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ISOMETRIES AND DISCRETE ISOMETRY SUBGROUPS OF HYPERBOLIC SPACES

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Abstract. Let $\mathbb{H}^n$ be the n-dimensional hyperbolic space with $n \geq 2$. Suppose that G is a discrete, sense-preserving subgroup of $Isom\mathbb{H}^n$, the isometry group of $\mathbb{H}^n$. Let p be the projection map from $\mathbb{H}^n$ to the quotient space $M = \mathbb{H}^n/G$. The first goal of this paper is to prove that for any $a \in \partial\mathbb{H}^n$ (the sphere at infinity of $\mathbb{H}^n$), there exists an open neighbourhood U of a in $\mathbb{H}^n \cup \partial\mathbb{H}^n$ such that p is an isometry on $U \cap \mathbb{H}^n$ if and only if $a \in {}^o\Omega(G)$ (the domain of proper discontinuity of G). This is a generalization of the main result discussed in the work by Y. D. Kim (A theorem on discrete, torsion free subgroups of Isom$\mathbb{H}^n$, *Geometriae Dedicata* **109** (2004), 51–57). The second goal is to obtain a new characterization for the elements of $Isom\mathbb{H}^n$ by using a class of hyperbolic geometric objects: hyperbolic isosceles right triangles. The proof is based on a geometric approach.

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1. Introduction. Let $n \geq 2$, $\mathbb{H}^n$ be the n-dimensional hyperbolic space and $\mathbb{B}^n$ be the Poincaré ball model of $\mathbb{H}^n$, that is, $\mathbb{B}^n = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : |x| < 1\}$ with length differential $ds = \frac{2|dx|}{1-|x|^2}$. Let $\partial\mathbb{H}^n$ denote the sphere at infinity of $\mathbb{H}^n$. We use $\mathbb{S}^{n-1}$ to denote $\partial\mathbb{B}^n$ and $Isom\mathbb{H}^n$ the full group of the isometries of $\mathbb{H}^n$.

In this paper, G always denotes a sense-preserving subgroup of $Isom\mathbb{H}^n$. The action of G on $\mathbb{H}^n$ extends to a continuous action on the compactification of $\mathbb{H}^n$ by the sphere at infinity $\partial\mathbb{H}^n$. As in [**6**], let $\Lambda(G)$ and $\Omega(G)$ denote the limit set and the domain of discontinuity of G, respectively.

In [**10**, Section 12.1], the following is obtained:

Proposition 1.1. *Suppose that G is discrete and $a \in \Omega(G)$. Then there exists an open neighbourhood U of a in $\mathbb{H}^n \cup \Omega(G)$ such that for each $f \in G$, either $U \cap f(U) = \emptyset$ or $U = f(U)$ and $f(a) = a$.*

For $a \in \partial\mathbb{H}^n$, if there exists an open neighbourhood U of a in $\mathbb{H}^n \cup \partial\mathbb{H}^n$ such that for each non-trivial element $f \in G$, $U \cap f(U) = \emptyset$, then a is called *a properly discontinuous*

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point of G. The set of all properly discontinuous points of G, which is called *the domain of proper discontinuity*, is denoted by ${}^{o}\Omega(G)$ (see [**9**] for the case $n = 3$). It is obvious that ${}^{o}\Omega(G) \subset \Omega(G)$ and $\Omega(G) \backslash {}^{o}\Omega(G)$ consists of only fixed points of some elliptic elements of G. If G is discrete and not finite, then $\Omega(G) \neq \emptyset$ if and only if ${}^{o}\Omega(G) \neq \emptyset$. These imply following:

PROPOSITION 1.2. *If G is discrete and torsion free, then ${}^{o}\Omega(G) = \Omega(G)$.*

PROPOSITION 1.3. *Suppose that G is discrete and $a \in {}^{\circ}\Omega(G)$. Then there exists an open neighbourhood U of a in $\mathbb{H}^n \cup {}^{\circ}\Omega(G)$ such that for each non-trivial element $f \in G$, $U \cap f(U) = \emptyset$.*

