On Determinants of Symmetric Functions.

By Dr A. C. Aitken.

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§ 1. Introductory.

The result of dividing the alternant $|a^\alpha b^\beta c^\gamma \ldots|$ by the simplest alternant $|a^0 b^1 c^2 \ldots|$ (the difference-product of $a, b, c, \ldots$) is known to be a symmetric function expressible in two distinct ways, (1) as a determinant having for elements the elementary symmetric functions C_r of $a, b, c, \ldots$, (2) as a determinant having for elements the complete homogeneous symmetric functions H_r. For example

$$\frac{|a^0 b^2 c^5 d^6|}{|a^0 b^1 c^2 d^3|} = \begin{vmatrix} H_0 & H_2 & H_5 & H_6 \\ 0 & H_1 & H_4 & H_5 \\ 0 & H_0 & H_3 & H_4 \\ 0 & 0 & H_2 & H_3 \end{vmatrix} = \begin{vmatrix} C_2 & C_3 & C_5 \\ C_1 & C_2 & C_4 \\ C_0 & C_1 & C_3 \end{vmatrix}.$$

The formation of the (historically earlier) H-determinant is evident. The suffixes in the first row are the indices of the alternant; those of the other rows decrease by unit steps. This result is due to Jacobi.[1]

A simple rule for obtaining the C-determinant has been given by Muir,[2] as follows: *the indices which do not appear in the alternant are* 1, 3, 4; *their defects from the highest index* 6 *are* 5, 3, 2; *these, reversed in order, are the suffixes in the first row of the C-determinant*, the other rows being formed as before. This result is due to Naegelsbach.[3]

Dismissing for the present the alternants, let us examine the identity between the remaining determinants (called by Muir "bi-alternants"). Since by convention $C_0 = H_0 = 1$, we have

$$\begin{vmatrix} H_1 & H_4 & H_5 \\ H_0 & H_3 & H_4 \\ 0 & H_2 & H_3 \end{vmatrix} = \begin{vmatrix} C_2 & C_3 & C_5 \\ C_1 & C_2 & C_4 \\ C_0 & C_1 & C_3 \end{vmatrix}$$

(The determinants are *not in general of the same order*.)

[1] De functionibus alternantibus. *J. für Math.*, 22 (1841), pp. 370-371.

[2] *Theory of Determinants*, vol. III, pp. 145-146.

[3] Ueber eine Classe symmetrischen Functionen. Sch.-Programm, Zweibrücken, 1871.

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It will be seen that Muir's rule applies not only to first rows but also to *last columns*, for *e.g.* the suffixes not appearing in 3, 4, 5 are 0, 1, 2 and their defects from 5, reversed, are 3, 4, 5.

A second relation was observed and proved by Kostka.[1] The diagonal suffixes in the H-determinant are 1, 3, 3, a "partition" of the integer 7 which may be represented by a Ferrers-Sylvester diagram of rows of asterisks, as in Fig. I.

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Fig. I.	Fig. II.

The "conjugate partition," obtained as in Fig. II by interchanging axes, is 2, 2, 3, and these are the suffixes in the diagonal of the equivalent C-determinant. This important fact links up the determinantal theory of symmetric functions with the combinatory theory.

Conjugacy of the kind described is of course a reciprocal property, so that any identity between bi-alternants remains an identity when C's and H's are interchanged. (This is also evident from other considerations, such as the symmetry in H and C of Wronski's well-known recurrence relations, or the fact that the generating functions of C_r and H_r are reciprocal.)

Finally we refer to a type of ordered partition introduced by MacMahon[2], and called by him a "composition" of an integer. These are represented by zigzag diagrams of asterisks, the "conjugate composition" being obtained as before by interchange of axes. Here again MacMahon[1] finds important identities between C-determinants and H-determinants, *e.g.*

$$\begin{vmatrix} H_1 & H_3 & H_6 \\ H_0 & H_2 & H_5 \\ 0 & H_0 & H_3 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_4 & C_6 \\ C_0 & C_1 & C_3 & C_5 \\ 0 & C_0 & C_2 & C_4 \\ 0 & 0 & C_0 & C_2 \end{vmatrix}.$$

[1] Bemerkungen über symmetrischen Funktionen. *J. für Math.* 132 (1907), pp. 159, 161.

