we still give a proof. It will be used in Section 3.2.5. We need to fix some notation to state and prove it. Let k be a finite field of characteristic p, A be an abelian scheme over $W(k)$, σ be the ring automorphism $W(k) \to W(k)$ which lifts the p-Frobenius isomorphism on k. Denote by M the module $\mathrm{H}^1_{\mathrm{dR}}(A/W(k)) \cong \mathrm{H}^1_{\mathrm{cris}}(A_k/W(k))$ (see [6, 3.4.b], and this isomorphism is functorial in A). Then the absolute Frobenius on A_k induces a σ-linear map $\varphi\colon M \to M$ (see [6, 2.5.3, 3.4.2]) whose linearization will be denoted by φ^{lin}. Let $M \supseteq M^1$ be the Hodge filtration. We know that M^1 is a direct summand of M, and its reduction modulo p gives the kernel of Frobenius $\bar{\varphi}$ on $\mathrm{H}^1_{\mathrm{dR}}(A \otimes k/k)$. This implies that $\varphi(M^1) \subseteq pM$, and hence $\varphi/p\colon M^1 \to M$ is well defined.

Lemma 3.2.2 *For any splitting $M = M^0 \oplus M^1$, the linear map*

$$\alpha : M^{(\sigma)} := M \otimes_{W(k),\sigma} W(k) = M^{0(\sigma)} \oplus M^{1(\sigma)} \xrightarrow{\varphi^{\mathrm{lin}}|_{M^{0(\sigma)}} + (\frac{\varphi}{p})^{\mathrm{lin}}|_{M^{1(\sigma)}}} M$$

is an isomorphism.

Proof Let F (resp. V) be the Frobenius (resp. Verschiebung) on M_k. Then $(\varphi/p)^{\mathrm{lin}}|_{M_k^{1(\sigma)}}$ is induced by $V^{\mathrm{lin},-1}\colon \mathrm{Im}(V^{\mathrm{lin}}) \to M_k/\operatorname{Ker}(V^{\mathrm{lin}})$. So α_k is an isomorphism as $\operatorname{Ker}(F) = \mathrm{Im}(V)$ and $\operatorname{Ker}(V) = \mathrm{Im}(F)$, but then α is an isomorphism by Nakayama's lemma. ∎

Notations 3.2.3 Now we will fix some notation that will be used later. Notation as in Section 3.2.2, we will write W for $W(\kappa)$ for simplicity. By the discussions there, the orbit $[\varpi_x]$ gives a cocharacter $\mathbb{G}_m \to G_{\mathbb{Z}_p} \otimes W$ which is unique up to $G_{\mathbb{Z}_p}(W)$-conjugacy. Its inverse will be denoted by μ. The (contragredient) representation on $V^\vee_{\mathbb{Z}_{(p)}} \otimes W$ induced by μ has weights 0 and 1. Since we are interested in reductions of integral canonical models, we will work either over W or over κ. So we will simply write $\mathscr{S}$ for $\mathscr{S}_K(G,X) \otimes_{O_{E,(v)}} W$, and $\mathscr{S}_0$ for the special fiber of $\mathscr{S}$. We will write $\mathcal{A}$ for the pull back to $\mathscr{S}$ of the universal abelian scheme on $\mathscr{A}_{g,d,K'}$. We will still denote by $\mathcal{V}^\circ$ (resp. s_{dR}) the pullback to $\mathscr{S}$ of $\mathcal{V}^\circ$ (resp. s_{dR}) on $\mathscr{S}_K(G,K)$ as in Theorem 3.1.2, and $\overline{\mathcal{V}^\circ}$ (resp. $\overline{s_{\mathrm{dR}}}$) for the pull back to $\mathscr{S}_0$ of $\mathcal{V}^\circ$ (resp. s_{dR}) on $\mathscr{S}$.

3.2.4 Basic Properties of $s_{\mathrm{dR},\widetilde{x}}$ and φ

Now we will discuss some basic properties of $s_{\mathrm{dR},\widetilde{x}}$ related to the Frobenius on $\mathcal{V}^\circ_{\widetilde{x}}$ and the filtration on $\mathcal{V}^{\circ\otimes}_{\widetilde{x}}$ induced by the Hodge filtration. We will keep the notation as in Theorem 3.1.2(iii)(b). In particular, there is an element

$$t \in \mathbf{Isom}_{W(k(x))}\big((V^\vee_{\mathbb{Z}_p} \otimes W(k(x)), s \otimes 1), (\mathcal{V}^\circ_{\widetilde{x}}, s_{\mathrm{dR},\widetilde{x}}) \big)(W(k(x))).$$

The element t will be fixed once and for all in our discussion. Also, we will introduce some new notation as follows. Let

$$\mu'\colon \mathbb{G}_{m,W(k(x))} \longrightarrow G_{\mathbb{Z}_p} \otimes W(k(x))$$

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be a cocharacter such that $\mu'_t := t\mu' t^{-1}$ induces the Hodge filtration on $\mathcal{V}^\circ_{\tilde{x}}$. (The existence of μ' follows from Theorem 3.1.2(iii)(b).) Note that μ' induces a $W(k(x))$-point of C introduced after the proof of Proposition 3.2.1, as the Hodge filtration on $\mathcal{V}^\circ_{\tilde{x}} \otimes \mathbb{C}$ is always induced by a cocharacter conjugate to $\mu_{\mathbb{C}}$ (via the contragredient representation). In particular, μ' is $G_{\mathbb{Z}_p}(W(k(x)))$-conjugate to $\mu_{W(k(x))}$. We will write φ for the Frobenius on $\mathcal{V}^\circ_{\tilde{x}}$ and $\mathcal{V}^\circ_{\tilde{x}} = (\mathcal{V}^\circ_{\tilde{x}})^0 \oplus (\mathcal{V}^\circ_{\tilde{x}})^1$ for the splitting induced by μ'_t, with $(\mathcal{V}^\circ_{\tilde{x}})^i$ the sub-module of weight i. The filtration on $\mathcal{V}^\circ_{\tilde{x}}$ induces a filtration on $\mathcal{V}^{\circ\otimes}_{\tilde{x}}$ by the constructions at the beginning of Section 2.1. There is a Frobenius that is defined, not on $\mathcal{V}^{\circ\otimes}_{\tilde{x}}$, but on $(\mathcal{V}^\circ_{\tilde{x}}[\frac{1}{p}])^{\otimes}$, as follows. It is the tensor product of φ on $\mathcal{V}^\circ_{\tilde{x}}[\frac{1}{p}]$ and

$$^{\vee}\varphi\colon \big(\mathcal{V}^\circ_{\tilde{x}}[1/p])\big)^{\vee} \longrightarrow \big(\mathcal{V}^\circ_{\tilde{x}}[1/p]\big)^{\vee}, \quad f \longmapsto \sigma(f\circ\varphi^{-1})\ \forall\, f \in \mathcal{V}^{\circ\vee}_{\tilde{x}},$$

on $(\mathcal{V}^\circ_{\tilde{x}}[1/p]))^{\vee}$. The induced Frobenius on $(\mathcal{V}^\circ_{\tilde{x}}[1/p])^{\otimes}$ will still be denoted by φ. It is known that $s_{\mathrm{dR},\tilde{x}} \in \mathcal{V}^{\circ\otimes}_{\tilde{x}}$ actually lies in $\mathrm{Fil}^0\,\mathcal{V}^{\circ\otimes}_{\tilde{x}} \subseteq \mathcal{V}^{\circ\otimes}_{\tilde{x}}$, the submodule of non-negative weights, and that $s_{\mathrm{dR},\tilde{x}}$ is φ-invariant ([7, 1.3.3], and we view $s_{\mathrm{dR},\tilde{x}}$ as an element in $(\mathcal{V}^\circ_{\tilde{x}}[1/p])^{\otimes}$ when considering the φ-action).

We have the following better description.

Proposition 3.2.4 *The Frobenius φ takes integral value on $\mathrm{Fil}^0\,\mathcal{V}^{\circ\otimes}_{\tilde{x}}$. Let $(\mathcal{V}^{\circ\otimes}_{\tilde{x}})^0$ be the submodule of $\mathcal{V}^{\circ\otimes}_{\tilde{x}}$ such that $\mu'_t(\mathbb{G}_m)$ acts trivially, then $s_{\mathrm{dR},\tilde{x}} \in (\mathcal{V}^{\circ\otimes}_{\tilde{x}})^0$.*

Proof We use notation from Section 3.2.4. To see the first statement, note that we have

$$\mathrm{Fil}^0(\mathcal{V}^{\circ\otimes}_{\tilde{x}}) = \oplus_{i\geq 0}(\mathcal{V}^{\circ\otimes}_{\tilde{x}})^i,$$

where $(\mathcal{V}^{\circ\otimes}_{\tilde{x}})^i$ is the submodule whose elements are of weight i with respect to the cocharacter μ'_t. And elements in $(\mathcal{V}^{\circ\otimes}_{\tilde{x}})^i$ are images of sums of elements from

$$\big((\mathcal{V}^\circ_{\tilde{x}})^0\big)^{\otimes a} \otimes \big((\mathcal{V}^\circ_{\tilde{x}})^1\big)^{\otimes b} \otimes \big((\mathcal{V}^{\circ\vee}_{\tilde{x}})^{-1}\big)^{\otimes c} \otimes \big((\mathcal{V}^{\circ\vee}_{\tilde{x}})^0\big)^{\otimes d}$$

such that $b - c = i$. The σ-linear map φ induces well defined σ-linear maps

$$\varphi|_{(\mathcal{V}^\circ_{\tilde{x}})^0}\colon (\mathcal{V}^\circ_{\tilde{x}})^0 \longrightarrow \mathcal{V}^\circ_{\tilde{x}} \quad \text{and} \quad {}^{\vee}\varphi|_{(\mathcal{V}^{\circ\vee}_{\tilde{x}})^0}\colon ((\mathcal{V}^{\circ\vee}_{\tilde{x}})^0 \longrightarrow \mathcal{V}^{\circ\vee}_{\tilde{x}}.$$

But

$$(\varphi|_{(\mathcal{V}^\circ_{\tilde{x}})^1})^{\otimes b} \otimes ({}^{\vee}\varphi|_{(\mathcal{V}^{\circ\vee}_{\tilde{x}})^{-1}})^{\otimes c}\colon \big((\mathcal{V}^\circ_{\tilde{x}})^1\big)^{\otimes b} \otimes \big((\mathcal{V}^{\circ\vee}_{\tilde{x}})^{-1}\big)^{\otimes c} \longrightarrow (\mathcal{V}^\circ_{\tilde{x}})^{\otimes b} \otimes (\mathcal{V}^{\circ\vee}_{\tilde{x}})^{\otimes c}$$

is also defined as

$$(\varphi|_{(\mathcal{V}^\circ_{\tilde{x}})^1})^{\otimes b} \otimes ({}^{\vee}\varphi|_{(\mathcal{V}^{\circ\vee}_{\tilde{x}})^{-1}})^{\otimes c} = p^{b-c}\cdot\Big(\frac{\varphi}{p}|_{(\mathcal{V}^\circ_{\tilde{x}})^1}\Big)^{\otimes b} \otimes (p\cdot{}^{\vee}\varphi|_{(\mathcal{V}^{\circ\vee}_{\tilde{x}})^{-1}})^{\otimes c},$$

while $\varphi/p|_{(\mathcal{V}^\circ_{\tilde{x}})^1}$ and $p\cdot{}^{\vee}\varphi|_{(\mathcal{V}^{\circ\vee}_{\tilde{x}})^{-1}}$ are well defined. So φ is defined on $\mathrm{Fil}^0(\mathcal{V}^{\circ\otimes}_{\tilde{x}})$.

