Canad. Math. Bull. Vol. **52** (4), 2009 pp. 535–543

A Note on Locally Nilpotent Derivations and Variables of $k[X, Y, Z]$

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Abstract. We strengthen certain results concerning actions of $(\mathbb{C}, +)$ on $\mathbb{C}^3$ and embeddings of $\mathbb{C}^2$ in $\mathbb{C}^3$, and show that these results are in fact valid over any field of characteristic zero.

Introduction

The Lefschetz principle [15] suggests that any result, which has been proved over the field $\mathbb{C}$ of complex numbers and which involves a finite number of points and of varieties, remains valid over any universal domain (*i.e.*, over an algebraically closed field with infinite transcendence degree over the prime field) of characteristic zero. In this form the principle was proved by Eklof [3]. Furthermore, any statement of the first order predicate calculus true over $\mathbb{C}$ is valid over any algebraically closed field of characteristic zero, without restriction on the transcendence degree (see [11]). However, the situation changes when one tries to extend results to *all* fields of characteristic zero, which is what the present paper does for the following results.

(1) Let $B = \mathbb{C}[X, Y, Z]$, let $D\colon B \to B$ be a locally nilpotent derivation and let $A = \ker D$. Then the map $\operatorname{Spec} B \to \operatorname{Spec} A$, determined by the inclusion $A \hookrightarrow B$, is surjective (see [1]).
(2) Every free algebraic $\mathbb{C}_+$-action on $\mathbb{C}^3$ is a translation in a suitable polynomial coordinate system (see [5]).
(3) Every polynomial $f \in \mathbb{C}[X, Y, Z]$ whose general fiber is isomorphic to $\mathbb{C}^2$ is a variable of $\mathbb{C}[X, Y, Z]$ (see [4]).

Our ability to go beyond the Lefschetz principle in (1) and (3) is essentially based on Kambayashi's theorem (see 1.3) in combination with other serious results. We also present slightly stronger versions of results (1) and (3) and we show (see Corollary 3.3, Proposition 4.9 and Corollary 4.12) that under some additional assumptions the analogue of Kambayashi's theorem holds in dimension 3.

In the stronger version of (3) the assumption on the general fiber is replaced by the condition that there exists a Zariski-dense subset U of $\mathbb{C}$ such that the fiber of f over any point of U is an affine plane. We shall verify in Section 2 that this weaker hypothesis implies the stronger one, but it is worth mentioning that this implication

Received by the editors February 4, 2004; revised July 3, 2006.
The first author's research was supported by a grant from NSERC Canada. The second author's research was supported by the NSA grant MDA904-03-1-0009.
AMS subject classification: Primary: 14R10; secondary: 14R20, 14R25, 13N15.
Keywords: locally nilpotent derivations, group actions, polynomial automorphisms, variable, affine space.

is also a corollary to the fact that for any morphism $\varphi\colon X \to Y$ of algebraic varieties, $\{y \in Y \mid \varphi^{-1}(y) \cong \mathbb{A}^2\}$ is a constructible subset of Y. Actually, consider the following more general question:

If $\varphi\colon X \to Y$ is a morphism of algebraic varieties and S is an affine algebraic surface, is $\{y \in Y' \mid \varphi^{-1}(y) \cong S\}$ a constructible subset of Y' (where Y' is the set of closed points of Y)?

Classification of surfaces S with this property will be given in a subsequent paper.

1 Locally Nilpotent Derivations of $\mathbf{k}[X, Y, Z]$

Definition 1.1 • If B is an algebra over a ring A, then the notation $B = A^{[n]}$ means that B is A-isomorphic to a polynomial ring in n variables over A.

