ASSOCIATED REGULAR SPACES

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1. Introduction. Let $(X,\mathscr{T})$ be any topological space. In this paper, we show that there is a unique regular topology $\mathscr{T}_*$ on X which is coarser than $\mathscr{T}$ such that if Y is any regular space, the continuous maps $X \to Y$ are the same for $\mathscr{T}$ and $\mathscr{T}_*$. We shall call $\mathscr{T}_*$ the regular topology associated with $\mathscr{T}$ and $(X,\mathscr{T}_*)$ the regular space associated with $(X,\mathscr{T})$. In order to prove these results, we shall develop two operators, ultraclosure and weak ultraclosure, which are related to the closure operator. We shall also define a new separation axiom, called T_{2b}, and show that it is between T_{2a} (Urysohn) and T_3 (regular Hausdorff).

In what follows, X will denote a set, A and B subsets of X, and $\mathscr{T}$, $\mathscr{T}_1$, $\mathscr{T}_\alpha, \mathscr{T}_*, \ldots$, topologies on X. If $\mathscr{A}$ is a set of subsets of X, by $\Phi(\mathscr{A})$ we shall mean the topology generated by $\mathscr{A}$ as a sub-base. We shall denote the closure operator by Cl and the interior operator by Int. If it is necessary to distinguish between topologies, we shall use subscripts; e.g., $\mathrm{Cl}_2 A$ means the closure of A with respect to topology $\mathscr{T}_2$.

The author wishes to thank E. E. Enochs for his assistance in writing this paper and the referee for his many helpful suggestions. In particular, the first proof of Theorem 1, which is much shorter than the author's, is due to the referee. The author's proof is also included, since it seems to exhibit more of the topological structure involved. Parts of the paper were done under a research grant from the Charlotte College Foundation.

2. The principal theorem.

THEOREM 1. *If $\mathscr{T}$ is a topology on X, then there is a unique regular topology $\mathscr{T}_*$, coarser than $\mathscr{T}$, such that if Y is any regular space, the continuous maps $(X,\mathscr{T}) \to Y$ are the continuous maps $(X,\mathscr{T}_*) \to Y$. Furthermore, $\mathscr{T}_*$ is the least upper bound of the regular topologies coarser than $\mathscr{T}$.*

Proof (by the referee). Define $\mathscr{T}_*$ to be the family of all $G \subset X$ for which there exist a regular space Z, a continuous map f_z: $(X,\mathscr{T}) \to Z$, and an open $U \subset Z$ for which $G = f^{-1}(U)$. We first prove that $\mathscr{T}_*$ is a topology. To show that $\phi \in \mathscr{T}_*$ and $X \in \mathscr{T}_*$, let Z be the set X with the trivial (indiscrete) topology and let f_z be the identity map. If $\{G_i\}$ is a family of elements of $\mathscr{T}_*$ and f_i, Z_i, U_i the corresponding maps, regular spaces and subsets, let $Z = \prod_i Z_i$ and let f_z: $(X,\mathscr{T}) \to Z$ such that $f_z(x) = (f_i(x))_i$.

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For each i let $V_i \subset Z$ be the subset of Z consisting of those elements of Z whose ith coordinate is in U_i. Then $\bigcup_i G_i = f_z^{-1}(\bigcup_i V_i) \in \mathscr{T}_*$. If $G_1, G_2 \in \mathscr{T}_*$, then with the obvious notation, $f_1 \times f_2$ maps continuously into $Z_1 \times Z_2$ and $G_1 \cap G_2 = (f_1 \times f_2)^{-1}(U_1 \times U_2) \in \mathscr{T}_*$. Thus $\mathscr{T}_*$ is a topology on X. It is not difficult to check that $\mathscr{T}_*$ is regular. Now if Y is any regular space, it follows easily that f is continuous $(X,\mathscr{T}) \to Y$ if and only if f is continuous $(X,\mathscr{T}_*) \to Y$. If $\mathscr{T}_\#$ is any other topology on X having the desired properties, $\mathscr{T}_\# = \mathscr{T}_*$ since the identity maps $(X,\mathscr{T}) \to (X,\mathscr{T}_*)$ and $(X,\mathscr{T}) \to (X,\mathscr{T}_\#)$ are continuous. In a similar way it can be shown that $\mathscr{T}_*$ is the least upper bound of the regular topologies coarser than $\mathscr{T}$.

Note. The fact that the least upper bound of the regular topologies finer than $\mathscr{T}$ is itself regular was shown in (**1**).