To see that $s_{\mathrm{dR},\tilde{x}} \in (\mathcal{V}^{\circ\otimes}_{\tilde{x}})^0$, one only needs to use the fact that $s \in V^{\otimes}_{\mathbb{Z}_p}$ is $G_{\mathbb{Z}_p}$-invariant, and hence $s_{\mathrm{dR},\tilde{x}}$ is also $G_{\mathbb{Z}_p}$-invariant via t. In particular, it is of weight 0 with respect to the cocharacter μ'_t. ■

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3.2.5 Constructing Some Torsors Over $W(k(x))$

Now we will show that using the Frobenius φ and the splitting induced by μ'_t (see Section 3.2.4 for the definition of μ' and μ'_t), we can get an element g_t of $G_{\mathbb{Z}_p}(W(k(x)))$.

Construction 3.2.5 Let $\sigma\colon W(k(x)) \to W(k(x))$ be as in Section 3.2.3, and let ξ be the $W(k(x))$-linear isomorphism

$$V^{\vee}_{\mathbb{Z}_p} \otimes W(k(x)) \longrightarrow \big(V^{\vee}_{\mathbb{Z}_p} \otimes W(k(x))\big)^{(\sigma)}$$

given by $v \otimes w \mapsto v \otimes 1 \otimes w$ and $t^{(\sigma)}$ be the pull back of

$$t \in \mathbf{Isom}_{W(k(x))}\big(\big(V^{\vee}_{\mathbb{Z}_p} \otimes W(k(x)), s \otimes 1\big), (\mathcal{V}^{\circ}_{\tilde{x}}, s_{\mathrm{dR},\tilde{x}})\big)\big(W(k(x))\big)$$

via σ. Let $\xi_t = t^{(\sigma)} \circ \xi$, and g be the $W(k(x))$-linear map

$$\mathcal{V}^{\circ(\sigma)}_{\tilde{x}} = (\mathcal{V}^{\circ}_{\tilde{x}})^{0(\sigma)} \oplus (\mathcal{V}^{\circ}_{\tilde{x}})^{1(\sigma)} \xrightarrow{\varphi^{\mathrm{lin}}|_{(\mathcal{V}^{\circ}_{\tilde{x}})^{0(\sigma)}} + (\frac{\varphi}{p})^{\mathrm{lin}}|_{(\mathcal{V}^{\circ}_{\tilde{x}})^{1(\sigma)}}} \mathcal{V}^{\circ}_{\tilde{x}}\,.$$

We define g_t to be the composition $t^{-1} \circ g \circ \xi_t$, and $(\mathcal{V}^{\circ}_{\tilde{x}})_0$ (resp. $(\mathcal{V}^{\circ}_{\tilde{x}})_1$) to be the sub $W(k(x))$-module of $\mathcal{V}^{\circ}_{\tilde{x}}$ generated by $\varphi((\mathcal{V}^{\circ}_{\tilde{x}})^0)$ (resp. $\frac{\varphi}{p}((\mathcal{V}^{\circ}_{\tilde{x}})^1)$).

We have the following proposition.

Proposition 3.2.6

(i) *The linear map g_t is an element of $G_{\mathbb{Z}_p}(W(k(x)))$.*

(ii) *The splitting*

$$V^{\vee}_{\mathbb{Z}_p} \otimes W\big(k(x)\big) = t^{-1}\big((\mathcal{V}^{\circ}_{\tilde{x}})_0\big) \oplus t^{-1}\big((\mathcal{V}^{\circ}_{\tilde{x}})_1\big)$$

is induced by the cocharacter $\nu = g_t \mu'^{(\sigma)} g_t^{-1}$ of $G_{\mathbb{Z}_p} \otimes W(k(x))$, i.e., *$t^{-1}\big((\mathcal{V}^{\circ}_{\tilde{x}})_i\big)$ is of weight i with respect to ν.*

Proof By Lemma 3.2.2, $g_t \in \mathrm{GL}(V^{\vee}_{\mathbb{Z}_p})(W(k(x)))$. So, to prove (i), it suffices to check that the induced map

$$g_t\colon V^{\otimes}_{\mathbb{Z}_p} \otimes W\big(k(x)\big) \longrightarrow V^{\otimes}_{\mathbb{Z}_p} \otimes W(k(x))$$

maps $s \otimes 1$ to itself. Now we compute $g_t(s \otimes 1)$. First, $\xi(s \otimes 1) = s \otimes 1 \otimes 1$ and $t^{(\sigma)}(s \otimes 1 \otimes 1) = s_{\mathrm{dR},\tilde{x}} \otimes 1$. We decompose $\mathcal{V}^{\circ\otimes}_{\tilde{x}} = \oplus_i (\mathcal{V}^{\circ\otimes}_{\tilde{x}})^i$ via the weights of the cocharacter μ'_t introduced before. Then $(\mathcal{V}^{\circ\otimes}_{\tilde{x}})^{(\sigma)} = \oplus_i ((\mathcal{V}^{\circ\otimes}_{\tilde{x}})^i)^{(\sigma)}$.

Note that $s_{\mathrm{dR},\tilde{x}} \in (\mathcal{V}^{\circ\otimes}_{\tilde{x}})^0$ by Proposition 3.2.4, so

$$g^{\otimes} = \sum_i p^{-i} (\varphi^{\mathrm{lin}})^{\otimes}|_{(\mathcal{V}^{\circ\otimes}_{\tilde{x}})^i}\colon \bigoplus_i \big((\mathcal{V}^{\circ\otimes}_{\tilde{x}})^i\big)^{(\sigma)} \to \mathcal{V}^{\circ\otimes}_{\tilde{x}}$$

maps $s_{\mathrm{dR},\tilde{x}} \otimes 1$ to $s_{\mathrm{dR},\tilde{x}}$, as it is φ-invariant. And hence,

$$g_t(s \otimes 1) = t^{-1} \circ g \circ \xi_t(s \otimes 1) = s \otimes 1,$$

as t^{-1} takes $s_{\mathrm{dR},\tilde{x}}$ to $s \otimes 1$. This proves (i).

For (ii), we look at the commutative diagram

$$\begin{tikzcd} V_{\mathbb{Z}_p}^{\vee}\otimes W(k(x)) \arrow[r, "t^{(\sigma)}\circ\xi"] \arrow[dr, "g_t"] & (\mathcal{V}_{\widetilde{x}}^{\circ})^{(\sigma)} \arrow[r, equal] & (\oplus_i \mathcal{V}_{\widetilde{x}}^{\circ})^{i(\sigma)} \arrow[d, "\Sigma_i p^{-i}\varphi^{\mathrm{lin}}|_{(\mathcal{V}_{\widetilde{x}}^{\circ})^{i(\sigma)}}"] \\ & V_{\mathbb{Z}_p}^{\vee}\otimes W(k(x)) & \mathcal{V}_{\widetilde{x}}^{\circ}. \arrow[l, "t^{-1}"'] \end{tikzcd}$$

It shows directly that

$$\nu(m)(v) = g_t\sigma\big(\mu'(m)\big)g_t^{-1}(v), \quad \forall\, m\in\mathbb{G}_m(W(k(x))), \quad \forall\, v\in V_{\mathbb{Z}_p}^{\vee}\otimes W\big(k(x)\big).$$

∎

Corollary 3.2.7 *Let $\mu\colon\mathbb{G}_{m,W}\to G_{\mathbb{Z}_p}\otimes W$ be the cocharacter as in Notations 3.2.3. Let $C^{\bullet}$ be the descending filtration on $V_{\mathbb{Z}_p}^{\vee}\otimes W$ such that C^i is the sub-module of elements of weights $\geq i$ with respect to μ, and let $D_{\bullet}$ be the ascending filtration on $V_{\mathbb{Z}_p}^{\vee}\otimes W$ such that D_i is the sub-module of elements of weights $\leq i$ with respect to $\mu^{(\sigma)}$.*

Let P_+ (resp. P_-) be the stabilizer in $G_{\mathbb{Z}_p}\otimes W$ of $C^{\bullet}$ (resp. $D_{\bullet}$), and let $I_{\widetilde{x}}$ be

$$\mathbf{Isom}_{W(k(x))}\big((V_{\mathbb{Z}_p}^{\vee}\otimes W(k(x)), s\otimes 1), (\mathcal{V}_{\widetilde{x}}^{\circ}, s_{\mathrm{dR},\widetilde{x}})\big).$$

(i) *The closed subscheme*

$$I_{\widetilde{x},+} := \mathbf{Isom}_{W(k(x))}\Big((V_{\mathbb{Z}_p}^{\vee}\otimes W(k(x)), s\otimes 1, C^{\bullet}), \big(\mathcal{V}_{\widetilde{x}}^{\circ}, s_{\mathrm{dR},\widetilde{x}}, \mathcal{V}_{\widetilde{x}}^{\circ}\supseteq(\mathcal{V}_{\widetilde{x}}^{\circ})^1\big)\Big)\subseteq I_{\widetilde{x}}$$

is a trivial P_+-torsor.

(ii) *The closed subscheme*

$$I_{\widetilde{x},-} := \mathbf{Isom}_{W(k(x))}\Big((V_{\mathbb{Z}_p}^{\vee}\otimes W(k(x)), s\otimes 1, D_{\bullet}), \big(\mathcal{V}_{\widetilde{x}}^{\circ}, s_{\mathrm{dR},\widetilde{x}}, (\mathcal{V}_{\widetilde{x}}^{\circ})_0\subseteq\mathcal{V}_{\widetilde{x}}^{\circ}\big)\Big)\subseteq I_{\widetilde{x}}$$

is a trivial $P_-^{(\sigma)}$-torsor.

Proof To prove (i), take a $g_1\in G_{\mathbb{Z}_p}(W(k(x)))$ such that $g_1\big(\mu\otimes W(k(x))\big)g_1^{-1}=\mu'$, then we have $I_{\widetilde{x},+}=t\cdot g_1\big(P_+\otimes W(k(x))\big)$.

For (ii), by Proposition 3.2.6, $I_{\widetilde{x},-}=t\cdot g_t g_1^{(\sigma)}\big(P_-\otimes W(k(x))\big)^{(\sigma)}$. ∎

3.2.6 The G-zip Attached to a Filtered F-crystal

Notation as above, let L be the centralizer of μ. Let $\overline{\mu}$, $\overline{G}$, $\overline{P_+}$, $\overline{P_-}$, and let $\overline{L}$ be the reduction modulo p of μ, $G_{\mathbb{Z}_p}$, P_+, P_-, and L, respectively. For simplicity, we still write ξ, $C^{\bullet}$, and $D_{\bullet}$ for their reductions. The map ξ induces isomorphisms

$$\phi_0\colon (V_{\mathbb{Z}_p}^{\vee}\otimes\kappa)^{(p)}/\big((V_{\mathbb{Z}_p}^{\vee}\otimes\kappa)^1\big)^{(p)}\overset{\mathrm{pr}_2}{\to}\big((V_{\mathbb{Z}_p}^{\vee}\otimes\kappa)^0\big)^{(p)}\xrightarrow{\xi^{-1}}(V_{\mathbb{Z}_p}^{\vee}\otimes\kappa)_0,$$

$$\phi_1\colon \big((V_{\mathbb{Z}_p}^{\vee}\otimes\kappa)^1\big)^{(p)}\xrightarrow{\xi^{-1}}(V_{\mathbb{Z}_p}^{\vee}\otimes\kappa)_1\simeq(V_{\mathbb{Z}_p}^{\vee}\otimes\kappa)/(V_{\mathbb{Z}_p}^{\vee}\otimes\kappa)_0,$$

and hence induces σ-linear maps φ_0', φ_1' after pre-composing the natural map $V_{\mathbb{Z}_p}^{\vee}\otimes\kappa\to(V_{\mathbb{Z}_p}^{\vee}\otimes\kappa)^{(p)}$. The tuple $(V_{\mathbb{Z}_p}^{\vee}\otimes\kappa, C^{\bullet}, D_{\bullet}, \varphi_{\bullet}')$ is an F-zip. The $\overline{G}$-zip associated with $(\overline{G},\overline{\mu})$ is isomorphic to $\underline{I}_{\mathrm{id}}$ (here we use notations as at the end of Section 2.2).

