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# Positive Solutions of the Falkner–Skan Equation Arising in the Boundary Layer Theory

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*Abstract.* The well-known Falkner–Skan equation is one of the most important equations in laminar boundary layer theory and is used to describe the steady two-dimensional flow of a slightly viscous incompressible fluid past wedge shaped bodies of angles related to $\lambda\pi/2$, where $\lambda \in \mathbb{R}$ is a parameter involved in the equation. It is known that there exists $\lambda^* < 0$ such that the equation with suitable boundary conditions has at least one positive solution for each $\lambda \geq \lambda^*$ and has no positive solutions for $\lambda < \lambda^*$. The known numerical result shows $\lambda^* = -0.1988$. In this paper, $\lambda^* \in [-0.4, -0.12]$ is proved analytically by establishing a singular integral equation which is equivalent to the Falkner–Skan equation. The equivalence result provides new techniques to study properties and existence of solutions of the Falkner–Skan equation.

## 1 Introduction

The well-known Falkner–Skan equation arising in the boundary layer problems:

$$f'''(\eta) + f(\eta)f''(\eta) + \lambda[1 - (f')^2(\eta)] = 0 \quad \text{on } \eta \in (0, \infty) \tag{1.1}$$

subject to the boundary condition

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1, \tag{1.2}$$

and the side condition

$$0 < f'(\eta) < 1 \quad \text{for } \eta \in (0, \infty), \tag{1.3}$$

has been used to describe the steady two-dimensional flow of a slightly viscous incompressible fluid past wedge shaped bodies of angles related to $\lambda\pi/2$, where $\eta$ is the similarity boundary-layer ordinate, $f(\eta)$ is the similarity stream function and $f'(\eta)$ and $f''(\eta)$ are the velocity and the shear stress, respectively. If $\lambda \in [-2, 0]$, the corresponding flow is called the corner flow and if $\lambda \in [0, 2]$, the flow is the wedge flow.

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When $\lambda = 0$, the wedge reduces to a flat plate and the Falkner–Skan equation becomes the well-known Blasius equation and when $\lambda = 1/2$, it is called the Homann equation. We refer to [2, 12, 14] for more physical significance on (1.1).

Equation (1.1), (1.2), (1.3) has been widely studied analytically, for example in [1, 4–9, 16, 17, 19–21] and the references therein. We refer to [3, 11, 13] for the numerical treatments of (1.1) and (1.2) and to [9, 10] for the study of other solutions of (1.1).

It is well known that (1.1), (1.2), (1.3) has a unique solution for each $\lambda \geq 0$ (see [6, Theorems 6.1, 8.1]) and there exists $\lambda^* < 0$ such that (1.1), (1.2), (1.3) has at least one solution for each $\lambda \in (\lambda^*, 0)$, has a unique solution for $\lambda = \lambda^*$ and has no solutions for $\lambda < \lambda^*$ (see [6, Theorem 7.1], [7, Proposition 1.1, Theorem 1.1], and [8, Theorem]). Moreover, in all cases mentioned above, the solutions $f$ satisfy the following condition:

$$f''(\eta) > 0 \quad \text{for } \eta \in (0, \infty). \tag{1.4}$$

(see [6, Theorems 6.1, 7.1, 8.1]). Therefore, (1.1), (1.2), (1.3) is equivalent to (1.1), (1.2), (1.4). The main approaches used are the fixed point theorems for compact maps in suitable spaces [8, 18] or by considering trajectories in the three-dimensional phase spaces [4, 6, 7]. For example, Hastings [8] employed the Schauder–Tychonov theorem for compact self-maps defined in closed convex subsets of Fréchet spaces.

Recently, the existence of solutions of (1.1) and (1.2) with (1.4) when $\lambda \geq 0$ was studied in [17] by considering a singular integral equation of the form

$$z(t) = \int_t^1 \frac{(1-s)(\lambda + \lambda s + s)}{z(s)}\, ds + (1-t)\int_0^t \frac{s}{z(s)}\, ds \quad \text{for } t \in [0, 1], \tag{1.5}$$

where it was implicitly assumed that $\lim_{t\to 0^+}(1-t)\int_0^t \frac{s}{z(s)}\, ds = 0$. It was proved in [17] that if $\lambda \geq 0$, then (1.1), (1.2), (1.4) has a unique solution and has no such solutions when $\lambda \leq -1/2$. Some comparison principles and the Schauder fixed point theorems in Banach spaces were used in [17]. Moreover, the condition $z(0) > 0$ is used in an essential way. Wang, Gao, and Zhang pointed out that their techniques [17] cannot be applied to treat the existence of solutions of (1.1)–(1.2) and positive solutions of (1.5) when $\lambda \in (-1/2, 0)$. The main difficulty in treating the case when $\lambda \in (-1/2, 0)$ is that the integrand in the first integral of (1.5) takes negative values on an interval $[0, \delta)$ and $z(0)$ may be zero.

Yang [19] studied (1.1) and (1.2) with (1.4) when $\lambda \in (-1/2, 0)$ by considering the existence of positive solutions of a sequence of integral equations related to (1.5). The approach is to employ the Schauder fixed point theorem to the integral equations and the Helly selection principle, *i.e.,* there exists a pointwise convergent subsequence for a bounded infinite sequence of functions of bounded variation whose variations are bounded (see [15, Corollary 3.2]). It is shown in [19] that there exists $\lambda_1 \in (-1/2, 0)$ such that (1.1) and (1.2) with (1.4) have at least one positive solution for each $\lambda \in (\lambda_1, 0)$. Hence, $\lambda^* \leq \lambda_1$. An open question is: what exactly is $\lambda^*$? The numerical method shows $\lambda^* = -0.1988$ (see [3, 11, 13]).

