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APPROXIMATION BY (PLURI) SUBHARMONIC FUNCTIONS: FUSION AND LOCALIZATION

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ABSTRACT. Let u_1 and u_2 be subharmonic (plurisubharmonic) on overlapping sets K_1 and K_2 respectively. We seek to approximate u_1 and u_2 simultaneously by a single subharmonic (plurisubharmonic) function u.

1. **Introduction.** Let M be a paracompact complex manifold or a C^∞ connected Riemannian manifold. In this paper we study properties which are common to both plurisubharmonic functions on complex manifolds and to subharmonic functions on Riemannian manifolds. In order not to say everything twice, we shall say that u is a (pluri) subharmonic function on a set $E \subset M$. This means that either M is complex and u is plurisubharmonic on (an open neighbourhood of) E or M is Riemannian and u is subharmonic on (an open neighbourhood of) E. We denote by $S(E)$ the class of continuous (pluri) subharmonic functions on E. We shall, by abuse of notation, say that $u \in \bar{S}(E)$ if u is the uniform limit on E of functions in $S(E)$ restricted to E. Let $A(E)$ denote the functions which are continuous on E and (pluri) subharmonic on the interior E^o of E. If u is a function defined on a set E, we write $\|u\|_E = \sup\{|u(x)| : x \in E\}$. Since our interest is (pluri) subharmonic functions, we may and shall assume that M is not compact.

We distinguish between two types of (pluri) subharmonic approximation which we might call respectively *extension-type* approximation and *smoothing-type* approximation. In the former, we are given a function u on a set E and we seek to approximate u by a function v (pluri) subharmonic on a larger open set U containing E. This type of approximation is related to the problem of (pluri) subharmonic extensions (hence the nomenclature). In fact, if we can find a (pluri) subharmonic function v on U which actually agrees with u on E, then, of course, v is a very good approximation indeed, for the error function $u - v$ is identically zero on E. For some recent results on subharmonic extensions and their relation to approximation, see [G].

Smoothing-type approximation picks up where extension-type approximation leaves off. Namely, we suppose that we already have a (pluri) subharmonic function on an open set U and we seek to approximate it on the same set U (or a large portion thereof) by very nice (pluri) subharmonic functions, for example by smooth ones or, in the complex case, by functions of the type $\log|f|$, where f is holomorphic. In recent years, many such results have been obtained by Soviet mathematicians.

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In this note, we shall deal only with extension-type approximation. More precisely, given a function u on a set E, we seek to approximate u by functions in $S(E)$. It is natural and practical to try to ascertain just how well a function u can be approximated by functions in $S(E)$:

$$d\big(u, S(E)\big) = \inf_{v\in S(E)} \sup_{x\in E} |u(x) - v(x)|$$

This is clearly related to the notion of best approximation by (pluri) subharmonic functions for which we refer to [WZ].

A particular case of the above general problem, and the only one which will concern us in the present paper, is to determine whether a given function u is at zero distance from $S(E)$. If this is so, we say that u can be approximated by (pluri) subharmonic functions or notationally, by functions in $\bar{S}(E)$.

For approximation by subharmonic functions on a compact set E, a complete answer has been given by Bliedtner and Hansen [BH]. Namely, $u \in \bar{S}(E)$ if and only if u is continuous on E and finely subharmonic on the fine interior of E. Recently, this result was extended to closed sets E by Ladouceur and the author [GL]. From this follows a localization theorem: a function u on a closed subset $E \subset \mathbb{R}^n$ can be approximated by continuous subharmonic functions if and only if each point $x \in E$ has a neighbourhood $U = U_x$ such that $u|(E\cap U)$ can be approximated by continuous subharmonic functions.

In this paper we prove a so-called fusion lemma which (among other consequences) yields a localization theorem for (pluri) subharmonic functions.

Recently, Fornaess and Wiegerinck [FW] have shown the beautiful theorem that if Ω is a smoothly bounded domain in $\mathbb{C}^n$, then

$$\bar{S}(\bar{\Omega}) = A(\bar{\Omega}).$$

Sibony (*op. cit.*) has remarked that the proof in [FW] actually yields a localization theorem for plurisubharmonic functions. This means more or less that if a given function can be approximated locally by plurisubharmonic functions, then it can be approximated globally. However, the localization result in [FW] is (very slightly) misstated.

