European Journal of Applied Mathematics (2024), 1–13
doi:10.1017/S0956792524000792

CAMBRIDGE
UNIVERSITY PRESS

PAPER

Global well-posedness and uniform boundedness of 2D urban crime models with nonlinear advection enhancement

Wenjing Jiang and Qi Wang

Department of Mathematics, Southwestern University of Finance and Economics, Chengdu, Sichuan, China
Corresponding author: Qi Wang; Email: qwang@swufe.edu.cn

Received: 12 January 2024; **Revised:** 25 July 2024; **Accepted:** 14 October 2024

Keywords: Urban crime model; nonlinear advection; global existence; boundedness

2020 Mathematics Subject Classification: 35K57 (Primary); 91C20, 91D25, 35A01 (Secondary)

Abstract
We study the global well-posedness and uniform boundedness of a two-dimensional reaction–advection–diffusion system with nonlinear advection. This strongly coupled system of nonlinear partial differential equations represents the continuum of a 2D lattice model designed to describe residential burglary, where each location is characterised by a tractability value that varies in both space and time. We show that the model with sublinear advection enhancement is globally well-posed, with a unique solution that is classical and uniformly bounded in time. Our results provide valuable insights into the development of urban crime models with nonlinear advection enhancements, making them suitable for broader applications, including nonlocal or heterogeneous near-repeat victimisation effects.

1. Introduction

This paper investigates the well-posedness and boundedness of the solution $(\rho, A) = (\rho(\mathbf{x}, t), A(\mathbf{x}, t))$ to a reaction–advection–diffusion system, where $\mathbf{x}$ represents location and t denotes time, in the following form:

$$\begin{cases} \partial_t A = \eta \Delta A - A + A\rho + \bar{A}, & \mathbf{x} \in \Omega, t > 0, \\ \partial_t \rho = \nabla \cdot (\nabla \rho - \phi(\rho) \nabla \ln A) - A\rho + \bar{B}, & \mathbf{x} \in \Omega, t > 0, \\ \dfrac{\partial A}{\partial \mathbf{n}} = \dfrac{\partial \rho}{\partial \mathbf{n}} = 0, & \mathbf{x} \in \partial\Omega, t > 0, \\ A(\mathbf{x}, 0) = A_0(\mathbf{x}) > 0, \rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}) \geq, \not\equiv 0, & \mathbf{x} \in \Omega. \end{cases} \tag{1.1}$$

Here, Ω is a bounded domain in $\mathbb{R}^2$ with a piecewise smooth boundary $\partial\Omega$ endowed with unit outer normal $\mathbf{n}$. $\phi(\rho)$ is a continuously differentiable function, and $\eta, \bar{A}$ and $\bar{B}$ are arbitrary positive constants. The main result of our paper is Theorem 1.1 which proves that system (1.1) is globally well-posed, and both A and ρ are bounded in $L^\infty(\mathbb{R}^+; L^\infty(\Omega))$.

1.1. Urban criminal activities

It is widely believed that human behaviour is influenced by a multitude of factors and is therefore too complex to obey the same laws that govern physics and other natural sciences. In particular, individual behaviour is difficult to describe with mechanistic models or to predict with quantitative methods. At the group level, however, human behaviour can exhibit spatiotemporal patterns that are more amenable to mathematical analysis. A prime example of this is the urban crime hotspot phenomenon, in which certain neighbourhoods experience higher crime rates compared to surrounding regions with lower rates. In addition, criminal activity may also fluctuate over time, with certain periods characterised by increased disorder. Such spatiotemporal clustering suggests that while individual actions are unpredictable, collective behaviour follows discernible patterns that can be studied mathematically (see [4, 14]).

1.2. Mathematical models of residential urban crimes

In 2008, a group at UCLA [55] developed a mathematical model to explore the spatiotemporal dynamics of urban crime hotspots. Their model is based on two key criminological assumptions: the broken-windows effect ([51, 61]), which suggests that visible signs of disorder promote further crime, and the repeat and near-repeat victimisation effect ([3, 5, 15, 27, 54]), which argues that areas previously victimised are more likely to experience subsequent crimes. Consequently, they constructed a 2D agent-based lattice model that not only simulates the movement of criminals but also incorporates the dynamics of attractiveness values, representing locations more prone to criminal activity. The spatial continuum of the lattice model is governed by strongly coupled partial differential equations of parabolic type, expressed in the following form

$$\begin{cases} A_t = \eta \Delta A - A + A\rho + \bar{A}, & \mathbf{x} \in \Omega, t > 0, \\ \rho_t = \nabla \cdot (\nabla \rho - \chi \rho \nabla \ln A) - A\rho + \bar{B}, & \mathbf{x} \in \Omega, t > 0. \end{cases} \tag{1.2}$$

where functions $A(\mathbf{x}, t)$ and $\rho(\mathbf{x}, t)$ represent the area attractiveness and the population density of criminal agents at the space-time location $(\mathbf{x}, t)$, respectively; Δ and ∇ denote the Laplace and gradient operators, while η and $\bar{B}$ are positive constants. The term $\bar{A}$ represents a time-independent positive function of location $\mathbf{x}$, capturing the static attractiveness of an area. The attractiveness $A(\mathbf{x}, t)$ consists of two components with $A(\mathbf{x}, t) = \bar{A} + B(\mathbf{x}, t)$, $\bar{A}$ representing the static part and B the dynamic part. Diffusion rate η reflects the intensity of the near-repeat victimisation effect, whereas the broken-window effect is incorporated through the dynamics of $A\rho$ in the first equation. The positive coefficient χ represents the strength of the migration rate of criminal agents towards attractive sites, driven by the gradient of the logarithm of perceived house attractiveness. We refer the reader to [20, 55] for the derivation, justification and extension of (1.2) and to [13, 18, 33] for a review of agent-based urban crime modelling.

Over the last 15 years, a growing interest in mathematical analysis of the urban crime model (1.2) has been driven by the work of [55]. Weak linear analysis by [52, 53] reveals the emergence of stable hotspots, while rigorous bifurcation analysis by [11, 20] provides deeper quantitative insights into these dynamics. Furthermore, stationary large-amplitude peaks in the continuum model (1.2) are investigated by [8, 9, 28, 34, 38]. The 2D lattice model and its continuum not only capture aggregation phenomena in urban criminal activity but also exhibit rich spatiotemporal dynamics and complex regular and irregular spatial patterns that urge further analytical and numerical studies. It seems necessary to mention that the migration of criminal agents towards attractive targets, as modelled here, is analogous to the movement of chemotactic organisms in response to uneven chemical distribution. See [22, 23, 60] for some surveys on the mathematical modelling of chemotaxis.

Mathematical modelling of system (1.2) has been extended in several directions. For instance, Gu, Wang, and Yi in [20] explored spatial heterogeneity in both the near-repeat victimisation effect and the dispersal strategy of criminal agents, proposing a class of reaction–advection–diffusion systems with nonlinear diffusion. Additionally, works such as [10, 29, 41] model the dispersal of criminal agents using the Lévy process, which leads to a fractional Laplacian diffusion problem in the continuum limit of the agent-based model. Other modelling developments can be seen in the consideration of age-structured

https://doi.org/10.1017/S0956792524000792 Published online by Cambridge University Press

populations ([50]) and geographic profiling ([39]), and we refer to [7, 9, 13, 40] for further developments in this direction. It is also noteworthy that the authors of [26, 42] independently introduced an additional equation to (1.2) to describe police deployment and focused patrols, examining the effects of law enforcement on the dynamics of criminal activity. This three-component reaction-diffusion system has been further analysed and developed by [6, 42, 44, 47, 52, 53, 58, 66].

