Bull. Aust. Math. Soc. **108** (2023), 422–427
doi:10.1017/S0004972722001599

A NOTE ON SUBNORMAL SUBGROUPS IN DIVISION RINGS CONTAINING SOLVABLE SUBGROUPS

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(Received 5 November 2022; accepted 17 November 2022; first published online 11 January 2023)

Abstract

Let D be a division ring and N be a subnormal subgroup of the multiplicative group D^*. We show that if N contains a nonabelian solvable subgroup, then N contains a nonabelian free subgroup.

2020 *Mathematics subject classification*: primary 16K20; secondary 16S85, 16S36, 20E05.

Keywords and phrases: division ring, free subgroup, solvable subgroup, subnormal subgroup.

1. Introduction

Let D be a division ring and D^* denote the multiplicative group of D. Recall that a subgroup N of D^* is *subnormal* in D^* if there is a finite sequence of subgroups

$$N = N_r \trianglelefteq N_{r-1} \trianglelefteq \cdots N_1 \trianglelefteq N_0 = D^*.$$

The motivation of this paper is a result of Tits for linear groups. Tits showed that if N is a finitely generated subgroup of the general linear group $\mathrm{GL}_n(F)$ over a field F, then N contains either a nonabelian free subgroup or a nilpotent normal subgroup H of finite index in N [13]. This famous result of Tits is now referred to as the Tits Alternative. For general linear groups over division rings, Lichtman [10] proved that there exists a division ring D such that D^* contains a finitely generated group which is not solvable-by-finite. Therefore, the Tits Alternative fails even for matrices of degree one over D^*. Also in [10], Lichtman remarked that it was not known whether the multiplicative group of a noncommutative division ring contains a nonabelian free subgroup. One year later, Lichtman showed in [11] that if N is a normal subgroup of D^* such that N contains a nonabelian nilpotent-by-finite subgroup, then N contains a nonabelian free subgroup. After that, Gonçalves and Mandel proposed the following conjecture.

Conjecture 1.1 [5, Conjecture 2]. For a division ring D with centre F and a subnormal subgroup N of D^*, if $N \not\subseteq F$, then N contains a nonabelian free subgroup.

The first author is funded by Vietnam National University Ho Chi Minh City (VNUHCM) under grant number T2022-18-03.

We give an affirmative answer to this conjecture in the case when N contains a nonabelian solvable subgroup. This result can be seen as a generalisation of Lichtman's result in [11].

THEOREM 1.2 (Main Theorem). *Let D be a division ring and N a subnormal subgroup of the multiplicative group D^*. If N contains a nonabelian solvable subgroup, then N contains a nonabelian free subgroup.*

We note that many special cases of the Main Theorem have been studied by many authors. For example, a subgroup N of D^* will contain nonabelian free subgroups if N satisfies any of the following assumptions:

- N is normal in D^* and N contains a nonabelian nilpotent-by-finite subgroup (Lichtman [11]);
- N is normal in D^* and N contains a nonabelian locally solvable-by-locally-finite subgroup (Bell and Gonçalves [1]);
- N is subnormal in D^*, N contains a nonabelian locally solvable subgroup and the transcendence degree of the centre of D over its prime subfield is infinite (Bien and Hai [2]).

For convenience, throughout this paper, the phrase 'free subgroup' means 'nonabelian free subgroup'. The idea of the proof of the Main Theorem comes from [1]. We can sketch the proof as follows. Assume that N is a subnormal subgroup of D^* containing a nonabelian solvable subgroup. First, we will study the existence of free subgroups in subnormal subgroups of the division ring of fractions $K(x, \sigma)$ of the skew polynomial ring $K[x, \sigma]$ in a single-variable x twisted by a nonidentity automorphism σ over a field K. Namely, we will prove that every subnormal subgroup of $K(x, \sigma)^*$ containing x contains a free subgroup. After that, we shall build a subring of D which is isomorphic to $K[x, \sigma]$, and based on the result on $K(x, \sigma)$, the Main Theorem will follow.

Some special cases for the existence of free subgroups have also been considered. For example, the following conjecture was proposed by Hai and Thin [8].

CONJECTURE 1.3 [8, Conjecture 1]. Every locally solvable subnormal subgroup of D^* is central.