In this note, we prove a more general fusion lemma from which localization easily follows. Roughly speaking, a fusion lemma gives conditions on two sets E_1 and E_2 which insure that there exists a constant $c > 0$ such that for any two functions $u_j \in S(E_j)$, $j = 1, 2$, there exists $u \in S(E_1 u E_2)$ for which

$$|u - u_j|_{E_j} \leq c|u_1 - u_2|_{E_1\cap E_2}, \quad j = 1, 2.$$

Such fusion lemmas were pioneered by Alice Roth [R] who first obtained fusion for rational functions. Later, harmonic functions were fused by Hengartner and the author [GH].

Following the lead of Alice Roth in the case of rational approximation, we shall show that fusion yields, not only a localization theorem on compact sets, but also a technique for approximating on unbounded sets.

Approximation on unbounded sets has been and continues to be applied to obtain profound results on the boundary behaviour of holomorphic functions.

In [G], subharmonic approximation on unbounded sets is used to show that a recent maximum principle for unbounded domains is sharp.

2. **Fusion.** The following lemma is the principal result of this paper. It allows one to approximate two functions simultaneously.

FUSION LEMMA. *Let δ be a strictly (pluri) subharmonic function on a manifold M. Let U_1 and U_2 be open subsets of M with $\bar{U}_1 \cap \bar{U}_2 = \emptyset$ and $\bar{U}_1$ compact. Then, there is a constant $C > 0$ such that if $K \subset M$ is any compact set, $\varepsilon > 0$, and h_j are functions such that $h_j \in S(U_j \cup K)$ for $j = 1,2$, then there exists $v \in S(U_1 \cup K \cup U_2)$ such that, for all $x \in U_j \cup K$,*

$$|v(x) - h_j(x)| \leq C \cdot \max\{|\delta(x)|, 1\} \cdot \max\{\|h_1 - h_2\|_K, \varepsilon\} \tag{1}$$

REMARKS. If we are only interested in finding v (pluri) subharmonic in $U_1 \cup K^o \cup U_2$ rather than $U_1 \cup K \cup U_2$, then, in case $\|h_1 - h_2\|_K = 0$, (1) is trivial, for of course h_1 and h_2 are then (pluri) subharmonic extensions of each other and we may set $v = h_j$ on $U_j \cup K^o$.

Formula (1) can be simplified for most applications. Indeed, if

a) $\|h_1 - h_2\|_K \neq 0$, then $|v(x) - h_j(x)| \leq C \cdot \max\{|\delta(x)|, 1\} \cdot \|h_1 - h_2\|_K$;

b) $\|h_1 - h_2\|_K = 0$, then $|v(x) - h_j(x)| \leq \varepsilon \cdot \max\{|\delta(x)|, 1\}$;

c) δ is bounded on $U_1 \cup K \cup U_2$, then $|v(x) - h_j(x)| \leq C \max\{\|h_1 - h_2\|_K, \varepsilon\}$;

a,c) $\|h_1 - h_2\|_K \neq 0$ and δ is bounded on $U_1 \cup K \cup U_2$, then

$$|v(x) - h_j(x)| \leq C \cdot \|h_1 - h_2\|_K;$$

b,c) $\|h_1 - h_2\|_K = 0$ and δ is bounded on $U_1 \cup K \cup U_2$, then

$$|v(x) - h_j(x)| \leq \varepsilon.$$

In particular, from c) we see that if we are assured of the existence of a bounded strictly (pluri) subharmonic function on $U_1 \cup K \cup U_2$, then we may omit any mention of δ in the lemma. For example, we have the following.