1.3. Global well-posedness of the PDE models and main result

Theoretical and numerical studies of system (1.2) have attracted considerable academic attention, as seen in works such as [6–9, 11, 19, 28, 34–36, 38, 52, 55, 57]. These works demonstrate that this system exhibits rich and complex spatiotemporal dynamics, with nontrivial patterns that successfully capture the characteristic crime hotspots associated with urban residential burglary. Depending on the choice of parameters, the models can produce either dynamic or static concentrating profiles, representing the spatial hotspots of criminal activity. Rodríguez and Bertozzi [46] established the local well-posedness theory for system (1.2), proving both the existence and the uniqueness of its solution. They also highlighted the possibility of finite time blow-up in a modified version of (1.2). In the 1D case, Wang, Wang and Feng [59] extended this well-posedness theory to global time and prove the uniform boundedness of the solution for any $\chi > 0$ (see also [49]). However, in high dimensions, the global existence and boundedness of (1.2) remain open. Successful attempts have been made in special cases, including settings with small parameters or small data ([16, 25, 30, 56]), in a weak sense ([17, 21, 31, 63]), or in variants of (1.2) with superlinear diffusion and/or sublinear advection enhancement of criminal agents ([48, 64, 65]) as well as superlinear dissipation of criminal agents ([21, 32, 37, 43, 45]).

This paper extends the existing literature by proving global existence and uniform boundedness for the 2D urban crime model. We assume that there are some positive constants M and θ such that

$$0 \le \phi(\rho) \le M\rho^{\theta}, \theta < \frac{3}{4} \qquad \forall \rho \ge 0. \tag{1.3}$$

Then our main result is as follows.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$. Suppose that $\phi \in C^1(\mathbb{R})$ and (1.3) holds, then for any nonnegative initial data $(\rho_0, A_0) \in C(\bar{\Omega}) \times W^{1,\infty}(\Omega)$, there exists a unique pair (ρ, A) of functions each belonging to $C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$ that solves (1.1) classically. Moreover, this classical solution is strictly positive in $\bar{\Omega}$, and it is uniformly bounded such that $\|\rho(\cdot, t)\|_{L^\infty(\Omega)} + \|A(\cdot, t)\|_{L^\infty(\Omega)} \le C$ for all $t > 0$.*

This paper proves the well-posedness of the original system (1.2), for which we assume the sublinearity (1.3) of the sensitivity function for technical reasons. While this assumption deviates from linearity, it reflects the diminishing effect where criminal agents are less inclined to move to attractive regions that are already densely populated by other agents. Throughout the paper, we use C to represent a generic positive constant that may vary from line to line. For clarity and simplicity, we omit the differential "$d\mathbf{x}$" in calculations whenever possible.

2. Local existence and preliminary results

The mathematical analysis of the global well-posedness of system (1.1) is delicate since the maximum principle does not apply to the ρ equation. However, the local well-posedness can be easily established using the fundamental theory developed by Amann [2] (see also [46]) and standard parabolic regularity theory.

Proposition 1. *Let all assumptions in Theorem 1.1 hold. Then there exist $T_{\max} \in (0, \infty]$ and a unique couple (ρ, A) of nonnegative functions from $C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))$ solving (1.1) classically in $\Omega \times (0, T_{\max})$. Moreover, $\rho(\mathbf{x}, t) > 0$ and $A(\mathbf{x}, t) > 0$ in $\Omega \times (0, T_{\max})$ and the following dichotomy*

https://doi.org/10.1017/S0956792524000792 Published online by Cambridge University Press

holds:

$$\text{either} \quad T_{\max} = \infty \qquad \text{or} \qquad T_{\max} < \infty \text{ and } \limsup_{t \nearrow T_{\max}^-} \|\rho(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

Next, we collect some properties of the local solution obtained in Proposition 1. To begin with, for any space dimension, Lemma 2.1 and Lemma 2.2 in [59] ensure the existence of positive constants δ and C, which depend on $A_0(\mathbf{x})$, such that

$$\min_{\mathbf{x}\in\bar{\Omega}} A(\mathbf{x}, t) \geq \delta > 0 \qquad \forall t \in (0, T_{\max}) \tag{2.1}$$

and

$$\int_\Omega A(\mathbf{x}, t)d\mathbf{x} + \int_\Omega \rho(\mathbf{x}, t)d\mathbf{x} \leq C \qquad \forall t \in (0, T_{\max}). \tag{2.2}$$

Lemma 2.1. *Let (ρ, A) be a nonnegative classical solution of (1.1) in $\Omega \times (0, T_{\max})$. If $A\rho \in L^p(\Omega)$ for some $p \in [1, \infty)$, then there exists a positive constant C dependent on $\|A_0\|_{L^p(\Omega)}$ and $|\Omega|$ such that*

$$\|A(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C\Big(1 + \sup_{s\in(0,t)} \|(A\rho)(\cdot, s)\|_{L^p(\Omega)}\Big) \qquad \forall t \geq 0, \tag{2.3}$$

where $q \in [1, \frac{2p}{2-p})$ if $p \in [1, 2)$, $q \in [1, \infty)$ if $p = 2$ and $q = \infty$ if $p > 2$.

Proof. We first write the following abstract formula of A

$$A(\cdot, t) = e^{(\eta\Delta-1)t}A_0 + \int_0^t e^{(\eta\Delta-1)(t-s)}\big((A\rho)(\cdot, s) + \bar{A}\big)ds,$$

Thanks to the L^p–L^q estimates between semigroups $\{e^{t\Delta}\}_{t\geq 0}$ (cf. Lemma 1.3 of [62] with $N = 2$), we can find positive constants C_{21}, C_{22} and C_{23} such that

$$\begin{aligned}
&\|A(\cdot, t)\|_{W^{1,q}} \\
=&\Big\|e^{(\eta\Delta-1)t}A_0 + \int_0^t e^{(\eta\Delta-1)(t-s)}\big((A\rho)(\cdot, s) + \bar{A}\big)ds\Big\|_{W^{1,q}} \\
\leq&C_{21}\|A_0\|_{L^p} + C_{21}\int_0^t e^{-\nu(t-s)}(t-s)^{-\frac{1}{2}-(\frac{1}{p}-\frac{1}{q})}\|(A\rho)(\cdot, s) + \bar{A}\|_{L^p}ds \\
\leq&C_{22} + C_{23}\int_0^t e^{-\nu(t-s)}(t-s)^{-\frac{1}{2}-(\frac{1}{p}-\frac{1}{q})}\|(A\rho)(\cdot, s) + 1\|_{L^p}ds \\
\leq&C_{22} + C_{23}\Big(\int_0^t e^{-\nu(t-s)}(t-s)^{-\frac{1}{2}-(\frac{1}{p}-\frac{1}{q})}ds\Big) \sup_{s\in(0,t)} \|(A\rho)(\cdot, s) + 1\|_{L^p},
\end{aligned} \tag{2.4}$$

where ν is the first Neumann eigenvalue of $-\eta\Delta$. On the other hand, under the conditions on q after (2.3), we have that

$$\sup_{t\in(0,\infty)} \int_0^t e^{-\nu(t-s)}(t-s)^{-\frac{1}{2}-(\frac{1}{p}-\frac{1}{q})}ds < \infty,$$

and therefore (2.3) follows from (2.4). □

2.1. Boundedness of $\|\nabla A(\cdot, t)\|_{L^2(\Omega)}$

As we will see, establishing the boundedness of $\|\nabla A(\cdot, t)\|_{L^\infty(\Omega)}$ is essential for proving the boundedness of $\|\rho(\cdot, t)\|_{L^\infty(\Omega)}$. To achieve this, we first focus on establishing a weaker regularity condition for the boundedness of $\|\nabla A(\cdot, t)\|_{L^2(\Omega)}$.

