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A NOTE ON INHOMOGENEOUS DIOPHANTINE APPROXIMATION IN BETA-DYNAMICAL SYSTEM

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Abstract

We study the distribution of the orbits of real numbers under the beta-transformation T_β for any $\beta > 1$. More precisely, for any real number $\beta > 1$ and a positive function $\varphi : \mathbb{N} \to \mathbb{R}^+$, we determine the Lebesgue measure and the Hausdorff dimension of the following set:

$$E(T_\beta, \varphi) = \{(x, y) \in [0, 1] \times [0, 1] : |T_\beta^n x - y| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

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1. Introduction

In 1957, Rényi [13] introduced the beta-expansions of real numbers as a generalisation of the familiar integer base expansions. Since then, the study of the beta-expansion has attracted considerable interest. The corresponding beta-dynamical system has recently received much attention. One of the most important problems of the beta-dynamical system is to study the distribution of the orbits.

Let $\beta > 1$ be a real number and $T_\beta : [0, 1] \to [0, 1]$ the transformation defined by

$$T_\beta(x) = \beta x \pmod 1 \quad \text{for any } x \in [0, 1].$$

This map generates the beta-dynamical system $([0, 1], T_\beta)$. Since T_β is ergodic for the well-known Parry measure ν_β on $[0, 1]$ (see Section 2), equivalent to the Lebesgue measure $\mathcal{L}$, Birkhoff's ergodic theorem yields that for $\mathcal{L}$-almost all $x \in [0, 1]$, the orbit is normally distributed in $[0, 1]$ with respect to ν_β. Therefore, for any $x_0 \in [0, 1]$ and $\mathcal{L}$-almost all $x \in [0, 1]$,

$$\liminf_{n\to\infty} |T_\beta^n x - x_0| = 0. \tag{1.1}$$

It is a natural question to ask about the speed of convergence in (1.1). This leads to the study of the Diophantine properties of the orbits in the beta-dynamical system

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in analogy with the classical theory of Diophantine approximation. This study contributes to a better understanding of the distribution of the orbits in the beta-dynamical system.

In 1967, Philipp [12] proved that for any $\beta > 1$, the transformation T_β is not only strongly mixing, but also the dynamical Borel–Cantelli lemma holds. More precisely, given a sequence of balls $\{B(x_0, r_n)\}_{n\geq 1}$ with centre $x_0 \in [0, 1]$ and shrinking radius $\{r_n\}_{n\geq 1}$, let

$$D(T_\beta, \{r_n\}_{n\geq 1}, x_0) = \{x \in [0, 1] : |T_\beta^n x - x_0| < r_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

Philipp proved that

$$\mathcal{L}(D(T_\beta, \{r_n\}_{n\geq 1}, x_0)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{+\infty} r_n < +\infty, \\ 1 & \text{if } \sum_{n=1}^{+\infty} r_n = +\infty. \end{cases}$$

This is a typical example of the shrinking target problem [6] related to the Diophantine properties of the orbits in a dynamical system.

In the case that $\sum_{n=1}^{+\infty} r_n < +\infty$, the set $D(T_\beta, \{r_n\}_{n\geq 1}, x_0)$ consists of points whose orbits have good approximation properties near the point x_0 and has null measure. Inspired by the Jarník–Besicovitch theorem [1, 7], Shen and Wang [17] studied the Hausdorff dimension of the set $D(T_\beta, \{r_n\}_{n\geq 1}, x_0)$ when $\sum_{n=1}^{+\infty} r_n < +\infty$, and found that its size is related to the sequence $\{r_n\}_{n\geq 1}$ in the sense that

$$\dim_H D(T_\beta, \{r_n\}_{n\geq 1}, x_0) = \frac{1}{1+\alpha} \quad \text{with } \alpha = \liminf_{n\to\infty} \frac{\log_\beta r_n^{-1}}{n}.$$

Notice that in the above results about $D(T_\beta, \{r_n\}_{n\geq 1}, x_0)$, the point x_0 is always assumed to be fixed. One can then ask, what will happen if the point x_0 is not fixed? In particular, what can one say about the metric properties of the set

$$\{(x, y) \in [0, 1] \times [0, 1] : |T_\beta^n x - y| < r_n \text{ for infinitely many } n \in \mathbb{N}\}$$

in the sense of measure and in the sense of dimension? Let $\beta > 1$ be any real number and let $\varphi : \mathbb{N} \to \mathbb{R}^+$ be a positive function. In this note, we determine the Lebesgue measure and the Hausdorff dimension of the set

$$E(T_\beta, \varphi) = \{(x, y) \in [0, 1] \times [0, 1] : |T_\beta^n x - y| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

The main results are the following theorems.