3. $\mathscr{T}_*$ by means of the ultraclosure operator.

Definition 1. By the ultraclosure of A, denoted Ucl A, we shall mean the intersection of all closed sets B such that $A \subset B$ and such that if $c \in X - B$, there are disjoint open $\mathscr{T}$ neighbourhoods of B and c. If $A = \operatorname{Ucl} A$ we say that A is ultraclosed.

THEOREM 2. Ucl *is a Kuratowski closure operator on* X.

The proof is straightforward and will be left to the reader.

Example 1. A T_{2a}-space such that the topology $\mathscr{T}_1$ determined by Ucl is neither regular nor Hausdorff:

$A = [0, 1] \cap$ rationals,
$B = [0, 1] \cap \{p/q + \sqrt{2},\ p \text{ and } q \text{ integers}, q \neq 0\}$,
$C = [0, 1] \cap \{p/q + \sqrt{3},\ p \text{ and } q \text{ integers}, q \neq 0\}$,
$D = [0, 1] \cap \{p/q + \sqrt{5},\ p \text{ and } q \text{ integers}, q \neq 0\}$,
$E = \{1'\} \cup ([0, 1] \cap \{p/q + \sqrt{7},\ p \text{ and } q \text{ integers}, q \neq 0\}),\ 1' \neq 1$,

and let $X = A \cup B \cup C \cup D \cup E$. Let a base for $\mathscr{T}$ consist of all sets of the following forms:

$(a, b) \cap A$,
$(a, b) \cap (A \cup B \cup C),\ b < 1$,
$(a, b) \cap C$,
$(a, b) \cap (C \cup D \cup E)$,
$((a, b) \cap E) \cup \{1'\},\ a < 1 < b$,
$(a, b) \cap E,\ b < 1$.

Then $\{1\}$ and $\{1'\}$ are ultraclosed for $\mathscr{T}$, hence closed for $\mathscr{T}_1$. But neighbourhoods of 1 and $1'$ always intersect. Indeed, $\mathscr{T}_1 \mid [0, 1)$ is the interval topology, while sets of the form $(a, 1]$ and $(a, 1) \cup \{1'\}$ are base neighbourhoods of 1 and $1'$, respectively. Thus 1 and $1'$ have the same systems of deleted neighbourhoods. If the ultraclosure operator is applied to $(X,\mathscr{T}_1)$ we obtain a topology $\mathscr{T}_2$ which differs from the interval topology only in that 1 and $1'$ have the same system of neighbourhoods. Thus $\mathscr{T}_2$ is not T_0.

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Definition 2. If $\mathscr{T}_0 = \mathscr{T}$ and if $\mathscr{T}_\alpha$ has been defined for each ordinal less than β, define $\mathscr{T}_\beta$ to be the topology determined by Ucl as applied to $\bigcap_{\alpha<\beta}\mathscr{T}_\alpha$. We call $\mathscr{T}_\beta$ the βth topology associated with $\mathscr{T}$ and $\bigcap_{\alpha<\beta}\mathscr{T}_\alpha$ the β^-th topology associated with $\mathscr{T}$, denoted by $\mathscr{T}_{\beta^-}$.

LEMMA 1. *$(X,\mathscr{T})$ is regular if and only if* Ucl A = Cl A *for each* $A \subset X$.

The proof is left to the reader.

LEMMA 2. *There exists an ordinal β such that $\mathscr{T}_{\beta+1} = \mathscr{T}_\beta$.*

Proof. Otherwise, choose δ of cardinality greater than that of $P(X)$. Then with each $\alpha < \delta$ we can associate $A_\alpha \in \mathscr{T}_{\alpha^-} - \mathscr{T}_\alpha$. But this is impossible, for this would imply a subset of $P(X)$ having cardinality δ.

LEMMA 3. *Let β be as in Lemma 2. Then $\mathscr{T}_\beta$ is regular.*

Proof. If A is closed for $\mathscr{T}_\beta$, it is closed for $\mathscr{T}_{\beta+1}$, that is, $\mathrm{Ucl}_\beta A = A = \mathrm{Cl}_\beta A$. Thus $\mathscr{T}_\beta$ is regular by Lemma 1.

LEMMA 4. *Let Y be a regular space and let $f: (X,\mathscr{T}) \to Y$ be continuous. Then if $D \subset Y$ is closed, $f^{-1}(D)$ is ultraclosed in $(X,\mathscr{T})$.*

The proof is straightforward and will be left to the reader.