https://doi.org/10.1017/S0956792524000792 Published online by Cambridge University Press

First of all, let us calculate using the A-equation as follows:

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{|\nabla A|^2}{A} =&2\int_\Omega \frac{1}{A}\nabla A\cdot\nabla A_t-\int_\Omega\frac{|\nabla A|^2}{A^2}\cdot A_t\\ =&2\int_\Omega\frac{\nabla A}{A}\cdot\nabla(\eta\Delta A-A+A\rho+\bar{A})-\int_\Omega\frac{|\nabla A|^2}{A^2}\cdot(\eta\Delta A-A+A\rho+\bar{A})\\ =&2\eta\int_\Omega\frac{\nabla A}{A}\cdot\nabla\Delta A+2\int_\Omega\frac{\nabla A}{A}\cdot\nabla(-A+A\rho)\\ &-\eta\int_\Omega\frac{|\nabla A|^2}{A^2}\Delta A+\int_\Omega\frac{|\nabla A|^2}{A}-\int_\Omega\frac{\rho}{A}|\nabla A|^2-\bar{A}\int_\Omega\frac{|\nabla A|^2}{A^2}.\end{aligned}$$

Then, we apply the identity $\nabla A\cdot\nabla\Delta A=\frac{1}{2}\Delta|\nabla A|^2-|D^2A|^2$ and integrate by parts to obtain that

$$\begin{aligned}\frac{d}{dt}\int_\Omega\frac{|\nabla A|^2}{A}=&\eta\int_\Omega\frac{1}{A}\Delta|\nabla A|^2-2\eta\int_\Omega\frac{|D^2A|^2}{A}-2\int_\Omega\frac{|\nabla A|^2}{A}+2\int_\Omega\frac{\rho}{A}|\nabla A|^2+2\int_\Omega\nabla A\cdot\nabla\rho\\ &-\eta\int_\Omega\frac{|\nabla A|^2}{A^2}\Delta A+\int_\Omega\frac{|\nabla A|^2}{A}-\int_\Omega\frac{\rho}{A}|\nabla A|^2-\bar{A}\int_\Omega\frac{|\nabla A|^2}{A^2}\\ =&\eta\int_{\partial\Omega}\frac{1}{A}\frac{\partial|\nabla A|^2}{\partial\mathbf{n}}+\eta\int_\Omega\frac{1}{A^2}\nabla A\cdot\nabla|\nabla A|^2-2\eta\int_\Omega\frac{|D^2A|^2}{A}-\eta\int_\Omega\frac{|\nabla A|^2}{A^2}\Delta A\\ &-\int_\Omega\frac{|\nabla A|^2}{A}+\int_\Omega\frac{\rho}{A}|\nabla A|^2+2\int_\Omega\nabla A\cdot\nabla\rho-\bar{A}\int_\Omega\frac{|\nabla A|^2}{A^2}\\ =&\eta\overbrace{\int_{\partial\Omega}\frac{1}{A}\frac{\partial|\nabla A|^2}{\partial\mathbf{n}}}^{I_\partial}+\eta\overbrace{\Big(\int_\Omega\frac{1}{A^2}\nabla A\cdot\nabla|\nabla A|^2-2\int_\Omega\frac{|D^2A|^2}{A}-\int_\Omega\frac{|\nabla A|^2}{A^2}\Delta A\Big)}^{I_{01}}\\ &-\int_\Omega\frac{|\nabla A|^2}{A}-\bar{A}\int_\Omega\frac{|\nabla A|^2}{A^2}\overbrace{-2\int_\Omega A\rho\Delta\ln A}^{I_{02}}-\int_\Omega\frac{\rho}{A}|\nabla A|^2.\end{aligned}\tag{2.5}$$

To estimate the boundary integral I_∂ in (2.5), we apply the uniform lower bound of A in (2.1) along with the pointwise inequality $\frac{\partial|\nabla A|^2}{\partial\mathbf{n}}\le C_\Omega|\nabla A|^2$ (cf. inequality (2.4) in [24]) to find that, with C_Ω being a positive constant depending only on the curvatures of $\partial\Omega$, the following holds

$$\int_{\partial\Omega}\frac{1}{A}\frac{\partial|\nabla A|^2}{\partial\mathbf{n}}\le C_\Omega\int_{\partial\Omega}\frac{|\nabla A|^2}{A}:=4C_\Omega\|\nabla A^{\frac{1}{2}}\|^2_{L^2(\partial\Omega)}.\tag{2.6}$$

The Sobolev trace embedding (cf. (1.9), Lemma 2.3 and Lemma 2.4 in [24]) implies that there exists a positive constant C such that

$$4C_\Omega\|\nabla A^{\frac{1}{2}}\|_{L^2(\partial\Omega)}\le C_{24}\|\nabla A^{\frac{1}{2}}\|_{W^{\frac{3}{4},2}(\Omega)};\tag{2.7}$$

moreover, applying the fractional Gagliardo–Nirenberg interpolation inequality to (2.7) finds that

https://doi.org/10.1017/S0956792524000792 Published online by Cambridge University Press

$$
\begin{aligned}
&C_{24}\|\nabla A^{\frac{1}{2}}\|^2_{W^{\frac{3}{4},2}(\Omega)}\\
\le& C_{25}\|\nabla\cdot\nabla A^{\frac{1}{2}}\|^{\frac{3}{2}}_{L^2(\Omega)}\|\nabla A^{\frac{1}{2}}\|^{\frac{1}{2}}_{L^2(\Omega)}+C_{25}\|\nabla A^{\frac{1}{2}}\|^2_{L^2(\Omega)}\\
=& C_{25}\Big\|\frac{\Delta A}{A}-\frac{|\nabla A|^2}{A^{\frac{3}{2}}}\Big\|^{\frac{3}{2}}_{L^2(\Omega)}\|\nabla A^{\frac{1}{2}}\|^{\frac{1}{2}}_{L^2(\Omega)}+C_{25}\|\nabla A^{\frac{1}{2}}\|^2_{L^2(\Omega)}\\
\le&\sqrt{2}C_{25}\Big\|\frac{\Delta A}{A}\Big\|^{\frac{3}{2}}_{L^2(\Omega)}\|\nabla A^{\frac{1}{2}}\|^{\frac{1}{2}}_{L^2(\Omega)}+\sqrt{2}C_{25}\Big\|\frac{|\nabla A|^2}{A^{\frac{3}{2}}}\Big\|^{\frac{3}{2}}_{L^2(\Omega)}\|\nabla A^{\frac{1}{2}}\|^{\frac{1}{2}}_{L^2(\Omega)}+C_{25}\|\nabla A^{\frac{1}{2}}\|^2_{L^2(\Omega)},
\end{aligned}
$$

where to derive the second inequality we have used the inequality that $(a+b)^{\frac{3}{2}}\le\sqrt{2}(a^{\frac{3}{2}}+b^{\frac{3}{2}})$ for any positive real numbers $a,b>0$. We then further apply Cauchy's inequality to find that for any arbitrary positive constant ϵ, there exists a positive constant C_ϵ such that