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To get a $\overline{G}$-zip from $\mathcal{V}^\circ_{\widetilde{x}}$, one needs to "compare" the above F-zip and the one coming from $\mathcal{V}^\circ_{\widetilde{x}}$. Let $\varphi_0\colon \mathcal{V}^\circ_x/(\mathcal{V}^\circ_x)^1 \to (\mathcal{V}^\circ_x)_0$ be the reduction mod p of $\varphi|_{(\mathcal{V}^\circ_{\widetilde{x}})^0}$, and

$$\varphi_1\colon (\mathcal{V}^\circ_x)^1 \longrightarrow (\mathcal{V}^\circ_x)_1 \cong \mathcal{V}^\circ_x/(\mathcal{V}^\circ_x)_0$$

be the reduction mod p of $\frac{\varphi}{p}|_{(\mathcal{V}^\circ_{\widetilde{x}})^1}$. Let $I_{x,+}$, $I_{x,-}$ and let I_x be the reduction mod p of $I_{\widetilde{x},+}$, $I_{\widetilde{x},-}$, and $I_{\widetilde{x}}$, respectively. For simplicity, we still write t, g_1, g_t for their reductions. By the proof of Corollary 3.2.7, $I_{x,+}$ and $I_{x,-}$ are $\overline{P_{+}}_{,k(x)}$-torsor and $\overline{P_{-}}^{(p)}_{,k(x)}$-torsor, respectively. For any $k(x)$-algebra R, an element $\beta \in I_{x,+}(R)$ is an isomorphism

$$\beta\colon \big(V^\vee_{\mathbb{Z}_p} \otimes k(x), \bar{s} \otimes 1, C^\bullet(V^\vee_{\mathbb{Z}_p} \otimes k(x))\big) \otimes R \longrightarrow \big(\mathcal{V}^\circ_x, \overline{s_{\mathrm{dR},x}}, C^\bullet(\mathcal{V}^\circ_x)\big) \otimes R.$$

It induces an isomorphism

$$\oplus \operatorname{gr}^i_C \big((V^\vee_{\mathbb{Z}_p} \otimes k(x))^{(p)} \otimes R\big) \xrightarrow{\sim} \oplus \operatorname{gr}^i_C(\mathcal{V}^{\circ\,(p)}_x \otimes R)$$

which will still be denoted by $\beta^{(p)}$. But then $\beta^{(p)}$ is an element in $(I^{(p)}_{x,+}/U^{(p)}_+)(R)$, and any element of $(I^{(p)}_{x,+}/U^{(p)}_+)(R)$ is of this form (by [2, XXVI Corollary 2.2], as U_+ is unipotent).

Let $\iota\colon I^{(p)}_{x,+}/U^{(p)}_+ \to I_{x,-}/U^{(p)}_-$ be the morphism taking $\beta^{(p)}$ to

$$\begin{array}{ccc} \oplus \operatorname{gr}^D_i((V^\vee_{\mathbb{Z}_p} \otimes k(x)) \otimes R) & & \oplus \operatorname{gr}^D_i(\mathcal{V}^\circ_x \otimes R) \\ \Big\downarrow{\scriptstyle (\phi_0^{-1}\oplus\phi_1^{-1})\otimes 1} & & \Big\uparrow{\scriptstyle \varphi^{\mathrm{lin}}_\bullet\otimes 1} \\ \oplus \operatorname{gr}^i_C((V^\vee_{\mathbb{Z}_p} \otimes k(x))^{(p)} \otimes R) & \xrightarrow{\beta^{(p)}} & \oplus \operatorname{gr}^i_C(\mathcal{V}^{\circ\,(p)}_x \otimes R). \end{array}$$

We claim that ι is an isomorphism of $\overline{L}^{(p)}$-torsors. First note that

$$\phi_0^{-1} \oplus \phi_1^{-1}\colon \oplus \operatorname{gr}^D_i \big(V^\vee_{\mathbb{Z}_p} \otimes k(x)\big) \to \oplus \operatorname{gr}^i_C \big((V^\vee_{\mathbb{Z}_p} \otimes k(x))^{(p)}\big)$$

and $\varphi^{\mathrm{lin}}_\bullet\colon \oplus \operatorname{gr}^i_C(\mathcal{V}^{\circ\,(p)}_x) \to \oplus \operatorname{gr}^D_i(\mathcal{V}^\circ_x)$ are isomorphisms, and so are their base changes to R. This implies that ι is an isomorphism. We only need to show that ι is $L^{(p)}$-equivariant. But this follows from the fact that $\phi_0^{-1} \oplus \phi_1^{-1}$ is $L^{(p)}$-equivariant. So the tuple $(I_x, I_{x,+}, I_{x,-}, \iota)$ is a $\overline{G}$-zip of type $\overline{\mu}$ over $k(x)$.

Remark 3.2.8 One can describe $(I_x, I_{x,+}, I_{x,-}, \iota)$ explicitly. We have $\beta = tg_1p$ for some $q \in \overline{P_{+}}_{k(x)}(R)$ and

$$\iota(\beta^{(p)}) = \varphi^{\mathrm{lin}}_\bullet \circ t^{(p)} g_1^{(p)} q^{(p)} \circ (\phi_0^{-1} \oplus \phi_1^{-1}) = t g_t g_1^{(p)} q^{(p)}.$$

Using notations and constructions in the discussion after Theorem 2.2.2, we have

$$(I_x, I_{x,+}, I_{x,-}, \iota) \cong \underline{I}_{q^{-1} g_1^{-1} g_t g_1^{(p)} q^{(p)}} \cong \underline{I}_{g_1^{-1} g_t g_1^{(p)}}.$$

If we replace t by tg_1, then $(I_x, I_{x,+}, I_{x,-}, \iota) \cong \underline{I}_{g_t}$.

3.3 Construction of the G-zip over a Complete Local Ring

We want to globalize the above point-wise results to $\mathscr{S}_0$. But to do so, we need first to work at completions of stalks at closed points. And to study the $\overline{G}$-zip structure at the complete local rings, we need Faltings's deformation theory. For simplicity, we assume that t is such that $t\mu t^{-1}$ induces the Hodge filtration.

3.3.1 Faltings's Deformation Theory and Complete Local Rings of the Integral Model

Now we will describe Faltings's deformation theory for p-divisible groups following [14, 4.5] and its relation with Shimura varieties following [7, 1.5, 2.3].

Let k be a perfect field of characteristic p, and $W(k)$ be the ring of Witt vectors. Let H be a p-divisible group over $W(k)$ with special fiber H_0. The formal deformation functor for H_0 is represented by a ring R of formal power series over $W(k)$. More precisely, let (M_0, M_0^1, φ_0) be the filtered Dieudonné module associated with H, and L be a Levi subgroup of $P = \operatorname{stab}(M_0 \supseteq M_0^1)$. Let U be the opposite unipotent of P; then R is isomorphic to the completion at the identity section of U. Let u be the universal element in $U(R)$, and let $\sigma\colon R \to R$ be the homomorphism that is the Frobenius on $W(k)$ and p-th power on variables; then the filtered Dieudonné module of the universal p-divisible group over R is the tuple $(M, M^1, \varphi, \nabla)$, where $M = M_0 \otimes R$, $M^1 = M_0^1 \otimes R$, $\varphi = u \cdot (\varphi_0 \otimes \sigma)$, and ∇ is an integrable connection, which we do not want to specify, but just refer the reader to [14, Chapter 4].

More generally, let $G \subseteq \mathrm{GL}(M)$ be a reductive group defined by a tensor $s \in \mathrm{Fil}^0(M^{\otimes}) \subseteq M^{\otimes}$, which is φ_0-invariant. Assume that the filtration $M_0 \supseteq M_0^1$ is induced by a cocharacter μ of G. Let R_G be the completion along the identity section of the opposite unipotent of the parabolic subgroup $P_G = \operatorname{stab}_G(M_0 \supseteq M_0^1)$ of G, and u_G be the universal element in $U_G(R_G)$ which is also the pull back to R_G of u. Then R_G parametrizes deformations of H such that the horizontal continuation of s remains a Tate tensor (see [14, Proposition 4.9]).

For any closed point $x \in \mathscr{S}_0$, let $\widehat{O_{\mathscr{S}_0,x}}$ and $\widehat{O_{\mathscr{S},x}}$ be the completions of $O_{\mathscr{S}_0,x}$ and $O_{\mathscr{S},x}$ with respect to the maximal ideals defining x respectively. Clearly, $\widehat{O_{\mathscr{S},x}}/p\widehat{O_{\mathscr{S},x}} = \widehat{O_{\mathscr{S}_0,x}}$. Let $\widetilde{x}$ be a $W(k(x))$-point of $\mathscr{S}$ lifting x, and μ' be a cocharacter of $G_{\mathbb{Z}_p} \otimes W(k(x))$ as in the proof of Proposition 3.2.6, which induces the Hodge filtration on $\mathcal{V}^{\circ}_{\widetilde{x}}$ via t as introduced in Theorem 3.1.2(iii)(b). Let R_G be as above and $\sigma\colon R_G \to R_G$ be the morphism which is Frobenius on $W(k(x))$ and p-th power on variables. We will simply write u for u_G. Then by the proof of [7, Proposition 2.3.5], the p-divisible group $\mathcal{A}[p^\infty]|_{\widehat{O_{\mathscr{S},x}}}$ gives a formal deformation of $\mathcal{A}[p^\infty]\big|_x$, and induces an isomorphism $R_G \to \widehat{O_{\mathscr{S},x}}$. Moreover, if we take the Frobenius on $\widehat{O_{\mathscr{S},x}}$ to be the one on R_G, then the Dieudonné module of $\mathcal{A}[p^\infty]|_{\widehat{O_{\mathscr{S},x}}}$ is of the form $(\mathcal{V}^{\circ}_{\widetilde{x}} \otimes \widehat{O_{\mathscr{S},x}}, (\mathcal{V}^{\circ}_{\widetilde{x}})^1 \otimes \widehat{O_{\mathscr{S},x}}, \varphi, \nabla)$, where φ is the composition

$$\mathcal{V}^{\circ}_{\widetilde{x}} \otimes \widehat{O_{\mathscr{S},x}} \xrightarrow{\varphi\otimes\sigma} \mathcal{V}^{\circ}_{\widetilde{x}} \otimes \widehat{O_{\mathscr{S},x}} \xrightarrow{u_t} \mathcal{V}^{\circ}_{\widetilde{x}} \otimes \widehat{O_{\mathscr{S},x}},$$

with $u_t = tut^{-1}$, and ∇ is given by restricting the connection on the universal deformation to the closed sub formal scheme $\operatorname{Spf}(\widehat{O_{\mathscr{S},x}})$ (see [14, 4.5]). Note that by [7, 1.5.4, proof of Corollary 2.3.9], $s_{\mathrm{dR},\widetilde{x}} \otimes 1 = s_{\mathrm{dR}} \otimes 1$ in $\mathcal{V}^{\circ\otimes} \otimes \widehat{O_{\mathscr{S},x}}$.

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Lemma 3.3.1

(i) *The scheme*

$$\mathbb{I} := \mathbf{Isom}_{\mathrm{Spec}(\widehat{O_{\mathscr{S},x}})}\big((V_{\mathbb{Z}_p}^{\vee}\otimes W, s)\otimes_W \widehat{O_{\mathscr{S},x}},\ (\mathcal{V}^{\circ}, s_{\mathrm{dR}})\otimes \widehat{O_{\mathscr{S},x}}\big)$$

is a trivial $G_{\mathbb{Z}_p}$-torsor over $\widehat{O_{\mathscr{S},x}}$.

