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# An analogue of Banach's contraction principle for 2-metric spaces

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In this paper we establish a fixed point theorem for 2-metric spaces. Some interesting particular cases of this theorem are also obtained.

## 1.

Just as a metric abstracts the properties of the length function, a 2-metric space has its topology given by a real function of point triples which abstracts the properties of the area function for euclidean triangles. Let $X$ be a set consisting of at least three points. A 2-metric on $X$ is a mapping $\rho$ of $X \times X \times X$ into the set of real numbers $R$ that satisfies the following conditions:

(1.1) if at least two of $a, b, c$ are equal, then $\rho(a, b, c) = 0$ and, for any $a \neq b$, there exists a point $c$ such that $\rho(a, b, c) \neq 0$ ;

(1.2) $\rho(a, b, c) = \rho(b, c, a) = \rho(c, a, b)$ for all $a, b, c$ in $X$ ;

(1.3) $\rho(a, b, c) \leq \rho(a, b, d) + \rho(a, d, c) + \rho(d, b, c)$ for all $a, b, c, d$ in $X$ .

The pair $(X, \rho)$ is called a 2-metric space. A sequence $\langle x_n \rangle$ in $(X, \rho)$ is said to be a Cauchy sequence if $\rho(x_m, x_n, a) \to 0$ as $m$ and $n \to \infty$ for every $a \in X$ . The sequence $\langle x_n \rangle$ is said to converge to the point $x \in X$ if $\lim \rho(x_n, x, a) = 0$ for every $a \in X$ . A 2-metric

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space $(X, \rho)$ is said to be complete if every Cauchy sequence in it is convergent. Further the 2-metric space $(X, \rho)$ is said to be bounded if there exists a constant $K$ such that $\rho(a, b, c) \le K$ for all $a, b, c$ in $X$ ([2], [5]).

## 2.

As is well-known, there are a large number of generalisations of Banach's contraction principle and analogous results in the literature (see for example [1], [3], [6]). In this paper we establish the following analogue of Banach's contraction principle for 2-metric spaces.

THEOREM. *Let* $(X, \rho)$ *be a complete 2-metric space and* $\Phi_1$ *and* $\Phi_2$ *two self-maps on* $X$ *such that for all* $x, y, a$ *in* $X$ ,

$$(2.1)\quad \rho\big(\Phi_1(x), \Phi_2(y), a\big) \le a_1\rho\big(x, \Phi_1(x), a\big) + a_2\rho\big(y, \Phi_2(y), a\big) + a_3\rho\big(x, \Phi_2(y), a\big) + a_4\rho\big(y, \Phi_1(x), a\big) + a_5\rho(x, y, a) ,$$

*where* $a_1, a_2, a_3, a_4$ , *and* $a_5$ *are non-negative numbers such that* $\sum_{i=1}^{5} a_i < 1$ *and* $(a_1-a_2)(a_3-a_4) \ge 0$ . *Then* $\Phi_1$ *and* $\Phi_2$ *have a unique common fixed point.*

## 3.

Proof of the theorem. Let $[x]$ denote the integral part of $x$ and write

$$\frac{a_1+a_3+a_5}{1-a_2-a_3} = \alpha \quad \text{and} \quad \frac{a_2+a_4+a_5}{1-a_1-a_4} = \beta .$$

Take any $x_0 \in X$ and define

$$x_{2n+1} = \Phi_1(x_{2n}) \quad \text{and} \quad x_{2n+2} = \Phi_2(x_{2n+1}) \quad (n = 0, 1, 2, \ldots) .$$

For any non-negative integer $n$ we have

$$\rho(x_{2n}, x_{2n+1}, x_{2n+2}) = \rho\big(\Phi_1(x_{2n}), \Phi_2(x_{2n+1}), x_{2n}\big) \le a_2\rho(x_{2n+1}, x_{2n+2}, x_{2n}) ,$$

using (1.1) and condition (2.1) of the theorem. As $a_2 < 1$ , in view of

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(1.2) the above inequality gives