$$
\begin{aligned}
&C_{24}\|\nabla A^{\frac{1}{2}}\|^2_{W^{\frac{3}{4},2}(\Omega)}\\
\le&\frac{\epsilon}{4}\Big\|\frac{\Delta A}{A}\Big\|^2_{L^2(\Omega)}+\frac{\epsilon}{2}\Big\|\frac{|\nabla A|^2}{A^{\frac{3}{2}}}\Big\|^2_{L^2(\Omega)}+2C_\epsilon\|\nabla A^{\frac{1}{2}}\|^2_{L^2(\Omega)}+C_{25}\|\nabla A^{\frac{1}{2}}\|^2_{L^2(\Omega)}\\
=&\frac{\epsilon}{4}\int_\Omega\frac{|\Delta A|^2}{A^2}+\frac{\epsilon}{2}\int_\Omega\frac{|\nabla A|^4}{A^3}+C_{26}\int_\Omega|\nabla A^{\frac{1}{2}}|^2\\
\le&\frac{\epsilon}{2}\int_\Omega\frac{|D^2A|^2}{A^2}+\frac{\epsilon}{2}\int_\Omega\frac{|\nabla A|^4}{A^3}+C_{26}\int_\Omega|\nabla A^{\frac{1}{2}}|^2.
\end{aligned}\tag{2.8}
$$

To estimate $C_{26}\int_\Omega|\nabla A^{\frac{1}{2}}|^2$ in (2.8), we have from Young's inequality and the Gagliardo–Nirenberg interpolation inequality

$$
\begin{aligned}
&C_{26}\int_\Omega|\nabla A^{\frac{1}{2}}|^2\\
=&4C_{26}\int_\Omega|\nabla A^{\frac{1}{4}}|^2A^{\frac{1}{2}}\\
\le&\frac{\epsilon}{4}\int_\Omega\frac{|\nabla A|^4}{A^3}+C_\epsilon\int_\Omega A^2\\
\le&\frac{\epsilon}{4}\int_\Omega\frac{|\nabla A|^4}{A^3}+C_{27}\|\nabla A^{\frac{1}{4}}\|^2_{L^4(\Omega)}\|A^{\frac{1}{4}}\|^6_{L^4(\Omega)}+C_{27}\|A^{\frac{1}{4}}\|^8_{L^4(\Omega)}\\
\le&\frac{\epsilon}{2}\int_\Omega\frac{|\nabla A|^4}{A^3}+C_{28}.
\end{aligned}\tag{2.9}
$$

Therefore, we have from (2.6)-(2.9) that

$$
I_\partial\le\frac{\epsilon\eta}{2}\int_\Omega\frac{|D^2A|^2}{A^2}+\epsilon\eta\int_\Omega\frac{|\nabla A|^4}{A^3}+\eta C_{28}\le\frac{\varepsilon}{2}\int_\Omega\frac{|D^2A|^2}{A^2}+\varepsilon\int_\Omega\frac{|\nabla A|^4}{A^3}+C_{29},\tag{2.10}
$$

where $\varepsilon:=\epsilon\eta$ is an arbitrary positive constant. To estimate I_{01} in (2.5), we first integrate by parts to have that

$$
\begin{aligned}
I_{01}=&-\int_\Omega\nabla\frac{\nabla A}{A^2}\cdot|\nabla A|^2-2\int_\Omega\frac{|D^2A|^2}{A}-\int_\Omega\frac{|\nabla A|^2}{A^2}\Delta A\\
=&-2\int_\Omega\frac{|D^2A|^2}{A}-2\int_\Omega\frac{1}{A}|\nabla A|^2\Delta A+2\int_\Omega\frac{|\nabla A|^4}{A^3}.
\end{aligned}\tag{2.11}
$$

https://doi.org/10.1017/S0956792524000792 Published online by Cambridge University Press

We then use the identity $\nabla|\nabla A|^2 = 2D^2A\cdot\nabla A$ and integrate by parts again to find that

$$\begin{aligned}\int_\Omega A|D^2\ln A|^2 &= \int_\Omega \frac{|D^2A|^2}{A} - 2\int_\Omega \frac{\nabla A}{A^2}\cdot(D^2A\cdot\nabla A) + \int_\Omega \frac{|\nabla A|^4}{A^3}\\ &= \int_\Omega \frac{|D^2A|^2}{A} - \int_\Omega \frac{1}{A}\nabla A\cdot\nabla|\nabla A|^2 + \int_\Omega \frac{|\nabla A|^4}{A^3}\\ &= \int_\Omega \frac{|D^2A|^2}{A} + \int_\Omega \frac{1}{A^2}|\nabla A|^2\Delta A - \int_\Omega \frac{|\nabla A|^4}{A^3}. \end{aligned} \tag{2.12}$$

Thanks to (2.11) and (2.12), we find that $I_{01} = -2\int_\Omega A|D^2\ln A|^2$. Moreover, we have that for the same arbitrarily small $\varepsilon > 0$ as in (2.10)

$$\begin{aligned} I_{01} &= -(2-\varepsilon)\int_\Omega A|D^2\ln A|^2 + \frac{\varepsilon}{2}I_{01}\\ &= -(2-\varepsilon)\int_\Omega A|D^2\ln A|^2 + \frac{\varepsilon}{2}\Big(\int_\Omega \frac{1}{A^2}\nabla A\cdot\nabla|\nabla A|^2 - 2\int_\Omega \frac{|D^2A|^2}{A} - \int_\Omega \frac{|\nabla A|^2}{A^2}\Delta A\Big)\\ &= -(2-\varepsilon)\int_\Omega A|D^2\ln A|^2 + \frac{\varepsilon}{2}\Big(2\int_\Omega \frac{|\nabla A|^4}{A^3} - 2\int_\Omega \frac{|\nabla A|^2}{A^2}\Delta A - 2\int_\Omega \frac{|D^2A|^2}{A}\Big)\\ &= -(2-\varepsilon)\int_\Omega A|D^2\ln A|^2 - \varepsilon\int_\Omega \frac{|D^2A|^2}{A} - \varepsilon\int_\Omega \frac{|\nabla A|^2}{A^2}\Delta A + \varepsilon\int_\Omega \frac{|\nabla A|^4}{A^3}. \end{aligned}$$

Moreover, in the light of the following inequality due to Young's inequality

$$-\varepsilon\int_\Omega \frac{|\nabla A|^2}{A^2}\Delta A \le \varepsilon\int_\Omega \frac{|\nabla A|^4}{A^3} + \frac{\varepsilon}{4}\int_\Omega \frac{|\Delta A|^2}{A} \le \varepsilon\int_\Omega \frac{|\nabla A|^4}{A^3} + \frac{\varepsilon}{2}\int_\Omega \frac{|D^2A|^2}{A},$$

we have that

$$I_{01} \le -(2-\varepsilon)\int_\Omega A|D^2\ln A|^2 - \frac{\varepsilon}{2}\int_\Omega \frac{|D^2A|^2}{A} + 2\varepsilon\int_\Omega \frac{|\nabla A|^4}{A^3}. \tag{2.13}$$

To estimate I_{02}, we apply the pointwise identity $|\Delta\ln A|^2 \le 2|D^2\ln A|^2$ and use Young's inequality

$$I_{02} \le 2\sqrt{2}\int_\Omega A\rho|D^2\ln A| \le \int_\Omega A|D^2\ln A|^2 + 2\int_\Omega A\rho^2. \tag{2.14}$$