LEMMA 5. *If γ is an ordinal and Y is a regular space, the continuous maps $(X,\mathscr{T}_\gamma) \to Y$ are continuous $(X,\mathscr{T}) \to Y$.*

Proof. Clearly, the lemma is true if $\gamma = 0$. Suppose it is true for each $\alpha < \gamma$. Let F be closed in Y. Then $f^{-1}(F)$ is closed in $(X,\mathscr{T}_\alpha)$ for each $\alpha < \gamma$, hence closed for $\bigcap_{\alpha<\gamma}\mathscr{T}_\alpha = \mathscr{T}_{\gamma^-}$. Hence, by Lemma 4, $f^{-1}(F)$ is ultraclosed for $\mathscr{T}_{\gamma^-}$, therefore closed for $\mathscr{T}_\gamma$.

Proof of Theorem 1 (by the author). Let β be an ordinal such that $\mathscr{T}_\beta = \mathscr{T}_{\beta+1}$ (Lemma 2) and let $\mathscr{T}_* = \mathscr{T}_\beta$. Then $\mathscr{T}_*$ is regular (Lemma 3), and continuous maps $(X,\mathscr{T}_*) \to Y$ are continuous $(X,\mathscr{T}) \to Y$ (Lemma 5). Clearly, continuous maps $(X,\mathscr{T}) \to Y$ are continuous $(X,\mathscr{T}_*) \to Y$. Proof of uniqueness and of the fact that $\mathscr{T}_*$ is the least upper bound of the regular topologies coarser than $\mathscr{T}$ is the same as in the referee's proof.

4. Ultraclosure by means of the weak ultraclosure operator. In order to obtain a better understanding of the ultraclosure operator, we shall now develop it by means of the weak ultraclosure operator which we now define.

Definition 3. The weak ultraclosure of A, denoted Wucl A is $\{x \in X|$ every closed neighbourhood of x intersects $A\}$.

Remark. Both Ucl A and Wucl A are closed. If $(X,\mathscr{T})$ is regular, then Cl A = Ucl A = Wucl A.

Wucl is not a closure operator since it may be false that Wucl (Wucl A) = Wucl A as we shall see in Example 2. This fact motivates Definition 4.

Definition 4. Let the 0th weak ultraclosure of A be A, denoted by $\mathrm{Wucl}^0 A$. If the αth weak ultraclosure of A has been defined for each $\alpha < \gamma$, define the γth weak ultraclosure of A, denoted $\mathrm{Wucl}^\gamma A$ to be $\mathrm{Wucl}(\bigcup_{\alpha<\gamma}\mathrm{Wucl}^\alpha A)$.

Example 3. A subset A of a T_{2a}-space $(X,\mathscr{T})$ such that $\mathrm{Wucl}^\alpha A \neq \mathrm{Wucl}^\beta A$ for each $\alpha < \beta \leqq \omega$, where ω denotes the just infinite ordinal. Let $(X,\mathscr{T}_\#)$ be the real numbers with the order topology. Let $B_0 = X$. For each $n \in N$ let B_n, $B_{n-1} - B_n$ be a resolution of B_{n-1} into disjoint dense sets. Let

$$\mathscr{T} = \Phi(\mathscr{T}_\# \cup \{B_1, B_0 - B_2, B_3, B_0 - B_4, \ldots\}).$$

Let $A = B_0 - B_1$.

We first prove, by induction, that $\mathrm{Wucl}^n A = B_0 - B_{2n+1}$ for each $n \geqq 0$. Clearly, $\mathrm{Wucl}^0 A = A = B_0 - B_1 = B_0 - B_{2\cdot 0+1}$, therefore, suppose that

$$\mathrm{Wucl}^k A = B_0 - B_{2k+1}.$$

We can write $B_0 - B_{2k+3} = \mathrm{Wucl}^k A \cup (B_{2k+1} - B_{2k+2}) \cup (B_{2k+2} - B_{2k+3})$. Clearly, $\mathrm{Wucl}^k A \subset \mathrm{Wucl}^{k+1} A$. Let $x \in B_{2k+1} - B_{2k+2}$ and let V be any open neighbourhood of x. Then $(a, b) \cap B_{2k+2} \cap (B_0 - B_{2k+2}) \subset V$ for some $a < x < b$, since B_{2k+1} and $B_0 - B_{2k+2}$ are the minimal sub-base elements of $\mathscr{T}$ which are not in $\mathscr{T}_\#$ and which contain x. Let

$$y \in (a, b) \cap \mathrm{Wucl}^k A \cap B_{2k} = (a, b) \cap (B_{2k} - B_{2k+1}).$$

