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Nagoya Math. J.
Vol. 185 (2007), 143–150

SOME ESTIMATES FOR THE BERGMAN KERNEL AND METRIC IN TERMS OF LOGARITHMIC CAPACITY

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Abstract. For a bounded domain Ω on the plane we show the inequality $c_\Omega(z)^2 \le 2\pi K_\Omega(z)$, $z \in \Omega$, where $c_\Omega(z)$ is the logarithmic capacity of the complement $\mathbb{C} \setminus \Omega$ with respect to z and K_Ω is the Bergman kernel. We thus improve a constant in an estimate due to T. Ohsawa but fall short of the inequality $c_\Omega(z)^2 \le \pi K_\Omega(z)$ conjectured by N. Suita. The main tool we use is a comparison, due to B. Berndtsson, of the kernels for the weighted complex Laplacian and the Green function. We also show a similar estimate for the Bergman metric and analogous results in several variables.

§1. Introduction

Let Ω be a bounded domain in $\mathbb{C}$. Suita [S] conjectured that

$$c_\Omega(z)^2 \le \pi K_\Omega(z), \quad z \in \Omega, \tag{1.1}$$

where

$$K_\Omega(z) = \sup\left\{ \frac{|f(z)|^2}{\int_\Omega |f|^2} : f \text{ holomorphic in } \Omega,\ f \not\equiv 0 \right\}$$

is the Bergman kernel and $c_\Omega(z)$ the logarithmic capacity of the complement $\mathbb{C} \setminus \Omega$ with respect to z, that is

$$c_\Omega(z) = \exp \lim_{\zeta \to z} (G_\Omega(\zeta, z) - \log|\zeta - z|),$$

where G_Ω is the (negative) Green function. If true, this estimate would be optimal, since for simply connected Ω we have equality in (1.1). Ohsawa [O1], using the methods of the $\bar\partial$-equation, showed that

$$c_\Omega^2 \le 750\pi K_\Omega. \tag{1.2}$$

Received May 17, 2005.
Revised October 31, 2005.
2000 Mathematics Subject Classification: 30C40, 31A35.
Partially supported by KBN Grant #2 P03A 03726.

In fact, as noticed in [O1] and explored in [O2], the Suita conjecture seems to be closely related to the Ohsawa-Takegoshi theorem on extension of L^2 holomorphic functions [OT]. In [O2] Ohsawa proved a general result which covered in particular the extension theorem, as well as the estimate (1.2) (with the constant $2^8\pi$). Berndtsson [B3], using the methods of his proof of the Ohsawa-Takegoshi theorem from [B2] improved the constant in the Ohsawa estimate to 6π. The author has also recently found the paper by B.-Y. Chen [C], where he shows the estimate with the constant $\alpha\pi$, where $\alpha = 2(1+\sqrt{5})e^{a+1-\sqrt{5}}$ and a is the solution of $a + \log a = 0$ (then $\alpha \approx 3.3155$).

One of the goals of this note is to show an estimate (see (2.4) below) from which it follows in particular that

$$c_\Omega^2 \le 2\pi K_\Omega. \tag{1.3}$$

We do not aim at merely improving the constant in the Ohsawa estimate but also to present a slightly modified approach to the problem, where we more or less precisely construct a holomorphic function in Ω with specified value at a given point and an appropriate bound for the L^2 norm. We will use the kernel for the weighted complex Laplacian and the main tool will be a bound for this kernel in terms of the Green function due to Berndtsson [B1].

Our method yields also the following inequality for the Bergman metric

$$c_\Omega^4 \le \pi K_\Omega B_\Omega, \tag{1.4}$$

where

$$B_\Omega = \frac{\partial^2}{\partial z \partial \overline{z}} \log K_\Omega.$$

The methods we use can be also applied in the same way for arbitrary Riemann surface which admits a Green function and estimates (1.3) and (1.4) are also valid. We also show that from one-dimensional case and the extension theorem of Ohsawa-Takegoshi [OT] one can easily deduce corresponding estimates in several complex variables.