We then combine (2.10), (2.13) and (2.14) with (2.5) to obtain that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{|\nabla A|^2}{A} &\le -(1-\varepsilon)\int_\Omega A|D^2\ln A|^2 + 3\varepsilon\int_\Omega \frac{|\nabla A|^4}{A^3} - \int_\Omega \frac{|\nabla A|^2}{A}\\ &\quad - \bar{A}\int_\Omega \frac{|\nabla A|^2}{A^2} - \int_\Omega \frac{\rho}{A}|\nabla A|^2 + 2\int_\Omega A\rho^2 + C_{29}\\ &\le -(1-4\varepsilon)\int_\Omega \frac{|\nabla A|^4}{A^3} - \int_\Omega \frac{|\nabla A|^2}{A} - \bar{A}\int_\Omega \frac{|\nabla A|^2}{A^2}\\ &\quad - \int_\Omega \frac{\rho}{A}|\nabla A|^2 + 2\int_\Omega A\rho^2 + C_{29}\\ &\le -(1-4\varepsilon)\int_\Omega \frac{|\nabla A|^4}{A^3} - \int_\Omega \frac{|\nabla A|^2}{A} + 2\int_\Omega A\rho^2 + C_{29}, \end{aligned} \tag{2.15}$$

where the second inequality follows from the inequality $\int_\Omega \frac{|\nabla A|^4}{A^3} \le \int_\Omega A|D^2\ln A|^2$, and the last inequality skips anything redundant for our future estimates.

https://doi.org/10.1017/S0956792524000792 Published online by Cambridge University Press

On the other hand, we first apply the ρ-equation and then use Young's inequality with the same ε as (2.10) to find that for a positive constant C_ε we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^2 = \int_\Omega \rho\rho_t &= -\int_\Omega |\nabla\rho|^2 + \int_\Omega \phi(\rho)\nabla\rho\cdot\nabla\ln A - \int_\Omega A\rho^2 + \bar{B}\int_\Omega \rho \\ &\le -\frac{1}{2}\int_\Omega |\nabla\rho|^2 + \varepsilon\int_\Omega |\nabla\ln A|^4 + C_\varepsilon\int_\Omega \rho^{4\theta} - \int_\Omega A\rho^2 + C_{210} \\ &\le -\frac{1}{2}\int_\Omega |\nabla\rho|^2 + \frac{\varepsilon}{\delta}\int_\Omega \frac{|\nabla A|^4}{A^3} + \varepsilon\int_\Omega \rho^3 - \int_\Omega A\rho^2 + C_{211}, \end{aligned} \tag{2.16}$$

where the last inequality is obtained by (1.3) and Lemma 2.1. By Gagliardo–Nirenberg interpolation inequality, there exists a positive constant C such that

$$\int_\Omega \rho^3 \le C(\|\rho\|_{L^1})\int_\Omega |\nabla\rho|^2 + C(\|\rho\|_{L^1}). \tag{2.17}$$

We are now ready to present the following uniform-in-time boundedness of ∇A and ρ in their L^2 norms.

Lemma 2.2. *There exists a positive constant C such that*

$$\|\nabla A(\cdot,t)\|_{L^2(\Omega)} + \|\rho(\cdot,t)\|_{L^2(\Omega)} \le C \qquad \forall t\in(0,T_{\max}). \tag{2.18}$$

Proof. Substituting (2.17) into (2.16) gives us

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^2 \le -\frac{1}{4}\int_\Omega |\nabla\rho|^2 + \frac{\varepsilon}{\delta}\int_\Omega \frac{|\nabla A|^4}{A^3} - \int_\Omega A\rho^2 + C_{212}. \tag{2.19}$$

We use Gagliardo–Nirenberg interpolation inequality to estimate $\int_\Omega \rho^2$

$$\begin{aligned}\|\rho\|^2_{L^2(\Omega)} \le& C\|\nabla\rho\|_{L^2(\Omega)}\|\rho\|_{L^1(\Omega)} + C(\|\rho\|_{L^1(\Omega)}) \\ \le& \varepsilon\|\nabla\rho\|^2_{L^2(\Omega)} + C_\varepsilon\|\rho\|^2_{L^1(\Omega)} + C(\|\rho\|_{L^1(\Omega)}) \\ \le& \varepsilon\|\nabla\rho\|^2_{L^2(\Omega)} + C_{213}\end{aligned}$$

This, together with (2.15) and (2.19), gives us that

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega \frac{|\nabla A|^2}{A} + \int_\Omega \rho^2\Big) &\le -\Big(\int_\Omega \frac{|\nabla A|^2}{A} + \frac{1}{2}\int_\Omega |\nabla\rho|^2\Big) + C_{214} \\ &\le -\Big(\int_\Omega \frac{|\nabla A|^2}{A} + \int_\Omega \rho^2\Big) + C_{214}\end{aligned} \tag{2.20}$$

where ε is chosen small so that $1-4\varepsilon \ge \frac{2\varepsilon}{\delta}$. Then solving (2.20) easily implies the boundedness of $\int_\Omega \frac{|\nabla A|^2}{A}$ and $\int_\Omega \rho^2$.

To prove the boundedness of $\int_\Omega |\nabla A|^2$, we choose $q=2$ and $p=\frac{3}{2}$ in (2.3) to conclude from the Young's inequality that

$$\|A(\cdot,t)\|_{W^{1,2}(\Omega)} \le C_{215}\Big(1+\sup_{s\in(0,t)}\|(A\rho)(\cdot,s)\|_{L^{\frac{3}{2}}(\Omega)}\Big) \le C_{216}\bigg(1+\sup_{s\in(0,t)}\Big(\int_\Omega A^6 + \int_\Omega \rho^2\Big)\bigg). \tag{2.21}$$

Due to the fact that $A^{\frac{1}{2}}\in W^{1,2}$, we readily have from the Sobolev embedding that

$$\|A^{\frac{1}{2}}\|^{12}_{L^{12}(\Omega)} \le C\|\nabla A^{\frac{1}{2}}\|^{10}_{L^2(\Omega)}\|A^{\frac{1}{2}}\|^2_{L^2(\Omega)} + C\|A^{\frac{1}{2}}\|^{12}_{L^2(\Omega)} \le C,$$

We can iterate the process to find that $\|A\|_{L^p}$ is bounded for any $p\in(1,\infty)$, which then leads us to (2.18) thanks to the GN inequality. □

Remark 1. The boundedness of $\|\nabla A\|_{L^2}$ is not enough and we need to prove that of $\|\nabla A\|_{L^q}$ for q sufficiently large to show the boundedness of ρ in L^∞. Indeed, we can show that $\|\nabla A\|_{L^\infty}$. To see this,

https://doi.org/10.1017/S0956792524000792 Published online by Cambridge University Press

in the light of (2.18), we can show that (2.3) holds for some $p > 1$, which then implies the boundedness of A in $W^{1,p}$ for some $p > 1$, hence $A \in L^\infty$ as claimed.