(ii) *The closed subscheme $\mathbb{I}_+ \subseteq \mathbb{I}$ defined by*

$$\mathbb{I}_+ = \mathbf{Isom}_{\mathrm{Spec}(\widehat{O_{\mathscr{S},x}})}\big((V_{\mathbb{Z}_p}^{\vee}\otimes W, C^{\bullet}, s)\otimes_W \widehat{O_{\mathscr{S},x}},\ (\mathcal{V}^{\circ}, \mathcal{V}^{\circ}\supseteq(\mathcal{V}^{\circ})^1, s_{\mathrm{dR}})\otimes \widehat{O_{\mathscr{S},x}}\big)$$

is a trivial P_+-torsor over $\widehat{O_{\mathscr{S},x}}$.

Proof (i) follows from $(\mathcal{V}^{\circ}, s_{\mathrm{dR}})\otimes\widehat{O_{\mathscr{S},x}} \cong (\mathcal{V}^{\circ}_{\widetilde{x}}, s_{\mathrm{dR},\widetilde{x}})\otimes\widehat{O_{\mathscr{S},x}}$ and Theorem 3.1.2(iii)(a). And (ii) follows from

$$\big(\mathcal{V}^{\circ}, \mathcal{V}^{\circ}\supseteq(\mathcal{V}^{\circ})^1, s_{\mathrm{dR}}\big)\otimes\widehat{O_{\mathscr{S},x}} \cong \big(\mathcal{V}^{\circ}_{\widetilde{x}}, \mathcal{V}^{\circ}_{\widetilde{x}}\supseteq(\mathcal{V}^{\circ}_{\widetilde{x}})^1, s_{\mathrm{dR},\widetilde{x}}\big)\otimes\widehat{O_{\mathscr{S},x}}$$

and Corollary 3.2.7(i). ∎

Let t be as in Section 3.3.1, and $\widehat{g}_t$ be the composition of

$$\xi: V_{\mathbb{Z}_p}^{\vee}\otimes\widehat{O_{\mathscr{S},x}} \longrightarrow (V_{\mathbb{Z}_p}^{\vee}\otimes\widehat{O_{\mathscr{S},x}})^{(\sigma)};\quad v\otimes s\longmapsto v\otimes 1\otimes s,$$
$$(t\otimes 1)^{(\sigma)}: (V_{\mathbb{Z}_p}^{\vee}\otimes W(k(x))\otimes\widehat{O_{\mathscr{S},x}})^{(\sigma)} \longrightarrow (\mathcal{V}^{\circ}_{\widetilde{x}}\otimes\widehat{O_{\mathscr{S},x}})^{(\sigma)},$$

with $\widehat{g}$,

$$(\mathcal{V}^{\circ}_{\widetilde{x}}\otimes\widehat{O_{\mathscr{S},x}})^{(\sigma)} =$$
$$\big(((\mathcal{V}^{\circ}_{\widetilde{x}})^0\oplus(\mathcal{V}^{\circ}_{\widetilde{x}})^1\big)_{\widehat{O_{\mathscr{S},x}}})^{(\sigma)} \xrightarrow{u_t\circ(((\varphi|_{(\mathcal{V}^{\circ}_{\widetilde{x}})^0}+\frac{\varphi}{p}|_{(\mathcal{V}^{\circ}_{\widetilde{x}})^1})\otimes\sigma)^{\mathrm{lin}})} \mathcal{V}^{\circ}_{\widetilde{x}}\otimes\widehat{O_{\mathscr{S},x}}$$

and $(t\otimes 1)^{-1}$. We have the following lemma.

Lemma 3.3.2 *The $\widehat{O_{\mathscr{S},x}}$-linear map $\widehat{g}_t$ is an element of $G_{\mathbb{Z}_p}(\widehat{O_{\mathscr{S},x}})$. Let $\widehat{\mathcal{V}^{\circ}}_0$ (resp. $\widehat{\mathcal{V}^{\circ}}_1$) be the module $\widehat{g}\big(((\mathcal{V}^{\circ}_{\widetilde{x}})^0\otimes\widehat{O_{\mathscr{S},x}})^{(\sigma)}\big)$ (resp. $\widehat{g}\big(((\mathcal{V}^{\circ}_{\widetilde{x}})^1\otimes\widehat{O_{\mathscr{S},x}})^{(\sigma)}\big)$), then the scheme $\mathbb{I}_-$ given by*

$$\mathbf{Isom}_{\mathrm{Spec}(\widehat{O_{\mathscr{S},x}})}\big((V_{\mathbb{Z}_p}^{\vee}\otimes W, D_{\bullet}, s)\otimes_W\widehat{O_{\mathscr{S},x}},\ (\mathcal{V}^{\circ}\otimes\widehat{O_{\mathscr{S},x}}, \widehat{\mathcal{V}^{\circ}}_0\subseteq\mathcal{V}^{\circ}\otimes\widehat{O_{\mathscr{S},x}}, s_{\mathrm{dR}}\otimes 1)\big)$$

is a trivial $P_-^{(\sigma)}$-torsor over $\widehat{O_{\mathscr{S},x}}$.

Proof To prove the first statement, we need to show that $\widehat{g}_t^{\otimes}(s\otimes 1) = s\otimes 1$. We have the following commutative diagram

$$\begin{tikzcd}
V^\vee_{\mathbb{Z}_p}\otimes \widehat{O_{\mathscr{S},x}} \arrow[r, "(t\otimes 1)^{(\sigma)}\circ\xi"] \arrow[dr, "g_t\otimes 1"] \arrow[ddr, "\widehat{g}_t"'] & (\mathcal{V}^\circ_{\widetilde{x}}\otimes_{W(k(x))} \widehat{O_{\mathscr{S},x}})^{(\sigma)} \arrow[r, equal] & \left((\oplus_i(\mathcal{V}^\circ_{\widetilde{x}})^i)\otimes \widehat{O_{\mathscr{S},x}}\right)^{(\sigma)} \arrow[d, "\Sigma_i(p^{-i}\varphi\otimes\sigma)^{\mathrm{lin}}"] \\
 & V^\vee_{\mathbb{Z}_p}\otimes \widehat{O_{\mathscr{S},x}} \arrow[d, "u"] & \mathcal{V}^\circ_{\widetilde{x}}\otimes \widehat{O_{\mathscr{S},x}} \arrow[l, "t^{-1}\otimes 1"'] \arrow[d, "u_t"] \\
 & V^\vee_{\mathbb{Z}_p}\otimes \widehat{O_{\mathscr{S},x}} & \mathcal{V}^\circ_{\widetilde{x}}\otimes \widehat{O_{\mathscr{S},x}}. \arrow[l, "t^{-1}\otimes 1"']
\end{tikzcd}$$

We know from Proposition 3.2.6 that $g_t^{\otimes}(s\otimes 1) = s\otimes 1$. But $u^{\otimes}(s\otimes 1) = s\otimes 1$ by definition. So $\widehat{g}_t^{\otimes}(s\otimes 1) = s\otimes 1$.

To prove the second statement, we use the same method as in the proof of Corollary 3.2.7. Let let $I_{\widetilde{x},+}$, $I_{\widetilde{x},-}$ be as in Corollary 3.2.7 (but with t replaced by tg_1). Then by the proof of Lemma 3.3.1, we have

$$\mathbb{I}_+ = \Big(t\cdot\big(P_+\otimes W(k(x))\big)\Big)\times \mathrm{Spec}(\widehat{O_{\mathscr{S},x}}) = I_{\widetilde{x},+}\times\mathrm{Spec}(\widehat{O_{\mathscr{S},x}}).$$

By the proof of Proposition 3.2.6(ii) and the commutative diagram above, the splitting

$$V^\vee_{\mathbb{Z}_p}\otimes \widehat{O_{\mathscr{S},x}} = (t\otimes 1)^{-1}(\widehat{\mathcal{V}^\circ}_0)\oplus(t\otimes 1)^{-1}(\widehat{\mathcal{V}^\circ}_1)$$

is induced by the cocharacter $u(\nu\otimes 1)u^{-1}$. So $\mathbb{I}_- = t\cdot u(t^{-1}I_{\widetilde{x},-}\times\mathrm{Spec}(\widehat{O_{\mathscr{S},x}}))$, and hence it is a trivial $P_-^{(\sigma)}$-torsor over $\mathrm{Spec}(\widehat{O_{\mathscr{S},x}})$. ■

3.3.2 Description of the G-zip over the Complete Local Ring

From now on, we only need to use mod p parts of previous result, so we simplify notation as follows. We will write G (resp. V) for the special fiber of $G_{\mathbb{Z}_p}$ (resp. $V_{\mathbb{Z}_p}$), and μ, s, R_G, $\mathbb{I}$, $\mathbb{I}_+$, $\mathbb{I}_-$, P_+, P_-, U_+, U_-, L, t, u, and g_t for their reductions mod p. To get a G-zip structure on $R_G\cong \widehat{O_{\mathscr{S}_0,x}}$, we only need to construct an isomorphism $\iota{:}\,\mathbb{I}_+^{(p)}/U_+^{(p)}\to\mathbb{I}_-/U_-^{(p)}$ of $L^{(p)}$-torsors over R_G. The construction is the same as in Section 3.2.6. Let $\mathcal{V}$ be reduction mod p of $\mathcal{V}^\circ\otimes\widehat{O_{\mathscr{S},x}}$ in Lemma 3.3.1, $\mathcal{C}^\bullet$ be the filtration on $\mathcal{V}$ given by reduction mod p of $(\mathcal{V}^\circ\supseteq(\mathcal{V}^\circ)^1)\otimes\widehat{O_{\mathscr{S},x}}$ there, and $\mathcal{D}_\bullet$ be the filtration on $\mathcal{V}$ given by reduction mod p of $\widehat{\mathcal{V}^\circ}_0\subseteq\mathcal{V}^\circ\otimes\widehat{O_{\mathscr{S},x}}$ in Lemma 3.3.2. For $\beta\in\mathbb{I}_+(R)$ with R a R_G-algebra, denote by $\beta^{(p)}$ its image of Frobenius pull back in $(\mathbb{I}_+^{(p)}/U_+^{(p)})(R)$, then the composition

$$\oplus\,\mathrm{gr}_i^D(V_R^\vee)\xrightarrow{(\phi_0^{-1}\oplus\phi_1^{-1})\otimes 1}\oplus\,\mathrm{gr}_C^i(V_R^{\vee,(p)})\xrightarrow{\beta^{(p)}}\oplus\,\mathrm{gr}_{\mathcal{C}}^i(\mathcal{V}_R^{(p)})\xrightarrow{\varphi_\bullet^{\mathrm{lin}}\otimes 1}\oplus\,\mathrm{gr}_i^{\mathcal{D}}(\mathcal{V}_R)$$

is in $\mathbb{I}_-/U_-^{(p)}(R)$. This gives the morphism $\iota{:}\,\mathbb{I}_+^{(p)}/U_+^{(p)}\to\mathbb{I}_-/U_-^{(p)}$, which is $L^{(p)}$-equivariant as $\phi_0^{-1}\oplus\phi_1^{-1}$ is so. Moreover, we have $(\mathbb{I},\mathbb{I}_+,\mathbb{I}_-,\iota)\cong\underline{I}_{ug_t}$.