§2. Proofs of one-dimensional estimates

Without loss of generality we may assume that Ω is smooth and bounded in $\mathbb{C}$, and $0 \in \Omega$. We will always denote $G = G_\Omega(\cdot, 0)$. We also use the notation (slightly different than the one from several variables)

$$\partial\alpha = \frac{\partial \alpha}{\partial z}, \quad \overline{\partial}\alpha = \frac{\partial \alpha}{\partial \overline{z}}.$$

If φ is smooth in $\overline{\Omega}$ then the adjoint to $\overline{\partial}$ with respect to the scalar product in $L^2(\Omega, e^{-\varphi})$ is given by

$$\overline{\partial}^* \alpha = -e^{\varphi}\partial(e^{-\varphi}\alpha) = -\partial\alpha + \alpha\partial\varphi.$$

The complex Laplacian in $L^2(\Omega, e^{-\varphi})$ is defined by

$$\Box\alpha = -\overline{\partial}\,\overline{\partial}^*\alpha = \partial\overline{\partial}\alpha - \partial\varphi\overline{\partial}\alpha - \alpha\partial\overline{\partial}\varphi.$$

The basic relation to the standard Laplacian is given by the following formula of Berndtsson [B1]:

$$\partial\overline{\partial}(|\alpha|^2 e^{-\varphi}) = \left(2\,\mathrm{Re}(\overline{\alpha}\,\Box\alpha) + |\overline{\partial}\alpha|^2 + |\overline{\partial}^*\alpha|^2 + |\alpha|^2\partial\overline{\partial}\varphi\right)e^{-\varphi}. \tag{2.1}$$

If φ is subharmonic then in particular we can find $N \in C^\infty(\overline{\Omega}\setminus\{0\}) \cap L^1(\Omega)$ such that

$$\Box N = \frac{\pi}{2}e^{\varphi(0)}\delta_0, \quad N = 0 \ \text{ on } \ \partial\Omega.$$

(The constant $\pi/2$ is chosen so that $N = G$ if $\varphi \equiv 0$.) The estimate of Berndtsson [B1] asserts that

$$|N|^2 \le e^{\varphi+\varphi(0)}G^2. \tag{2.2}$$

Remark. Berndtsson in [B1] shows using (2.1) that for any C^2 smooth α and $\varepsilon > 0$ one has

$$\partial\overline{\partial}(|\alpha|^2 e^{-\varphi} + \varepsilon)^{1/2} \ge -|\Box\alpha|e^{-\varphi/2}.$$

Now by approximation one can easily deduce that

$$\partial\overline{\partial}\left(-|N|e^{-(\varphi+\varphi(0))/2}\right) \le \frac{\pi}{2}\delta_0 = \partial\overline{\partial}G$$

from which (2.2) immediately follows. In a particular case when Ω is simply connected and φ harmonic we have

$$N = e^{g+\overline{g(0)}}G,$$

where g is a holomorphic function in Ω such that $\mathrm{Re}\,g = \varphi/2$. Therefore, in this case we have equality in (2.2).

https://doi.org/10.1017/S0027763000025782 Published online by Cambridge University Press

As in [B3] we shall use the weight

$$\varphi := 2(\log|z| - G). \tag{2.3}$$

Note that φ is harmonic in Ω, smooth on $\overline{\Omega}$ and

$$e^{-\varphi(0)} = c_\Omega(0)^2.$$

For harmonic weights the operators $\overline{\partial}$ and its adjoint commute

$$\Box = -\overline{\partial}\,\overline{\partial}^* = -\overline{\partial}^*\overline{\partial}.$$

Therefore

$$\overline{\partial}(e^{-\varphi}\partial\overline{N}) = \overline{\partial}(-e^{-\varphi(0)}\overline{\partial}^* N) = \frac{\pi}{2}\delta_0.$$

It follows that the functions

$$f := ze^{-\varphi}\partial\overline{N}, \quad g := -ze^{-\varphi(0)}\overline{\partial}^* N$$

are holomorphic in Ω, smooth on $\overline{\Omega}$, and $f(0) = g(0) = 1/2$.