Let $p : \mathbb{H}^n \to M = \mathbb{H}^n/G$ be the projection map, where G is discrete, $d_{\mathbb{H}}$ be the hyperbolic metric of $\mathbb{H}^n$ and d be defined on M as follows:

$$d(p(x), p(y)) = \inf_{f \in G} d_{\mathbb{H}}(x, f(y)) \quad \text{for} \quad x,\ y \in \mathbb{H}^n.$$

As the main result of [**6**], Kim proved the following:

THEOREM K. *Suppose that G is a discrete, torsion-free subgroup and $a \in \Omega(G)$. Then there exists an open neighbourhood U of a in $\mathbb{H}^n \cup \Omega(G)$ such that*

$$d(p(x), p(y)) = d_{\mathbb{H}}(x, y) \quad \text{for} \quad x,\ y \in U \cap \mathbb{H}^n.$$

Firstly, we will prove the following:

THEOREM 1.4. *Suppose that G is a discrete subgroup. Then for any $a \in \partial\mathbb{H}^n$, there exists an open neighbourhood U of a in $\mathbb{H}^n \cup \partial\mathbb{H}^n$ such that*

$$d(p(x), p(y)) = d_{\mathbb{H}}(x, y) \quad \textit{for} \quad x,\ y \in U \cap \mathbb{H}^n$$

if and only if $a \in {}^{o}\Omega(G)$.

As a corollary of Theorem 1.4 and Proposition 1.2, we can easily get the following:

COROLLARY 1.5. *Suppose that G is a discrete, torsion-free subgroup and $a \in \partial\mathbb{H}^n$. Then there exists an open neighbourhood U of a in $\mathbb{H}^n \cup \partial\mathbb{H}^n$ such that*

$$d(p(x), p(y)) = d_{\mathbb{H}}(x, y) \quad \textit{for} \quad x,\ y \in U \cap \mathbb{H}^n$$

if and only if $a \in \Omega(G)$.

REMARK 1.1. Corollary 1.5 shows that Theorem 1.4 is a generalization of Theorem K.

A map f of $\mathbb{H}^n$ to itself is called *r-hyperplane preserving* if the image of any r-dimensional hyperplane in $\mathbb{H}^n$ under f is still an r-dimensional hyperplane. When $r = 1$, we call the corresponding map f to be *a geodesic-preserving map* in $\mathbb{H}^n$. The relation between isometries and r-hyperplane preserving maps in $\mathbb{H}^n$ has been studied by many authors. For instance, in [**5**], Jeffers proved

THEOREM Je ([**5**, Theorem 3.6]). *Suppose that $f : \mathbb{H}^n \to \mathbb{H}^n$ is a bijection. If f is geodesic preserving, then f is an isometry, i.e., $f \in Isom\mathbb{H}^n$.*

Recently, Li, Wang and Yao [**7**, **8**] studied this relation too and obtained the following generalization:

THEOREM LWY$_1$ ([**7**, Theorem 2] and [**8**, Theorem 3]). *Suppose that $f : \mathbb{H}^n \to \mathbb{H}^n$ is an r-hyperplane preserving map. Then f is an isometry if and only if f is non-degenerate.*

Here, f is called *degenerate* if the image $f(\mathbb{H}^n)$ of $\mathbb{H}^n$ under f is an r-hyperplane.

The second goal of this paper is to study this relation further. By using a class of hyperbolic geometric objects: hyperbolic isosceles right triangles, we get the following:

THEOREM 1.6. *Suppose $f : \mathbb{H}^n \to \mathbb{H}^n$ is a continuous bijection. Then f is an isometry in $\mathbb{H}^n$ if and only if f preserves hyperbolic isosceles right triangles in $\mathbb{H}^n$.*

Here, we say that a map $f : \mathbb{H}^n \to \mathbb{H}^n$ preserves hyperbolic isosceles right triangles in $\mathbb{H}^n$ if for every hyperbolic isosceles right triangle in $\mathbb{H}^n$, its image under f is still a hyperbolic isosceles right triangle in $\mathbb{H}^n$ and vertices correspond to vertices under f.