In this paper, we shall provide the upper and lower bounds for $\lambda^*$; in particular, we show $\lambda^* \in [-0.4, -0.12]$. The approach is to study (1.1) and (1.2) with (1.4) via the following singular integral equation of the form

$$z(t)=\int_t^1\frac{(1-s)(\lambda+\lambda s+s)}{z(s)}\,ds+(1-t)\int_0^t\frac{s}{z(s)}\,ds\quad\text{for } t\in(0,1). \tag{1.6}$$

Unlike [17, 19], we do not consider the end-points $t=0,1$ in (1.6) because it is not clear that if the limit $\lim_{t\to 1^-}(1-t)\int_0^t\frac{s}{z(s)}\,ds$ always is 0 and if the first integral with $t=0$ in (1.6) converges. We shall show that if $z$ is a solution in $C[0,1]$ with $z(t)>0$ for $t\in(0,1)$, then

$$\lim_{t\to 1^-}\int_0^t\frac{s}{z(s)}\,ds=\infty,\qquad \lim_{t\to 1^-}(1-t)\int_0^t\frac{s}{z(s)}\,ds=0.$$

We shall prove some properties of positive solutions of (1.6) and one property of solutions of (1.1), (1.2), (1.4) which shows the behaviour of $f''(\eta)$ as $\eta\to\infty$. Using these properties we prove the equivalence of (1.6) and (1.1), (1.2), (1.4). Some explicit expressions between positive solutions of (1.6) and solutions of (1.1), (1.2), (1.4) are given. The new equivalence result plays an important role in estimating $\lambda^*$ and studying properties of solutions of (1.1), (1.2), (1.4).

## 2 Properties of Positive Solutions of a Singular Integral Equation

In this section, we give new properties of positive solutions of the singular integral equation of (1.6) and prove that (1.6) is equivalent to a first order differential equation for each $\lambda\in(-1/2,\infty)$ and to a second order differential equation for each $\lambda\in[0,\infty)$.

We denote by $C[0,1]$ the Banach space of continuous functions defined on $[0,1]$ with the maximum norm $\|z\|=\max\{|z(t)|:t\in[0,1]\}$. Let

$$Q=\{z\in C[0,1]:z(t)>0;\text{for } t\in(0,1)\}\quad\text{and}\quad \delta:=\delta(\lambda)=\frac{-\lambda}{1+\lambda}.$$

It is clear that $\delta\in(0,1)$ if and only if $-1/2<\lambda<0$. We define

$$Az(t)=\int_t^1 f_z(s)\,ds\ \text{ for } t\in[0,1]\quad\text{and}\quad Bz(t)=\int_0^t\frac{s}{z(s)}\,ds\ \text{ for } t\in[0,1),$$

where $z\in Q$ and $f_z(s):=\frac{(1-s)(\lambda+\lambda s+s)}{z(s)}$ for $s\in(0,1)$. It is easy to verify that if $\lambda\in(-1/2,0)$, then

$$f_z(s)\le 0 \text{ for } s\in(0,\delta)\quad\text{and}\quad f_z(s)\ge 0\text{ for } s\in[\delta,1) \tag{2.1}$$

and if $z\in Q$ and the improper integral $Az(t)$ converges for $t\in[0,1)$, then $f_z$ is Lebesgue integrable on $(t,1)$.

The following result provides a range of $\lambda$ for which (1.6) has no solutions in $Q$.

***Theorem 2.1*** *If* $(\lambda,z)\in\mathbb{R}\times Q$ *satisfies* (1.6), *then* $\lambda>-1/2$.

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**Proof** The proof is by contradiction. Assume that there exist $\lambda \in (-\infty, -1/2]$ and $z \in Q$ such that (1.6) holds. Since $f_z(t) \leq 0$ and $(Az)'(t) = -f_z(t) > 0$ for $t \in (0,1)$, $Az$ is strictly increasing on $t \in (0,1)$. This implies $\lim_{t\to 0^+} Az(t) < (Az)(t_0) \leq 0$ for $t_0 \in (0,1)$. Since $z(t) = (Az)(t) + (1-t)Bz(t)$ and $z$ is continuous from the right at 0, we obtain $0 \leq \lim_{t\to 0^+} z(t) = \lim_{t\to 0^+} Az(t) < 0$, which is a contradiction. ■

By Theorem 2.1, we obtain that if $\lambda \leq -1/2$, then (1.5) has no solutions in $Q$. This result generalizes [17, Theorem 2.6] which shows that if $\lambda \leq -1/2$, (1.6) has no solutions satisfying $z(t) > 0$ for all $t \in [0,1)$.

If a function $z\colon [0,1] \to [0,\infty)$ satisfies (1.6), then $z \in C(0,1)$. However, $z$ may not be continuous at the end-points. The following result shows that the limits of $z$ at the end-points exist under suitable conditions on $z$ and $Az(0)$ and the limit $\lim_{t\to 1^-}(1-t)Bz(t)$ is an indeterminate form of type $0 \times \infty$.

***Proposition 2.2*** *Let $\lambda \in (-1/2, \infty)$ and let $z\colon (0,1) \to [0,\infty)$ be bounded. Assume that $(\lambda, z)$ satisfies (1.6) and $Az(0) \in [0,\infty)$. Then the following assertions hold.*

(i) $Az(t) \geq 0$ *for* $t \in [0,1]$.
(ii) $\lim_{t\to 1^-}(1-t)Bz(t) = 0$.
(iii) $\lim_{t\to 1^-} Bz(t) = \infty$.
(iv) $\lim_{t\to 0^+} z(t) = Az(0)$ *and* $\lim_{t\to 1^-} z(t) = 0$.

**Proof** (i) If $\lambda \in (-1/2, 0)$, then it follows from (2.1) and $(Az)'(s) = -f_z(s)$ for $s \in (0,1)$ that

$$Az \text{ is increasing on } (0,\delta) \text{ and decreasing on } [\delta, 1). \tag{2.2}$$

Noting that $Az(0) \geq 0$, $Az(1) = 0$, and (2.2), we have

$$Az(t) \geq \min\{Az(0), Az(1)\} = Az(1) = 0 \quad \text{for } t \in [0,1].$$

If $\lambda \geq 0$, then $f_z(s) \geq 0$ for $s \in (0,1)$ and $Az$ is decreasing on $(0,1)$. This implies $Az(t) \geq Az(1) = 0$ and (i) holds.