COROLLARY. *Let $U_j \subset\subset M, j = 1,2$, be open in a Stein manifold M with $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. Then, there is a constant $C > 0$ such that if K is any compact set in M, $\varepsilon > 0$, and $h_j \in S(U_j \cup K)$, $j = 1,2$, there exists $v \in S(U_1 \cup K \cup U_2)$ such that*

$$\|v - h_j\|_{U_j \cup K} \leq C \max\{\|h_1 - h_2\|_K, \varepsilon\}.$$

In particular, if $\|h_1 - h_2\|_K \neq 0$,

$$\|v - h_j\|_{U_j \cup K} \leq C \cdot \|h_1 - h_2\|_K;$$

if $\|h_1 - h_2\|_K = 0$,

$$\|v - h_j\|_{U_j \cup K} \leq \varepsilon.$$

PROOF OF FUSION LEMMA. Let $\chi_1 \in C_0^\infty(M)$ such that $-1 \leq \chi_1 \leq 0$, $\chi_1 = -1$ on $\bar{U}_2$ and $\chi_1 = 0$ on $\bar{U}_1$. Set $\chi_2 = -1 - \chi_1$. Choose λ positive and so small that $\delta + \lambda\chi_j$ are both (pluri) subharmonic for $j = 1, 2$. Choose a neighbourhood V of K such that $h_j \in S(\bar{V})$ and

$$\|h_1 - h_2\|_V < 2\max\{\varepsilon, \|h_1 - h_2\|_K\}.$$

We define a positive constant η by

$$\lambda\eta = 2\max\{\varepsilon, \|h_1 - h_2\|_K\}.$$

Now set

$$f_j = h_j + \eta(\delta + \lambda\chi_j)$$

on $U_j \cup \bar{V}$ and $f_j \equiv -\infty$ elsewhere. Finally, we set

$$v = \max(f_1, f_2).$$

Clearly v is continuous and (pluri) subharmonic on $(U_1 \cup V \cup U_2) \setminus \partial V$. Suppose $x^0 \in \partial V \cap U_1$. Since $\chi_1 = 0$ on U_1 and $\chi_2 = -1$ on U_1, if x is near x^0 and $x \in \bar{V}$, then

$$f_2(x) = h_2(x) + \eta(\delta - \lambda) = f_1(x) + [h_2(x) - h_1(x)] - \lambda\eta < f_1(x).$$

Since $f_2(x) = -\infty$ for $x \in U_1 \setminus \bar{V}$, we have that $f_2(x) \leq f_1(x)$ for all x near x^0. The point x^0 was an arbitrary point of $\partial V \cap U_1$ and so it follows that v is continuous and (pluri) subharmonic on a neighbourhood of $\partial V \cap U_1$ and, by a similar argument, on a neighbourhood of $\partial V \cap U_2$. Thus, $v \in S(U_1 \cup V \cup U_2)$.

There remains to verify that v has the required approximation property.

On $U_1 \setminus \bar{V}$,

$$\begin{aligned}|v(x) - h_1(x)| &= |f_1(x) - h_1(x)| = \eta|\delta(x) + \lambda\chi_1(x)| \\ &= \eta|\delta(x)| = 2\lambda^{-1}\max\{\varepsilon, \|h_1 - h_2\|_K\}\delta(x).\end{aligned}$$

On $U_1 \cap \bar{V}$, we consider two cases. First of all, if $|v(x) - h_1(x)| = v(x) - h_1(x)$, then, since $v \leq \max(h_1, h_2) + \eta(|\delta| + \lambda)$, we have

$$\begin{aligned}|v(x) - h_1(x)| &\leq \max\Big\{\big(h_1(x), h_2(x)\big) - h_1(x) + \eta\big(|\delta(x)| + \lambda\big)\Big\} \\ &\leq 2\max\{\varepsilon, \|h_1 - h_2\|_K\} + \lambda^{-1}2\max\{\varepsilon, \|h_1 - h_2\|_K\}\big(|\delta(x)| + \lambda\big).\end{aligned}$$

If, on the other hand, $|v(x) - h_1(x)| = h_1(x) - v(x)$, then $|v(x) - h_1(x)| = h_1(x) - \max\big(f_1(x), f_2(x)\big) \leq h_1(x) - \max\big[h_1(x) - \eta\big(|\delta(x)| + \lambda\big), h_2(x) - \eta\big(|\delta(x)| + \lambda\big)\big] = h_1(x) - \max\big(h_1(x), h_2(x)\big) + \eta\big(|\delta(x)| + \lambda\big)$, which yields the same estimate as in the first case. Thus, for $C = 4 + 2\lambda^{-1}$, (1) holds on $U_1 \cap \bar{V}$. The estimates on $U_2 \cap \bar{V}$ are similar. This completes the proof of the fusion lemma.