(3.1) $$\rho(x_{2n}, x_{2n+1}, x_{2n+2}) = 0 .$$

Similarly

(3.2) $$\rho(x_{2n+1}, x_{2n+2}, x_{2n+3}) = 0 .$$

For any $a \in X$ ,

$$\begin{aligned}\rho(x_{2n+1}, x_{2n+2}, a) \le\ & a_1\rho(x_{2n}, x_{2n+1}, a) + a_2\rho(x_{2n+1}, x_{2n+2}, a) \\ & + a_3\{\rho(x_{2n}, x_{2n+1}, a)+\rho(x_{2n}, x_{2n+2}, x_{2n+1})+\rho(x_{2n+1}, x_{2n+2}, a)\} \\ & + a_4\rho(x_{2n+1}, x_{2n+1}, a) + a_5\rho(x_{2n}, x_{2n+1}, a) ,\end{aligned}$$

and therefore, using (3.1), we get

$$\rho(x_{2n+1}, x_{2n+2}, a) \le \alpha\rho(x_{2n}, x_{2n+1}, a) .$$

Similarly, **using** (3.2), we get

$$\rho(x_{2n+2}, x_{2n+3}, a) \le \beta\rho(x_{2n+1}, x_{2n+2}, a) .$$

With the help of the above two inequalities it follows that

$$\rho(x_{2n+1}, x_{2n+2}, a) \le (1+\alpha)(\alpha\beta)^{[(2n+1)/2]}\rho(x_0, x_1, a) ,$$

and

$$\rho(x_{2n+2}, x_{2n+3}, a) \le (1+\alpha)(\alpha\beta)^{[(2n+2)/2]}\rho(x_0, x_1, a) .$$

Hence

(3.3) $$\rho(x_m, x_{m+1}, a) \le (1+\alpha)(\alpha\beta)^{[m/2]}\rho(x_0, x_1, a) .$$

Note that

(3.4) $$\rho(x_0, x_1, x_m) = 0$$

for $m = 0, 1, 2, \ldots$ . This is true for $m = 0$ and $m = 1$ . Suppose now that it holds for every $m$ in $2 \le m \le k-1$ . Using (1.3) we have

$$\begin{aligned}\rho(x_0, x_1, x_k) &\le \rho(x_0, x_1, x_{k-1}) + \rho(x_0, x_{k-1}, x_k) + \rho(x_{k-1}, x_k, x_1) \\ &\le (1+\alpha)(\alpha\beta)^{[(k-1)/2]}[\rho(x_0, x_1, x_0)+\rho(x_0, x_1, x_1)] ,\end{aligned}$$

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and this proves (3.4).

Since

$$\rho(x_m, x_{m+1}, x_n) \le (1+\alpha)(\alpha\beta)^{[m/2]}\rho(x_0, x_1, x_n) ,$$

it follows that

(3.5) $$\rho(x_m, x_{m+1}, x_n) = 0 ,$$

for all non-negative integers $m$ and $n$.

Note that for any $a \in X$ and $m < n$ we have

$$\rho(x_m, x_n, a) \le \rho(x_m, x_{m+1}, a) + \rho(x_m, x_{m+1}, x_n) + \rho(x_{m+1}, x_n, a) ,$$

and therefore in view of (3.5) and (3.3) we have

$$\begin{aligned}\rho(x_m, x_n, a) &\le \rho(x_m, x_{m+1}, a) + \rho(x_{m+1}, x_{m+2}, a) + \dots + \rho(x_{n-1}, x_n, a)\\ &\le (1+\alpha)\left((\alpha\beta)^{[m/2]} + (\alpha\beta)^{[(m+1)/2]} + \dots + (\alpha\beta)^{[(n-1)/2]}\right)\rho(x_0, x_1, a) .\end{aligned}$$

As $\alpha\beta < 1$, the right hand side of the above inequality tends to zero as $m \to \infty$. Hence $\langle x_n \rangle$ is a Cauchy sequence and it converges to some $x \in X$.

Since

$$\begin{aligned}\rho(x, \Phi_1(x), a) &\le \rho(x_{2n+2}, x, \Phi_1(x)) + \rho(x_{2n+2}, x, a) + a_1\rho(x, \Phi_1(x), a)\\ &+ a_2\rho(x_{2n+1}, x_{2n+2}, a) + a_3\rho(x_{2n+2}, x, a)\\ &+ a_4\{\rho(x_{2n+1}, \Phi_1(x), x)+\rho(x_{2n+1}, x, a)+\rho(x, \Phi_1(x), a)\} + a_5\rho(x_{2n+1}, x, a) ,\end{aligned}$$

taking the limit as $n \to \infty$, the above inequality gives

$$\rho(x, \Phi_1(x), a) \le (a_1+a_4)\rho(x, \Phi_1(x), a) ,$$

and therefore

$$\rho(x, \Phi_1(x), a) = 0 ,$$

for all $a \in X$. Axiom (1.1) now gives $\Phi_1(x) = x$. Similarly $\Phi_2(x) = x$. If $y$ is a fixed point of $\Phi_2$, then using (2.1) we get $\rho(x, y, a) = 0$ for all $a \in X$. Hence $x = y$. This proves that $x$ is a unique fixed

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point of $\Phi_1$ and similarly it is a unique fixed point of $\Phi_2$ as well. Hence the theorem.