(ii) Let $M = \sup\{z(t) : t \in (0,1)\}$. Then $M < \infty$ since $z$ is bounded. By (1.6) and Proposition 2.2(i), we have $M > 0$ and

$$z(t) \geq (1-t)Bz(t) \geq \frac{(1-t)}{M}\int_0^t s\,ds = \frac{(1-t)t^2}{2M} \quad \text{for } t \in (0,1). \tag{2.3}$$

Hence, we have for $\gamma \in (0,1)$ and $t \in (\gamma, 1)$,

$$Bz(t) \leq Bz(\gamma) + 2M\int_\gamma^t \frac{1}{s(1-s)}\,ds = Bz(\gamma) + 2M\Big[\ln t - \ln(1-t) - \ln\frac{\gamma}{1-\gamma}\Big].$$

This, together with $\lim_{t\to 1^-}(1-t)\ln(1-t) = 0$, implies $\lim_{t\to 1^-}(1-t)Bz(t) = 0$.

(iii) Let $\sigma \in [\delta, 1)$ if $\lambda \in (-1/2, 0)$ and $\sigma \in (0,1)$ if $\lambda \geq 0$. Since $0 \leq \lambda + \lambda s + s \leq 2\lambda + 1$ for $s \in [\sigma, 1)$, it follows from (2.3) that $f_z(s) \leq 2M(2\lambda+1)/s^2$ for $s \in [\sigma, 1)$ and

$$Az(t) \leq 2M(2\lambda+1)\int_t^1 \frac{1}{s^2}\,ds \leq \frac{2M(2\lambda+1)}{\sigma^2}(1-t) \quad \text{for } t \in [\sigma, 1]. \tag{2.4}$$

By (2.3), we have

$$\int_{\sigma}^{t} \frac{s}{z(s)}\, ds \le 2M \int_{\sigma}^{t} \frac{1}{(1-s)s}\, ds \le -2M\ln(1-t) - 2M\ln\frac{\sigma}{1-\sigma}.$$

This, together with (1.6) and (2.4) implies that

$$z(t) = Az(t) + (1-t)\Big[Bz(\sigma) + \int_{\sigma}^{t} \frac{s}{z(s)}\, ds\Big] \le (1-t)[c - 2M\ln(1-t)],$$

where $c = \frac{2M(2\lambda+1)}{\sigma} + Bz(\sigma) - 2M\ln\frac{\sigma}{1-\sigma}$. Let $u(t) = c - 2M\ln(1-t)$ for $t \in [\sigma, 1)$. Then $du = \frac{2M}{1-t}\, dt$ and $\frac{1}{z(t)} \ge \frac{1}{(1-t)u(t)}$ for $t \in [\sigma, 1)$. This implies

$$\int_{\sigma}^{1} \frac{s}{z(s)}\, ds \ge \sigma \int_{\sigma}^{1} \frac{1}{(1-s)u(s)}\, ds = \frac{\sigma}{2M}\int_{u(\sigma)}^{\infty} \frac{1}{u}\, du = \infty,$$

and this implies

$$\int_{0}^{1} \frac{s}{z(s)}\, ds = \lim_{\sigma\to 0^+} \int_{\sigma}^{1} \frac{s}{z(s)}\, ds = \infty.$$

(iv) By (1.6) and (ii), we have $\lim_{t\to 1^-} z(t) = 0$. Since $Az(0) \in [0, \infty)$ and $\lim_{t\to 0^+}(1-t)Bz(t) = 0$, it follows from (1.6) that $\lim_{t\to 0^+} z(t) = Az(0)$. ■

If $z$ is continuous at the end-points, then by Lemma 2.2(iv) we have the following.

***Corollary 2.3*** *Assume that* $(\lambda, z) \in (-1/2, \infty) \times Q$ *satisfies* (1.6). *Then* $z(0) = Az(0)$ *and* $z(1) = 0$.

In some cases, it is convenient to change (1.6) into a first-order or a second order differential equation with suitable boundary conditions. In fact, we can prove that they are equivalent.

***Theorem 2.4*** (i) *Let* $(\lambda, z) \in (-1/2, \infty) \times Q$. *Then* $(\lambda, z)$ *satisfies* (1.6) *if and only if* $z(1) = 0$ *and*

$$z'(t) = \frac{-\lambda(1-t^2)}{z(t)} - Bz(t) \quad \text{for } t \in (0,1). \tag{2.5}$$

(ii) *Let* $(\lambda, z) \in [0, \infty) \times Q$. *Then* $(\lambda, z)$ *satisfies* (1.6) *if and only if* $(\lambda, z)$ *is a solution of the following second order differential equation of the form*

$$z''(t) = -\lambda\Big(\frac{1-t^2}{z(t)}\Big)' - \frac{t}{z(t)} \quad \text{for } t \in (0,1) \tag{2.6}$$

*subject to the boundary condition:*

$$z(0) > 0, \quad z(1) = 0 \quad z'(0) = -\lambda/z(0). \tag{2.7}$$

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**Proof** (i) Assume that $(\lambda, z) \in (-1/2, \infty) \times Q$ satisfies (1.6). Differentiating (1.6) implies (2.5) and it follows from Proposition 2.2(iv) that $z(1) = 0$. Conversely, we have for $t \in (0, 1)$,

$$\int_t^1 \int_0^s \frac{\xi}{z(\xi)}\, d\xi ds = \int_0^t \Big[ \int_t^1 \frac{\xi}{z(\xi)}\, ds \Big]\, d\xi + \int_t^1 \Big[ \int_\xi^1 \frac{\xi}{z(\xi)}\, ds \Big]\, d\xi$$
$$= (1-t)Bz(t) + \int_t^1 (1-s)\frac{s}{z(s)}\, ds.$$

Integrating (2.5) from $t$ to 1, we have

$$\begin{aligned}\int_t^1 z'(s)\, ds &= -\lambda \int_t^1 \frac{1-s^2}{z(s)}\, ds - \int_t^1 \int_0^s \frac{\xi}{z(\xi)}\, d\xi ds \\ &= -\lambda \int_t^1 \frac{1-s^2}{z(s)}\, ds - (1-t)Bz(t) - \int_t^1 (1-s)\frac{s}{z(s)}\, ds \\ &= -Az(t) - (1-t)Bz(t).\end{aligned}$$

This, together with $z(1) = 0$, implies (1.6).