Corollary 2.1. *For any $q \in (1, \infty)$, there exists a constant C_q dependent on q such that*

$$\|\nabla A(\cdot, t)\|_{L^q(\Omega)} \leq C_q \quad \forall t \in (0, T_{max}); \tag{2.22}$$

moreover, there exists a positive constant C such that

$$\|A(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \forall t \in (0, T_{max}); \tag{2.23}$$

Proof. Let $p = \frac{2}{2\varepsilon+1}$ in (2.3), $\varepsilon > 0$ being arbitrary, then we have that for any $q \in (1, \frac{1}{\varepsilon})$

$$\begin{aligned}
\|A(\cdot, t)\|_{W^{1,q}(\Omega)} &\leq C_{217}\Big(1 + \sup_{s\in(0,t)} \|(A\rho)(\cdot, s)\|_{L^{\frac{2}{2\varepsilon+1}}(\Omega)}\Big) \\
&\leq C_{217}\Big(1 + \sup_{s\in(0,t)} \|A(\cdot, s)\|_{L^{\frac{1}{\varepsilon}}(\Omega)} + \sup_{s\in(0,t)} \|\rho(\cdot, s)\|_{L^2(\Omega)}\Big) \\
&\leq C_q.
\end{aligned}$$

We infer from (2.22) the boundedness of A in $W^{1,q}$ for any q, which eventually leads to the uniform boundedness of A. □

3. Global well-posedness of the fully parabolic system

According to Proposition 1, in order to prove Theorem 1.1, it is sufficient to show that $\|\rho(\cdot, t)\|_{L^\infty(\Omega)}$ is bounded for each $t \in (0, T_{\max})$ and therefore $T_{\max} = \infty$ and the solution is global. In fact, we will show that $\|\rho(\cdot, t)\|_{L^\infty(\Omega)}$ is uniformly bounded in $t \in (0, \infty)$. To this end, it suffices to prove that $\|\rho(\cdot, t)\|_{L^m(\Omega)}$ is bounded for some m large according to (2.3), now that we already have the boundedness of ρ in L^2. Choosing $m > 2$, one obtains from (2.3) the boundedness of $\|A\|_{W^{1,q}}$ with q being sufficient large. One can then find that the boundedness of $\|\rho\|_{L^\infty}$ follows from the standard Moser–Alikakos iteration in [1].

3.1. A priori estimates

For any $m > 1$, we multiply the ρ-equation in (1.1) by ρ^{m-1} and then integrate it over Ω by parts to have that

$$\begin{aligned}
&\frac{1}{m}\frac{d}{dt}\int_\Omega \rho^m \\
&= \int_\Omega \rho^{m-1}\nabla\cdot\nabla\rho - \int_\Omega \rho^{m-1}\nabla\cdot(\phi(\rho)\nabla \ln A) - \int_\Omega \rho^m A + \bar{B}\int_\Omega \rho^{m-1} \\
&= -(m-1)\int_\Omega \rho^{m-2}|\nabla\rho|^2 + (m-1)\int_\Omega \phi(\rho)\rho^{m-2}\nabla\rho\cdot\nabla\ln A - \int_\Omega \rho^m A + \bar{B}\int_\Omega \rho^{m-1} \\
&= -\frac{4(m-1)}{m^2}\int_\Omega |\nabla\rho^{\frac{m}{2}}|^2 + (m-1)\int_\Omega \phi(\rho)\rho^{m-2}\nabla\rho\cdot\nabla\ln A - \int_\Omega \rho^m A + \bar{B}\int_\Omega \rho^{m-1},
\end{aligned} \tag{3.1}$$

https://doi.org/10.1017/S0956792524000792 Published online by Cambridge University Press

where the second identity holds as $\rho^{m-2}|\nabla\rho|^2 = \frac{4}{m^2}|\nabla\rho^{\frac{m}{2}}|^2$. Moreover, Young's inequality implies that

$$
\begin{aligned}
&(m-1)\int_\Omega \phi(\rho)\rho^{m-2}\nabla\rho\cdot\nabla(\ln A)\\
\leq&\frac{m-1}{2}\int_\Omega \rho^{m-2}|\nabla\rho|^2+\frac{m-1}{2}\int_\Omega \rho^{m-2}\phi^2(\rho)\Big|\frac{\nabla A}{A}\Big|^2\\
\leq&\frac{2(m-1)}{m^2}\int_\Omega |\nabla\rho^{\frac{m}{2}}|^2+\frac{M^2(m-1)}{2}\int_\Omega \rho^{m+2\theta-2}\Big|\frac{\nabla A}{A}\Big|^2,
\end{aligned}
\tag{3.2}
$$

and

$$
-\int_\Omega \rho^m A+\bar{B}\int_\Omega \rho^{m-1}\leq -\delta\int_\Omega \rho^m+\bar{B}\int_\Omega \rho^{m-1}\leq -\frac{\delta}{2}\int_\Omega \rho^m + C_{31}, \tag{3.3}
$$

where C_{31} is a positive constant dependent on m. Thanks to (3.2) and (3.3), we have from (3.1) that

$$
\frac{1}{m}\frac{d}{dt}\int_\Omega \rho^m+\frac{\delta}{2}\int_\Omega \rho^m+\frac{2(m-1)}{m^2}\int_\Omega |\nabla\rho^{\frac{m}{2}}|^2\leq \frac{M^2(m-1)}{2}\int_\Omega \rho^{m+2\theta-2}\Big|\frac{\nabla A}{A}\Big|^2+C_{31}. \tag{3.4}
$$

Since ρ is uniformly bounded in L^2, we infer from (3.4) that it is also bounded in L^4. This allows us to iterate the *a priori* estimates concerning the boundedness of $\|\nabla A\|_{L^q}$ for arbitrary q. We will present this iteration in the final section.

3.2. Global existence and uniform boundedness

Proof of Theorem 1.1. First of all, we have the uniform boundedness of $\|A(\cdot,t)\|_{L^\infty(\Omega)}$ from (2.23). The verification of the boundedness of $\|\rho\|_{L^\infty}$ follows from the standard L^p−iteration sketches as follows. By (3.4), we have from the uniform lower bound of A given in (2.1) that

$$
\begin{aligned}
&\frac{1}{m}\frac{d}{dt}\int_\Omega \rho^m+\frac{2(m-1)}{m^2}\int_\Omega |\nabla\rho^{\frac{m}{2}}|^2\\
\leq&\frac{M^2(m-1)}{2\delta^2}\int_\Omega \rho^{m+2\theta-2}|\nabla A|^2+C_{31}\\
\leq&\frac{M^2(m-1)}{2}\int_\Omega \rho^{(m+2\theta-2)\cdot\frac{m}{m+2\theta-2}}+C_\delta\int_\Omega |\nabla A|^{2\cdot\frac{m}{2-2\theta}}+C_{31}\\
\leq&\frac{M^2(m-1)}{2}\int_\Omega \rho^m+C_{32},
\end{aligned}
\tag{3.5}
$$

where apply (2.22) to derive the last inequality. Then we apply Gagliardo–Ladyzhenskaya–Nirenberg inequality (cf. Corollary 1 in [12] with $d=2$) and the Young's inequality to estimate (3.5) such that

$$
\begin{aligned}
&\frac{M^2(m-1)}{m}\int_\Omega \rho^2\\
=&\frac{2(m-1)}{m^2\varepsilon}\int_\Omega \rho^m-\frac{M^2(m-1)}{2}\int_\Omega \rho^m\\
\leq&\frac{2(m-1)}{m^2}\int_\Omega |\nabla\rho^{\frac{m}{2}}|^2+\frac{2(m-1)\kappa(1+\varepsilon^{-1})}{m^2\varepsilon}\Big(\int_\Omega \rho^{\frac{m}{2}}\Big)^2-\frac{M^2(m-1)}{2}\int_\Omega \rho^m,
\end{aligned}
\tag{3.6}
$$

where we choose $\varepsilon := \frac{2}{M^2m^2}$ and $\frac{1+\varepsilon^{-1}}{\varepsilon}=(1+\frac{M^2m^2}{2})\frac{M^2m^2}{2}$. In the light of (3.6), (3.5) becomes

$$
\frac{d}{dt}\int_\Omega \rho^m\leq -\frac{M^2m(m-1)}{2}\int_\Omega \rho^m+M^2m(m-1)\kappa(1+\frac{M^2m^2}{2})\Big(\int_\Omega \rho^{\frac{m}{2}}\Big)^2 \tag{3.7}
$$