- Let B be a ring. A derivation $D\colon B \to B$ is *locally nilpotent* if for each $b \in B$ there exists an integer $r > 0$ satisfying $D^r(b) = 0$.
- A subring A of an integral domain B is called an *inert subring* of B if the conditions $x, y \in B \setminus \{0\}$ and $xy \in A$ imply $x, y \in A$. It is well known that if B is an integral domain of characteristic zero and $D\colon B \to B$ is a locally nilpotent derivation, then $\ker(D)$ is an inert subring of B; it follows that $\ker(D)$ contains all units of B and, consequently, that if $\mathbf{k}$ is any field contained in B then D is a $\mathbf{k}$-derivation.

We begin with the following simple observation.

Lemma 1.2 *Let $\mathbf{k}'/\mathbf{k}$ be a field extension, B a $\mathbf{k}$-algebra and $B' = \mathbf{k}' \otimes_{\mathbf{k}} B$.*

(i) *The map* $\operatorname{Spec} B' \to \operatorname{Spec} B$*, corresponding to the natural map $B \hookrightarrow B'$, is surjective.*

(ii) *Suppose that $D'\colon B' \to B'$ is a $\mathbf{k}'$-derivation, $D\colon B \to B$ is a $\mathbf{k}$-derivation and D' extends D. Then* $\ker(D') \cong \mathbf{k}' \otimes_{\mathbf{k}} \ker(D)$*. Consequently,* $\operatorname{Spec}(\ker D') \to \operatorname{Spec}(\ker D)$ *is surjective.*

Proof Since $\mathbf{k}'$ is free over $\mathbf{k}$, B' is a free, hence faithfully flat, B-module; assertion (i) follows (see for instance [7, p. 28]. Applying the exact functor $\mathbf{k}' \otimes_{\mathbf{k}} -$ to the exact sequence $0 \to \ker(D) \to B \xrightarrow{D} B$ gives $\ker(D') \cong \mathbf{k}' \otimes_{\mathbf{k}} \ker(D)$. Then assertion (i) implies that $\operatorname{Spec}(\ker D') \to \operatorname{Spec}(\ker D)$ is surjective. ■

We need the following result of Kambayashi (see [6], or [12] for a different proof).

Proposition 1.3 *Let $\mathbf{k}'/\mathbf{k}$ be a separable field extension and let A be a $\mathbf{k}$-algebra. If $\mathbf{k}' \otimes_{\mathbf{k}} A = \mathbf{k}'^{[2]}$ then $A = \mathbf{k}^{[2]}$.*

We also require the following result of Miyanishi:

Proposition 1.4 *Let $\mathbf{k}$ be a field of characteristic zero, $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$ and $0 \neq D : B \to B$ a locally nilpotent derivation. Then* $\ker D = \mathbf{k}^{[2]}$.

Actually, Miyanishi [8] proved the case $\mathbf{k} = \mathbb{C}$, and the general case 1.4 follows from Miyanishi's result by a straightforward application of Kambayashi's result 1.3. We include the proof for the sake of completeness.

https://doi.org/10.4153/CMB-2009-054-5 Published online by Cambridge University Press

Proof Choose a finite subset F of $\mathbf{k}$ which contains all coefficients of the polynomials DX, DY and DZ and define the field $\mathbf{k}_0 = \mathbb{Q}(F)$ and the algebra $B_0 = \mathbf{k}_0[X, Y, Z]$; then D restricts to a nonzero locally nilpotent derivation $D_0 : B_0 \to B_0$. Since $\ker D = \mathbf{k} \otimes_{\mathbf{k}_0} \ker D_0$ by 1.2, it suffices to show that $\ker D_0 = \mathbf{k}_0^{[2]}$. Since $\mathbf{k}_0$ is isomorphic to a subfield of $\mathbb{C}$, it suffices to prove 1.4 in the case where $\mathbf{k} \subseteq \mathbb{C}$.

Assume that $\mathbf{k} \subseteq \mathbb{C}$, let $B' = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and consider the extension $D' : B' \to B'$ of D. By Miyanishi's result we have $\ker(D') = \mathbb{C}^{[2]}$, so Lemma 1.2 gives $\mathbb{C} \otimes_{\mathbf{k}} \ker(D) = \mathbb{C}^{[2]}$. Then Kambayashi's result 1.3 gives $\ker D = \mathbf{k}^{[2]}$. ■

Next we prove the following generalization of Bonnet's result.