3. **Localization.** The following result asserts that if a function u can be locally uniformly approximated on a closed set by (pluri) subharmonic functions, then global uniform weighted approximation of u is feasible. As a weight, any strictly (pluri) subharmonic function is admissible. Let $C^+(E)$ denote the set of positive continuous functions on a set E.

LOCALIZATION THEOREM. *Suppose u is a function defined on a closed set $E \subset M$ and for each $x \in E$, there is a neighbourhood $B = B_x$ such that $u|_{E\cap\bar{B}} \in \bar{S}(E \cap \bar{B})$. Then, if δ is any strictly (pluri) subharmonic function on E and $\varepsilon \in C^+(E)$, there exists $v \in S(E)$ such that*

$$|u - v| < \max\{|\delta|, \varepsilon\}. \tag{2}$$

The role of the function ε is to avoid interpolation at the zeros of δ, in other words, to insure that we have a positive weight. If δ is already zero-free, we have the following reformulation.

COROLLARY 1. *Under the hypotheses of the localization theorem, if δ has no zeros, then for each positive constant $\varepsilon > 0$, we may replace (2) by*

$$|u - v| < \varepsilon|\delta|.$$

The strength of Corollary 1 depends on which strictly (pluri) subharmonic functions δ we can come up with.

EXAMPLE 1. In $\mathbb{C}^n$, we may set, for example $\delta(z) = |z|^2 + 1$.

EXAMPLE 2. In $\mathbb{C}^1$, we may set $\delta(z) = |z - z_0|^{-2m}$, where $m \neq 0$ and $z_0 \notin E$.

EXAMPLE 3. In $\mathbb{R}^n$, for $n \geq 3$, we may set $\delta(x) = -(1+|x|^2)^{-\alpha}$, for $0 < \alpha \leq n/2-1$.

EXAMPLE 4. In $\mathbb{R}^n$, we may also set $\delta(x) = -(1 + |x|^4)^{-\alpha}$, for $0 < \alpha \leq n/2$.

EXAMPLE 5. If M is any (non-compact) Riemannian manifold, then there always exists a global strictly subharmonic function δ on M. Indeed, M admits a global fundamental solution $e(x, y)$ for the Laplacian Δ (see *e.g.* [BB]). Hence, if $\varphi \in C_0^\infty(M)$, then there exists a solution $\psi \in C^\infty(M)$ for the Poisson equation $\Delta\psi = \varphi$. Let $\{\varphi_j\}$ be a partition of unity on M and for each j, $\psi_j \in C^\infty(M)$ be a solution of $\Delta\psi_j = \varphi_j$. We may construct a sequence $\{\lambda_j\}$ of positive numbers decreasing to zero so rapidly, that the series $\delta = \sum_{j=1}^\infty \lambda_j\psi_j$ is in $C^\infty(M)$ and can be differentiated term by term any number of times. In particular, $\Delta\delta = \Sigma\lambda_j\varphi_j > 0$ and so δ is strictly subharmonic.

If the Riemannian manifold is hyperbolic, then the above global strictly subharmonic function δ may be constructed so as to be positive and bounded. Thus, if E is a closed subset of a (not necessarily hyperbolic) Riemannian manifold, there always exists such a δ on (a neighbourhood of) E.

Note that with Examples 2, 3 and 4, we actually obtain better-than-uniform approximation in Corollary 1. That is, the error function $\varepsilon|\delta|$ tends to zero as we approach infinity.

The following corollary is a more familiar form of localization.