(ii) Let $(\lambda, z) \in [0, \infty) \times Q$ satisfy (1.6). It follows from (i) and (2.5) that (2.6) holds and $z(1) = 0$. Since $\lambda \geq 0$, $z(0) = Az(0) > 0$. By (2.5), we have $z'(0) = -\lambda/z(0)$ and (2.7) holds. Conversely, integrating (2.6) from 0 to $t$ implies (2.5). ■

By (2.6), (2.7), we obtain $z''(0) = -\lambda^2/z^3(0)$, so $z \in C^2[0, 1)$ and (2.6) holds for $t \in [0, 1)$. This shows that (2.6), (2.7) is equivalent to [17, (1.6),(1.7)].

Now, we prove properties of positive solutions of (1.6) when $\lambda \in (-1/2, \infty)$ which provide upper and lower bounds of positive solutions.

***Proposition 2.5*** *Assume that $(\lambda, z) \in (-1/2, \infty) \times Q$ satisfies* (1.6). *Then the following assertions hold:*

(P1) $z(t) \geq \frac{1}{2\|z\|}(1-t)t^2$ *for* $t \in [0, 1]$.
(P2) *If* $\lambda \in (-1/2, 0)$, *then* $2/27 \leq \|z\| \leq 1$.
(P3) *If* $\lambda \geq 0$, *then* $\sqrt{(1+4\lambda)/6} \leq \|z\| \leq \sqrt{(1+4\lambda)/3}$.

**Proof** (P1): By (1.6) and Proposition 2.2(i), we have

$$z(t) \geq (1-t)Bz(t) \geq \frac{1-t}{\|z\|} \int_0^t s\, ds \geq \frac{1}{2\|z\|}(1-t)t^2 \quad \text{for } t \in (0, 1).$$

This implies (P1).

(P2): We define a function $h\colon (-1/2, 0] \to [0, \infty)$ by

$$h(\lambda) = \int_{\delta(\lambda)}^1 (1-t)(\lambda + \lambda t + t)\, dt. \tag{2.8}$$

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Then $h'(\lambda) = \int_{\delta(\lambda)}^{1}(1-t^2)\,dt > 0$ for $\lambda \in (-1/2, 0]$ and

$$h(\lambda) \le h(0) = \int_0^1 (1-t)t\,dt = 1/6 \quad \text{for } \lambda \in (-1/2, 0].$$

Assume that $(\lambda, z) \in (-1/2, 0) \times Q$ satisfies (1.6). Then $z(t) > 0$ and $z(t) \ge (Az)(t)$ for $t \in (0,1)$. By (2.1), $f_z(t) \ge 0$ for $t \in [\delta, 1)$ and we have for $t \in [\delta, 1)$,

$$-(Az)'(t)(Az)(t) = f_z(t)(Az)(t) \le f_z(t)z(t) = (1-t)(\lambda + \lambda t + t).$$

Integrating the above inequality from $\delta$ $(= \delta(\lambda))$ to 1 and using $Az(1) = 0$, we have

$$(Az)(\delta) \le \sqrt{2h(\lambda)} \le \sqrt{2h(0)} = \sqrt{3}/3.$$

By (2.2), we have $(Az)(t) \le (Az)(\delta)$ for $t \in [0,1]$ and $z(0) = (Az)(0) \le \sqrt{3}/3$. By (2.5) and the continuity of $z$, we obtain $z(t)z'(t) \le -\lambda(1-t^2)$ for $t \in [0,1]$. Integrating the inequality from 0 to $t$ implies

$$\frac{1}{2}[z^2(t) - z^2(0)] \le -\lambda \int_0^t (1-t^2)dt \le -\lambda(2/3) \le 1/3.$$

This implies $z^2(t) \le 2/3 + z^2(0) \le 1$ for $t \in [0,1]$ and so, $\|z\| \le 1$. This, together with (P1) implies $\|z\| \ge \max\{\frac{1}{2}(1-t)t^2 : t \in [0,1]\} = 2/27$.

(P3): Since $z(t) \ge Az(t)$ and $f_z(t) \ge 0$ for $t \in (0,1)$, we have

$$-(Az)'(t)(Az)(t) \le (1-t)(\lambda + \lambda t + t) \quad \text{for } t \in [0,1).$$

Integrating the above inequality from 0 to 1 and noting that $Az(1) = 0$, we have

$$\frac{1}{2}(Az)^2(0) \le \int_0^1 (1-t)(\lambda + \lambda t + t)\,dt = (2/3)\lambda + 1/6.$$

This implies $z(0) = Az(0) \le \sqrt{(1+4\lambda)/3}$. Since $z$ is decreasing on $[0,1]$, $\|z\| \le \sqrt{(1+4\lambda)/3}$. Let $g(t) = \int_t^1 (1-s)(\lambda + \lambda s + s)\,ds + (1-t)\int_0^t s\,ds$ for $t \in [0,1)$. Then $g(t) \le g(0) = (2\lambda)/3 + 1/6$ for $t \in [0,1]$. Since $z(t) = Az(t) + (1-t)Bz(t) \ge \|z\|^{-1}g(t)$ for $t \in [0,1]$. This implies $\|z\| \ge \|z\|^{-1}g(0)$ and $\|z\| \ge \sqrt{g(0)}$. ∎

## 3 Equivalence between the Falkner–Skan Equation and the Singular Integral Equation

We denote by $\Gamma$ the set of solutions of (1.1), (1.2), (1.4), that is,

$$\Gamma = \{(\lambda, f) \in \mathbb{R} \times C^2(\mathbb{R}_+) : (\lambda, f) \text{ satisfies (1.1), (1.2), (1.4)}\}.$$

We first provide a necessary condition for $(\lambda, f)$ to be a solution of (1.1), (1.2), (1.4) and will use the result to prove the equivalence of (1.6) and (1.1), (1.2), (1.4).