Theorem 1.5 *Let $\mathbf{k}$ be a field of characteristic zero, $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$, $D\colon B \to B$ a locally nilpotent derivation and $A = \ker D$.*

(i) *B is faithfully flat as an A-module.*
(ii) *The ideal $A \cap D(B)$ of A is principal.*

Remarks. (i) Faithful flatness has the following standard consequences, (see [7, pp. 28, 46]).

(a) The map $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$, determined by the inclusion $A \hookrightarrow B$, is submersive, *i.e.*, π is surjective and a subset E of $\operatorname{Spec} A$ is closed if and only if $\pi^{-1}(E)$ is closed.
(b) Each ideal I of A satisfies $A \cap IB = I$.

(ii) The fact that B is flat over A is well known for any field $\mathbf{k}$ of characteristic zero (if $D \neq 0$, one can show that every nonempty fiber of π has dimension 1). Thus, *π is also an open map.*

(iii) Bonnet also proved (but he did not publish this part) that the ideal $A \cap D(B)$ is principal when $\mathbf{k} = \mathbb{C}$; our proof of this fact is a simplification of his argument.

2 Proof of Theorem 1.5

Consider the following condition on a field $\mathbf{k}$:

($\mathcal{B}$) Let $B = \mathbf{k}^{[3]}$; then for every locally nilpotent derivation $D\colon B \to B$ the map $\operatorname{Spec} B \to \operatorname{Spec}(\ker D)$ is surjective.

As mentioned in the introduction, P. Bonnet proved the following.

Proposition 2.1 *$\mathbb{C}$ satisfies ($\mathcal{B}$).*

We begin with the following Lemma.

Lemma 2.2 *If $\mathbf{k}'$ is a field of characteristic zero which satisfies ($\mathcal{B}$), then every subfield of $\mathbf{k}'$ satisfies ($\mathcal{B}$).*

Proof Let $\mathbf{k}$ be a subfield of $\mathbf{k}'$, X, Y, Z indeterminates over $\mathbf{k}'$, and

$$B = \mathbf{k}[X, Y, Z] \subseteq B' = \mathbf{k}'[X, Y, Z].$$

Consider a locally nilpotent derivation $D\colon B \to B$ and let $A = \ker D$. Note that D extends uniquely to a locally nilpotent derivation $D'\colon B' \to B'$. Let $A' = \ker D'$. In

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the commutative diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[d, "\pi"'] & \operatorname{Spec} B' \arrow[l, "\beta"'] \arrow[d, "\pi'"] \\ \operatorname{Spec} A & \operatorname{Spec} A' \arrow[l, "\alpha"'] \end{tikzcd}$$

the map π' is surjective because $\mathbf{k}'$ satisfies $(\mathcal{B})$ by assumption, and α is surjective by Lemma 1.2. It follows that π is surjective, as desired. ■

Lemma 2.3 *Every field of characteristic zero satisfies $(\mathcal{B})$.*

Proof By 2.2, it suffices to prove that every algebraically closed field of characteristic zero satisfies $(\mathcal{B})$. This follows from 2.1 and the Lefschetz Principle, but we include the proof for the sake of completeness (actually, verifying that the hypothesis of Eklof's Theorem [3] is satisfied is roughly equivalent to the following argument).