COROLLARY 2. *If E is a closed subset of M and there exists a bounded strictly (pluri) subharmonic function on E, then $u \in \bar{S}(E)$ if and only if for each $x \in E$, there is a neighbourhood $B = B_x$ such that*

$$u|_{E\cap\bar{B}} \in \bar{S}(E \cap \bar{B}).$$

From the examples of strictly (pluri) subharmonic functions which we have given earlier, we see that the hypotheses of Corollary 2 are satisfied if E is any (proper) closed subset of a Riemannian manifold or compact subset of a Stein manifold.

PROOF OF LOCALIZATION THEOREM. Let δ be strictly (pluri) subharmonic on an open set $W \supset E$. We may construct a locally finite open cover $\{B_j\}$ of E such that for each j, $B_j \subset\subset W$ and $u|_{E\cap\bar{B}_j} \in \bar{S}(E \cap \bar{B}_j)$. For every j, we can find compact sets $K_{j,k}$, $k \neq j$, only finitely many of which are non-empty, such that $K_{j,k} \subset B_k$ and

$$\partial B_j \cap E \subset \bigcup_{k\neq j} K_{j,k}.$$

Set

$$K_k = \bigcup_j K_{j,k};$$

then

$$K_k \subset\subset B_k.$$

Fix any metric on M, and let

$$d_k = \text{distance}(K_k, \partial B_k).$$

For every k, there exists $\chi_k \in C^\infty(M)$ with $-1 \leq \chi_k \leq 0$, $\chi_k(x) = 0$ for $\text{dist}(x, K_k) \leq d_k/2$, while $\chi_k = -1$ outside of B_k. There exists a constant $\eta_k^0 = \eta_k^0(\delta) > 0$ such that for $0 < \eta_k < \eta_k^0$, the function $\delta + \eta_k\chi_k$ is continuous and (pluri) subharmonic on an open set $V_k, B_k \subset\subset V_k \subset\subset W$.

Let $\{\varepsilon_j\}$ be, for the moment, any sequence of positive numbers such that, for each $x \in E$,

$$2 \max_{x\in\bar{B}_j} \varepsilon_j < \min_{x\in\bar{B}_j} \eta_j. \tag{3}$$

From the hypotheses of the localization theorem, for each j, there is an open set U_j, with $E \cap B_j \subset\subset U_j \subset\subset V_j$ and a function $h_j \in S(U_j)$ such that

$$|u - h_j| < \varepsilon_j \text{ on } E \cap \bar{B}.$$

We set, for each j,

$$\tilde{f}_j = h_j + \delta, \text{ on } U_j,$$

and on $(U_j \setminus E) \cup (E \cap \bar{B}_j)$, we define

$$f_j = \tilde{f}_j + \eta_j \chi_j.$$

Elsewhere, we set $f_j = -\infty$.

Finally, we define

$$v = \max f_j.$$

Let us now verify that v approximates u on E. For $x \in E$,

$$\begin{aligned}
|u(x) - v(x)| &= \left|u(x) - \max_k f_k(x)\right| \\
&= \left|u(x) - \max_{x \in E \cap \bar{B}_k} f_k(x)\right| \\
&= \left|u(x) - \max_{x \in E \cap \bar{B}_k} [h_k(x) + \delta(x) + \eta_k \chi_k(x)]\right| \\
&\leq \max_{x \in E \cap \bar{B}_k} \eta_k + \left|u(x) - \max_{x \in E \cap \bar{B}_k} h_k(x)\right| + |\delta(x)|.
\end{aligned}$$

Since the cover $\{\bar{B}_k\}$ of E is locally finite, and since the function ε is locally bounded away from zero, it follows that we may choose, first the sequence $\{\eta_j\}$ and then the sequence $\{\varepsilon_j\}$ such that

$$|u - v| < |\delta| + \varepsilon$$

everywhere on E. Since ε is an arbitrary function in $C^+(E)$ and we may multiply δ by a small positive number, this is equivalent to the assertion of the theorem, provided of course that $v \in S(E)$, which we now show.

Fix $x \in E$. If x is not on the boundary of some B_j, then each of the (finitely many) f_j which is not $-\infty$ at x is continuous and (pluri) subharmonic in a neighbourhood of x. Thus, the same is true of their maximum v.