Let us denote $\lambda := \frac{M^2m(m-1)}{2}$, then for each $T \in (0, \infty)$ we can solve the differential inequality (3.7) and obtain for all $t \in (0, T)$

$$\int_\Omega \rho^m \le e^{-\lambda t} \int_\Omega \rho_0^m + M^2 m(m-1)\kappa(1 + \frac{M^2m^2}{2}) \int_0^t e^{-\lambda(t-s)} (\int_\Omega \rho^{\frac{m}{2}})^2 ds$$

$$\le \int_\Omega \rho_0^m + 2\lambda\kappa(1 + \frac{M^2m^2}{2}) \sup_{t\in(0,T)} \left(\int_\Omega \rho^{\frac{m}{2}} \right)^2 \quad \forall m \ge 2. \tag{3.8}$$

Finally, the L^∞ boundedness of ρ follows from the Moser–Alikakos iteration following (3.8). This concludes the proof of Theorem 1.1. □

Acknowledgements. QW is partially supported by the Guanghua Talent Program at SWUFE. Both authors thank all the three referees for their helpful comments and suggestions.

Competing interest. The authors have no competing financial and/or non-financial interests in relation to this work.

References

[1] Alikakos, N. D. (1979) Lp bounds of solutions of reaction-diffusion equations, *Comm. Partial Differential Equations* **4**, 827–868.
[2] Amann, H. (1993). *Nonhomogeneous Linear and Quasilinear Elliptic and Parabolic Boundary Value Problems, Function Spaces, Differential Operators and Nonlinear Analysis*, Teubner, Stuttgart, Leipzig, **133**, 9–126.
[3] Anselin, L., Cohen, J., Cook, D., Gorr, W. & Tita, G. (2000) Spatial analyses of crime. *Crim. Just.* **4**, 212–262.
[4] Brantingham, P. J. & Brantingham, P. L., Patterns in crime, Macmillan New York, 1984
[5] Bowers, K. J. & Johnson, S. D. (2005) Domestic burglary repeats and space-time clusters the dimensions of risk. *Eur. J. Criminol.* **2**(1), 67–92.
[6] Buttenschoen, A., Kolokolnikov, T., Ward, M. & Wei, J. (2020) Cops-on-the-dots: The linear stability of crime hotspots for a 1-D reaction-diffusion model of urban crime, *Eur. J. Appl. Math.* **31**, 871–917.
[7] Berestycki, H. & Nadal, J. (2010) Self-organised critical hot spots of criminal activity, *Eur. J. Appl. Math.* **21**, 371–399.
[8] Berestycki, H., Rodriguez, N. & Ryzhik, L. (2013) Traveling wave solutions in a reaction-diffusion model for criminal activity. *Multiscale Model. Simul.* **11**, 1097–1126.
[9] Berestycki, H., Wei, J. & Winter, M. (2014) Existence of Symmetric and Asymmetric Spikes for a Crime Hotspot Model. *SIAM J. Math. Anal.* **46**, 691–719.
[10] Chaturapruek, S., Breslau, J., Yazdi, D., Kolokolnikov, T. & McCalla, S. (2013) Crime Modeling with Lévy Flights. *SIAM J. Appl. Math.* **73**, 1703–1720.
[11] Cantrell, R., Cosner, C. & Manásevich, R. (2012) Global Bifurcation of Solutions for Crime Modeling Equations. *SIAM J. Math. Anal.* **44**, 1340–1358.
[12] Chertock, A., Kurganov, A., Wang, X. & Wu, Y. (2012) On a chemotaxis model with saturated chemotactic flux. *Kinet. Relat. Models* **5**, 51–95.
[13] D'Orsogna, M. & Perc, M. (2015) Statistical physics of crime: A review. *Phys. Life Rev.* **12**, 1–21.
[14] Eck, J. & Weisburd, D. L. (2015) Crime places in crime theory. *Crime and Place: Crime Prev. Stud.* **4**, 1–33.
[15] Felsen, M. (1987) Routine activities and crime prevention in the developing metropolis. *Criminology* **25**, 911–932.
[16] Freitag, M. (2018) Global solutions to a higher-dimensional system related to crime modeling. *Math. Methods Appl. Sci.* **41**, 6326–6335.
[17] Fuest, M. & Heihoff, F. (2024) Unboundedness phenomenon in a model of urban crime. *Commun. Contemp. Math.* **26**(7), 1.
[18] Groff, E. R., Johnson, S. D. & Thornton, A. (2019) State of the art in agent-based modeling of urban crime: An overview. *J. Quant. Criminol.* **35**, 155–193.
[19] García-Huidobro, M., Manásevich, R. & Mawhin, J. (2014) Solvability of a nonlinear Neumann problem for systems arising from a burglary model. *Appl. Math. Lett.* **35**, 102–108.
[20] Gu, Y., Wang, Q. & Yi, G. (2017) Stationary patterns and their selection mechanism of urban crime models with heterogeneous near-repeat victimization effect. *Eur. J. Appl. Math.* **28**, 141–178.
[21] Heihoff, F. (2020) Generalized solutions for a system of partial differential equations arising from urban crime modeling with a logistic source term. *Z. Angew. Math. Phys.* **71**, 80.
[22] Hillen, T. & Painter, K. (2009) A user's guide to PDE models for chemotaxis. *J. Math. Biol.* **58**, 183–217.
[23] Horstmann, D. & From, 1970) Until present: The keller–Segel model in chemotaxis and its consequences I. *Jahresber* **105**(2003), 103–165.
[24] Ishida, S., Seki, K. & Yokota, T. (2014) Boundedness in quasilinear Keller–Segel systems of parabolic–parabolic type on non-convex bounded domains. *J. Differential Equations* **256**, 2993–3010.
[25] Jiang, Y. & Yang, L. (2022) Global solvability and stabilization in a three-dimensional cross-diffusion system modeling urban crime propagation. *Acta Appl. Math.* **178**(11), 40**.**