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero, $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$ and $D\colon B \to B$ a locally nilpotent derivation; let $A = \ker D$. In order to show that $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, we may assume that $D \neq 0$; then $A = \mathbf{k}^{[2]}$ by 1.4. Let $\mathfrak{p} \in \operatorname{Spec} A$. If $\mathfrak{p} = 0$, then clearly $\mathfrak{p}$ is in the image of π. If $\operatorname{ht} \mathfrak{p} = 1$, then $\mathfrak{p} = pA$ for some prime element p of A; since A is an inert subring of B, it follows that $A \cap pB = pA$ and that pB is a prime ideal of B, so again $\mathfrak{p}$ is in the image of π. So we may assume that $\mathfrak{p}$ is a maximal ideal of A. Since $A = \mathbf{k}^{[2]}$ and $\mathbf{k}$ is algebraically closed, we may choose f, g such that $A = \mathbf{k}[f, g]$ and $\mathfrak{p} = (f, g)A$.

Consider a finite subset F of $\mathbf{k}$ which contains all coefficients of DX, DY, DZ, f and g; let $\mathbf{k}_1 = \mathbb{Q}(F)$ and $B_1 = \mathbf{k}_1[X, Y, Z] \subseteq B$. Then D restricts to a nonzero locally nilpotent derivation $D_1\colon B_1 \to B_1$; let $A_1 = \ker D_1$, then $A_1 = \mathbf{k}_1^{[2]}$ by 1.4; actually we have $A_1 = \mathbf{k}_1[f, g]$, because $f, g \in B_1$.

$$\begin{tikzcd} \operatorname{Spec} B_1 \arrow[d, "\pi_1"'] & \operatorname{Spec} B \arrow[l, "\beta"'] \arrow[d, "\pi"] \\ \operatorname{Spec} A_1 & \operatorname{Spec} A \arrow[l, "\alpha"'] \end{tikzcd}$$

Note that β is surjective by Lemma 1.2(i). Since $\mathbf{k}_1$ is a finitely generated field extension of $\mathbb{Q}$, it is isomorphic to some subfield of $\mathbb{C}$. By 2.1 and 2.2, it follows that $\mathbf{k}_1$ satisfies $(\mathcal{B})$, so π_1 is surjective. Consequently, $\alpha \circ \pi$ is surjective. Let $\mathfrak{p}_1 = (f, g)A_1 \in \operatorname{Spec}(A_1)$. Then the fibre of α over $\mathfrak{p}_1$ is $\{\mathfrak{p}\}$ (because $\mathfrak{p}_1 A = \mathfrak{p}$ is maximal). This and the fact that $\alpha \circ \pi$ is surjective imply that $\mathfrak{p}$ belongs to the image of π, which shows that π is surjective and consequently that $\mathbf{k}$ satisfies $(\mathcal{B})$. ■

We may now finish the proof of Theorem 1.5.

Let $\mathbf{k}$, B, D and A be as in the statement of the theorem and let $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the map determined by $A \hookrightarrow B$. Since B is flat over A and (by 2.3) π is surjective, it follows that B is faithfully flat over A [7, p. 28, Theorem 3]. As mentioned in the remark following the statement of the theorem, it also follows that each ideal I of A satisfies $A \cap IB = I$.

Next, we show that if f_1, f_2 are nonzero elements of $A \cap D(B)$, then $\gcd(f_1, f_2) \in A \cap D(B)$; of course, this implies that $A \cap D(B)$ is a principal ideal. (*Remark:* Given $x, y \in A$, $\gcd_A(x, y) = \gcd_B(x, y)$ because A is an inert subring of B.)

For each $i \in \{1, 2\}$, choose $s_i \in B$ such that $D(s_i) = f_i$. Define $I = (f_1, f_2)A$. Then clearly $s_1 f_2 - s_2 f_1 \in IB$; since $D(s_1 f_2 - s_2 f_1) = 0$, we have in fact $s_1 f_2 - s_2 f_1 \in A \cap IB = I$, so there exist $a_1, a_2 \in A$ such that $s_1 f_2 - s_2 f_1 = a_1 f_2 - a_2 f_1$. Thus $(s_1 - a_1) f_2 = (s_2 - a_2) f_1$. Write $f_i = f_i' d$ where $d = \gcd(f_1, f_2)$. Then $(s_1 - a_1) f_2' = (s_2 - a_2) f_1'$ and $\gcd(f_1', f_2') = 1$. So if we define

$$s = \frac{s_1 - a_1}{f_1'} = \frac{s_2 - a_2}{f_2'},$$

then $s \in B$ and clearly $Ds = d \in A$, whence $d \in A \cap D(B)$. This shows that $A \cap D(B)$ is a principal ideal of A.