In [7, Theorem 2.2], Hai and Thin proved that Conjecture 1.3 is true in the case of a locally nilpotent subnormal subgroup in an arbitrary division ring D and, in [7, Theorem 2.4], for the case of a division ring which is algebraic over its centre. After that, in [8], Conjecture 1.3 holds if D is a *weakly locally finite* division ring, that is, D satisfies the condition: for every finite subset S of D, the division subring $P(S)$ generated by S and a prime subfield P of D is finite-dimensional over the centre $Z(P(S))$. Recently, Conjecture 1.3 was fully confirmed in [3, Theorem 1]. By applying the Main Theorem, we can also show that Conjecture 1.3 is true.

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2. Proof of the main theorem

The fundamental result used in proving the Main Theorem is Theorem 2.6. Our approach to the proof of Theorem 2.6 comes from [1]. However, thanks to insights from [2], our proof is more straightforward than that in [1].

Let K be a field and σ be a field automorphism of K. Let us denote by $K[x, \sigma]$ the skew polynomial ring in an indeterminate x twisted by σ over K. Then $K[x, \sigma]$ is a principal left ideal domain and also a principal right ideal domain, that is, every one-sided ideal of $K[x, \sigma]$ can be generated by one element (see [6, Theorem 2.8]). The division ring of fractions of $K[x, \sigma]$ is denoted by $K(x, \sigma)$. Every element of $K(x, \sigma)$ is of the form $f(x) \cdot g(x)^{-1}$, where $f(x)$ and $g(x)$ are in $K[x, \sigma]$. Additionally, we denote by $K((x, \sigma))$ the skew Laurent series ring in the indeterminate x twisted by σ over K. It is well known that $K((x, \sigma))$ is a division ring (see [9, Example 1.8]). Clearly, $K[x, \sigma]$ can be embedded into $K((x, \sigma))$, and so $K(x, \sigma)$ can be seen as a division subring of $K((x, \sigma))$.

Let D be a division ring with the centre F. We can regard D as an F-algebra. If $\dim_F(D) < \infty$, then D is called *centrally finite*. From [9, Proposition 14.2], $K((x, \sigma))$ is centrally finite if and only if the order of σ is finite. As a corollary, if the order of σ is finite, then $K(x, \sigma)$ is centrally finite. We give some results about the existence of free subgroups in $K(x, \sigma)$. We need the following property.

LEMMA 2.1 [4, Theorem 2.1]. *Let D be a noncommutative centrally finite division ring and N a noncentral subnormal subgroup of D^*. Then N contains a free subgroup.*

We immediately have the following result.

COROLLARY 2.2. *Let $K(x, \sigma)$ be the division ring of fractions of $K[x, \sigma]$ and N a noncentral subnormal subgroup of $K(x, \sigma)^*$. If σ has a finite order, then N contains a free subgroup.*

For an arbitrary automorphism σ of K, whether a subnormal subgroup N of $K(x, \sigma)^*$ contains a free subgroup is an open question. In this section, we shall answer the question in case σ is a nonidentity and $x \in N$.

LEMMA 2.3 [2]. *Let K be a field and σ a nonidentity automorphism of K with fixed subfield $k = \{a \in K : \sigma(a) = a\}$. Assume that there exists an element $a \in K \setminus k$ such that $k(a, \sigma(a), \sigma^2(a), \ldots)$ is not a finitely generated extension of k. If N is a subnormal subgroup of $K(x, \sigma)^*$ containing x, then N contains a free subgroup.*

PROOF. This is the assertion (i) of [2, Theorem 2.5]. □

With the notation as in Lemma 2.3, we shall now consider the remaining case, that is, $k(a, \sigma(a), \sigma^2(a), \ldots)$ is a finitely generated extension of k. The next lemma is the key lemma to resolve this case.

LEMMA 2.4. *Let K be a field and σ an automorphism of K with fixed subfield $k = \{a \in K : \sigma(a) = a\}$. Assume that there exists an element $a \in K \setminus k$ such that $E = k(a, \sigma(a), \sigma^2(a), \ldots)$ is a finitely generated extension of k. Then, the following hold.*

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(i) $E = k(\sigma^{n_1}(a), \sigma^{n_2}(a), \dots, \sigma^{n_r}(a))$, *where the* n_i *are integers with* $0 \le n_1 < n_2 < \cdots < n_r$.