Theorem 1.1. *Let $\varphi : \mathbb{N} \to \mathbb{R}^+$ be a positive function. For any $\beta > 1$,*

$$\mathcal{L}^2(E(T_\beta, \varphi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{+\infty} \varphi(n) < +\infty, \\ 1 & \text{if } \sum_{n=1}^{+\infty} \varphi(n) = +\infty, \end{cases}$$

where $\mathcal{L}^2$ denotes the two-dimensional Lebesgue measure.

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THEOREM 1.2. *Let $\varphi : \mathbb{N} \to \mathbb{R}^+$ be a positive function with $\sum_{n=1}^{+\infty} \varphi(n) < +\infty$. For any $\beta > 1$,*

$$\dim_H E(T_\beta, \varphi) = 1 + \frac{1}{1+\alpha}, \quad \text{where } \alpha = \liminf_{n\to\infty} \frac{\log_\beta \varphi(n)^{-1}}{n}.$$

We would like to make a remark about our motivation. Besides the Jarník–Besicovitch theorem, many classical results of metric Diophantine approximation can find their traces in the beta-dynamical system. For any $x_0 \in [0, 1]$ and $\psi : \mathbb{N} \to \mathbb{R}^+$ a nonincreasing positive function, let

$$F(\psi, x_0) = \{x \in [0, 1] : \|nx - x_0\| < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\},$$

where $\|x\|$ denotes the distance of the real number x to the closest integer. By appealing to Schmidt's very general form of the Khintchine–Groshev theorem (see [15] and [16]), the Lebesgue measure of $F(\psi, x_0)$ can be determined by

$$\mathcal{L}(F(\psi, x_0)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{+\infty} \psi(n) < +\infty, \\ 1 & \text{if } \sum_{n=1}^{+\infty} \psi(n) = +\infty. \end{cases}$$

In the case $\sum_{n=1}^{+\infty} \psi(n) < +\infty$, Levesley [8] proved a general inhomogeneous Jarník–Besicovitch theorem, namely

$$\dim_H F(\psi, x_0) = \frac{2}{1+\gamma} \quad \text{with } \gamma = \liminf_{n\to\infty} \frac{\log \psi(n)^{-1}}{\log n}.$$

When the point x_0 is no longer assumed to be fixed, Dodson [3] studied the set

$$\widetilde{F}(\psi) = \{(x, y) \in [0, 1] \times [0, 1] : \|nx - y\| < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}$$

and proved that

$$\dim_H \widetilde{F}(\psi) = 1 + \frac{2}{1+\gamma} \quad \text{with } \gamma = \liminf_{n\to\infty} \frac{\log \psi(n)^{-1}}{\log n}.$$

The above discussion indicates that there is a natural correspondence between the metrical properties of the sets in metric Diophantine approximation and those for the beta-dynamical Diophantine approximation.

For more results related to the orbits in the beta-dynamical system, the reader is referred to the papers of Schmeling [14], Persson and Schmeling [11], Tan and Wang [18], Li *et al.* [9] and the references therein.

The rest of this paper is organised as follows: in the next section, we give some basic facts about beta-expansion and the beta-dynamical system. Theorems 1.1 and 1.2 will be proved in the last section.

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2. Properties of beta-expansion and the beta-dynamical system

Let $\beta > 1$ be a real number. The beta-expansion of a real number $x \in [0, 1]$ in base β is an infinite sequence $\varepsilon(x, \beta) = (\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \ldots)$ of integers with $0 \leq \varepsilon_i(x, \beta) \leq \beta$ for all i, defined by

$$\varepsilon_i(x, \beta) = \lfloor \beta T_\beta^{i-1} x \rfloor \quad \text{for all } i \geq 1,$$

where $\lfloor x \rfloor$ denotes the integral part of the real number x.

For any $x \in [0, 1]$ and $n \in \mathbb{N}$, by the definition of beta-expansion (see [13]),

$$x = \frac{\varepsilon_1(x, \beta)}{\beta} + \frac{\varepsilon_2(x, \beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(x, \beta)}{\beta^n} + \frac{T_\beta^n x}{\beta^n}. \tag{2.1}$$

Let $\Omega_\beta^n = \{0, 1, \ldots, \lfloor \beta \rfloor\}^n$ for all $n \in \mathbb{N}$ and

$$\Sigma_\beta^n = \{(\varepsilon_1, \ldots, \varepsilon_n) \in \Omega_\beta^n : \text{there exists } x \in [0, 1] \text{ such that } \varepsilon_i(x, \beta) = \varepsilon_i \text{ for all } 1 \leq i \leq n\}.$$

Lemma 2.1 [13]. *For any $\beta > 1$,*

$$\beta^n \leq \#\Sigma_\beta^n \leq \frac{\beta^{n+1}}{\beta - 1},$$

where # denotes the cardinality of a finite set.