The proof of Theorem 1.5 is complete. ∎

3 Free Locally Nilpotent Derivations of $\mathbf{k}[X, Y, Z]$

Definition 3.1 Let B be a ring. A locally nilpotent derivation $D\colon B \to B$ is *free* if the ideal of B generated by $D(B)$ is equal to B.

Consider the case $B = \mathbf{k}^{[3]}$, where $\mathbf{k}$ is a field of characteristic zero. There is a natural bijective correspondence between the set of locally nilpotent derivations of B and the set of G_a-actions on $\operatorname{Spec}(B) = \mathbb{A}^3_{\mathbf{k}}$ (see [10]). Consider an action α and the corresponding derivation D; the following facts are well known.

- A maximal ideal $\mathfrak{m}$ of B is a fixed point of α if and only if $D(B) \subseteq \mathfrak{m}$; so D is free if and only if α is free of fixed points. As an example, consider the locally nilpotent derivation D of $\mathbb{Q}[X, Y, Z]$ given by $D(X) = D(Y) = 0$ and $D(Z) = X^2 - 2$; then D is not free and the corresponding action on $\mathbb{A}^3_{\mathbb{Q}}$ has many fixed points, but no fixed point is in $\mathbb{Q}^3$.
- The condition $D(B) = B$ is equivalent to α being a translation in a suitable polynomial coordinate system of $\mathbb{A}^3_{\mathbf{k}}$ (*cf.* [16, Proposition 2.1]).

Theorem 3.2 *If $\mathbf{k}$ is a field of characteristic zero and D a free locally nilpotent derivation of $B = \mathbf{k}^{[3]}$, then $D(B) = B$. (Equivalently, every fixed point free G_a-action on $\mathbb{A}^3_{\mathbf{k}}$ is a translation in a suitable polynomial coordinate system.)*

Proof By [5] the case $\mathbf{k} = \mathbb{C}$ is true, so [3] implies that the theorem is true whenever $\mathbf{k}$ is a universal domain of characteristic zero. Suppose that $\mathbf{k}$, B and D satisfy the hypothesis of the theorem. Consider a field extension $\mathbf{k}'/\mathbf{k}$ where $\mathbf{k}'$ is a universal domain. Then D extends uniquely to a locally nilpotent derivation D' on $B' = \mathbf{k}' \otimes_{\mathbf{k}} B = (\mathbf{k}')^{[3]}$. As D is free, it is clear that D' is free; since the theorem is true for $\mathbf{k}'$,

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$D'\colon B' \to B'$ is a surjective map. Note that D' is obtained by applying the functor $\mathbf{k}' \otimes_{\mathbf{k}} -$ to D; since $\mathbf{k}'$ is a free (hence faithfully flat) $\mathbf{k}$-module, $D\colon B \to B$ must be surjective. ■

Corollary 3.3 *Let B be an algebra over a field $\mathbf{k}$ of characteristic zero such that $B \otimes_{\mathbf{k}} \mathbf{k}' \simeq (\mathbf{k}')^{[3]}$ for some field extension $\mathbf{k}'/\mathbf{k}$. If $D\colon B \to B$ is a free locally nilpotent derivation, then $B = \mathbf{k}^{[3]}$ and $D(B) = B$.*