(ii) *The restriction of* σ *to the subfield* E *is an automorphism of* E *of finite order.*

PROOF. (i) Since E/k is a finitely generated extension, $E = k(a_1, a_2, \dots, a_s)$ for some $a_1, a_2, \dots, a_s$ in E. For each $i \in \overline{1, s}$, there exist f_i and g_i in $k[a, \sigma(a), \sigma^2(a), \dots]$ such that $a_i = f_i g_i^{-1}$. We can select a finite number of elements $\sigma^{n_{i,1}}(a), \sigma^{n_{i,2}}(a), \dots, \sigma^{n_{i,\ell_i}}(a)$ such that f_i and g_i are in $k[\sigma^{n_{i,1}}(a), \sigma^{n_{i,2}}(a), \dots, \sigma^{n_{i,\ell_i}}(a)]$. It follows that $a_i \in k(\sigma^{n_{i,1}}(a), \sigma^{n_{i,2}}(a), \dots, \sigma^{n_{i,\ell_i}}(a))$. We have

$$E = k(a_i \mid 1 \le i \le s) \subseteq k(\sigma^{n_{i,j}}(a) \mid 1 \le i \le s, 1 \le j \le \ell_i) \subseteq k(a, \sigma(a), \sigma^2(a), \dots) = E.$$

This leads to $E = k(\sigma^{n_{i,j}}(a) \mid 1 \le i \le s, 1 \le j \le \ell_i)$. Hence, we can write $E = k(\sigma^{n_1}(a), \sigma^{n_2}(a), \dots, \sigma^{n_r}(a))$ for $0 \le n_1 < n_2 < \cdots < n_r$.

(ii) We have $\sigma(E) \subseteq E$ because $\sigma(\sigma^n(a)) = \sigma^{n+1}(a) \in E$. From assertion (i), $E = k(\sigma^{n_1}(a), \sigma^{n_2}(a), \dots, \sigma^{n_r}(a))$ for $0 \le n_1 < n_2 < \cdots < n_r$. Put $t_i = \sigma^{n_i}(a)$, and so $E = k(t_1, t_2, \dots, t_r)$. Remark that

$$\sigma^{n_i - n_j}(t_j) = \sigma^{n_i - n_j}(\sigma^{n_j}(a)) = \sigma^{n_i}(a) = t_i \quad \text{for } 1 \le i, j \le r.$$

Put $n = (n_1 - n_2)(n_2 - n_3) \dots (n_{r-1} - n_r)(n_r - n_1)$. For each $i = \overline{1, r}$,

$$n = (n_i - n_{i+1})(n_{i+1} - n_{i+2}) \dots (n_{r-1} - n_r)(n_r - n_1)(n_1 - n_2) \dots (n_{i-1} - n_i),$$

and so

$$\begin{aligned}\sigma^n(t_i) &= \sigma^{(n_i - n_{i+1})(n_{i+1} - n_{i+2}) \cdots (n_{r-1} - n_r)(n_r - n_1)(n_1 - n_2) \cdots (n_{i-1} - n_i)}(t_i) \\ &= \sigma^{n_i - n_{i+1}}(\sigma^{n_{i+1} - n_{i+2}}(\cdots(\sigma^{n_{r-1} - n_r}(\sigma^{n_r - n_1}(\sigma^{n_1 - n_2}(\dots(\sigma^{n_{i-1} - n_i}(t_i))) \cdots) = t_i.\end{aligned}$$

Thus, the restriction $\sigma|_E$ has finite order. Suppose that the order of $\sigma|_E$ is $d > 0$. Since $\sigma(E) \subseteq E$, we have

$$E = \sigma^d(E) \subseteq \sigma^{d-1}(E) \subseteq \cdots \subseteq \sigma^1(E).$$

Hence, $\sigma|_E$ is an automorphism of E. □

Lemma 2.4 enables us to prove the next lemma.

LEMMA 2.5. *Let* K *be a field and* σ *a nonidentity automorphism of* K *with fixed subfield* $k = \{a \in K : \sigma(a) = a\}$. *Assume that there exists* $a \in K \setminus k$ *such that* $E = k(a, \sigma(a), \sigma^2(a), \dots)$ *is a finitely generated extension of* k. *If* N *is a subnormal subgroup of* $K(x, \sigma)^*$ *containing* x, *then* N *contains a free subgroup.*

PROOF. By Lemma 2.4, the restriction $\sigma|_E$ is an automorphism of E of finite order. Since $\sigma|_E(a) = \sigma(a) \ne a$, the automorphism $\sigma|_E$ is a nonidentity. Thus, $R = E(x, \sigma|_E)$ is a noncommutative centrally finite division subring of $K(x, \sigma)$. Because N is subnormal in $K(x, \sigma)^*$, also $H = N \cap R^*$ is subnormal in R^*. However, H is noncentral since $x \in H$. According to Lemma 2.2, H contains a nonabelian free subgroup. Hence, N contains a free subgroup. □