3.4 Construction of the G-zip on the Reduction of the Integral Model

Let us write $\mathcal{A}$ (resp. $\mathcal{V}, s_{\mathrm{dR}}$) for the restriction of $\mathcal{A}$ (resp. $\mathcal{V}^{\circ}, s_{\mathrm{dR}}$) to $\mathscr{S}_0$. Notation as in Section 3.3.2, we will now explain how to get a G-zip on $\mathscr{S}_0$ using $\mathcal{A}[p]$. Let $\varphi: \mathcal{A}[p] \to \mathcal{A}[p]^{(p)}$ and $v: \mathcal{A}[p]^{(p)} \to \mathcal{A}[p]$ the Frobenius and Verschiebung respectively; then the sequences

$$\mathcal{A}[p] \xrightarrow{\varphi} \mathcal{A}[p]^{(p)} \xrightarrow{v} \mathcal{A}[p] \quad \text{and} \quad \mathcal{A}[p]^{(p)} \xrightarrow{v} \mathcal{A}[p] \xrightarrow{\varphi} \mathcal{A}[p]^{(p)}$$

are exact. After applying the contravariant Dieudonné functor, we get exact sequences

$$\mathcal{V} \xrightarrow{v} \mathcal{V}^{(p)} \xrightarrow{\varphi} \mathcal{V} \quad \text{and} \quad \mathcal{V}^{(p)} \xrightarrow{\varphi} \mathcal{V} \xrightarrow{v} \mathcal{V}^{(p)}.$$

Let $\delta: \mathcal{V} \to \mathcal{V}^{(p)}$ the Frobenius semi-linear map $x \mapsto x \otimes 1$, we write $\mathcal{C}^{\bullet}$ for the descending filtration given by

$$\mathcal{C}^0 := \mathcal{V} \supseteq \mathcal{C}^1 := \mathrm{Ker}(\varphi \circ \delta) \supseteq \mathcal{C}^2 := 0,$$

and $\mathcal{D}_{\bullet}$ for the ascending filtration given by

$$\mathcal{D}_{-1} := 0 \subseteq \mathcal{D}_0 := \mathrm{Im}(\varphi) \subseteq \mathcal{D}_1 := \mathcal{V}.$$

Let $\varphi_0: \mathcal{C}^0/\mathcal{C}^1 \to \mathcal{D}_0$ be the natural map induced by $\varphi \circ \delta$. Note that v induces an isomorphism $\mathcal{V}/\mathrm{Im}(\varphi) \xrightarrow{\simeq} \mathrm{Ker}(\varphi)$, whose inverse will be denoted by v^{-1}. Let $\varphi_1: \mathcal{C}^1 \to \mathcal{D}_1/\mathcal{D}_0$ be the map $v^{-1} \circ (\delta|_{\mathcal{C}^1})$; then the tuple $(\mathcal{V}, \mathcal{C}^{\bullet}, \mathcal{D}_{\bullet}, \varphi_{\bullet})$ is an F-zip over $\mathscr{S}_0$.

Let $C^{\bullet}$ and $D_{\bullet}$ be filtrations on $V_{\kappa}^{\vee}$ introduced at the beginning of Section 3.2.6.

Theorem 3.4.1

(i) *Let $I \subseteq \mathbf{Isom}_{\mathscr{S}_0}(V_{\kappa}^{\vee} \otimes O_{\mathscr{S}_0}, \mathcal{V})$ be the closed subscheme defined as*

$$I := \mathbf{Isom}_{\mathscr{S}_0}\big((V_{\kappa}^{\vee}, s) \otimes O_{\mathscr{S}_0},\ (\mathcal{V}, s_{\mathrm{dR}})\big).$$

Then I is a G_{κ}-torsor over $\mathscr{S}_0$.

(ii) *Let $I_+ \subseteq I$ be the closed subscheme*

$$I_+ := \mathbf{Isom}_{\mathscr{S}_0}\big((V_{\kappa}^{\vee}, s, C^{\bullet}) \otimes O_{\mathscr{S}_0},\ (\mathcal{V}, s_{\mathrm{dR}}, \mathcal{C}^{\bullet})\big).$$

Then I_+ is a P_+-torsor over $\mathscr{S}_0$.

(iii) *Let $I_- \subseteq I$ be the closed subscheme*

$$I_- := \mathbf{Isom}_{\mathscr{S}_0}\big((V_{\kappa}^{\vee}, s, D_{\bullet}) \otimes O_{\mathscr{S}_0},\ (\mathcal{V}, s_{\mathrm{dR}}, \mathcal{D}_{\bullet})\big).$$

Then I_- is a $P_-^{(p)}$-torsor over $\mathscr{S}_0$.

(iv) *The σ-linear maps φ_0 and φ_1 induce an isomorphism*

$$\iota: I_+^{(p)}/U_+^{(p)} \longrightarrow I_-/U_-^{(p)}$$

of $L^{(p)}$-torsors over $\mathscr{S}_0$.

Hence the tuple (I, I_+, I_-, ι) is a G-zip over $\mathscr{S}_0$.

Proof By construction, $G(S)$ acts simply transitively on $I(S)$ for any $\mathscr{S}_0$-scheme S, so if $I(S) \neq \varnothing$, the morphism $I_S \times_S G_S \to I_S \times_S I_S$, $(t, g) \mapsto (t, t\cdot g)$ is an isomorphism. To prove (i), it suffices to show that I is smooth over $\mathscr{S}_0$ with non-empty fibers. The non-emptiness of I_x for a closed point $x \in \mathscr{S}_0$ follow from Theorem 3.1.2(iii)(a). For smoothness, by Lemma 3.3.1(i), $I \to \mathscr{S}_0$ is smooth after base-change to the complete

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local rings at stalks of closed points. And hence $I \to \mathscr{S}_0$ is smooth at the stalk of each closed point of $\mathscr{S}_0$. But this implies that it is smooth at an open neighborhood for each closed point, and hence smooth.

(ii) follows from Corollary 3.2.7(i) and Lemma 3.3.1(ii) using the same strategy.

To prove (iii), we also use the same strategy. Take a point $x \in \mathscr{S}_0$, we consider $I_- \times_{\mathscr{S}_0} \operatorname{Spec}(\widehat{O_{\mathscr{S}_0,x}})$. We claim that

$$I_- \times_{\mathscr{S}_0} \operatorname{Spec}(\widehat{O_{\mathscr{S}_0,x}}) \cong \mathbb{I}_-|_{\operatorname{Spec}(\widehat{O_{\mathscr{S}_0,x}})}.$$

To see this, using notations in Lemma 3.3.2, we only need to show that

$$(\mathcal{V}, s_{\mathrm{dR}}, \mathcal{D}_\bullet) \otimes \widehat{O_{\mathscr{S}_0,x}} \cong (\mathcal{V}^\circ \otimes \widehat{O_{\mathscr{S},x}}, s_{\mathrm{dR}} \otimes 1, \widehat{\mathcal{V}^\circ}_0 \subseteq \mathcal{V}^\circ \otimes \widehat{O_{\mathscr{S},x}}) \otimes \widehat{O_{\mathscr{S}_0,x}}.$$

But by our construction, $\widehat{\mathcal{V}^\circ}_0 \subseteq \mathcal{V}^\circ \otimes \widehat{O_{\mathscr{S},x}}$ is the submodule generated by $\varphi(\widehat{\mathcal{V}^\circ}^0)$, and the composition

$$\widehat{\mathcal{V}^\circ}^0 \otimes \widehat{O_{\mathscr{S}_0,x}} \subseteq \mathcal{V}^\circ \otimes \widehat{O_{\mathscr{S}_0,x}} \twoheadrightarrow (\mathcal{V}^\circ \otimes \widehat{O_{\mathscr{S}_0,x}})/(\widehat{\mathcal{V}^\circ}^1 \otimes \widehat{O_{\mathscr{S}_0,x}})$$

is an isomorphism, as it has an inverse pr_1. So $\widehat{\mathcal{V}^\circ}_0 \otimes \widehat{O_{\mathscr{S}_0,x}} = \mathfrak{I}(\varphi)$ in $\mathcal{V}^\circ \otimes \widehat{O_{\mathscr{S}_0,x}}$, and this proves (iii).

For (iv), the same argument as before Remark 3.2.8 works. For $\beta \in I_+(R)$ with $\operatorname{Spec}(R)$ an affine scheme over $\mathscr{S}_0$, denote by $\beta^{(p)}$ its image of Frobenius pull back in $(I_+^{(p)}/U_+^{(p)})(R)$, the composition

$$\oplus \operatorname{gr}_i^D(V_R^\vee) \xrightarrow{(\phi_0^{-1} \oplus \phi_1^{-1}) \otimes 1} \oplus \operatorname{gr}_C^i(V_R^{\vee,(p)}) \xrightarrow{\beta^{(p)}} \oplus \operatorname{gr}_{\mathcal{C}}^i(\mathcal{V}_R^{(p)}) \xrightarrow{\varphi_\bullet^{\mathrm{lin}} \otimes 1} \oplus \operatorname{gr}_i^{\mathcal{D}}(\mathcal{V}_R)$$

is in $I_-/U_-^{(p)}(R)$. This induces a morphism $\iota: I_+^{(p)}/U_+^{(p)} \to I_-/U_-^{(p)}$, which is $L^{(p)}$-equivariant, as $\phi_0^{-1} \oplus \phi_1^{-1}$ is so. ∎

4 Ekedahl–Oort Strata for Shimura Varieties of Hodge Type

4.1 Basic Properties of Ekedahl–Oort Strata

In this section, we will define Ekedahl–Oort strata for Shimura varieties of Hodge type, and study their basic properties. Let G, V, μ, P_+, P_-, and L be as in Section 3.3.2, and let (I, I_+, I_-, ι) be the G-zip constructed in the revious theorem.

Definition 4.1.1 The G-zip (I, I_+, I_-, ι) on $\mathscr{S}_0$ induces a morphism of smooth algebraic stacks $\zeta: \mathscr{S}_0 \to G\text{-}\mathsf{Zip}_\kappa^\mu$. For a point x in the topological space of $G\text{-}\mathsf{Zip}_\kappa^\mu \otimes \bar{\kappa}$, the Ekedahl–Oort stratum in $\mathscr{S}_0 \otimes \bar{\kappa}$ associated with x is defined to be $\zeta^{-1}(x)$.

Now we will state our main result.

Theorem 4.1.2 *The morphism $\zeta: \mathscr{S}_0 \to G\text{-}\mathsf{Zip}_\kappa^\mu$ is smooth.*

Proof By Theorem 2.2.5, $G_\kappa \to G\text{-}\mathsf{Zip}^\mu_\kappa$ is an $E_{G,\mu}$-torsor. Let k be $\overline{\kappa}$, to prove that ζ is smooth, it suffices to prove that in the cartesian diagram

$$\begin{tikzcd} \mathscr{S}^\#_{0,k} \arrow[r] \arrow[d, "\zeta^\#"] & \mathscr{S}_{0,k} \arrow[d] \\ G_k \arrow[r] & G\text{-}\mathsf{Zip}^\mu_\kappa \otimes k, \end{tikzcd}$$

the morphism $\zeta^\#$ is smooth. Note that $\mathscr{S}^\#_{0,k}$ and G_k are both smooth over k, so to show that $\zeta^\#$ is smooth, it suffices to show that the tangent map at each closed point is surjective (see [5, Chapter 3, Theorem 10.4]).

