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Artinian Local Cohomology Modules

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Abstract. Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R and M a finitely generated R-module. Let t be a non-negative integer. It is known that if the local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(M)$ is finitely generated for all $i < t$, then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is finitely generated. In this paper it is shown that if $\mathrm{H}^i_{\mathfrak{a}}(M)$ is Artinian for all $i < t$, then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ need not be Artinian, but it has a finitely generated submodule N such that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))/N$ is Artinian.

1 Introduction

Throughout this paper R is a commutative Noetherian ring. Grothendieck [G] conjectured the following: For any ideal $\mathfrak{a}$ and any finite R-module M, the module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ is finite (*i.e.,* finitely generated) for all j. Although this conjecture is not true in general, *cf.* [H, Example 1], there are some attempts to show that for some non-negative integer t, the module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is finite. For example, Asadollahi, Khashyarmanesh and Salarian [AKS] proved the following: Let $\mathfrak{a}$ be an ideal of R and let M be a finite R-module. Let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is a finite R-module for all $i < t$; then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is finite. This result implies that the set of associated primes of the module $\mathrm{H}^t_{\mathfrak{a}}(M)$ is finite, see also [BL, KS]

Now it is natural to ask the following question: Let $\mathfrak{a}$ be an ideal of R and let M be a finite R-module. Let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is an Artinian R-module for all $i < t$. Is the module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ Artinian?

Although we give a negative answer to this question (see Proposition 2.4), it is shown that there is a finite submodule N such that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))/N$ is Artinian (see Theorem 2.2. This result implies that the set of associated primes of $\mathrm{H}^t_{\mathfrak{a}}(M)$ is finite.

Suppose that E is an injective R-module. An R-module M is called reflexive with respect to E if the canonical injection $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E))$ is an isomorphism.

In Section 2, we show the following, see Theorem 2.5. Let M be a finite R-module such that M is reflexive with respect to a minimal injective cogenerator E in the category of R-modules. Let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is a reflexive R-module for $i < t$. Then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is reflexive. This implies that not only is the set of associated primes of $\mathrm{H}^t_{\mathfrak{a}}(M)$ finite, but also that the Bass numbers of $\mathrm{H}^t_{\mathfrak{a}}(M)$ are finite. We use terminology and notation of [BS].

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2 Main Results

Recall that an R-module M is called minimax if there is a finite submodule N of M, such that M/N is Artinian, *cf.* [Z]. The class of minimax modules includes all finite and all Artinian modules. Moreover it is closed under taking submodules, quotients and extensions, *i.e.,* it is a Serre subcategory of the category of R-modules, [R, Z]. Obviously this class is strictly larger than the class of all finite modules and also Artinian modules, [BER, Theorem 12]. Keep in mind that a minimax R-module has only finitely many associated primes.

Let $\mathfrak{a}$ be an ideal of R and let M be a finite R-module. Let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is Artinian for all $i < t$ and $\mathrm{H}^t_{\mathfrak{a}}(M)$ is not Artinian. The integer t is equal to the filter depth, $\text{f-depth}_{\mathfrak{a}}(M)$ of M in $\mathfrak{a}$, *i.e.,* the length of a maximal filter regular sequence of M in $\mathfrak{a}$ [M, Theorem 3.1]. In this section we will show that for $s = \text{f-depth}_{\mathfrak{a}}(M)$ the module $\mathrm{Hom}(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ is not Artinian, but it is minimax.

Lemma 2.1 *Let M be a minimax R-module and let $\mathfrak{a}$ be an ideal of R. Then M is $\mathfrak{a}$-torsion free if and only if $\mathfrak{a}$ contains an M-regular element.*

Proof Follows immediately by the same proof as [BS, Lemma 2.1.1]. ■

Theorem 2.2 *Let $\mathfrak{a}$ be an ideal of R and let t be a non-negative integer. Let M be an R-module such that $\mathrm{Ext}^t_R(R/\mathfrak{a}, M)$ is a minimax R-module. If $\mathrm{H}^i_{\mathfrak{a}}(M)$ is minimax for all $i < t$, then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is a minimax module. Furthermore, if L is a finite R-module such that $\mathrm{Supp}(L) \subseteq V(\mathfrak{a})$, then $\mathrm{Hom}_R(L, \mathrm{H}^t_{\mathfrak{a}}(M))$ is a minimax module.*

Proof We use induction on t. If $t = 0$, then $\mathrm{H}^0_{\mathfrak{a}}(M) \cong \Gamma_{\mathfrak{a}}(M)$ and

$$\mathrm{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$$

is equal to the minimax R-module $\mathrm{Hom}_R(R/\mathfrak{a}, M)$. So, the assertion holds.