***Lemma 3.1*** *Assume that* $(\lambda, f) \in \Gamma$. *Then* $\lim_{\eta \to \infty} f''(\eta) = 0$.

**Proof** Let $\omega = \inf\{f''(\eta) : \eta \in [1, +\infty)\}$. It follows from (1.4) that $\omega \geq 0$. By the mean value theorem, we have that for each $n \in \mathbb{N}$, there exists $\xi_n \in (n, n+1)$ such that

$$f'(n+1) - f'(n) = f''(\xi_n)[(n+1) - n] = f''(\xi_n) \geq \omega \geq 0.$$

Since $f'(\infty) = 1$, taking limit as $n \to \infty$ implies $\omega = 0$.

We prove that there exists $r > 0$ such that $f$ is decreasing on $(r, \infty)$. In fact, if $\lambda \geq 0$, it follows from (1.1) that $f'''(\eta) \leq 0$ for $\eta \in (0, \infty)$ and $f''$ is decreasing on $(0, \infty)$. If $\lambda < 0$, then since $\lim_{\eta \to 0^+} f'''(\eta) = -\lambda > 0$, there exists $\mu \in (0, 1)$ such that $f'''(\eta) > 0$ for $\eta \in (0, \mu]$ and $f''$ is strictly increasing on $[0, \mu]$. Let $c \in (0, \mu)$. Then $f''(c) > 0$ and $f''(\eta) > f''(c)$ for $\eta \in (c, \mu)$. Since $\omega = 0$ and $f''(\eta) > 0$ for $\eta \in (0, \infty)$, there exists $r > 1$ such that $f''(r) < f''(c)$. Let $\eta_0 \in (c, r)$ be such that

$$f''(\eta_0) = \max\{f''(\eta) : \eta \in [c, r]\}.$$

Then $f''(\eta_0) > f''(c) > f''(r)$. We prove that $f''$ is decreasing on $(r, \infty)$. In fact, if not, there exist $\eta_1, \eta_2 \in (r, \infty)$ with $\eta_1 < \eta_2$ such that $f''(\eta_1) < f''(\eta_2)$. Let $\eta_3 \in [\eta_0, \eta_2]$ be such that $f''(\eta_3) = \min\{f''(\eta) : \eta \in [\eta_0, \eta_2]\}$. Then $\eta_3 < \eta_2$ since $f''(\eta_1) < f''(\eta_2)$ and $\eta_3 > \eta_0$ since $f''(\eta_3) \leq f''(r) < f''(\eta_0)$. It follows from Fermat's theorem that $f'''(\eta_3) = 0$. We show that $f^{(4)}(\eta_3) \geq 0$. In fact, if not, then $f^{(4)}(\eta_3) < 0$ and there exists $a, b \in (\eta_0, \eta_2)$ with $a < \eta_3 < b$ such that $f^{(4)}(\eta) < 0$ for $\eta \in [a, b]$. By Taylor's theorem there exists $\eta^* \in (a, \eta_3)$ such that

$$f''(a) - f''(\eta_3) = f'''(\eta_3)(a - \eta_3) + \frac{1}{2!} f^{(4)}(\eta^*)(a - \eta_3)^2 = \frac{1}{2!} f^{(4)}(\eta^*)(a - \eta_3)^2.$$

Since $f''(a) - f''(\eta_3) \geq 0$, we have $f^{(4)}(\eta^*) \geq 0$, a contradiction. Hence, $f^{(4)}(\eta_3) > 0$. By (1.1), we have for $\eta \in (0, \infty)$,

$$f^{(4)}(\eta) = -f'(\eta)f''(\eta) - f(\eta)f'''(\eta) + 2\lambda f'(\eta)f''(\eta).$$

This implies $0 \leq f^{(4)}(\eta_3) = (-1 + 2\lambda)f'(\eta_3)f''(\eta_3) < 0$, a contradiction. Hence, $f''$ is decreasing on $(r, \infty)$. It follows that $\lim_{\eta \to \infty} f''(\eta)$ exists whenever $\lambda \geq 0$ or $\lambda < 0$. Since $f''(\eta) > 0$ for $\eta > 0$ and $\omega = 0$, it follows that $\lim_{\eta \to \infty} f''(\eta) = 0$. ■

Now, we are in a position to prove that (1.6) is equivalent to (1.1), (1.2), (1.4).

***Theorem 3.2*** (i) *If* $(\lambda, f) \in \Gamma$, *then* $(\lambda, z)$ *satisfies* (1.6), *where* $z\colon [0, 1] \to [0, \infty)$ *is defined by*

$$z(t) = \begin{cases} f''((f')^{-1}(t)) & \text{if } t \in [0, 1), \\ 0 & \text{if } t = 1. \end{cases} \tag{3.1}$$

(ii) *If* $(\lambda, z) \in (-1/2, \infty) \times Q$ *satisfies* (1.6), *then* $(\lambda, f) \in \Gamma$, *where* $f\colon [0, \infty) \to [0, \infty)$ *is defined by*

$$f(\eta) = \int_0^{g^{-1}(\eta)} \frac{s}{z(s)}\, ds \tag{3.2}$$

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*and* $g\colon [0,1) \to [0,\infty)$ *is defined by*

$$g(t) = \int_0^t \frac{1}{z(s)}\, ds.$$

**Proof** (i) Assume that $(\lambda, f) \in \Gamma$. Then $f'(0) = 0$, $f'(\infty) = 1$ and $f''(\eta) > 0$ for $\eta \in (0,\infty)$. It follows that $f'$ is strictly increasing on $[0,\infty)$ and its inverse $(f')^{-1}\colon [0,1) \to [0,\infty)$ is strictly increasing with $(f')^{-1}(0) = 0$ and