Let $x^\# \in \mathscr{S}^\#_{0,k}$ be a closed point, its image in $\mathscr{S}_{0,k}$ is denoted by x, which is also a closed point. Let R_G be as in Section 3.3.2, which is the reduction modulo p of the universal deformation ring at x. Consider the cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\alpha"] & \mathrm{Spec}(R_G) \arrow[d] \\ G_k \arrow[r] & G\text{-}\mathsf{Zip}^\mu_\kappa \otimes k. \end{tikzcd}$$

The morphism $X \to \mathrm{Spec}(R_G)$ is a trivial $E_{G,\mu}$-torsor by our construction at the very end of Section 3.3.2: the G-zip over R_G is isomorphic to $\underline{I}_{ug_t}$ (see Constructions 2.2.3 and 2.3.4). The R_G-point ug_t of G gives a trivialization of the $E_{G,\mu}$-torsor X over R_G. This trivialization induces an isomorphism from $\mathrm{Spec}(R_G) \times_k (E_{G,\mu})_k$ to X that translates α into the morphism $\beta\colon \mathrm{Spec}(R_G) \times_k (E_{G,\mu})_k \to G_k$ that sends, for any k-scheme T, a point (u, l, u_+, u_-) to $lu_+ug_t(l^{(p)}u_-)^{-1}$ (see Equation (2.1) and the line following it, and note that as k-scheme, $E_{G,\mu} = L \times U_+ \times U_-^{(p)}$, and that R_G is the complete local ring of U_- at the origin).

Let $k[\epsilon] = k[x]/(x^2)$, and $g \in G(k[\epsilon])$ be a deformation of g_t. By viewing g_t as an element in $G(k[\epsilon])$, we get a $g_0 \in \mathrm{Lie}(G_k) = \mathrm{Ker}(G(k[\epsilon]) \to G(k))$ such that $g = g_0 g_t$. By [1, Chapter IV, Proposition 14.21(iii)], the product map $L \times U_{+,k} \times U_{-,k} \to G_k$ is an open immersion, so there exists $u \in \mathrm{Lie}(U_{-,k}) = R_G(k[\epsilon])$, $l \in \mathrm{Lie}(L_k)$ and $u_+ \in \mathrm{Lie}(U_{+,k})$ such that $lu_+u = g_0$. Noting that $l^{(p)} = \mathrm{id}$, we see that $\beta(u, l, u_+, \mathrm{id}) = g$, which proves the theorem. ∎

4.1.1 Dimension and Closure of a Stratum

Thanks to Theorem 4.1.2, the combinatory description for the topological space of $[E_{G,\mu}\backslash G_\kappa]$ developed in [20] can be used to describe Ekedahl–Oort strata for reduction of a Hodge type Shimura variety, and gives dimension formula and closure for each stratum. We will first present some notations and technical results following [20, 24], and then state how to use them.

Let $B \subseteq G$ be a Borel subgroup, and $T \subseteq B$ be a maximal torus. Note that such a B exists by [11, Theorem 2], and such a T exists by [2, XIV Theorem 1.1]. Let $W(B,T) := \mathrm{Norm}_G(T)(\overline{\kappa})/T(\overline{\kappa})$ be the Weyl group, and $I(B,T)$ be the set of simple reflections defined by $B_{\overline{\kappa}}$. Let φ be the Frobenius on G given by the p-th power. It induces an

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isomorphism $(W(B,T),W(B,T)) \to (W(B,T),W(B,T))$ of Coxeter systems still denoted by φ.

A priori the pair $(W(B,T),I(B,T))$ depends on the pair (B,T). However, any other pair (B',T') with $B' \subseteq G_{\overline{\kappa}}$ a Borel subgroup and $T' \subseteq B'$ a maximal torus is obtained on conjugating $(B_{\overline{\kappa}},T_{\overline{\kappa}})$ by some $g \in G(\overline{\kappa})$ which is unique up to right multiplication by $T_{\overline{\kappa}}$. So conjugation by g induces isomorphisms $W(B,T) \to W(B',T')$ and $I(B,T) \to I(B',T')$ that are independent of g. Moreover, the morphisms attached to any three of such pairs are compatible, so we will simply write (W,I) for $(W(T),I(B,T))$, and view it as 'the' Weyl group with 'the' set of simple reflections.

The cocharacter $\mu\colon \mathbb{G}_m \to G_\kappa$ as in Section 4.1 gives a parabolic subgroup P_+, and hence a subset $J \subseteq I$ by taking simple roots whose inverse are roots of P_+. Let W_J the subgroup of W generated by J, and ${}^JW$ be the set of elements w such that w is the element of minimal length in some coset W_Jw'. Note that there is a unique element in W_Jw' of minimal length, and each $w \in W$ can be uniquely written as $w = w_J{}^Jw$ with $w_J \in W_J$ and ${}^Jw \in {}^JW$. In particular, ${}^JW$ is a system of representatives of $W_J\backslash W$.

Furthermore, if K is a second subset of I, then for each w, there is a unique element in W_JwW_K which is of minimal length. We will denote by ${}^JW^K$ the set of elements of minimal length, and it is a set of representatives of $W_J\backslash W/W_K$.

Let ω_0 be the element of maximal length in W, set $K := {}^{\omega_0}\varphi(J)$. Here we write ${}^gJ$ for gJg^{-1}. Let $x \in {}^KW^{\varphi(J)}$ be the element of minimal length in $W_K\omega_0W_{\varphi(J)}$. Then x is the unique element of maximal length in ${}^KW^{\varphi(J)}$ (see [24, 5.2]). There is a partial order $\preceq$ on ${}^JW$, defined by $w' \preceq w$ if and only if there exists $y \in W_J$, $yw'x\varphi(y^{-1})x^{-1} \le w$ (see [24, Definition 5.8]). Here $\le$ is the Bruhat order (see [24, A.2] for the definition). The partial order $\preceq$ makes ${}^JW$ into a topological space.

Now we can state the the main result of Pink–Wedhorn–Ziegler that gives a combinatory description of the topological space of $[E_{G,\mu}\backslash G_\kappa]$ (and hence $G\text{-Zip}^{\mu}_{\kappa}$).

Theorem 4.1.3 *For $w \in {}^JW$, and $T' \subseteq B' \subseteq G_{\overline{\kappa}}$ with T' (resp. B') a maximal torus (resp. Borel) of $G_{\overline{\kappa}}$ such that $T' \subseteq L_{\overline{\kappa}}$ and $B' \subseteq P^{(p)}_{-,\overline{\kappa}}$, let $g,\dot{w} \in \mathrm{Norm}_{G_{\overline{\kappa}}}(T')$ be a representative of $\varphi^{-1}(x)$ and w respectively, and $G^w \subseteq G_{\overline{\kappa}}$ be the $E_{G,\mu}$-orbit of $gB'\dot{w}B'$. Then*

(i) *The orbit G^w does not depends on the choices of $\dot{w}$, T', B' or g.*
(ii) *The orbit G^w is a locally closed smooth subvariety of $G_{\overline{\kappa}}$. Its dimension is $\dim(P)+l(w)$. Moreover, G^w consists of only one $E_{G,\mu}$-orbit. So G^w is actually the orbit of $g\dot{w}$.*
(iii) *Denote by $\big|[E_{G,\mu}\backslash G_\kappa]\otimes\overline{\kappa}\big|$ the topological space of $[E_{G,\mu}\backslash G_\kappa]\otimes\overline{\kappa}$, and still write ${}^JW$ for the topological space induced by the partial order $\preceq$. Then the association $w \mapsto G^w$ induces a homeomorphism ${}^JW \to \big|[E_{G,\mu}\backslash G_\kappa]\otimes\overline{\kappa}\big|$.*

Proof By [20, Lemma 12.11], (B',T',g) is a frame of $(G_{\overline{\kappa}},P_{+,\overline{\kappa}},P^{(p)}_{-,\overline{\kappa}},\varphi)$ in the sense of [20, Definition 3.6]. Here $\varphi\colon P_{+,\overline{\kappa}}/U_{+,\overline{\kappa}} \to P^{(p)}_{-,\overline{\kappa}}/U^{(p)}_{-,\overline{\kappa}}$ is the morphism induced by the relative Frobenius of L. So the first statement is [20, Proposition 5.8], the second statement is [20, Theorem 1.3, Proposition 7.3, and Theorem 7.5], and the third statement is [20, Theorem 1.4]. ■

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The next statement (including its proof) is a word by word adaptation of results in [24] (to be more precise, Theorem 6.1, Corollary 10.2, and Proposition 10.3).

Proposition 4.1.4 *Let J be the type of P_+; then the Ekedahl–Oort strata are given by the finite set ${}^JW$. For $w \in {}^JW$, the stratum $\mathscr{S}_0^w$ is smooth and equi-dimensional of dimension $l(w)$ if $\mathscr{S}_0^w \neq \varnothing$. Moreover, the closure of $\mathscr{S}_0^w$ is the union of $\mathscr{S}_0^{w'}$ such that $w' \leq w$.*

Proof The first statement follows from our definition of Ekedahl–Oort strata and Theorem 4.1.3(iii). For the second one, note that by Theorem 4.1.3(ii), each G_w is equi-dimensional of codimension $\dim(U_-) - l(w)$ in G_κ, so each $\mathscr{S}_0^w$ is equi-dimensional of codimension $\dim(U_-) - l(w)$ in $\mathscr{S}_{0,\overline{\kappa}}$, as ζ is smooth by Theorem 4.1.2. So the dimension of $\mathscr{S}_0^w$ is $l(w)$, as $\dim(\mathscr{S}_0) = \dim(U_-)$.

The smoothness of each stratum follows from a direct adaption of the proof of [24, Proposition 10.3]. More precisely, let $w\colon \operatorname{Spec}(\overline{\kappa}) \to E_{G,\mu}\backslash G_\kappa$ be a point. Then its reduced gerbe $(E_{G,\mu}\backslash G_\kappa)^w$ is smooth. And hence $\zeta^{-1}((E_{G,\mu}\backslash G_\kappa)^w)$ is smooth. But $\mathscr{S}_0^w$ is reduced with the same topological space as $\zeta^{-1}((E_{G,\mu}\backslash G_\kappa)^w)$, so $\mathscr{S}_0^w$ is smooth. For the last statement, by Theorem 4.1.3(iii), the closure of $\{w\}$ in $\left|[E_{G,\mu}\backslash G_\kappa] \otimes \overline{\kappa}\right|$ is $\{w' \mid w' \leq w\}$. So $\overline{\mathscr{S}_0^w} = \zeta^{-1}(\overline{w})$ by the universally-openness of ζ. ∎

Remark 4.1.5 There is a unique maximal element in ${}^JW$; its corresponding stratum is called the ordinary stratum. It is open and non-empty by Theorem 4.1.2 and hence dense by Proposition 4.1.4. There is also a unique minimal element in ${}^JW$, namely the element 1. Its corresponding stratum is called the superspecial stratum which is expected to be non-empty (but we cannot prove it now)[1]. The non-emptiness of the superspecial stratum implies that every stratum is non-empty, as ζ is an open map by Theorem 4.1.2.

4.2 F-zips with Additional Structure

In this subsection, we will describe additional structure on F-zips associated with reductions of Shimura varieties of Hodge type, and show how to generalize the the theory in [24] to study Ekedahl–Oort strata for Shimura varieties of Hodge type.

4.2.1 Description of the additional structures

Let (G, V, μ, s) be as at the beginning of 4.2.1, and $C^\bullet$, $D_\bullet$ be the filtrations on $V_\kappa^\vee$ introduced at the beginning of 3.2.6.

Definition 4.2.1 Let S be a scheme over κ. By an F-zip of type (G, V, μ, s) with a Tate class s_{dR} over S, we mean an F-zip $(\mathcal{V}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \varphi_\bullet)$ over S equipped with a section $s_{\mathrm{dR}}\colon O_S \to \mathcal{V}^\otimes$, such that

[1]There are currently many announced proofs for non-emptiness of Newton strata (by Dong Uk Lee, Kisin-Madapusi Pera, and Chia-Fu Yu). Together with works of Kisin on the Langlands–Rapoport conjecture and those of Nie on fundamental elements, this implies the non-emptiness of the superspecial stratum.