## 4.

A point is an unique fixed point of a map $\Phi : X \to X$ if and only if it is an unique fixed point of any positive power of $\Phi$. This observation leads us to the following:

THEOREM A. *Let* $(X, \rho)$ *be a complete 2-metric space and* $\Phi_1$ *and* $\Phi_2$ *two self-maps on* $X$ *such that for all* $x, y, a$ *in* $X$ *and positive integers* $p, q$,

$$\rho\left(\Phi_1^p(x), \Phi_2^q(y), a\right) \le a_1\rho\left(x, \Phi_1^p(x), a\right) + a_2\rho\left(y, \Phi_2^q(y), a\right) + a_3\rho\left(x, \Phi_2^q(y), a\right) + a_4\rho\left(y, \Phi_1^p(x), a\right) + a_5\rho(x, y, a),$$

*where* $a_1, a_2, a_3, a_4$, *and* $a_5$ *are non-negative constants such that* $\sum_{i=1}^{5} a_i < 1$ *and* $(a_1-a_2)(a_3-a_4) \ge 0$. *Then* $\Phi_1$ *and* $\Phi_2$ *have a unique and common fixed point.*

COROLLARY 1. *Let* $(X, \rho)$ *be a complete 2-metric space and* $f_i$ $(i = 1, 2, 3, \ldots)$ *a family of mappings of* $X$ *into itself. Suppose there exists a sequence of positive integers* $\langle m_i \rangle$ *and non-negative numbers* $a_1, a_2, a_3, a_4, a_5$ *such that for all* $x, y, a$ *in* $X$ *and every pair* $i, j$, $i \ne j$,

$$\rho\left(f_i^{m_i}(x), f_j^{m_j}(y), a\right) \le a_1\rho\left(x, f_i^{m_i}(x), a\right) + a_2\rho\left(y, f_j^{m_j}(y), a\right) + a_3\rho\left(x, f_j^{m_j}(y), a\right) + a_4\rho\left(y, f_i^{m_i}(x), a\right) + a_5\rho(x, y, a),$$

*where* $\sum_{i=1}^{5} a_i < 1$ *and* $(a_1-a_2)(a_3-a_4) \ge 0$. *Then the sequence of mappings* $\langle f_i \rangle$ *has a unique common fixed point.*

It is interesting to note that the particular case $a_1 = a_2 = \alpha$,

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$a_3 = a_4 = 0$ and $a_5 = \beta$ of this result has been recently established [4] with the additional assumption that the 2-metric space is bounded.

Proof. Take any pair $i \neq j$. Then, by Theorem A, $f_i$ and $f_j$ have an unique and common fixed point. Since $i$ and $j$ are arbitrary, the corollary follows.

COROLLARY 2. *Let* $(X, \rho)$ *be a complete 2-metric space and* $\Phi_1$ *and* $\Phi_2$ *two self maps on* $X$ *satisfying the following conditions:*

*(a) there exist non-negative constants* $a_1$, $a_2$, *and* $a_3$ *such that* $2(a_1+a_2) + a_3 < 1$ *and*

$$\rho\left[\Phi_1^p\Phi_2^q(x),\ \Phi_1^p\Phi_2^q(y),\ a\right] \le a_1\left\{\rho\left[x,\ \Phi_1^p\Phi_2^q(x),\ a\right]+\rho\left[y,\ \Phi_1^p\Phi_2^q(y),\ a\right]\right\}$$
$$+ a_2\left\{\rho\left[x,\ \Phi_1^p\Phi_2^q(y),\ a\right]+\rho\left[y,\ \Phi_1^p\Phi_2^q(x),\ a\right]\right\} + a_3\rho(x,\ y,\ a)\ ,$$

*for all* $x, y, a$ *in* $X$ *and any positive integers* $p$ *and* $q$ *;*

*(b)* $\Phi_1$ *and* $\Phi_2$ *commute.*

*Then* $\Phi_1$ *and* $\Phi_2$ *have a unique common fixed point.*

Proof. By our theorem of Section 2, the map $\Phi_1^p\Phi_2^q$ has a unique fixed point say $u$. Now

$$\Phi_1(u) = \Phi_1\left(\Phi_1^p\Phi_2^q(u)\right) = \Phi_1^p\Phi_2^q(\Phi_1(u))\ ,$$

for $\Phi_1$ and $\Phi_2$ commute. Hence $\Phi_1(u)$ is a fixed point of $\Phi_1^p\Phi_2^q$ and so $\Phi_1(u) = u$. Similarly $\Phi_2(u) = u$. Observe that if $x$ is a common fixed point of $\Phi_1$ and $\Phi_2$ then $x$ is a fixed point of $\Phi_1^p\Phi_2^q$ and so $x = u$. Hence the result.

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