2. The proof of Theorem 1.4. For any non-trivial sense-preserving element $f \in \mathrm{Isom}\mathbb{H}^n$, f is called

(1) *elliptic* if it has a fixed point in $\mathbb{H}^n$;
(2) *parabolic* if it has only one fixed point in $\partial\mathbb{H}^n$ and none in $\mathbb{H}^n$;
(3) *loxodromic* if it has two fixed points in $\partial\mathbb{H}^n$ and none in $\mathbb{H}^n$.

Suppose f is loxodromic and its fixed points are x and y. We say that x is *attractive* if $f^r(z) \to x$ as $r \to +\infty$ for any $z \in \partial\mathbb{H}^n - \{y\}$. And y is called *repulsive* (cf. [**4**]). Then y is the attractive fixed point of f^{-1} and x the repulsive one.

2.1. Preliminary lemmas. As in [**14**], let Γ_n denote the n-dimensional Clifford group; see [**1**, **2**, **12–14**, **16**] etc. for the representation of sense-preserving Möbius transformations by using the Clifford numbers in Γ_n and its applications. It easily follows from [**1**, Theorem A] or [**2**, Vahlen's theorem] that

LEMMA 2.1. *Every sense-preserving element f in $Isom\mathbb{B}^n$ has the following representation:*

$$f = \begin{pmatrix} a & b \\ b' & a' \end{pmatrix},$$

where $a, b \in \Gamma_{n-1} \cup \{0\}$, ab^*, $\bar{a}b \in \bar{\mathbb{R}}^{n-1}$ and $|a|^2 - |b|^2 = 1$.

Let $f = \left(\begin{smallmatrix} a & b \\ b' & a' \end{smallmatrix}\right) \in Isom\mathbb{B}^n$ be sense preserving and $b \neq 0$, i.e., $f(\infty) \neq \infty$. Then

$$S(c_f, r_f) = \left\{x \in \mathbb{S}^{n-1} : |x - (b')^{-1}a'| = \frac{1}{|b|}\right\}$$

is called the isometric sphere of f, where $c_f = (b')^{-1}a'$ and $r_f = \frac{1}{|b|}$ are the centre and the radius of $S(c_f, r_f)$, respectively.

For any $z \in \mathbb{H}^n \cup \partial\mathbb{H}^n$, let

$$Stab_G(z) = \{g \in G : g(z) = z\},$$

which is called the *stabilizer* of z in G.

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By using Lemma 2.1, we can get the following generalization of [**6**, Proposition 1]:

LEMMA 2.2. *Suppose that G is a discrete subgroup of $Isom\mathbb{B}^n$ and $G\backslash Stab_G(O) \neq \emptyset$, where O denotes the origin of $\mathbb{B}^n$. For $f \in G\backslash Stab_G(O)$, let $f = A_f \circ i_f$ be the decomposition of f as in [**6**, Theorem 3], where i_f is the reflection in the sphere $S(c_f, r_f)$ (cf. [**3**]). Then*

$$\sup_{f \in G\backslash Stab_G(O)} r_f < \infty.$$

Proof. Suppose

$$\sup_{f \in G\backslash Stab_G(O)} r_f = \infty.$$

Then there is an infinite sequence $\{f_m\}$ in G such that

$$r_{f_m} \to \infty.$$

By Lemma 2.1, we may assume that

$$f_m = \begin{pmatrix} a_m & b_m \\ b'_m & a'_m \end{pmatrix},$$

where $b_m \neq 0$.

Then

$$|b_m|^{-1} = r_{f_m} \to \infty.$$

This yields

$$b_m \to 0 \text{ and } |a_m| \to 1$$

since $|a_m|^2 - |b_m|^2 = 1$.

It follows from

$$f_m(O) = \frac{a_m b_m^*}{|a_m|^2}$$

that

$$f_m(O) \to O \text{ as } m \to \infty.$$

This implies that $O \in \Lambda(G) \subset \mathbb{S}^{n-1}$. This is the desired contradiction. □

We recall the following result from [**10**].