$$\lim_{t\to 1^-} (f')^{-1}(t) = \infty.$$

Let $\eta := \eta(t) = (f')^{-1}(t)$ for $t \in [0,1)$. Then $f'(\eta) = t$ and by (3.1), we have $z(t) = f''(\eta)$ for $t \in [0,1)$. This implies that $z(t) > 0$ for $t \in (0,1)$ and $z$ is continuous on $[0,1)$. By (3.1) and Lemma 3.1, we see that $z$ is continuous from the left at 1. Hence, we have $z \in Q$. By using the chain rule to $z(t) = f''(\eta)$, we obtain $f'''(\eta)\frac{d\eta}{dt} = z'(t)$ and by the inverse function theorem, we have $\frac{d\eta}{dt} = \frac{1}{f''(\eta)} = \frac{1}{z(t)}$ for $t \in (0,1)$. This, together with $f'(\eta) = t$, implies

$$f'''(\eta) = z'(t)z(t) \quad \text{and} \quad f'(\eta)\frac{d\eta}{dt} = \frac{t}{z(t)} \quad \text{for } t \in (0,1).$$

Integrating the last equality from 0 to $s$ implies $f(\eta(s)) = Bz(s)$ for $s \in [0,1)$. Substituting $f(\eta)$, $f'(\eta)$, $f''(\eta)$ and $f'''(\eta)$ into (1.1) implies (2.5). It follows from Theorem 2.4(i) that $(\lambda, z)$ satisfies (1.6).

(ii) Assume that $(\lambda, z) \in (-1/2,\infty) \times Q$ satisfies (1.6). If $z(0) > 0$, then $z(s) > 0$ for $s \in [0,1)$. This implies $\int_0^t \frac{1}{z(s)}\, ds < \infty$ for $t \in [0,1)$ and $g(t)$ is well defined on $[0,1)$. If $z(0) = 0$, then $\lambda \in (-1/2, 0)$ and $Az(0) = z(0) = 0$. This implies $\int_0^1 f_z(s)\, ds = \int_0^{\delta/2} f_z(s)\, ds + \int_{\delta/2}^1 f_z(s)\, ds = 0$ and

$$Az(\delta/2) = -\int_0^{\delta/2} f_z(s)\, ds \geq -\frac{\lambda}{2}\Big(1 - \frac{\delta}{2}\Big) \int_0^{\delta/2} \frac{1}{z(s)}\, ds.$$

This implies $g(\delta/2) < \infty$ and $g$ is well-defined on $[0,1)$. It follows from Corollary 2.3 that $\lim_{t\to 1^-} g(t) = \infty$. It is obvious that $g$ is strictly increasing on $[0,1)$ with $g(0) = 0$ and its inverse $g^{-1}\colon [0,\infty) \to [0,1)$ is strictly increasing with $g^{-1}(0) = 0$ and $g^{-1}(\infty) = 1$. Let $t := t(\eta) = g^{-1}(\eta)$ for $\eta \in (0,\infty)$. Then $\frac{dt}{d\eta} = (g^{-1})'(\eta) = \frac{1}{g'(t)} = z(t)$. By (3.2), we have $f(\eta) = Bz(t)$. This implies

$$g^{-1}(\eta) = f'(\eta) = \frac{t}{z(t)}\frac{dt}{d\eta} = t, \quad f''(\eta) = z(t), \quad f'''(\eta) = z'(t)z(t).$$

This, together with (2.5), implies (1.1). It is easy to see that $f$ satisfies (1.2) and the result follows. ∎

## 4 Positive Solutions of the Falkner–Skan Equation

In this section, we prove properties of the positive solutions of the Falkner–Skan equation (1.1), (1.2), (1.3) and provide the estimates for $\lambda^*$ mentioned in the Introduction, which will be given below again.

We state the following result which can be derived from some well-known results in [6, 9]; see also [7, 8].

***Lemma 4.1*** (i) (1.1), (1.2), (1.3) *has a unique solution for* $\lambda \geq 0$. *Moreover, the solution satisfies* (1.4) *with* $\eta \in [0, \infty)$.
(ii) *There exists* $\lambda^* \in (-\infty, 0)$ *such that* (1.1), (1.2), (1.3) *has at least one solution for* $\lambda \in (\lambda^*, 0)$, *and the solutions are not unique and the solutions satisfy* (1.4).
(iii) (1.1), (1.2), (1.3) *has a unique solution* $\lambda = \lambda^*$ *and the solution satisfies* (1.4).
(iv) (1.1), (1.2), (1.3) *has no solutions for* $\lambda < \lambda^*$.

Lemma 4.1(i) is Theorems 6.1 and 8.1 with $\alpha = \beta = 0$ in [6]. The other results of Lemma 4.1 follow from both Theorem 7.1 with $\alpha = \beta = 0$ in [6] and [9, Theorem A].

*Remark* 4.2. By Theorems 2.1, 3.2 and Lemma 4.1, we have $\lambda^* \in (-1/2, 0)$ and by Lemma 4.1 we see that (1.1), (1.2), (1.3) is equivalent to (1.1), (1.2), (1.4).

Let $H\colon [-1/2, 0] \to \mathbb{R}$ be a function defined by

$$H(\lambda) = \frac{2(-4\lambda+3)\lambda}{-\lambda+\frac{1}{2}}\sqrt{\frac{-\lambda}{-\lambda+\frac{1}{2}}} + \frac{(2\lambda+1)^3}{(\lambda+1)^2} + \frac{3(2\lambda+1)}{\lambda+1}\Big(\frac{-\lambda}{\lambda+1}\Big)^2. \tag{4.1}$$

By computation [19], we have $H(0) = 1$ and $H(-\frac{1}{2}) = -\frac{5\sqrt{2}}{2}$. Hence, the set of solutions of $H(\lambda) = 0$ in $[-1/2, 0]$ is non-empty. Let

$$\lambda_1 = \max\{\lambda \in [-1/2, 0] : H(\lambda) = 0\}.$$

Then $\lambda_1 \in (-1/2, 0)$.

We need the following known result obtained by Yang [19], which shows $\lambda^* \leq \lambda_1$.