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(i) $I := \mathbf{Isom}_{\mathscr{S}_0}\big(\,(V_\kappa^\vee, s)\otimes O_{\mathscr{S}_0},\ (\mathcal{V}, s_{\mathrm{dR}})\big) \subseteq \mathbf{Isom}_{\mathscr{S}_0}(V_\kappa^\vee \otimes O_{\mathscr{S}_0}, \mathcal{V})$ is a G_κ-torsor over $\mathscr{S}_0$,

(ii) $I_+ := \mathbf{Isom}_{\mathscr{S}_0}\big(\,(V_\kappa^\vee, s, C^\bullet)\otimes O_{\mathscr{S}_0},\ (\mathcal{V}, s_{\mathrm{dR}}, \mathcal{C}^\bullet)\big) \subseteq I$ is a right P_+-torsor over $\mathscr{S}_0$,

(iii) $I_- := \mathbf{Isom}_{\mathscr{S}_0}\big(\,(V_\kappa^\vee, s, D_\bullet)\otimes O_{\mathscr{S}_0},\ (\mathcal{V}, s_{\mathrm{dR}}, \mathcal{D}_\bullet)\big) \subseteq I$ is a right $P_-^{(p)}$-torsor over $\mathscr{S}_0$,

(iv) $s_{\mathrm{dR}}\colon O_S \to \mathcal{V}^\otimes$ is a Tate sub F-zip of weight 0, *i.e.*, the F-zip structure on $\mathcal{V}^\otimes$ restricted to O_S makes it a Tate F-zip of weight 0.

Remark 4.2.2 Condition (i) in the above definition implies that $s_{\mathrm{dR}}\colon O_S \hookrightarrow \mathcal{V}^\otimes$ is a locally direct summand. As there is an fpqc-cover T of S, such that $I(T) \neq \varnothing$. An element $t \in I(T)$ identifies

$$(k \xrightarrow{s} V^\otimes)_T \quad \text{and} \quad (O_S \xrightarrow{s_{\mathrm{dR}}} \mathcal{V}^\otimes)_T,$$

so $(\mathcal{V}^\otimes/O_S)_T \cong (V^\otimes/k)_T$ is free. Noting that being locally free is local for the fpqc topology for finitely generated modules, we see that $O_S \hookrightarrow \mathcal{V}^\otimes$ is a locally direct summand. The embedding s_{dR} is then admissible in the sense of Definition 2.1.6.

We will simply call an 'F-zip of type (G, V, μ, s) with a Tate class s_{dR}' an 'F-zip with a Tate class s_{dR}' for short. We denoted by $F\text{-Zip}_{s_{\mathrm{dR}}}(S)$ the category whose objects are F-zips with a Tate class s_{dR} over S, and whose morphisms are isomorphisms of F-zips respecting Tate classes.

Construction 4.2.3 There is a functor $\mathfrak{Z}\colon G\text{-Zip}_\kappa^\mu(S) \to F\text{-Zip}_{s_{\mathrm{dR}}}(S)$ as follows. For $(I, I_+, I_-, \iota) \in G\text{-Zip}_\kappa^\mu(S)$, we define

$$\mathcal{V} = I \times^G V_S^\vee, \quad \mathcal{C}^1 = I_+ \times^{P_+} C_S^1,$$

$$\mathcal{D}_0 = I_- \times^{P_-^{(p)}} D_{0,S}, \quad \text{and} \quad \oplus\,\varphi_i\colon \oplus\,\mathrm{gr}_{\mathcal{C}}^i(\mathcal{V}) \to \oplus\,\mathrm{gr}_i^{\mathcal{D}}(\mathcal{V})$$

to be the σ-linear map whose linearization is the morphism

$$\iota \times (\phi_0 \oplus \phi_1)\colon I_+^{(p)}/U_+^{(p)} \times^{L^{(p)}} (\oplus\,\mathrm{gr}_C^i(V_\kappa^\vee))_S^{(p)} \longrightarrow I_-/U_-^{(p)} \times^{L^{(p)}} (\oplus\,\mathrm{gr}_i^D(V_\kappa^\vee))_S.$$

Here ϕ_0 and ϕ_1 are as in Section 3.2.6. The condition that ι is $L^{(p)}$-equivariant implies that $\iota \times (\phi_0 \oplus \phi_1)$ is well defined. The same argument as in the previous remark shows that $\mathcal{C}^1 \subseteq \mathcal{V}$ and $\mathcal{D}_0 \subseteq \mathcal{V}$ are locally direct summands. So $(\mathcal{V}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \varphi_\bullet)$ is an F-zip that satisfies the first three conditions of Definition 4.2.1. The section $s_{\mathrm{dR}} \in \mathcal{V}^\otimes$ is the image of $I \times \{s\}$ in $\mathcal{V}^\otimes = I \times^G (V_S^\otimes)$. Let $\mathcal{C}^\bullet(\mathcal{V}^\otimes)$ (resp. $C^\bullet(V_\kappa^\otimes)$) be the filtration induced by $\mathcal{C}^\bullet$ (resp. $C^\bullet$), then $\mathcal{C}^0(\mathcal{V}^\otimes) = I_+ \times^{P_+} (C^0(V_\kappa^\otimes)_S)$, and s_{dR} is in $\mathcal{C}^0(\mathcal{V}^\otimes)$ as $s \in C^0(V_\kappa^\otimes)$ is G-invariant. Similarly, $s_{\mathrm{dR}} \in \mathcal{D}_0(\mathcal{V}^\otimes)$, and it induces injections $O_S \to \mathcal{C}^0(\mathcal{V}^\otimes)/\mathcal{C}^1(\mathcal{V}^\otimes)$ and $O_S \to \mathcal{D}_0(\mathcal{V}^\otimes)/\mathcal{D}_{-1}(\mathcal{V}^\otimes)$. The F-zip $(\mathcal{V}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \varphi_\bullet)$ induces an F-zips structure on $\mathcal{V}^\otimes$, and in particular a σ-linear isomorphism $\Phi_0\colon \mathcal{C}^0(\mathcal{V}^\otimes)/\mathcal{C}^1(\mathcal{V}^\otimes) \to \mathcal{D}_0(\mathcal{V}^\otimes)/\mathcal{D}_{-1}(\mathcal{V}^\otimes)$. The linearization of Φ_0 is the identity on $O_S = \mathrm{Im}(s_{\mathrm{dR}})$, as $\phi_0 \oplus \phi_1$ is so on s. So $s_{\mathrm{dR}}\colon O_S \to \mathcal{V}^\otimes$ is a Tate sub F-zip of weight zero.

Corollary 4.2.4 *The functor* $\mathfrak{Z}\colon G\text{–Zip}_\kappa^\mu(S) \to F\text{–Zip}_{s_{\mathrm{dR}}}(S)$ *induces an equivalence of categories.*

Proof We only need to construct a quasi-inverse $\mathfrak{G}\colon F\text{–}\mathrm{Zip}_{s_{\mathrm{dR}}}(S) \to G\text{-}\mathrm{Zip}^{\mu}_{\kappa}(S)$ of $\mathfrak{Z}$. Let $(\mathcal{V}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \varphi_\bullet)$ be an F-zip with a Tate class s_{dR}. By Definition 4.2.1, we already have (I, I_+, I_-), and hence only need to construct an isomorphism of $L^{(p)}$-torsors $\iota\colon I_+^{(p)}/U_+^{(p)} \to I_-/U_-^{(p)}$. As in 3.2.6, for $\beta \in I_+(R)$ with $\mathrm{Spec}(R)$ an affine scheme over S, denote by $\beta^{(p)}$ its image of Frobenius pull back in $(I_+^{(p)}/U_+^{(p)})(R)$, then condition (iv) of Definition 4.2.1 implies that the composition

$$\oplus\, \mathrm{gr}_i^D(V_R^\vee) \xrightarrow{(\phi_0^{-1}\oplus\phi_1^{-1})\otimes 1} \oplus\, \mathrm{gr}^i_C(V_R^{\vee,(p)}) \xrightarrow{\beta^{(p)}} \oplus\, \mathrm{gr}^i_{\mathcal{C}}(\mathcal{V}_R^{(p)}) \xrightarrow{\varphi_\bullet^{\mathrm{lin}}\otimes 1} \oplus\, \mathrm{gr}_i^{\mathcal{D}}(\mathcal{V}_R)$$

is in $I_-/U_-^{(p)}(R)$. This induces a morphism $\iota\colon I_+^{(p)}/U_+^{(p)} \to I_-/U_-^{(p)}$, which is $L^{(p)}$-equivariant as $\phi_0^{-1}\oplus\phi_1^{-1}$ is so. One checks easily that $\mathfrak{G}$ is a quasi-inverse of $\mathfrak{Z}$. ■

4.2.2 Defining Ekedahl–Oort Strata Using F-zips

In this section, we will follow the construction in [17, 24] to show that the Ekedahl–Oort strata defined using G-zips are the same as those defined using F-zips with a Tate class. The main technical tool is still [20]. Fix the datum (G, V, μ, s) as before, let Z_μ be the Zariski sheafification of the presheaf that associates with a κ-scheme S the set of F-zip structures $(C^\bullet, D_\bullet, \varphi_\bullet)$ on V_S with Tate class $s \otimes 1$.

Let Z'_μ be the Zariski sheafification of the presheaf that associates with a κ-scheme S the set of triples $(P, Q, U_Q g U_{\varphi(P)})$, where $P \subset G_S$ is of type J (the type of P_+ defined before), $Q \subset G_S$ is of type $\varphi(J)$ and $g \in G(S)$ is such that Q and $g\varphi(P)g^{-1}$ are in opposite position. By [17, Corollary 4.3], Z'_μ is represented by a smooth κ-scheme. By [7, Lemma 1.1.1] and the proof of Proposition 1.1.5, the construction of [17, Lemma 5.1] induces an isomorphism $Z_\mu \cong Z'_\mu$. We remark that for an affine scheme S, $Z_\mu(S)$ (resp. $Z'_\mu(S)$) is precisely the set of triples described above. We also remark that our construction of Z_μ is slightly different from X_τ defined in [17, 5.2]. We insists to fix the type of cocharacters inducing the filtrations, rather than the type of the filtrations. This kills the problem mentioned before [17, Corollary 6.2].

Now we will construct a morphism $Z_\mu \to [E_{G,\mu}\backslash G_\kappa]$. By definition, to give such a morphism is the same as to give an $E_{G,\mu}$-torsor H over Z_μ, equipped with an $E_{G,\mu}$-equivariant morphism $H \to G_\kappa$.

The F-zip $(V_\kappa, C^\bullet, D_\bullet, \phi_\bullet)$ constructed in 3.2.6 is an element of $Z_\mu(\kappa)$. Using the proof of [20, Lemma 12.5], the group $G_\kappa \times G_\kappa$ acts on Z_μ transitively via

$$(g, h)\cdot(C^\bullet, D_\bullet, \varphi_\bullet) = (gC^\bullet, hD_\bullet, h\varphi_\bullet g^{-1}),$$

where $h\varphi_i g^{-1}$ is the composition

$$g(C^i)/g(C^{i+1}) \xrightarrow{g^{-1}} C^i/C^{i+1} \longrightarrow D_i/D_{i-1} \xrightarrow{h} h(D_i)/h(D_{i-1}).$$

Under the above action, the stabilizer of $(V_\kappa, C^\bullet, D_\bullet, \phi_\bullet)$ is $E_{G,\mu}$ (by the proof of [20, Lemma 12.5]), and hence the action induces an $E_{G,\mu}$-torsor $G_\kappa \times G_\kappa \to Z_\mu$ which is G_κ-equivariant with respect to the diagonal action on $G_\kappa \times G_\kappa$ and the restriction to diagonal on Z_μ. The morphism $m\colon G_\kappa \times G_\kappa \to G_\kappa$, $(g, h) \mapsto g^{-1}h$ is a G_κ-torsor which is $E_{G,\mu}$-equivariant. By the same reason as in [20, Theorem 12.7], we get an isomorphism of stacks $\beta : [G_\kappa\backslash Z_\mu] \simeq [E_{G,\mu}\backslash G_\kappa]$ after passing to quotients.