Using the fact that both the function $|N|^2 e^{-\varphi}$ and its derivative vanish on $\partial\Omega$, integration by parts and (2.1) give

$$\begin{aligned}\int_\Omega |N|^2 e^{-\varphi}\partial\overline{\partial}(|z|^2 e^{-\varphi}) &= \int_\Omega |z|^2(|\overline{\partial}N|^2 + |\overline{\partial}^* N|^2)e^{-2\varphi}\\ &= \int_\Omega |f|^2 + e^{2\varphi(0)}\int_\Omega |g|^2 e^{-2\varphi}.\end{aligned}$$

On the other hand, we have $|z|^2 e^{-\varphi} = e^{2G}$ and by (2.2)

$$\int_\Omega |N|^2 e^{-\varphi}\partial\overline{\partial}(|z|^2 e^{-\varphi}) \le e^{\varphi(0)}\int_\Omega G^2\partial\overline{\partial}e^{2G}.$$

We need the following simple lemma.

LEMMA. *For every summable* $\gamma : (-\infty, 0) \to \mathbb{R}$ *we have*

$$\int_\Omega \gamma\circ G\,|\nabla G|^2 = 2\pi\int_{-\infty}^0 \gamma(t)\,dt.$$

Proof. Let $\chi : (-\infty, 0) \to \mathbb{R}$ be such that $\chi' = \gamma$ and $\chi(-\infty) = 0$. Then

$$\int_\Omega \gamma\circ G\,|\nabla G|^2 = \int_\Omega \langle\nabla(\chi\circ G), \nabla G\rangle = \int_{\partial\Omega}\chi(0)\frac{\partial G}{\partial n} = 2\pi\chi(0). \qquad \square$$

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It follows that

$$\int_\Omega G^2 \partial\overline{\partial} e^{2G} = \int_\Omega G^2 e^{2G} |\nabla G|^2 = \frac{\pi}{2}$$

and

$$\int_\Omega |f|^2 + e^{2\varphi(0)} \int_\Omega |g|^2 e^{-2\varphi} \le \frac{\pi}{2} e^{\varphi(0)}.$$

We conclude that

$$\frac{1}{K_\Omega(0)} + \frac{1}{c_\Omega(0)^4 K_\Omega^{2\varphi}(0)} \le \frac{2\pi}{c_\Omega(0)^2}, \tag{2.4}$$

where φ is given by (2.3) and

$$K_\Omega^{2\varphi}(z) = \sup\left\{ \frac{|f(z)|^2}{\int_\Omega |f|^2 e^{-2\varphi}} : f \text{ holomorphic in } \Omega,\ f \not\equiv 0 \right\}$$

is the weighted Bergman kernel. In particular, we get (1.3).

To show (1.4) we first recall a well known formula for the Bergman metric

$$B_\Omega(z) = \frac{1}{K_\Omega(z)} \sup\left\{ \frac{|f'(z)|^2}{\int_\Omega |f|^2} : f \text{ holomorphic in } \Omega,\ f(z) = 0,\ f \not\equiv 0 \right\}.$$

We now proceed the same way as before, we only choose the weight

$$\varphi := 4(\log|z| - G),$$

so that

$$e^{-\varphi(0)} = c_\Omega(0)^4,$$

and the functions

$$f := z^2 e^{-\varphi} \partial \overline{N}, \quad g := -z^2 e^{-\varphi(0)} \overline{\partial}^* N,$$

so that they are holomorphic in Ω, $f(0) = g(0) = 0$, and $f'(0) = g'(0) = 1/2$. We will get

$$\int_\Omega |f|^2 + e^{2\varphi(0)} \int_\Omega |g|^2 e^{-2\varphi} \le e^{\varphi(0)} \int_\Omega G^2 \partial\overline{\partial} e^{4G} = \frac{\pi}{4} e^{\varphi(0)}$$

and (1.4) follows.

Remark. Similarly as for the Suita conjecture, it would be interesting to improve the constant in (1.4) to $\pi/2$ – it would then be optimal. It is also interesting whether a reverse to the Ohsawa estimate

$$K_\Omega \le C c_\Omega^2$$

holds for some constant C. This would have far reaching consequences: for example it would give another potential theoretic characterization of Bergman exhaustive domains (compare [Z2]). It would also provide a quantitative version of the following well known result (see e.g. [Co]): an (unbounded) domain Ω in $\mathbb{C}$ contains a non-vanishing square-integrable holomorphic function if and only if $\mathbb{C} \setminus \Omega$ is not polar. Another consequence would be an estimate

$$c_\Omega^2 \le C B_\Omega,$$

and, in higher dimensions, an estimate from below of the Bergman metric in terms of the Azukawa metric (see below).