Suppose that $t > 0$ and that the case $t - 1$ is settled. Since $\Gamma_{\mathfrak{a}}(M)$ is minimax, $\mathrm{Ext}^i_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is minimax for all i. Now by using the exact sequence $0 \to \Gamma_{\mathfrak{a}}(M) \to M \to M/\Gamma_{\mathfrak{a}}(M) \to 0$, we get that $\mathrm{Ext}^t_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is minimax. On the other hand, $\mathrm{H}^0_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = 0$ and $\mathrm{H}^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) \cong \mathrm{H}^i_{\mathfrak{a}}(M)$ for all $i > 0$. Thus we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$. Let E be an injective hull of M and put $N = E/M$. Then $\Gamma_{\mathfrak{a}}(E) = 0$ and $\mathrm{Hom}_R(R/\mathfrak{a}, E) = 0$. Consequently, $\mathrm{Ext}^i_R(R/\mathfrak{a}, N) \cong \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, M)$ and $\mathrm{H}^i_{\mathfrak{a}}(N) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(M)$ for all $i \geq 0$. Now the induction hypothesis yields that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{t-1}_{\mathfrak{a}}(N))$ is minimax and hence $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is minimax.

For the last assertion, since $\mathrm{Supp}(L) \subseteq \mathrm{V}(\mathfrak{a})$, by using Gruson's theorem there is a finite chain $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$ such that L_i/L_{i-1} is a homomorphic image of finitely many copies of $R/\mathfrak{a}$ for all $i = 1, 2, \ldots, n$. By induction, we may immediately reduce to the case where $n = 1$. Therefore, there is a short exact sequence $0 \to K \to (R/\mathfrak{a})^m \to L \to 0$ for some positive integer m and R-module K. Now, the exact sequence $0 \to \mathrm{Hom}_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)) \to \mathrm{Hom}_R((R/\mathfrak{a})^m, \mathrm{H}^t_{\mathfrak{a}}(M))$ shows that the R-module $\mathrm{Hom}_R(L, \mathrm{H}^t_{\mathfrak{a}}(M))$ is minimax. ■

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The following result is a generalization of [BL, Proposition 2.1] and [KS, Theorem B].

Corollary 2.3 *Let* $\mathfrak{a}$ *be an ideal of* R *and let* M *be a minimax* R*-module. Let* t *be a non-negative integer such that* $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is minimax for all* $i < t$. *Let* N *be a submodule of* $\mathrm{H}^t_{\mathfrak{a}}(M)$ *such that* $\mathrm{Ext}^1_R(R/\mathfrak{a}, N)$ *is minimax. Then* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ *is a minimax module. In particular,* $\mathrm{H}^t_{\mathfrak{a}}(M)/N$ *has finitely many associated primes.*

Proof Let N be a submodule of $\mathrm{H}^t_{\mathfrak{a}}(M)$ such that $\mathrm{Ext}^1_R(R/\mathfrak{a}, N)$ is minimax. The short exact sequence $0 \to N \to \mathrm{H}^t_{\mathfrak{a}}(M) \to \mathrm{H}^t_{\mathfrak{a}}(M)/N \to 0$ induces the following exact sequence $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)) \to \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)/N) \to \mathrm{Ext}^1_R(R/\mathfrak{a}, N)$. Since the left hand (by Theorem 2.2) and the right hand are minimax, we have that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ is minimax. On the other hand $\mathrm{Supp}\, \mathrm{H}^t_{\mathfrak{a}}(M)/N \subseteq \mathrm{Supp}\, \mathrm{H}^t_{\mathfrak{a}}(M) \subseteq \mathrm{V}(\mathfrak{a})$ and $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ has finitely many associated primes. Therefore the same holds for $\mathrm{H}^t_{\mathfrak{a}}(M)/N$. ■

Proposition 2.4 *Let* $(R, \mathfrak{m})$ *be a local ring. Let* $\mathfrak{a}$ *be an ideal of* R *and let* M *be a finite* R*-module such that* $\mathrm{Supp}(M/\mathfrak{a}M) \nsubseteq \{\mathfrak{m}\}$. *Then* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$, *where* $s = \text{f-depth}_{\mathfrak{a}}(M)$, *is not Artinian but is minimax.*

Proof By [M, Theorem 3.1], the module $\mathrm{H}^s_{\mathfrak{a}}(M)$ is not Artinian and so by [M, Theorem 1.1], the module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ is not Artinian. Whereas,

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$$

is minimax by Theorem 2.2. ■

Suppose that E is the minimal injective cogenerator of the category of R-modules. An R-module M is called reflexive with respect to E if the canonical injection

$$M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E))$$

is an isomorphism. It is well known that an R-module M is reflexive (with respect to E) if and only if M is minimax and $R/\mathrm{Ann}(M)$ is a complete semilocal ring, *cf.* [BER, Theorem 2]. Recall that if N is an arbitrary submodule of a module M, then M is reflexive if and only if both N and M/N are reflexive, *cf.* [BER, Lemma 5]. Consequently, a finite direct sum of modules is reflexive if and only if each direct summand is reflexive.