Proof Extend D to a locally nilpotent derivation D' of $B' = \mathbf{k}' \otimes_{\mathbf{k}} B = (\mathbf{k}')^{[3]}$. Then D' is free, so D' is surjective by Theorem 3.2 and $D(B) = B$ by faithful flatness. By [16, Proposition 2.1], the surjectivity of D implies that $B = A^{[1]}$ where $A = \ker(D)$. We have $\mathbf{k}' \otimes_{\mathbf{k}} A = \ker(D')$, and $\ker(D') = (\mathbf{k}')^{[2]}$ by 1.4. So $A = \mathbf{k}^{[2]}$ by Kambayashi's result 1.3 and this implies that $B = \mathbf{k}^{[3]}$. ■

4 Variables of $\mathbf{k}^{[3]}$

Definition 4.1 Let $\mathbf{k}$ be a field, $B = \mathbf{k}^{[3]}$, and $f \in B$. We say that f is a *variable* of B if $B = \mathbf{k}[f, g, h]$ for some g, h.

Notation Suppose that R is a subring of a ring B. If $\mathfrak{p} \in \operatorname{Spec} R$, we write $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ for the residue field at $\mathfrak{p}$ and we consider the $\kappa(\mathfrak{p})$-algebra $B \otimes_R \kappa(\mathfrak{p})$. Define $U(R, B) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid B \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})^{[2]}\}$.

Suppose that $f \in B = \mathbf{k}^{[3]}$ is such that $U(\mathbf{k}[f], B) \neq \varnothing$, *i.e.,* at least one fiber of f is an affine plane. Then the Abhyankar–Sathaye embedding conjecture asserts that f is a variable of B. We shall prove a weaker statement:

Theorem 4.2 *Let $B = \mathbf{k}^{[3]}$, where $\mathbf{k}$ is a field of characteristic* 0. *If $f \in B$ is such that $U(\mathbf{k}[f], B)$ is a dense subset of* $\operatorname{Spec} \mathbf{k}[f]$, *then f is a variable of B.*

Let us first recall some known facts.

Proposition 4.3 *Let $f \in B = \mathbb{C}^{[3]}$ and suppose that one of the following conditions is satisfied.*

(i) *The zero ideal of $\mathbb{C}[f]$ belongs to $U(\mathbb{C}[f], B)$ (*i.e., *the generic fiber of f is an affine plane);*
(ii) *$U(\mathbb{C}[f], B)$ contains all closed points of* $\operatorname{Spec} \mathbb{C}[f]$, *except possibly a finite number of them,* i.e., *the general closed fiber of f is an affine plane.*

Then f is a variable of B.

Proposition 4.4 *Let $\varphi\colon X \to Y$ be a morphism of complex irreducible quasi-projective algebraic varieties. Then there exists a Zariski-dense open subset Y_0 of Y so that the fibers of φ over any two closed points of Y_0 are homeomorphic in the standard topology.*

Proposition 4.5 *Let S be a smooth complex algebraic variety homeomorphic to $\mathbb{R}^4$ in the standard topology. Then S is isomorphic to $\mathbb{A}^2_{\mathbb{C}}$ as an algebraic variety.*

Case (ii) of Proposition 4.3 is given in [4]; it follows that case (i) also holds, as it is easy to see that if the generic fiber of f is an affine plane, then so is the general

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fiber. Proposition 4.4 is a special case of the Varchenko equisingularity theorem [14, Theorem 5.2] and Proposition 4.5 is a consequence of Ramanujam's result [9]. Now Propositions 4.3–4.5 imply the following.

Proposition 4.6 *Theorem 4.2 is valid in the case $\mathbf{k} = \mathbb{C}$.*

So our task consists in showing that validity over $\mathbb{C}$ implies validity over any field of characteristic zero. First note that Kambayashi's result, Proposition 1.3, has the following consequence.