Suppose now that $x \in \partial B_j \cap E$. Then, for some k, $x \in E \cap K_k \subset E \cap B_k$. For this j and this k,

$$\begin{aligned}
f_j(x) &= h_j(x) + \delta(x) + \eta_j \chi_j(x) = h_j(x) + \delta(x) - \eta_j \\
&= [h_j(x) - h_k(x)] + [h_k(x) + \delta(x) + \eta_k \cdot 0] - \eta_j \\
&= f_k(x) + [h_j(x) - h_k(x)] - \eta_j.
\end{aligned}$$

Since $|h_j - h_k| \leq \varepsilon_j + \varepsilon_k$ on $E \cap \bar{B}_j \cap B_k$, it follows from (3) that $f_j(x) < f_k(x)$. Moreover, the inequality $f_j < f_k$ persists in a full neighbourhood of x. Thus, in calculating $v(y)$ in a neighbourhood of x, we may write

$$v(y) = \max_{k \neq j} f_k(y),$$

if $x \in \partial B_j \cap E$. Since the same argument applies to all j for which $x \in \partial B_j \cap E$, there is some neighbourhood $U = U_x$ such that

$$v(y) = \max_{U_x \subset B_k} f_k(y),$$

for all $y \in U_x$. Since there are only finitely many such B_k, $v \in S(U_x)$. We have shown that v is continuous and (pluri) subharmonic in a neighbourhood of each point of E. Thus, $v \in S(E)$. This completes the proof of the localization theorem.

The localization theorem tells us when an individual function u on a set E can be approximated by functions v in $S(E)$. Of course, such a function u is necessarily in the class $A(E)$. The following corollary gives a localization condition on a set E which insures that *every* function $u \in A(E)$ can be approximated by functions in $S(E)$.

COROLLARY 3. *Suppose E is a closed set in M and for each $x \in E$, there is a neighbourhood $U = U_x$ such that $A(E \cap \bar{U}) = \bar{S}(E \cap \bar{U})$. Then, if $u \in A(E)$, δ is strictly (pluri) subharmonic on E, and $\varepsilon \in C^+(E)$, there exists $v \in S(E)$ such that*

$$|u - v| < \max\{|\delta|, \varepsilon\}.$$

4. **Geometric conditions.** We shall say that an open set $\Omega \subset M$ is *smoothly bordered* if for each $x \in \partial\Omega$, there is a neighbourhood $U = U_x$ and a C^1-diffeomorphism $\varphi \colon U \longrightarrow B$ onto the unit ball ($|t| < 1$) in $\mathbb{R}^m$ such that $\varphi(x) = 0$ and

$$\varphi(U \cap \Omega) = \{t \in B : t_m > 0\}.$$

The techniques used in this paper are inspired by the proof of the following beautiful result of Fornaess and Wiegerinck.

THEOREM [FW]. *If Ω is a smoothly bordered bounded open set in $\mathbb{R}^n$ or $\mathbb{C}^n$, then*

$$\bar{S}(\bar{\Omega}) = A(\bar{\Omega}).$$

In [FW] the theorem is stated for domains in $\mathbb{C}^n$, but the proof is identical for domains of $\mathbb{R}^n$. Their proof is based on the observation that we can find a finite open cover $B_0, \ldots, B_m$ of $\bar{\Omega}$ with the following properties: $B_0 \subset\subset \Omega$; for every j, $1 \le j \le m$, there exists $x_j \in \partial\Omega \cap B_j$ such that for all sufficiently small $\nu > 0$;

$$\Omega \subset\subset \bigcup_{j=0}^{m} \tilde{B}_{j,\nu},$$

where

$$\tilde{B}_{j,\nu} = \{z = \zeta + \nu n_j, \zeta \in B_j \cap \Omega\}$$

with $n_j, j \ge 1$, the unit outward normal to $\partial\Omega$ at x_j, while $n_0 = 0$; and finally

$$\operatorname{dist}\big(\partial\Omega, (\partial\Omega \cap B_j) + \nu n_j\big) > \nu/2.$$

To construct a cover, we begin by considering the outward normal unit vector $n(x)$ at each point $x \in \partial\Omega$ and then use the compactness of $\bar{\Omega}$. The proof also works if at each

$x \in \partial\Omega$, we choose an arbitrary unit vector $n(x)$ in the outer half-space determined by the tangent space to $\partial\Omega$ at x.