LEMMA 2.3 [**10**, Theorem 5.5.1]. *If G is discrete and purely elliptic (that is, each non-trivial element of G is elliptic), then there exists $\eta \in \mathbb{H}^n$ such that $f(\eta) = \eta$ for each $f \in G$.*

REMARK 2.1. The condition 'G being discrete' in Lemma 2.3 cannot be removed (cf. [**15**]).

LEMMA 2.4. *Suppose that G is discrete. For any $a \in \partial\mathbb{H}^n$, if there exists an open neighbourhood U of a in $\mathbb{H}^n \cup \partial\mathbb{H}^n$ such that*

$$d(p(x), p(y)) = d_{\mathbb{H}}(x, y) \text{ for } x, y \in U \cap \mathbb{H}^n,$$

then $U \cap \partial\mathbb{H}^n \subset \Omega(G)$. In particular, $a \in \Omega(G)$.

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Proof. Suppose that there exists some $b \in \Lambda(G) \cap (U \cap \partial\mathbb{H}^n)$, for the contradiction. Since loxodromic fixed points are dense in the limit set (see, for example, [**11**, Theorem B1] or [**3**, Theorem 5.3.8]), we may assume that b is fixed by some $f \in G$ which is loxodromic or parabolic. Without loss of generality, we assume that b is the attractive fixed point of f if f is loxodromic. For any $x \in U \cap \mathbb{H}^n$, there exists a sufficiently large number $r > 0$ such that $f^r(x) \in U \cap \mathbb{H}^n$ and $f^r(x) \neq x$.

Let $y = f^r(x)$. Then

$$d(p(x), p(y)) = \inf_{g \in G} d_{\mathbb{H}}(x, g(y)) \leq d_{\mathbb{H}}(x, f^{-r}(y)) = 0 < d_{\mathbb{H}}(x, y).$$

This is the desired contradiction. □

2.2. The proof of Theorem 1.4. In the proof, we use the Poincaré ball model $\mathbb{B}^n$ of $\mathbb{H}^n$.

Since d_E (the topological Euclidean metric on $\mathbb{B}^n$) is invariant under the subgroup $Stab_G(O)$, it implies that, except for Theorem 2 and Proposition 1, all other theorems, propositions and lemmas used in the proof of [**6**, Theorem 1] (i.e., Theorem K) also hold in the case of G being only discrete. Hence, the proof of the sufficiency follows from Proposition 1.3, Lemma 2.2 and similar discussions as those in [**6**].

Here, we prove the necessity.

Since the assumptions in Lemma 2.4 are satisfied it follows that $a \in \Omega(G)$. Suppose $a \notin {}^o\Omega(G)$, for the contradiction. Then there exists some elliptic element $h \in G$ such that $h(a) = a$. Then $Stab_G(a)$ is non-trivial and purely elliptic. It follows from Lemma 2.3 that there is $\eta \in \mathbb{B}^n$ such that

$$g(\eta) = \eta \quad \text{for any} \quad g \in Stab_G(a).$$

Let A be the hyperbolic geodesic in $\mathbb{B}^n$ with the endpoint a passing through η. Let $\omega \in \mathbb{S}^{n-1}$ be the other endpoint of A. Then ω is also fixed by each element of $Stab_G(a)$. This implies that there exists a neighbourhood $V \subset \mathbb{S}^{n-1} \cup \mathbb{B}^n$ of a such that

$$V \subset U \quad \text{and} \quad g(V) = V \quad \text{for every} \quad g \in Stab_G(a).$$

We can find $x \in V \cap \mathbb{B}^n$ and $g \in Stab_G(a)$ such that $g(x) \neq x$. Let $y = g(x)$. Then $y \in V$ and

$$d_{\mathbb{B}}(x, y) > 0,$$

but

$$d(p(x), p(y)) = 0.$$

This contradiction completes the proof.