The next theorem is another main result of this paper.

Theorem 2.5 *Let* M *be a finite* R*-module and let* $R/\mathrm{Ann}(M)$ *be a reflexive* R *module. Let* t *be a non-negative integer such that* $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is a reflexive* R*-module for* $i < t$. *Then* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ *is reflexive.*

Proof We argue by induction on t. Set $t = 0$. By [X, Theorem 1.6], M is reflexive and so $\mathrm{H}^0_{\mathfrak{a}}(M)$ is reflexive. Suppose inductively $t > 0$. Inspired by the ideas of Lemma 2.1, we see that there exists $x \in \mathfrak{a} \setminus \mathrm{Z}(M)$. Consider the exact sequence $0 \to M \stackrel{.x}{\to} M \to M/xM \to 0$. From the induced exact sequence

$$\cdots \to \mathrm{H}^{i-1}_{\mathfrak{a}}(M) \to \mathrm{H}^{i-1}_{\mathfrak{a}}(M/xM) \to \mathrm{H}^{i}_{\mathfrak{a}}(M) \to \cdots,$$

it follows that $\mathrm{H}^i_{\mathfrak{a}}(M/xM)$ is reflexive for all $i \leq t-2$. Note that $R/\operatorname{Ann}(M/xM)$ is a quotient of $R/\operatorname{Ann}(M)$ and so is reflexive. Thus by induction

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{t-1}_{\mathfrak{a}}(M/xM))$$

is reflexive. Now the exact sequence

$$\mathrm{H}^{t-1}_{\mathfrak{a}}(M) \stackrel{g}{\to} \mathrm{H}^{t-1}_{\mathfrak{a}}(M/xM) \stackrel{f}{\to} \mathrm{H}^{t}_{\mathfrak{a}}(M) \stackrel{.x}{\to} \mathrm{H}^{t}_{\mathfrak{a}}(M)$$

induces the following exact sequence

$$0 \to \mathrm{Im}\, g \to \mathrm{H}^{t-1}_{\mathfrak{a}}(M/xM) \to Im f \to 0.$$

So we have the following exact sequence:

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{t-1}_{\mathfrak{a}}(M/xM)) \stackrel{h}{\to} \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Im}\, f) \stackrel{k}{\to} \mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{Im}\, g).$$

By using the facts that any subquotient of a reflexive module is again reflexive, and any finite direct sum of reflexive modules is reflexive, we obtain that $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{Im}\, g)$ is reflexive. On the other hand, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{t-1}_{\mathfrak{a}}(M/xM))$ is reflexive. Thus

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Im}\, f)$$

is reflexive. Now the assertion follows from the fact that

$$\mathrm{Hom}(R/\mathfrak{a}, \mathrm{Im}\, f) = \mathrm{Hom}(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)). \qquad \blacksquare$$

Corollary 2.6 *With the same assumption as Theorem* 2.5*, the Bass numbers of the R-module* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ *are all finite.*

Proof The assertion follows from [B, Lemma 2]. ■

We end the paper with the following question. Let M be a finite R-module. Grothendieck proved that when R is a homomorphic image of a regular local ring, the least integer t such that $\mathrm{H}^t_{\mathfrak{a}}(M)$ is not finite is

$$\mathrm{Min}\{\operatorname{depth} M_{\mathfrak{p}} + \operatorname{ht}((\mathfrak{a}+\mathfrak{p})/\mathfrak{p}) | \mathfrak{p} \not\supseteq \mathfrak{a}\}.$$

Melkersson [M] showed that when $\operatorname{Supp} M/\mathfrak{a}M \not\subseteq \{\mathfrak{m}\}$, the least integer t such that $\mathrm{H}^t_{\mathfrak{a}}(M)$ is not Artinian is

$$\mathrm{Min}\{\operatorname{depth}(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}}) | \mathfrak{p} \in \operatorname{Supp} M/\mathfrak{a}M \setminus \{\mathfrak{m}\}\}.$$

Now it is natural to ask "What is the least integer t such that $\mathrm{H}^t_{\mathfrak{a}}(M)$ is not minimax (resp. reflexive)?"

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