THEOREM 2.6. *Let K be a field and σ a nonidentity automorphism of K. Assume that N is a subnormal subgroup of $K(x,\sigma)^*$. If $x \in N$, then N contains a free subgroup.*

PROOF. Let $D = K(x,\sigma)$, where K is a field and σ is a nonidentity automorphism of K with fixed subfield $k = \{a \in K : \sigma(a) = a\}$. Select $a \in K \setminus k$. Put $E = k(a, \sigma(a), \sigma^2(a), \ldots)$, $R = E(x, \sigma|_E)$ and $H = N \cap E^*$. Then H is a subnormal subgroup of R^* containing x. It follows from Lemma 2.5 that if E/k is a finitely generated extension, then H contains a free subgroup. Otherwise, by Lemma 2.3, H also contains a free subgroup. Hence, in both cases, N contains a free subgroup. The proof is complete. □

PROOF OF THE MAIN THEOREM. Assume that N is a subnormal subgroup of D^* containing a nonabelian solvable subgroup G. Suppose that the derived length of G is $r \geq 1$ with the derived series

$$\langle 1 \rangle = G^{(r)} \trianglelefteq G^{(r-1)} \trianglelefteq \ldots \trianglelefteq G^{(1)} \trianglelefteq G^{(0)} = G.$$

Clearly, $G^{(r-2)}$ is a nonabelian solvable subgroup. Replace G by $G^{(r-2)}$. Then G' is a nontrivial abelian subgroup of G. By Zorn's lemma, the set

$$\mathcal{U} = \{A \leq G : A \text{ is abelian and } G' \subseteq A\}$$

has a maximal element M. Due to the maximality, $M \not\subseteq \mathrm{Z}(G)$. Additionally, since $G' \subseteq M$, M is normal in G. Select $g \in G$ and $u \in M$ such that $ug \neq gu$. The normality of M in G leads to $u_n := g^n u g^{-n} \in M$ for every $n \in \mathbb{Z}$, and so all these elements commute. Put $F = \mathrm{Z}(D)$. Then the division subring $K = F(u_n \mid n \in \mathbb{Z})$ is a field and the map $\sigma : K \to K$, $z \mapsto gzg^{-1}$ is a nonidentity automorphism of K.

Let $R = K[g]$ be the subring of D generated by $K \cup \{g\}$ and $K[x,\sigma]$ the ring of skew polynomials in an indeterminate x. We observe that the map $\phi : K[x,\sigma] \to R$, $\sum_i a_i x^i \mapsto \sum_i a_i g^i$ is a surjective F-algebra homomorphism. Put $I = \ker\phi$. We consider two cases.

Case 1: $I = (0)$. Then $R \cong K[x,\sigma]$, and thus the division ring of fractions $\overline{R}$ of R is isomorphic to $K(x,\sigma)$. The image of $N \cap \overline{R}^*$ is a subnormal subgroup of $K(x,\sigma)^*$ and contains x. By Theorem 2.6, this subgroup contains a free subgroup. Consequently, $N \cap \overline{R}^*$ and also N contain a free subgroup.

Case 2: $I \neq (0)$. Then $R \cong K[x,\sigma]/I$. This means $K[x,\sigma]/I$ is a domain and I is a prime ideal. Additionally, $K[x,\sigma]$ is a principal left ideal domain. This implies that $I = (p)$ for some irreducible polynomial p in $K[x,\sigma]$, that is, I is a maximal ideal of $K[x,\sigma]$, and so $K[x,\sigma]/I$ is a division ring. However, by the divide algorithm for $K[x,\sigma]$ (see [12, The Euclidean algorithm]), $K[x,\sigma]/I$ is a finite dimensional K-vector space. Therefore, R is a centrally finite division subring. By Lemma 2.1, $N \cap R^*$ contains a free subgroup. Hence, N also contains a free subgroup. □

Finally, the Main Theorem gives the affirmation of Conjecture 1.3 as follows.

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COROLLARY 2.7. *Let D be a division ring. Then every locally solvable subnormal subgroup of D^* is central.*

PROOF. Let D be a division ring and N be a locally solvable subnormal subgroup of D^*. Assume that N is noncentral. Then N contains a finitely generated subgroup that is nonabelian and solvable. It follows from the Main Theorem that N contains a free subgroup. This is a contradiction to the local solvability of N. Hence, N is central. $\square$

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https://doi.org/10.1017/S0004972722001599 Published online by Cambridge University Press