§3. Several dimensional analogues

Let now Ω be a bounded domain in $\mathbb{C}^n$. The pluricomplex Green function is defined by

$$G_\Omega(\,\cdot\,, w) = \sup\Big\{u \in PSH(\Omega) : u < 0,\ \overline{\lim_{z\to w}}\,(u(\zeta) - \log|z-w|) < \infty\Big\},\ \ w \in \Omega.$$

The multidimensional logarithmic capacity can be defined as

$$c_\Omega(w) = \exp \overline{\lim_{z\to w}}\,(G_\Omega(z,w) - \log|z-w|), \quad w \in \mathbb{C}^n.$$

The Azukawa metric can be regarded as a directional logarithmic capacity

$$A_\Omega(w;X) = \exp \overline{\lim_{\lambda\to 0}}\,(G_\Omega(w+\lambda X, w) - \log|\lambda|), \quad w \in \Omega,\ X \in \mathbb{C}^n.$$

For basic properties of A_Ω we refer to [Z1]. Recall also that the Bergman metric is defined as

$$B_\Omega(z;X) = \frac{\partial^2}{\partial\lambda\partial\overline{\lambda}} \log K_\Omega(z+\lambda X)\Big|_{\lambda=0}, \quad z \in \Omega,\ X \in \mathbb{C}^n.$$

We are now in position to formulate multidimensional analogues of the previous estimates.

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THEOREM. *For a bounded pseudoconvex domain* Ω *in* $\mathbb{C}^n$ *we have*

$$c_\Omega(w)^2 \le C K_\Omega(w), \quad A_\Omega(w;X)^4 \le C' K_\Omega(w) B_\Omega(w;X), \quad w \in \Omega,\ X \in \mathbb{C}^n,$$

where C, C' *depend only on* n *and the diameter of* Ω.

Proof. Since all the considered functions behave well under approximation, without loss of generality we may assume that Ω is sufficiently regular, even smooth, then $G_\Omega(\,\cdot\,, w)$ is continuous on $\overline{\Omega}\setminus\{w\}$. By [Z1] A_Ω is continuous (as a function on $\Omega\times\mathbb{C}^n$), $\overline{\lim}$ in the definition of A_Ω can be replaced with lim, and

$$c_\Omega(w) = A_\Omega(w;X)$$

for some $X \in \mathbb{C}^n$. Let

$$D := \{\lambda \in \mathbb{C} : w + \lambda X \in \Omega\}.$$

Then

$$G_\Omega(w+\lambda X, w) \le G_D(\lambda, 0), \quad \lambda \in D,$$

and thus

$$A_\Omega(w;X) \le c_D(0).$$

On the other hand, by the Ohsawa-Takegoshi extension theorem [OT] (see also [B2]),

$$K_D(\lambda) \le C K_\Omega(w+\lambda X), \quad \lambda \in D,$$

and the first inequality follows from the one-dimensional Ohsawa estimate.

The proof of the second inequality is similar. We have

$$B_\Omega(w;X) = \frac{1}{K_\Omega(w)} \sup\left\{\frac{|D_X f(w)|^2}{\int_\Omega |f|^2} : f \text{ holomorphic in } \Omega,\ f(w)=0,\ f\not\equiv 0\right\},$$

where

$$D_X f(w) = \sum_{j=1}^n X_j \frac{\partial f}{\partial z_j}(w).$$

Therefore, by the Ohsawa-Takegoshi theorem,

$$K_D(\lambda) B_D(\lambda) \le C K_\Omega(w+\lambda X) B_\Omega(w+\lambda X; X), \quad \lambda \in D,$$

and it is enough to use (1.4). □

https://doi.org/10.1017/S0027763000025782 Published online by Cambridge University Press

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