(Such a y exists by induction and the transitive property of density.) Then if W is any open neighbourhood of y we have that

$$(c, d) \cap (B_0 - B_{2k+2}) \cap B_{2k-1} \subset W$$

for some c, d such that $a < c < x < d < b$, since $B_0 - B_{2k+2}$ and B_{2k-1} are minimal sub-basis elements of $\mathscr{T}$ which are not in $\mathscr{T}_\#$ and which contain y. Hence

$$V \cup W \supset [(a, b) \cap B_{2k+1} \cap (B_0 - B_{2k+2})] \cap \\ [(c, d) \cap (B_0 - B_{2k+2}) \cap B_{2k-1}] = (c, d) \cap (B_{2k+1} - B_{2k+2}) \neq \emptyset$$

by transitivity of density. Thus $y \in \mathrm{Cl}\, V \cap \mathrm{Wucl}^k A$, which implies $x \in \mathrm{Wucl}^{k+1} A$. If $x \in B_{2k+2} - B_{2k+3}$ and V is any neighbourhood of x, an argument similar to the above will show that $x \in \mathrm{Wucl}^{k+1} A$. Thus,

$$B_0 - B_{2k+3} \subset \mathrm{Wucl}^{k+1} A.$$

We now show that $B_{2k+3} \subset B_0 - \mathrm{Wucl}^{k+1} A$. Let $x \in B_{2k+3}$. Then $x \in X - (B_0 - B_{2k+2})$ which is closed and

$$\mathrm{Wucl}^k A = B_0 - B_{2k+1} \subset B_0 - B_{2k+2},$$

therefore $\mathrm{Cl}\, B_{2k+3} \cap \mathrm{Wucl}^k A = \emptyset$. We have now completed the proof that $\mathrm{Wucl}^n A = B_0 - B_{2n+1}$ for each $n \in N \cup \{0\}$, which shows that

$$\mathrm{Wucl}^n A \neq \mathrm{Wucl}^m A \text{ if } m, n \in N,\ m \neq n.$$

From the above it is clear that $\text{Wucl}^{\omega}A \neq \text{Wucl}^{n}A$ for each n. This completes the proof.

LEMMA 6. *If for some ordinal β it is true that* $\text{Wucl}^{\beta}A = \text{Wucl}^{\beta+1}A$, *then* $\text{Wucl}^{\gamma}A = \text{Wucl}^{\beta}A$ *for each* $\gamma > \beta$.

Proof, by transfinite induction, is easy and will be left to the reader.

LEMMA 7. *There exists an ordinal β, independent of A, such that*

$$\text{Wucl}^{\beta}A = \text{Wucl}^{\gamma}A$$

whenever $\gamma \geqq \beta$.

Proof. Let β be an ordinal whose cardinality is greater than that of X, and suppose $\text{Wucl}^{\beta+1}A \neq \text{Wucl}^{\beta}A$. Then, by Lemma 6, for each $\alpha < \beta$, we may choose $x \in \text{Wucl}^{\alpha}A - \bigcup_{\delta<\alpha}\text{Wucl}^{\delta}A$. But then the cardinality of X is greater than or equal to that of β, which is impossible. It follows by Lemma 6 that $\text{Wucl}^{\tau}A = \text{Wucl}^{\beta}A$ whenever $\tau > \beta$.

THEOREM 3. *Let β be as in Lemma 7. Then* $\text{Wucl}^{\beta}A = \text{Ucl}\, A$.

Proof. Suppose $x \notin \text{Wucl}^{\beta}A$. Then there exists a neighbourhood V of x such that $\text{Cl}\, V \cap \text{Wucl}^{\beta}A = \emptyset$. For each $y \in V$, V is a neighbourhood of y which does not intersect $X - V \supset \text{Wucl}^{\beta}A \supset A$. Thus $x \notin \text{Ucl}\, A$ and therefore $\text{Ucl}\, A \subset \text{Wucl}^{\beta}A$. Suppose now that $\text{Wucl}^{\beta}A \not\subset \text{Ucl}\, A$. Let γ be the smallest ordinal for which $\text{Wucl}^{\gamma}A \not\subset \text{Ucl}\, A$ and let $z \in \text{Wucl}^{\gamma}A - \text{Ucl}\, A$. There is a neighbourhood W of z such that $\text{Cl}\, W \cap \text{Ucl}\, A = \emptyset$. But then for each $\alpha < \gamma$ we have $\text{Cl}\, W \cap \text{Wucl}^{\alpha}A = \emptyset$, since $\text{Wucl}^{\alpha}A \subset \text{Ucl}\, A$. Thus $z \notin \text{Wucl}^{\gamma}A$, a contradiction.