3. The proof of Theorem 1.6. Here, we also use the Poincaré ball model $\mathbb{B}^n$ of $\mathbb{H}^n$.

We always use A, B, C, ... to denote the points in $\mathbb{B}^n$. Also we denote by A', B', C', ... the images of A, B, C, ... under f, by $\widehat{AB}$ the geodesic segment between A and B, by ΔABC the hyperbolic triangle with vertices A, B and C, and by $\angle ABC$ the angle between $\widehat{AB}$ and $\widehat{BC}$. Recall that O denotes the origin of $\mathbb{B}^n$.

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Here, we assume that $f : \mathbb{B}^n \to \mathbb{B}^n$ is a continuous bijection that preserves the hyperbolic isosceles right triangles in $\mathbb{B}^n$ and fixes the origin O. For any hyperbolic triangle AOB, we use $\mathbb{B}^2_{\Delta AOB}$ to denote the intersection of the two-dimensional hyperplane in $\bar{\mathbb{R}}^n$ containing ΔAOB and $\mathbb{B}^n$, which is a two-dimensional unit disk with the centre O.

3.1. Preliminary lemmas.

LEMMA 3.1. *For any hyperbolic isosceles right triangle, it is uniquely determined by its acute angle.*

Proof. It easily follows from [**3**, Theorem 7.11.2]. □

LEMMA 3.2. *Suppose $\triangle AOB$ is a hyperbolic isosceles right triangle in $\mathbb{B}^n$ and $\angle AOB$ is the right angle. Then $\angle A'O'B'$ is the right angle in $\triangle A'O'B'$.*

Proof. Assume the contradiction. Without loss of generality, we may assume that $\angle O'A'B'$ is the right angle in $\triangle O'A'B'$. We may find a point $C \in \mathbb{B}^2_{\Delta AOB}$ which satisfies that $d_{\mathbb{B}}(O, A) = d_{\mathbb{B}}(A, C)$, $\angle OAC$ is a right angle and $\widehat{OC}$ intersects $\widehat{AB}$ with the intersection point D. Then D lies in the interior of $\widehat{AB}$ and $\triangle OAC$ is a hyperbolic isosceles right triangle. Since f is a bijection and preserves hyperbolic isosceles right triangles, we see that $\triangle O'A'C'$ is also a hyperbolic isosceles right triangle and D' is an interior point of $\widehat{A'B'}$. Obviously, $\angle O'A'C' > \frac{\pi}{2}$. It follows from [**3**, Theorem 7.16.2] that this is a contradiction. □

LEMMA 3.3. *Suppose $\triangle AOB$ is a hyperbolic isosceles right triangle in $\mathbb{B}^n$ and $\angle AOB$ is an acute angle. Then $\angle A'O'B'$ is also an acute angle.*

Proof. Assume the contradiction. Then $\angle A'O'B'$ is the right angle in $\Delta A'O'B'$. We may find a point C in $\mathbb{B}^2_{\Delta AOB}$ such that ΔAOC is a hyperbolic isosceles right triangle with $\angle AOC$ being the right angle and $\widehat{OB}$ intersects $\widehat{AC}$ with the intersection point D. Then $\triangle A'O'C' > \frac{\pi}{2}$ and D' is an interior point of $\widehat{O'B'}$. This is a contradiction by [**3**, Theorem 7.16.2]. □

LEMMA 3.4. *Suppose $\triangle AOB$ is a hyperbolic isosceles right triangle with $\angle OAB$ being the right angle. Then $\angle O'A'B'$ is a right angle.*

Proof. Assume the contradiction. By Lemma 3.3, we know that $\angle A'O'B'$ is an acute angle. Hence, $\angle O'B'A'$ is the right angle. Choose two points D and E in the interior of $\widehat{OA}$ and $\widehat{AB}$, respectively, such that $d_{\mathbb{B}}(D, A) = d_{\mathbb{B}}(A, E)$. Then $\triangle D'A'E'$ is a hyperbolic isosceles right triangle. Obviously, D' and E' are interior points in $\widehat{A'O'}$ and $\widehat{A'B'}$, respectively. By Lemma 3.1, this is the desired contradiction. □

LEMMA 3.5. *f preserves any angle with the vertex origin O.*

Proof. Let $\angle AOB$ be any angle in $\mathbb{B}^n$. We come to prove that $\angle AOB$ is the same as $\angle A'O'B'$. By Lemma 3.2 and the hypothesis f being a bijection, we may assume that $\angle AOB$ is an acute angle. Let us start our discussions with the following special cases.