Lemma 4.7 *Let $\mathbf{k}'/\mathbf{k}$ be an extension of fields of characteristic zero, let $R \subset A$ be $\mathbf{k}$-algebras and define $R' = \mathbf{k}' \otimes_{\mathbf{k}} R$ and $A' = \mathbf{k}' \otimes_{\mathbf{k}} A$. Then $U(R',A') = \pi^{-1}(U(R,A))$, where $\pi\colon \operatorname{Spec}(R') \to \operatorname{Spec}(R)$ is the surjective map determined by $R \hookrightarrow R'$.*

Proof Since R' is a free R-module, it is in particular faithfully flat and consequently π is surjective. Let $\mathfrak{q} \in \operatorname{Spec}(R')$; it suffices to show

$$\mathfrak{q} \in U(R',A') \iff \pi(\mathfrak{q}) \in U(R,A). \tag{4.1}$$

Let $\mathfrak{p} = \pi(\mathfrak{q}) \in \operatorname{Spec}(R)$ and write $\kappa = \kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $\kappa' = \kappa(\mathfrak{q}) = R'_{\mathfrak{q}}/\mathfrak{q}R'_{\mathfrak{q}}$; note that κ' is a separable extension of κ. We have

$$\begin{aligned}\kappa' \otimes_{\kappa} (\kappa \otimes_R A) &= \kappa' \otimes_R A = \kappa' \otimes_{R'} R' \otimes_R A = \kappa' \otimes_{R'} (\mathbf{k}' \otimes_{\mathbf{k}} R \otimes_R A)\\ &= \kappa' \otimes_{R'} (\mathbf{k}' \otimes_{\mathbf{k}} A) = \kappa' \otimes_{R'} A'.\end{aligned}$$

Thus, by 1.3, $\kappa' \otimes_{R'} A' = \kappa'^{[2]}$ if and only if $\kappa \otimes_R A = \kappa^{[2]}$; in other words, (4.1) holds. ■

We thank Sathaye for pointing out the following fact to us.

Proposition 4.8 **(Sathaye)** *Let B be a regular affine domain over a field $\mathbf{k}$ of characteristic zero. If $f \in B$ and $U(\mathbf{k}[f], B) = \operatorname{Spec}\mathbf{k}[f]$, then $B = \mathbf{k}^{[3]}$ and f is a variable of B.*

Proof If $\mathfrak{m}$ is a maximal ideal of $R = \mathbf{k}[f]$, then the assumption implies that $U(R_{\mathfrak{m}}, B_{\mathfrak{m}}) = \operatorname{Spec} R_{\mathfrak{m}}$, so the main result of [13] gives $B_{\mathfrak{m}} = R_{\mathfrak{m}}^{[2]}$. Then [2] implies that B is the symmetric algebra of a finitely generated projective R-module P but $R = \mathbf{k}^{[1]}$ implies that P is free and hence that $B = R^{[2]}$. ■

Combining the results of Kambayashi and Sathaye, we obtain the following.

Proposition 4.9 *Let $\mathbf{k}'/\mathbf{k}$ be an extension of fields of characteristic zero, let B be a $\mathbf{k}$-algebra and let $f \in B$. Suppose that $\mathbf{k}' \otimes_{\mathbf{k}} B = (\mathbf{k}')^{[3]}$ and that f is a variable of $\mathbf{k}' \otimes_{\mathbf{k}} B$. Then $B = \mathbf{k}^{[3]}$ and f is a variable of B.*

Proof Let $R = \mathbf{k}[f]$, $B' = \mathbf{k}' \otimes_{\mathbf{k}} B$, and $R' = \mathbf{k}'[f] = \mathbf{k}' \otimes_{\mathbf{k}} R$. Since f is a variable of B', we have $B' = R'^{[2]}$ so in particular $U(R', B') = \operatorname{Spec}(R')$; then $U(R,B) = \operatorname{Spec}(R)$ by 4.7. Since the assumption $\mathbf{k}' \otimes_{\mathbf{k}} B = (\mathbf{k}')^{[3]}$ implies that B is a regular affine domain over $\mathbf{k}$, the desired conclusion follows from Sathaye's result 4.8. ■

The following is immediate.