For any $n \in \mathbb{N}$ and $\omega = (\omega_1, \ldots, \omega_n) \in \Sigma_\beta^n$, write

$$I_n(\omega) = \{x \in [0, 1] : \varepsilon_i(x, \beta) = \omega_i \text{ for all } 1 \leq i \leq n\};$$

then

$$[0, 1] = \bigcup_{\omega \in \Sigma_\beta^n} I_n(\omega). \tag{2.2}$$

For the corresponding beta-dynamical system, it is well known (see, for example, [2, 5, 10, 13]) that for any real number $\beta > 1$, there exists a unique probability measure ν_β, equivalent to the Lebesgue measure $\mathcal{L}$ on $[0, 1]$, which is invariant under the beta-transformation T_β. Moreover, the transformation T_β is ergodic for the measure ν_β, which is usually called the Parry measure.

3. Inhomogeneous Diophantine approximation

Proof of Theorem 1.1. Fix an arbitrary point $y \in [0, 1]$. We consider the sequence of balls $\{B(y, \varphi(n))\}_{n \geq 1}$. Let

$$D(T_\beta, \{\varphi(n)\}_{n \geq 1}, y) = \{x \in [0, 1] : |T_\beta^n x - y| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

By Philipp's result (see Section 1),

$$\mathcal{L}(D(T_\beta, \{\varphi(n)\}_{n \geq 1}, y)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{+\infty} \varphi(n) < +\infty, \\ 1 & \text{if } \sum_{n=1}^{+\infty} \varphi(n) = +\infty. \end{cases}$$

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Thus, if we write $E = E(T_\beta, \varphi)$ and $D_y = D(T_\beta, \{\varphi(n)\}_{n\geq 1}, y)$ for simplicity, by using Fubini's theorem,

$$\mathcal{L}^2(E) = \int_0^1 \int_0^1 \chi_E((x,y))\,dx\,dy = \int_0^1 \int_0^1 \chi_{D_y}(x)\,dx\,dy = \int_0^1 \mathcal{L}(D_y)\,dy,$$

where χ_A is the characteristic function of the set A. Therefore,

$$\mathcal{L}^2(E(T_\beta, \varphi)) = \mathcal{L}^2(E) = \begin{cases} 0 & \text{if } \sum_{n=1}^{+\infty} \varphi(n) < +\infty, \\ 1 & \text{if } \sum_{n=1}^{+\infty} \varphi(n) = +\infty. \end{cases} \qquad \square$$

In the case $\sum_{n=1}^{+\infty} \varphi(n) < +\infty$, by the result of Shen and Wang (see Section 1), for any $y \in [0, 1]$,

$$\dim_H D(T_\beta, \{\varphi(n)\}_{n\geq 1}, y) = \frac{1}{1+\alpha} \quad \text{with } \alpha = \liminf_{n\to\infty} \frac{\log_\beta \varphi(n)^{-1}}{n}.$$

Then [4, Corollary 7.12] implies that

$$\dim_H E(T_\beta, \varphi) \geq 1 + \frac{1}{1+\alpha}.$$

Therefore, in order to prove Theorem 1.2, we only need to prove that

$$\dim_H E(T_\beta, \varphi) \leq 1 + \frac{1}{1+\alpha}.$$

PROOF OF THEOREM 1.2. For simplicity, we write $E = E(T_\beta, \varphi)$. For all $n \in \mathbb{N}$, let

$$E_n = \{(x, y) \in [0, 1] \times [0, 1] : |T_\beta^n x - y| < \varphi(n)\};$$

then

$$E = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n. \tag{3.1}$$

For all $n \in \mathbb{N}$, let $J_n(i) = [i\varphi(n)/\beta^n, ((i+1)\varphi(n))/\beta^n] \cap [0, 1]$ for all $0 \leq i \leq \lfloor \beta^n/\varphi(n) \rfloor$. Then

$$[0, 1] = \bigcup_{0\leq i\leq \lfloor \beta^n/\varphi(n)\rfloor} J_n(i).$$

Thus, by (2.2),

$$[0, 1] \times [0, 1] = \bigcup_{\omega\in\Sigma_\beta^n} \bigcup_{0\leq i\leq \lfloor \beta^n/\varphi(n)\rfloor} I_n(\omega) \times J_n(i).$$