Case I. $\angle AOB = \frac{\pi}{p}$ with $p > 4$.

By [**3**, Theorem 7.16.2], we may assume that ΔAOB is a hyperbolic isosceles right triangle with the angle $\angle OAB$ being right angle. In $\mathbb{B}^2_{\Delta AOB}$, let

$$K_1 = \{z \in \mathbb{B}^2_{\Delta AOB} : \ d_{\mathbb{B}}(O, z) = d_{\mathbb{B}}(O, A)\},$$
$$K_2 = \{z \in \mathbb{B}^2_{\Delta AOB} : \ d_{\mathbb{B}}(O, z) = d_{\mathbb{B}}(O, B)\}$$

and the rays r_i $(i = 1, 2, \ldots, 2p)$ from O satisfy that the $2p$ rays r_i are anticlockwise arranged from r_1 to r_{2p} and each angle formed by r_i and r_{i+1} is $\frac{\pi}{p}$, where we assume that A lies in r_1 and B in r_2.

We also let A_i be the intersection point of K_1 and r_i, and B_i the one of K_2 and r_i, where $i = 1, 2, \ldots, 2p$, $A_1 = A$ and $B_2 = B$.

Then each hyperbolic triangle ΔA_iOB_{i+1} is an isosceles right one $(i = 1, 2, \ldots, 2p)$, where $B_{2p+1} = B_1$, and the union of the closures of all ΔA_iOB_{i+1} $(i = 1, 2, \ldots, 2p)$ consists of a neighbourhood of O. By Lemmas 3.3 and 3.4, and the hypothesis f being a bijection, we know that $\angle A'O'B' = \angle AOB = \frac{\pi}{p}$.

Case II. $\angle AOB = \frac{\pi}{3}$.
By dividing $\angle AOB$ into two $\frac{\pi}{6}$-valued angles and Case I, we see that $\angle A'O'B' = \angle AOB$.

Case III. $\angle AOB = \frac{\pi}{4}$.
Similar discussions as in Case II show that $\angle A'O'B' = \frac{\pi}{4}$.

Case IV. $\angle AOB = \frac{q\pi}{p}$, where the two natural numbers p and q are prime.

Since $\angle AOB$ is acute, we see that $0 < 2q < p$. By the discussions as mentioned above, we may assume that $p > 4$. Let us divide $\angle AOB$ into q^2 many $\frac{\pi}{pq}$-valued angles. Then it follows from Case I that $\angle A'O'B' = \angle AOB$.

For general case, since f is continuous, it follows from Case IV that $\angle A'O'B' = \angle AOB$. The proof is complete. □

3.2. The proof of Theorem 1.6. The necessity is obvious. Hence, we only need to prove the sufficiency.

By composite with some element in $Isom\mathbb{B}^n$, we may assume that f fixes O. Let A be an arbitrary point in $\mathbb{B}^n$ which is different from O. Then we can find a hyperbolic isosceles right triangle ΔAOB such that $\widehat{OA}$ is a side of ΔAOB and $\angle AOB$ is an acute angle. It follows from Lemmas 3.1, 3.2 and 3.5 that $d_{\mathbb{B}}(O, A) = d_{\mathbb{B}}(O', A')$. Then for any points B and C in $\mathbb{B}^n$, we see that $d_{\mathbb{B}}(O', B') = d_{\mathbb{B}}(O, B)$, $d_{\mathbb{B}}(O', C') = d_{\mathbb{B}}(O, C)$ and by Lemma 3.5, we also see that $\angle A'O'B' = \angle AOB$. These imply that $d_{\mathbb{B}}(B', C') = d_{\mathbb{B}}(B, C)$. These mean that f is an isometry. This completes our proof.

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