[26] Jones, P. A., Brantingham, P. J. & Chayes, L. R. (2010) Statistical models of criminal behavior: The effects of law enforcement actions. *Math. Models Methods Appl. Sci.* **20**,1397–1423.
[27] Johnson, S. D., Bowers, K. & Hirschfield, A. (1997) New insights into the spatial and temporal distribution of repeat victimization. *Br. J. Criminol.* **37**, 224–241.
[28] Kolokolnikov, T., Ward, M. & Wei, J. (2014) The stability of steady-state hot-spot patterns for a reaction-diffusion model of urban crime. *Discrete Contin. Dyn. Syst. Ser. B* **19**, 1373–1410.
[29] Levajkovic, T., Mena, H. & Zarfl, M. (2016) Lévy processes, subordinators and crime modelling. *Novi. Sad. J. Math.* **46**, 65–86.
[30] Li, B., Wang, Z. & Xie, L. (2023) Regularization effect of the mixed-type damping in a higher-dimensional logarithmic Keller-Segel system related to crime modeling. *Math. Biosci. Engin.* **20**, 4532–4559.
[31] Li, B. & Xie, L. (2024) Generalised solution to a 2D parabolic-parabolic chemotaxis system for urban crime: Global existence and large-time behaviour. *Eur. J. Appl. Math.* **35**, 409–429.
[32] Li, H., Xiao, Y. & Xie, L. (2023) On a crime model in higher-dimensional setting: global generalized solvability and eventual smoothness. *Z. Angew. Math. Phys.* **74**, 160.
[33] Lindstrom, M. & Bertozzi, A. (2020) Qualitative features of a nonlinear, nonlocal, agent-based PDE model with applications to homelessness. *Math. Models Methods Appl. Sci.* **30**, 1863–1891.
[34] Lloyd, D. & O'Farrell, H. (2013) On localised hotspots of an urban crime model. *Phys. D* **253**, 23–39.
[35] Lloyd, D., Santitissadeekorn, N. & Short, M. (2016) Exploring data assimilation and forecasting issues for an urban crime model. *Eur. J. Appl. Math.* **27**, 451–478.
[36] Manásevich, R., Garcia-Huidobro, M. & Mawhin, J. (2013) Existence of solutions for a 1-D boundary value problem coming from a model for burglary. *Nonlinear Anal. Real World Appl.* **14**, 1939–1946.
[37] Manásevich, R., Phan, Q. & Souplet, P. (2013) Global existence of solutions for a chemotaxis-type system arising in crime modelling. *Eur. J. Appl. Math.* **24**, 273–296.
[38] Mei, L. & Wei, J. (2020) The existence and stability of spike solutions for a chemotax is system modeling crime pattern formation. *Math. Models Methods Appl. Sci.* **30**, 1727–1764.
[39] Mohler, G. & Short, M. (2012) Geographic Profiling from Kinetic Models of Criminal Behavior. *SIAM J. Appl. Math.* **72**, 163–180.
[40] Mohler, G., Short, M., Brantingham, P., Schoenberg, F. & Tita, G. (2011) Self-Exciting Point Process Modeling of Crime. *J. Am. Stat. Assoc.* **106**, 100–108.
[41] Pan, C., Li, B., Wang, C., et al. (2018) Crime modeling with truncated Lévy flights for residential burglary models. *Math. Models Methods Appl. Sci.* **28**, 1857–1880.
[42] Pitcher, A. (2010) Adding police to a mathematical model of burglary. *Eur. J. Appl. Math.* **21**, 401–419.
[43] Qiu, Z. & Li, B. (2023) Eventual smoothness of generalized solutions to a singular chemotaxis system for urban crime in space dimension. 2. *Electron. Res. Arch.* **31**, 3218–3244
[44] Ricketson, L. (2010), A continuum model of residential burglary incorporating law enforcement, preprint.
[45] Rodríguez, N. (2013) On the global well-posedness theory for a class of PDE models for criminal activity. *Phys. D* **260**, 191–200.
[46] Rodríguez, N. & Bertozzi, A. (2010) Local existence and uniqueness of solutions to a PDE model for criminal behavior. *Math. Models Methods Appl. Sci.* **20**, 1425–1457.
[47] Rodríguez, N., Wang, Q. & Zhang, L. (2022) Understanding the effects of on-and off-hotspot policing: Evidence of hotspot, oscillating, and chaotic activities. *SIAM J. Appl. Dyn. Syst.* **20**, 1882–1916.
[48] Rodríguez, N. & Winkler, M. (2020) Relaxation by nonlinear diffusion enhancement in a two-dimensional cross-diffusion model for urban crime propagation. *Math. Models Methods Appl. Sci.* **30**, 2105–2137.
[49] Rodríguez, N. & Winkler, M. (2022) On the global existence and qualitative behavior of one-dimensional solutions to a model for urban crime. *Eur. J. Appl. Math.* **33**, 919–959.
[50] Saldana, J., Aguareles, M., Avinyó, A. & M (2018) Pellicer and J.Ripoll, An age-structured population approach for the mathematical modeling of urban burglaries. *SIAM J. Appl. Dyn. Syst.* **17**, 2733–2760.
[51] Sampson, R. & Raudenbush, S. (2004) Seeing disorder: Neighborhood stigma and the social construction of broken windows. *Soc. Psychol. Q.* **67**, 319–342.
[52] Short, M., Bertozzi, A. & Brantingham, P. (2010) Nonlinear patterns in urban crime: Hotspots, bifurcations, and suppression. *SIAM J. Appl. Dyn. Syst.* **9**, 462–483.
[53] Short, M., Brantingham, P., Bertozzi, A. & Tita, G. (2010) Dissipation and displacement of hotspots in reaction-diffusion models of crime. *Proc. Natl. Acad. Sci.* **107**, 3961–3965.
[54] Short, M., D'Orsogna, M., Brantingham, P. & Tita, G. (2009) Measuring and modeling repeat and near-repeat burglary effects. *J. Quant. Criminol.* **25**, 325–339.
[55] Short, M., D'Orsogna, M., Pasour, V., Tita, G., Brantingham, P. & Bertozzi, A. (2008) A statistical model of criminal behavior. *Math. Models Methods Appl. Sci.* **18**, 1249–1267.
[56] Tao, Y. & Winkler, M. (2021) Global smooth solutions in a two-dimensional cross-diffusion system modeling propagation of urban crime. *Commun. Math. Sci.* **19**, 829–849.
[57] Tse, W. & Ward, M. (2016) Hotspot formation and dynamics for a continuum model of urban crime. *Eur. J. Appl. Math.* **27**, 583–624.
[58] Tse, W. & Ward, M. (2018) Asynchronous instabilities of crime hotspots for a 1-D reaction-diffusion model of urban crime with focused police patrol. *SIAM J. Appl. Dyn. Syst.* **17**, 2018–2075.

[59] Wang, Q., Wang, D. & Feng, Y. (2020) Global well-posedness and uniform boundedness of urban crime models: One-dimensional case. *J. Differential Equations* **269**, 6216–6235.
[60] Wang, Z. (2013) Mathematics of traveling waves in chemotaxis. *Discrete Contin. Dyn. Syst-Series B.* **18**, 601–641.
[61] Wilson, J. & Kelling, G. (1982) Broken windows: The police and neighborhood safety. *Atlantic Mon.* **249**, 29–38.
[62] Winkler, M. (2010) Aggregation vs. global diffusive behavior in the higher-dimensional keller–Segel model. *J. Differential Equations* **248**, 2889–2905.
[63] Winkler, M. (2019) Global solvability and stabilization in a two-dimensional cross-diffusion system modeling urban crime propagation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **36**, 1747–1790.
[64] Yang, L. & Yang, X., Global existence in a two-dimensional nonlinear diffusion model for urban crime propagation, Nonlinear Anal., 224 (2022), 113086, 32.
[65] Zhao, X., Boundedness and asymptotic behavior of solutions to one-dimensional urban crime system with nonlinear diffusion, Nonlinear Anal. Real World Appl., (2023), 103946.
[66] Zipkin, J., Short, M. & Bertozzi, A. (2014) Cops on the dots in a mathematical model of urban crime and police response. *Discrete Contin. Dyn. Syst. Ser. B.* **19**, 1479–1506.

Cite this article: Jiang W. and Wang Q. Global well-posedness and uniform boundedness of 2D urban crime models with nonlinear advection enhancement. *European Journal of Applied Mathematics*, https://doi.org/10.1017/S0956792524000792