***Lemma 4.3*** (1.1), (1.2), (1.3) *has at least one solution for each* $\lambda \in (\lambda_1, 0)$

In the following, we shall improve Lemma 4.3 and provide better estimates for $\lambda^*$. We first prove some useful properties of $H$ defined in (4.1).

***Lemma 4.4*** (h1) *The function H defined above is strictly increasing on* $(-1/6, 0]$.
(h2) $\lambda_1 \in (-0.14, -0.12)$.

**Proof** (h1): Let $u(\lambda) = \sqrt{\frac{-2\lambda}{1-2\lambda}}$. By computation, we have for each $\lambda \in (-1/2, 0)$,

$$H(\lambda) = 12\lambda\int_0^{u(\lambda)} (1-s^2)\,ds + 6\int_{\delta(\lambda)}^1 (1-s)(\lambda+\lambda s+s)\,ds + 6(1-\delta(\lambda))\int_0^{\delta(\lambda)} s\,ds,$$

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and $H'(\lambda) = H_1(\lambda) + 6H_2(\lambda)$, where

$$H_1(\lambda) = 12\int_0^{u(\lambda)} (1-s^2)\,ds + 12\lambda(1-u^2(\lambda))u'(\lambda)$$

and $H_2(\lambda) = \int_{\delta(\lambda)}^1 (1-s^2)\,ds - \lambda(1-\delta^2(\lambda))\delta'(\lambda) - \delta'(\lambda)\int_0^{\delta(\lambda)} s\,ds$. Since $u'(\lambda) < 0$ and $\delta'(\lambda) < 0$ for $\lambda \in (-1/2, 0)$, we have $H_1(\lambda) > 0$ for $\lambda \in (-1/6, 0)$ and

$$\begin{aligned} H_2(\lambda) &= 2/3 - \delta(\lambda)(1-\delta^2(\lambda))/3 - \delta(\lambda) + \delta^2(\lambda)(1-2\lambda)/2(1+\lambda^2) \\ &\geq 2/3 - 2\delta(\lambda) + \delta^3(\lambda)/3 \geq 2/3 - 2/5 > 0. \end{aligned}$$

This implies (h1).

(h2): By calculation, we have

$$\begin{aligned} H(-0.14) &= -\frac{623\sqrt{224}}{12800} + \frac{1135404}{1987675} < 0, \\ H(-0.12) &= -\frac{1044\sqrt{186}}{24025} + \frac{163723}{266200} > 0. \end{aligned}$$

The result follows. ∎

Let $k(\lambda) = \int_0^{\delta_\lambda} (1-t)(\lambda + \lambda t + t)\,dt$ and let $h$ be same as in (2.8), that is, $h(\lambda) = \int_{\delta(\lambda)}^1 (1-t)(\lambda + \lambda t + t)\,dt$. We define a function $\Phi\colon [-1/2, 0] \to \mathbb{R}$ by $\Phi(\lambda) = k(\lambda) + \sqrt{2h(\lambda)}$. Then

$$\Phi(\lambda) = \frac{1+4\lambda}{6} - \frac{(2\lambda+1)^3}{6(1+\lambda)^2} + \frac{\sqrt{3}(2\lambda+1)^{3/2}}{3(1+\lambda)}. \tag{4.2}$$

The following result gives the properties of $\Phi$.

***Lemma 4.5*** *The function $\Phi$ defined in* (4.2) *is strictly increasing on* $[-1/2, 0]$ *and there exists a unique* $\lambda_0 \in (-0.4, -0.38)$ *such that* $\Phi(\lambda_0) = 0$.

**Proof** Since $k'(\lambda) = \int_0^{\delta(\lambda)} (1-s^2)\,ds > 0$ and $h'(\lambda) = \int_{\delta(\lambda)}^1 (1-s^2)\,ds > 0$ for $\lambda \in (-1/2, 0)$, $\Phi$ is strictly increasing. By computation, we have

$$\Phi(-0.4) = -\frac{14}{135} + \frac{\sqrt{15}}{45} < 0 \quad \text{and} \quad \Phi(-0.38) = -\frac{13357}{144150} + \frac{12\sqrt{2}}{155} > 0.$$

The result follows. ∎

*Notation.* Let $\Gamma^* := \{(\lambda, f) \in [\lambda^*, 0] \times C^2[0,\infty) : (\lambda, f)$ satisfies (1.1), (1.2), (1.3)$\}$.

Now we prove the following new result which shows $\lambda^* \geq \lambda_0$.

***Theorem 4.6*** *If* $(\lambda, f) \in \Gamma^*$, *then* $\lambda \geq \lambda_0$.

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**Proof** Let $(\lambda, f) \in \Gamma^*$. By Theorem 3.2(i), there exists $z \in Q$ such that (1.6) holds. Since $\lambda + \lambda s + s \leq 0$ for $s \in (0, \delta(\lambda)]$, it follows from Proposition 2.5(P1) that $z(s) \leq 1$ for $s \in (0,1)$ and $f_z(s) \leq (1-s)(\lambda + \lambda s + s)$ for $s \in (0, \delta(\lambda)]$. Hence, we have $\int_0^{\delta(\lambda)} f_z(s)\, ds \leq k(\lambda)$. This, together with (1.6), implies

$$0 \leq z(0) = \int_0^{\delta(\lambda)} f_z(s)\, ds + Az(\delta(\lambda)) \leq \Phi(\lambda)$$

and hence, $\lambda \geq \lambda_0$. ■

By Theorem 4.6 and Lemma 4.3, we see that $\lambda^* \in [\lambda_0, \lambda_1]$. Moreover, by Lemmas 4.4 and 4.5, we obtain the following.

***Corollary 4.7*** (1.1), (1.2), (1.3) *has at least one solution for each* $\lambda \in [-0.12, 0)$ *and has no solutions for each* $\lambda \in (-\infty, -0.4]$.

We denote by $BC[0,\infty)$ the Banach space of continuous bounded functions defined on $[0,\infty)$ with the norm $\|f\| = \sup\{|f(x)| : x \in [0,\infty)\}$.