Therefore,

$$E_n = \bigcup_{\omega\in\Sigma_\beta^n} \bigcup_{0\leq i\leq \lfloor \beta^n/\varphi(n)\rfloor} \{(x, y) \in I_n(\omega) \times J_n(i) : |T_\beta^n x - y| < \varphi(n)\}.$$

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Given $\omega \in \Sigma_\beta^n$ and $0 \le i \le \lfloor \beta^n/\varphi(n) \rfloor$ and any $x \in I_n(\omega)$ and $y \in J_n(i)$, if $(x, y) \in E_n$, then

$$\left| T_\beta^n x - \frac{i\varphi(n)}{\beta^n} \right| \le |T_\beta^n x - y| + \left| y - \frac{i\varphi(n)}{\beta^n} \right| < \varphi(n) + \frac{\varphi(n)}{\beta^n} < 2\varphi(n).$$

Hence,

$$\begin{aligned} E_n &\subset \bigcup_{\omega \in \Sigma_\beta^n} \bigcup_{0 \le i \le \lfloor \beta^n/\varphi(n) \rfloor} \left\{ (x, y) \in I_n(\omega) \times J_n(i) : \left| T_\beta^n x - \frac{i\varphi(n)}{\beta^n} \right| < 2\varphi(n) \right\} \\ &= \bigcup_{\omega \in \Sigma_\beta^n} \bigcup_{0 \le i \le \lfloor \beta^n/\varphi(n) \rfloor} \left(\left\{ x \in I_n(\omega) : \left| T_\beta^n x - \frac{i\varphi(n)}{\beta^n} \right| < 2\varphi(n) \right\} \times J_n(i) \right). \end{aligned} \tag{3.2}$$

Notice that for any $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \Sigma_\beta^n$ and $x \in I_n(\omega)$, by (2.1),

$$x = \frac{\omega_1}{\beta} + \frac{\omega_2}{\beta^2} + \cdots + \frac{\omega_n}{\beta^n} + \frac{T_\beta^n x}{\beta^n}.$$

Then

$$\left| \left\{ x \in I_n(\omega) : \left| T_\beta^n x - \frac{i\varphi(n)}{\beta^n} \right| < 2\varphi(n) \right\} \right| \le \frac{4\varphi(n)}{\beta^n},$$

where $|A|$ denotes the diameter of the set A. Thus, for any $\omega \in \Sigma_\beta^n$ and $0 \le i \le \lfloor \beta^n/\varphi(n) \rfloor$,

$$\left| \left\{ x \in I_n(\omega) : \left| T_\beta^n x - \frac{i\varphi(n)}{\beta^n} \right| < 2\varphi(n) \right\} \times J_n(i) \right| < \frac{5\varphi(n)}{\beta^n}. \tag{3.3}$$

By (3.1) and (3.2), it is clear that for any $N \in \mathbb{N}$, the family

$$\left\{ \left\{ x \in I_n(\omega) : \left| T_\beta^n x - \frac{i\varphi(n)}{\beta^n} \right| < 2\varphi(n) \right\} \times J_n(i) : n \ge N, \omega \in \Sigma_\beta^n, 0 \le i \le \left\lfloor \frac{\beta^n}{\varphi(n)} \right\rfloor \right\}$$

is a cover of the set E. Recall that $\alpha = \liminf_{n\to\infty} (\log_\beta \varphi(n)^{-1}/n)$. Thus, for any $s > 1 + (1/(1 + \alpha))$, by (3.1)–(3.3) and Lemma 2.1,

$$\begin{aligned} \mathcal{H}^s(E) &\le \liminf_{N\to\infty} \sum_{n \ge N} \sum_{\omega \in \Sigma_\beta^n} \sum_{0 \le i \le \lfloor \beta^n/\varphi(n) \rfloor} \left| \left\{ x \in I_n(\omega) : \left| T_\beta^n x - \frac{i\varphi(n)}{\beta^n} \right| < 2\varphi(n) \right\} \times J_n(i) \right|^s \\ &\le \liminf_{N\to\infty} \sum_{n \ge N} \sum_{\omega \in \Sigma_\beta^n} \sum_{0 \le i \le \lfloor \beta^n/\varphi(n) \rfloor} \left(\frac{5\varphi(n)}{\beta^n} \right)^s \\ &\le \liminf_{N\to\infty} \sum_{n \ge N} \frac{\beta^{n+1}}{\beta - 1} \cdot \frac{2\beta^n}{\varphi(n)} \cdot \left(\frac{5\varphi(n)}{\beta^n} \right)^s < +\infty. \end{aligned}$$

This gives that

$$\dim_H E(T_\beta, \varphi) = \dim_H E \le 1 + \frac{1}{1 + \alpha}.$$

□

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