Let us say that a common boundary point $x \in \partial\Omega_1 \cap \partial\Omega_2$ is a *point of tangency* of two smoothly bordered open sets Ω_1 and Ω_2 if the outer half-spaces of Ω_1 and Ω_2 at x are disjoint. If Ω_1 and Ω_2 are free of points of tangency, then we can repeat the above construction for the domain $\Omega = \Omega_1 \cap \Omega_2$ by choosing at each $x \in \partial\Omega$, a vector $n(x)$ in the outer normal cone to $\partial\Omega$ at x. This same technique works for finite intersections and yields the following corollary of (the proof of) the theorem of Fornaess and Wiegerinck.

COROLLARY. If $\Omega_1, \dots, \Omega_k$ are finitely many smoothly bordered bounded open sets in $\mathbb{R}^n$ or $\mathbb{C}^n$ having, pairwise, no points of tangency and $\Omega = \Omega_1 \cap \cdots \cap \Omega_n$, then

$$\bar{S}(\bar{\Omega}) = A(\bar{\Omega}).$$

In the following theorem, we drop the assumption that $\bar{\Omega}$ be compact.

THEOREM. *Let Ω be a smoothly bordered open set in M. Then, for each $u \in A(\bar{\Omega})$, for each δ strictly (pluri) subharmonic on $\bar{\Omega}$, and for each $\varepsilon \in C^+(\bar{\Omega})$, there exists $v \in S(\bar{\Omega})$ such that*

$$|u - v| < \max\{|\delta|, \varepsilon\}.$$

PROOF. Let W be an open set, $W \supset \bar{\Omega}$, in which δ is strictly (pluri) subharmonic.

We shall say that an open set $B \subset M$ is a *parametric ball* if $B \subset\subset U$, where $\varphi: U \longrightarrow \varphi(U)$ is some chart of M, and $\varphi(B)$ is a ball.

Since Ω is smoothly bordered, each point $x \in \partial\Omega$ has a neighbourhood system $\{B_\alpha\}$ of smoothly bounded open sets such that for each B_α there is a $B_\beta \subset\subset B_\alpha$ and a smoothly bounded $G \subset\subset B_\alpha$ such that $\bar{\Omega} \cap \bar{B}_\beta = \bar{G} \cap \bar{B}_\beta$ and ∂G and ∂B_β meet transversally.

We may assume that each B_α is a (domain of a coordinate) chart for M. Hence, by the preceding corollary

$$\bar{S}(\bar{\Omega} \cap \bar{B}_\beta) = A(\bar{\Omega} \cap \bar{B}_\beta).$$

The theorem now follows from Corollary 3 of the localization theorem.

We may also prove all of the results of this section more directly. Indeed, let $u \in A(\bar{\Omega})$ and let $x \in \partial\Omega$. Then the translates of u in any direction (in any coordinate neighbourhood of x) approximate u uniformly. Now if Ω satisfies the geometric hypotheses of the theorems or the corollary of this section, then there is at least one (actually an entire cone) direction for which the translates of u are in $S(\bar{\Omega} \cap \bar{B})$, for a sufficiently small fixed neighbourhood B of x and for all sufficiently small translations. All of the results of this section now follow directly from the localization theorem.

5. **Next step.** The subject of our investigation has been the approximation of a given function u on a set E by functions (pluri) subharmonic on (a neighbourhood of) E. It is preferable that the approximating functions be (pluri) subharmonic on as large a neighbourhood of E as possible. This leads to the following problem. Given sets $A \subset B$, we seek to approximate functions in $S(A)$ by functions in $S(B)$, or better yet, extend functions in $S(A)$ to functions in $S(B)$. The subharmonic case is treated in [G]. The plurisubharmonic case is open.

ADDED IN PROOF. Recently, I received a manuscript from S. Gardiner on subharmonic extensions as well as a letter from J. Chaumat in which he mentions that a student of his has some results on plurisubharmonic approximation.

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