We end the section by giving properties of positive solution of (1.1), (1.2), (1.3). In particular, we give better estimates for $f''(0)$ when $\lambda \geq 0$. It is known that the initial value $f''(0)$ of $f''(\eta)$ is of importance in the hydrodynamical problem, where it determines the skin friction on the wedge involved.

***Theorem 4.8*** *Assume that* $(\lambda, f) \in [\lambda^*, \infty) \times C^2[0,\infty)$ *satisfies* (1.1), (1.2), (1.3). *Then $f$ has the following properties.*

(i) $f(\eta) < \eta$ *for* $\eta \in (0,\infty)$.
(ii) $\lim_{\eta\to\infty} f(\eta)/\eta = 1$.
(iii) *If* $\lambda \in [\lambda^*, 0)$, *then* $2/27 \leq \|f''\| \leq 1$.
(iv) *If* $\lambda \geq 0$, *then* $\sqrt{(1+4\lambda)/6} \leq f''(0) \leq \sqrt{(1+4\lambda)/3}$.

**Proof** (i): Let $\xi(\eta) = \eta - f(\eta)$ for $\eta \in [0,\infty)$. By the proof of Theorem 3.2, we have $f'(\eta) = g^{-1}(\eta)$ for $\eta \geq 0$. Since $\xi'(\eta) = 1 - g^{-1}(\eta) > 0$ for $\eta \in (0,\infty)$, we have $\xi(\eta) > \xi(0) = 0$ and $f(\eta) < \eta$ for $\eta \in (0,\infty)$.

(ii): Using L'Hopital's rule we have $\lim_{\eta\to\infty} f(\eta)/\eta = \lim_{\eta\to\infty} g^{-1}(\eta) = 1$.

(iii): By the proof of Theorem 3.2, we see that $z(t) = f''(\eta)$, where $f'(\eta) = t$. The result follows from Proposition 2.5(P2).

(iv): By the proof of Proposition 2.5(P3), we see that if $\lambda \geq 0$, then $z$ is decreasing on $[0,1]$ and $\|z\| = z(0)$. Hence, we have $f''(0) = \|z\|$ and the result follows from Proposition 2.5(P3). ■

*Remark* 4.9. When $\lambda \in [0, 1/4)$, Theorem 4.8(iv) generalizes a result in [4, p. 107] due to Weyl [18], where $f''(0) \geq \sqrt{4\lambda/3}$ and the second inequality in (iv) was established only for $\lambda \geq 1/2$.

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## References


[1] R. P. Agarwal and D. O'Regan, *Singular integral equations arising in Homann flow.* Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms **9**(2002), no. 4, 481–488.

[2] D. J. Acheson, *Elementary Fluid Dynamics.* Oxford Univ. Press, New York, 1990.

[3] S. N. Brown and K. Stewartson, *On the reversed flow solutions of the Falkner-Skan equations.* Mathematika **13**(1966), 1–6.

[4] W. A. Coppel, *On a differential equation of boundary-layer theory.* Phil. Trans. Roy. Soc. London, Ser. A **253**(1960), 101–136.

[5] A. H. Craven and L. A. Peletier, *On the uniqueness of solutions of the Falkner-Skan equation.* Mathematika **19**(1972), 129–133.

[6] P. Hartman, *Ordinary Differential Equations.* Classics in Applied Mathematics 38. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.

[7] ———, *On the existence of similar solutions of some boundary layer problems.* SIAM J. Math. Anal. **3**(1972), 120–147.

[8] S. P. Hastings, *An existence theorem for a class of nonlinear boundary value problems including that of Falkner and Skan.*, J. Differential Equations **9**(1971), 580–590.

[9] ———, *Reversed flow solutions of the Falkner-Skan equations.* SIAM J. Appl. Math. **22**(1972), 329–334.

[10] S. P. Hastings and W. C. Troy, *Osillating solutions of the Falkner-Skan equation for negative* $\beta$. SIAM J. Math. Anal. **18**(1987), no. 2, 422–429.

[11] P. A. Libby and T. M. Liu, *Further solutions of the Falkner-Skan equation.* AIAA J. **5**(1967), no. 5, 1040–1042.

[12] T. Y. Na, *Computational Methods in Engineering Boundary Value Problems.* Mathematics in Science and Engineering 145. Academic Press, New York, 1979.

[13] N. Riley and P. D. Weidman, *Multiple solutions of the Falkner-Skan equation for flow past a stretching boundary.* SIAM J. Appl. Math. **49**(1989), no. 5, 1350–1358.

[14] H. Schlichting and K. Gersten, *Boundary-Layer Theory.* Eighth edition. Springer-Verlag, Berlin, 2000.

[15] K. Schrader, *A generalization of the Helly selection theorem.* Bull. Amer. Math. Soc. **78**(1972), 415–419.

[16] K. K. Tam, *A note on the existence of a solution of the Falkner-Skan equation.* Canad. Math. Bull. **13**(1970), 125–127.

[17] J. Wang, W. Gao and Z. Zhang, *Singular nonlinear boundary value problems arising in boundary layer theory.* J. Math. Anal. Appl. **233**(1999), no. 1, 246–256.

[18] H. Weyl, *On the differential equations of the simplest boundary-layer problems.* Ann. of Math. **43**(1942), 381–407.

[19] G. C. Yang, *Existence of solutions to the third-order nonlinear differential equations arising in boundary layer theory.* Appl. Math. Lett. **16**(2003), no. 6, 827–832.

[20] ———, *A note on* $f''' + ff'' + \lambda(1 - f'^2) = 0$ *with* $\lambda \in (-\frac{1}{2}, 0)$ *arising in boundary layer theory.* Appl. Math. Lett. **17**(2004), no. 11, 1261–1265.

[21] G. C. Yang and J. Li, *A new result on* $f''' + ff'' + \lambda(1 - f'^2) = 0$ *arising in boundary layer theory.* Nonlinear Func. Anal. Appl. **10**(2005), no. 1, 117–122.



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