Corollary 4.10 *Let $\mathbf{k}'/\mathbf{k}$ be an extension of fields of characteristic zero, let X, Y, Z be indeterminates over $\mathbf{k}'$ and let $f \in \mathbf{k}[X, Y, Z] \subseteq \mathbf{k}'[X, Y, Z]$. Then*

$$f \text{ is a variable of } \mathbf{k}[X,Y,Z] \iff f \text{ is a variable of } \mathbf{k}'[X,Y,Z].$$

We make the trivial observation.

Lemma 4.11 *Let $\mathbf{k}'/\mathbf{k}$ be an extension of fields, let T be an indeterminate and let $\pi \colon \operatorname{Spec} \mathbf{k}'[T] \to \operatorname{Spec} \mathbf{k}[T]$ be the map defined by $\pi(\mathfrak{q}) = \mathfrak{q} \cap \mathbf{k}[T]$. For any subset E of $\operatorname{Spec} \mathbf{k}[T]$,*

$$E \text{ is dense in } \operatorname{Spec} \mathbf{k}[T] \iff \pi^{-1}(E) \text{ is dense in } \operatorname{Spec} \mathbf{k}'[T].$$

Proof of Theorem 4.2 Let $\mathbf{k}$ be a field of characteristic zero, let X, Y, Z be indeterminates and let $f \in B = \mathbf{k}[X, Y, Z]$ be such that $U(\mathbf{k}[f], B)$ is dense in $\operatorname{Spec} \mathbf{k}[f]$. Let F be a finite subset of $\mathbf{k}$ which contains all coefficients of f and define $\mathbf{k}_0 = \mathbb{Q}(F)$ and $B_0 = \mathbf{k}_0[X, Y, Z]$. Note that $f \in B_0$ and that (by 4.7 and 4.11) $U(\mathbf{k}_0[f], B_0)$ is dense in $\operatorname{Spec} \mathbf{k}_0[f]$. Fix a homomorphism of fields $\mathbf{k}_0 \hookrightarrow \mathbb{C}$ and regard f as an element of $B' = \mathbb{C}[X, Y, Z]$. Using 4.7 and 4.11 again, we find that $U(\mathbb{C}[f], B')$ is dense in $\operatorname{Spec} \mathbb{C}[f]$. Since Theorem 4.2 is known to be true over $\mathbb{C}$ (see 4.6), f is a variable of B'. By 4.10, f is a variable of B_0 and hence a variable of B. ■

Corollary 4.12 *Let B be an algebra over a field $\mathbf{k}$ of characteristic zero such that $B \otimes_{\mathbf{k}} \mathbf{k}' \simeq (\mathbf{k}')^{[3]}$ for some field extension $\mathbf{k}'/\mathbf{k}$. If $f \in B$ is such that $U(\mathbf{k}[f], B)$ is a dense subset of $\operatorname{Spec} \mathbf{k}[f]$, then $B = \mathbf{k}^{[3]}$ and f is a variable of B.*

Proof Let $B' = B \otimes_{\mathbf{k}} \mathbf{k}' = (\mathbf{k}')^{[3]}$. By 4.7 and 4.11, $U(\mathbf{k}'[f], B')$ is dense in $\operatorname{Spec} \mathbf{k}'[f]$, so Theorem 4.2 implies that f is a variable of B'. The desired conclusion follows from 4.9. ■

Remark. Suppose for a moment that the Abhyankar–Sathaye embedding conjecture is true over $\mathbb{C}$, *i.e.,* if $f \in B = \mathbb{C}^{[3]}$ and $U(\mathbb{C}[f], B) \neq \varnothing$, then f is a variable of B. Then the conjecture holds over any field of characteristic zero and 4.12 remains true if the word "dense" is replaced by "nonempty". To see this, simply replace "dense" by "nonempty" in 4.11 and in the proofs of Theorem 4.2 and of 4.12.

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https://doi.org/10.4153/CMB-2009-054-5 Published online by Cambridge University Press