Note. $\mathscr{T}$ is regular if and only if $\text{Wucl}\, A = \text{Cl}\, A$ for each $A \subset X$.

5. T_{2b}-spaces and other results.

THEOREM 4. *If $(X,\mathscr{T})$ is not T_{2a}, then $(X,\mathscr{T}_*)$ is not T_1.*

Proof. Let $x, y \in X$ such that every closed neighbourhood of x intersects every closed neighbourhood of y for $\mathscr{T}$. Then, if V is any $\mathscr{T}$ neighbourhood of x, $y \in \text{Ucl}\, V$. Thus for $\mathscr{T}_1$ every neighbourhood of y intersects V; hence $(X,\mathscr{T}_1)$ is not T_2. Since $\mathscr{T}_*$ is regular and coarser than $\mathscr{T}_1$, it follows that $\mathscr{T}_*$ is not T_1.

We noted that for the T_{2a}-space $(X,\mathscr{T})$ defined in Example 1, $\mathscr{T}_2 = \mathscr{T}_*$ was not T_0. On the other hand, we have the following example.

Example 4. A T_{2a}-space $(X,\mathscr{T})$ which is not T_3 and such that $(X,\mathscr{T}_*)$ is T_3. Let X be the reals, $\mathscr{T}_\#$ the order topology, Q the rationals and

$$\mathscr{T} = \Phi(\mathscr{T}_\# \cup \{Q\}).$$

Then $\mathscr{T}_* = \mathscr{T}_\#$.

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Theorem 4 and Examples 3 and 4 give rise to the following definition, which seems to be of interest in view of Theorem 1.

Definition 5. A topological space $(X, \mathscr{T})$ is said to be a T_{2b}-space if $\mathscr{T}_*$ is T_1 (hence T_3).

I know of no characterization of T_{2b}-spaces other than the defining one.

Example 5. $(\mathscr{T} \mid A)_*$ is not necessarily the same as $\mathscr{T}_* \mid A$. Let

$$X = \{a, b, c, d\}, \quad \mathscr{T} = \Phi(\{\{b\}, \{c\}, \{b, d\}, \{a, b, c\}\}), \quad A = \{b, c, d\}.$$

Then $\mathscr{T}_* \mid A$ is the trivial topology, while $(\mathscr{T}|A)_* = \Phi(\{c\}, \{b, d\})$.

THEOREM 5. *In order that X be connected for $\mathscr{T}$ it is necessary and sufficient that it be connected for $\mathscr{T}_*$.*

Proof. Necessity follows immediately from the fact that $\mathscr{T}_* \subset \mathscr{T}$. To show sufficiency, suppose A is not connected for $\mathscr{T}$. Then there is a continuous and onto map f: $(X, \mathscr{T}) \to Y$, where $Y = \{a, b\}$ with the discrete topology. But then, since Y is regular, we have, by Theorem 1, that f is continuous $(X, \mathscr{T}_*) \to Y$, hence $(X, \mathscr{T}_*)$ is not connected.

Note. Theorem 5 cannot be extended to subspaces. In Example 5, $(A, \mathscr{T}_*|A)$ is connected but $(A, \mathscr{T}|A)$ is not.

THEOREM 6. *Let $\mathscr{T}_\#$ be the semi-regular topology associated with $\mathscr{T}$* (see **2**). *Then $\mathscr{T}_* \subset \mathscr{T}_\# \subset \mathscr{T}$.*

Proof. The fact that $\mathscr{T}_\# \subset \mathscr{T}$ is stated as an exercise in (**2**). To show that $\mathscr{T}_* \subset \mathscr{T}_\#$, it suffices to show that the complement V of any ultraclosed set $X - V$ is the union of regular open sets. But, since $X - V$ is ultraclosed, for each $x \in V$ we can choose a neighbourhood V_x of x such that

$$\text{Cl } V_x \cap (X - V) = \emptyset.$$

Then $V = \bigcup_{x \in V} \text{Int Cl } V_x$.

THEOREM 7. *Let f, g be continuous functions $X \to Y$, where Y is T_3. If $f(x) = g(x)$ for each point of a subset $D \subset X$ which is dense in $(X, \mathscr{T}_*)$, then $f = g$.*

Proof. D determines continuous functions $(X, \mathscr{T}_*) \to Y$ by the principle of extension of identities. The theorem then follows from Theorem 1.

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