Let I be $\mathbf{Isom}_{\mathscr{S}_0}((V_\kappa, s) \otimes O_{\mathscr{S}_0}, (\mathcal{V}, s_{\mathrm{dR}}))$ as before. There is a G_κ-equivariant morphism from the G_κ-torsor I to Z_μ, given by mapping $t \in I$ to the pull back via t of the F-zip structure on $\mathcal{V}$. This induces a morphism $\zeta' : \mathscr{S}_0 \to [G_\kappa \backslash Z_\mu]$. Our Ekedahl–Oort strata are defined by the morphism $\zeta : \mathscr{S}_0 \to [E_{G,\mu} \backslash G_\kappa]$ constructed in Section 4.1. But by what we have seen, one can identify $[G_\kappa \backslash Z_\mu]$ with $[E_{G,\mu} \backslash G_\kappa]$ via β. So it is natural to ask whether they induce the same theory of Ekedahl–Oort strata.

Proposition 4.2.5 *We have an equality $\beta \circ \zeta' = \zeta$.*

Proof By [20, 12.6], there is a cartesian diagram

$$\begin{tikzcd} G_\kappa \times G_\kappa \arrow[r, "m"] \arrow[d, "n"] & G_\kappa \arrow[d] \\ Z_\mu \arrow[r] & {[E_{G,\mu} \backslash G_\kappa]} \end{tikzcd}$$

whose vertical arrows are G_κ-equivariant $E_{G,\mu}$-torsors and horizontal arrows are $E_{G,\mu}$-equivariant G_κ-torsors. One only needs to check that the pull back to $G_\kappa \times G_\kappa$ of $\mathscr{S}^\# \to G_\kappa$ and $I \to Z_\mu$ are $G_\kappa \times E_{G,\mu}$-equivariantly isomorphic over $G_\kappa \times G_\kappa$.

Let $\widetilde{\mathscr{S}_0}$ be the pull back

$$\begin{tikzcd} \widetilde{\mathscr{S}_0} \arrow[r] \arrow[d] & \mathscr{S}_0^\# \arrow[d] \\ G_\kappa \times G_\kappa \arrow[r, "m"] & G_\kappa. \end{tikzcd}$$

For any T/κ,

$$\widetilde{\mathscr{S}_0}(T) = \left\{ (g_1, g_2, a, b) \mid g_i \in G_\kappa(T), (a, d) \in \mathscr{S}_0(T) \text{ such that } g_1^{-1} g_2 = a^{-1} b \right\}.$$

For any $(g, p_1, p_2) \in G_\kappa \times E_{G,\mu}(T)$, the action is given by

$$(g, p_1, p_2) \cdot (g_1, g_2, a, b) = (g g_1 p_1^{-1}, g g_2 p_2^{-1}, a p_1^{-1}, b p_2^{-1}).$$

Let $\widetilde{I}$ be the pull back

$$\begin{tikzcd} \widetilde{I} \arrow[r] \arrow[d] & I \arrow[d] \\ G_\kappa \times G_\kappa \arrow[r, "n"] & Z_\mu. \end{tikzcd}$$

For any T/κ,

$$\begin{aligned} \widetilde{I}(T) = \big\{ (g_1, g_2, t) \mid g_i \in G_\kappa(T), t \in I(T) \text{ such that } & (g_1(C^\bullet_T), g_2(D_{\bullet,T}), g_2 \phi_\bullet g_1^{-1}) \\ & = t^{-1}(\mathcal{C}^\bullet_T, \mathcal{D}_{\bullet,T}, \varphi_\bullet) \big\}, \end{aligned}$$

where $(\mathcal{V}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \varphi_\bullet)$ is the F-zip on $\mathscr{S}_0$ introduced at the beginning of Section 3.4.

For any $(g, p_1, p_2) \in G_\kappa \times E_{G,\mu}(T)$, the action is given by

$$(g, p_1, p_2) \cdot (g_1, g_2, t) = (g g_1 p_1^{-1}, g g_2 p_2^{-1}, g \cdot t).$$

There is a $G_\kappa \times G_\kappa$-morphism $\widetilde{\mathscr{S}_0} \to \widetilde{I}$ mapping (g_1, g_2, a, b) to $(g_1, g_2, a g_1^{-1})$. This is clearly an isomorphism. One also checks easily that it is $G_\kappa \times E_{G,\mu}$-equivariant. ■

5 Ekedahl–Oort Strata for CSpin-varieties

We apply our main results to CSpin Shimura varieties, which are typical examples of Shimura varieties of Hodge type but not necessarily of PEL type.

5.1 CSpin-Shimura Varieties

We explain what CSpin-Shimura varieties are and their integral canonical models follow [12].

Let V be a $n+2$-dimensional $\mathbb{Q}$-vector space with a quadratic form Q of signature $(n, 2)$. We will always assume that $n > 0$. Let $p > 2$ be a prime and $L \subseteq V$ be a $\mathbb{Z}_{(p)}$-lattice such that Q is non-degenerate on $L_{\mathbb{Z}_{(p)}}$ (*i.e.*, the bilinear form attached to Q induces an isomorphism $L \to L^\vee$). Let $C(L)$ and $C^+(L)$ be the Clifford algebra and even Clifford algebra respectively (see [12, 1.1]). Note that there is an embedding $L \hookrightarrow C(L)$ and an anti-involution $*$ on $C(L)$ (see [12, 1.1]).

Let $\mathrm{CSpin}(L)$ be the stabilizer in $C^+(L)^\times$ of $L \hookrightarrow C(L)$ with respect to the conjugation action of $C^+(L)^\times$ on $C(L)$. Then $\mathrm{CSpin}(L)$ is a reductive group over $\mathbb{Z}_{(p)}$. Consider the left action of $\mathrm{CSpin}(L)$ on $C(L)$. There is a perfect alternating form ψ on $C(L)$, such that the embedding $\mathrm{CSpin}(L) \hookrightarrow \mathrm{GL}(C(L))$ factors through $\mathrm{GSp}(C(L), \psi)$ which induces an embedding of Shimura data

$$\big(\mathrm{CSpin}(V), X\big) \longrightarrow \big(\mathrm{GSp}(C(V), \psi), X'\big).$$

We refer to [12, 1.8, 1.9, 3.4, 3.5] for details. Here X is the space of oriented negative 2-planes in $V_{\mathbb{R}}$, and X' is the union of Siegel half-spaces attached to $\mathrm{GSp}(C(V), \psi)$.

The above construction shows that $(\mathrm{CSpin}(V), X)$ is a Shimura datum of Hodge type. Let $K_p = \mathrm{CSpin}(L)(\mathbb{Z}_p)$ and $K^p \subseteq \mathrm{CSpin}(V)(\mathbb{A}_f^p)$ be a compact open subgroup which is small enough. Let $K = K_p K^p$; then

$$\mathrm{Sh}_K := \mathrm{CSpin}(V)(\mathbb{Q})\backslash X \times \big(\mathrm{CSpin}(V)(\mathbb{A}_f)/K\big)$$

has a canonical model over $\mathbb{Q}$ that will again be denoted by Sh_K. Moreover, Kisin's main theorem on existence of integral canonical models implies that Sh_K has an integral canonical model $\mathscr{S}_K$ over $\mathbb{Z}_{(p)}$.

5.2 Ekedahl–Oort Strata for CSpin-varieties

Let $\mathscr{S}_0$ the special fiber of $\mathscr{S}_K$. The Shimura datum determines a cocharacter $\mu: \mathbb{G}_{m,\mathbb{Z}_p} \to \mathrm{CSpin}(L_{\mathbb{Z}_p})$, which is unique up to conjugation. The special fiber of μ will still be denoted by μ. The cocharacter μ determines a parabolic subgroup $P_+ \subseteq \mathrm{CSpin}(L_{\mathbb{F}_p})$, whose type will be denoted by J. Let W be the Weyl group of $\mathrm{CSpin}(L_{\mathbb{F}_p})$, and let ${}^J W$ be as in Section 4.1.1 The set ${}^J W$ is equipped with a partial order $\preceq$ (see Section 4.1.1, before Theorem 4.1.3). Then Proposition 4.1.4 implies that the structure of Ekedahl–Oort stratification on $\mathscr{S}_0$ is described by ${}^J W$ together with the partial order $\preceq$.

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All we need is a combinatorial description of $({}^J W, \preceq)$. But everything reduces to the computations in [28], after identifying the Weyl group of $\mathrm{CSpin}(L_{\mathbb{F}_p})$ with that of $\mathrm{SO}(L_{\mathbb{F}_p})$.

5.2.1 A Description of $({}^J W, \preceq)$

Let us recall the description of $({}^J W, \preceq)$ in [28]. Let m be the dimension of a maximal torus in $\mathrm{SO}(L_{\mathbb{F}_p})$.

(a) If n is odd, then the partial order $\preceq$ on ${}^J W$ is a total order, and the length function induces an isomorphism of totally ordered sets $({}^J W, \preceq) \to \{0, 1, 2, \dots, n\}$.

(b) If n is even, noting that in this case $n + 2 = 2m$, then W is generated by simple reflections $\{s_i\}_{i=1,\dots,m}$, where

$$s_i = \begin{cases} (i, i+1)(n-i+2, n-i+3), & \text{for } i = 1, \dots, m-1, \\ s_m = (m-1, m+1)(m, m+2), & \text{for } i = m. \end{cases}$$

Let

$$w_i = \begin{cases} s_1 s_2 \cdots s_i, & \text{for } i \leq m-1, \\ s_1 s_2 \cdots s_m, & \text{for } i = m, \\ s_1 s_2 \cdots s_m s_{m-2} \cdots s_{2m-i-1}, & \text{for } i \geq m+1. \end{cases}$$

and w'_{m-1} be $s_1 s_2 \cdots s_{m-2} s_m$. Then ${}^J W = \{w_i\}_{0 \leq i \leq n} \cup \{w'_{m-1}\}$, and the partial order $\preceq$ is given by

$$w_0 = \mathrm{id} \preceq w_1 \preceq \cdots \preceq w_{m-2} \preceq w_{m-1}, w'_{m-1} \preceq w_m \preceq \cdots \preceq w_n.$$

Now we can describe structure of the Ekedahl–Oort stratification on $\mathscr{S}_0$.

Corollary 5.2.1 *Let m and n be as before.*

(i) *There are at most $2m$ Ekedahl–Oort strata on $\mathscr{S}_0$.*

(ii) (a) *If n is odd, then for any integer $0 \leq i \leq n$, there is at most one stratum $\mathscr{S}_0^i$ such that $\dim(\mathscr{S}_0^i) = i$. These are all the Ekedahl–Oort strata on $\mathscr{S}_0$. Moreover, the Zariski closure of $\mathscr{S}_0^i$ is the union of all the $\mathscr{S}_0^{i'}$ such that $i' \leq i$.*

(b) *If n is even and positive, then for any integer i such that $0 \leq i \leq n$ and $i \neq n/2$, there is at most one stratum $\mathscr{S}_0^i$ such that $\dim(\mathscr{S}_0^i) = i$. There are at most 2 strata of dimension $n/2$. These are all the Ekedahl–Oort strata on $\mathscr{S}_0$. Moreover, the Zariski closure of the stratum $\mathscr{S}_0^w$ is the union of $\mathscr{S}_0^w$ with all the strata whose dimensions are smaller than $\dim(\mathscr{S}_0^w)$.*

Proof Apply Proposition 4.1.4 together with Section 5.2.1. ∎

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