[2] *Combinatory Analysis*, vol. I, p. 205.

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The diagonal suffixes 1, 2, 3 and 1, 1, 2, 2 are *conjugate compositions* of the integer 6. (Figs. III, IV.)

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Fig. III. Fig. IV.

Inspection of the determinants shows that here again Muir's rule holds for first rows and last columns.

§ 2. A General Identity.

It is natural to suspect the existence of a more general identity including the preceding as special cases and involving partitions of a wider kind. If in the bi-alternants we remove the restriction that suffixes from row to row are to move by unit steps, we obtain the more general determinant

$$\begin{vmatrix} H_{\alpha+\alpha'} & H_{\alpha+\beta'} & H_{\alpha+\gamma'} & . & . \\ H_{\beta+\alpha'} & H_{\beta+\beta'} & H_{\beta+\gamma'} & . & . \\ H_{\gamma+\alpha'} & H_{\gamma+\beta'} & H_{\gamma+\gamma'} & . & . \\ . & . & . & . & . \end{vmatrix}$$

where $\alpha > \beta > \gamma > \ldots$, $\alpha' < \beta' < \gamma' < \ldots$ This determinant is also of importance in the theory of symmetric functions.

First of all it will be shown that Muir's rule continues to hold. For example we shall have

$$\begin{vmatrix} C_2 & C_5 & C_6 \\ C_0 & C_3 & C_4 \\ 0 & C_1 & C_2 \end{vmatrix} = \begin{vmatrix} H_2 & H_3 & H_5 & H_6 \\ H_1 & H_2 & H_4 & H_5 \\ 0 & H_0 & H_2 & H_3 \\ 0 & 0 & H_0 & H_1 \end{vmatrix}.$$

As for Kostka's observation on partitions, and MacMahon's on compositions, we have now new forms; *e.g.* the identity just given will refer to the conjugacy of diagrams like Figs. V and VI.

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* *                 * *
  * * *               * *
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Fig. V. Fig. VI.

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The diagonal suffixes are still represented by rows of asterisks, but where the suffixes in any row of the determinant are less by r than those of the preceding row, the left asterisk of the corresponding row in the diagram is placed beneath the r^{th} from the left of the preceding row. The new diagrams are conjugable and give rise to the same conjugate arrays of suffixes as Muir's rule. They also have some intrinsic interest.

Certain simple relations between diagram and determinant are evident on inspection but important. A diagram has a northeast and a southwest border, each being a zigzag line of asterisks, in fact a diagram of compositions. If either border be removed, the part remaining represents the minor obtained by deleting the first row and the last column of the determinant. *Conjugate diagrams remain conjugate after borders have been removed.*

Diagrams identical with their conjugates may be called *self-conjugate.* They represent an interesting type of determinant which is invariant with respect to the interchange of C_r and H_r, *e.g.*

$$\begin{vmatrix} C_2 & C_4 & C_5 \\ C_1 & C_3 & C_4 \\ 0 & C_1 & C_2 \end{vmatrix} = \begin{vmatrix} H_2 & H_4 & H_5 \\ H_1 & H_3 & H_4 \\ 0 & H_1 & H_2 \end{vmatrix}.$$

§ 3. Proof of the Identity.

Let (λ) and $(\lambda)'$ represent arrays of suffixes conjugate in the sense of § 2, $C_{(\lambda)}$ and $H_{(\lambda)'}$ determinants of C's and H's having those suffixes. Let the minors obtained by deleting first rows and last columns be $C'_{(\lambda)}$ and $H'_{(\lambda)'}$, the second minors obtained by further deletion be $C''_{(\lambda)}$ and $H''_{(\lambda)'}$, and so on.

The proof for general arrays becomes prolix. It will be sufficient to indicate the steps by the example given in § 2.

Let Wronski's recurrence relation

$$0 = C_0H_r - C_1H_{r-1} + C_2H_{r-2} - \dots + (-)^r C_r H_0$$

be denoted by $\{r\}$. Take $\{6\}$, $\{5\}$, $\{3\}$, $\{1\}$, the numbers being the suffixes in the *last column* of $H_{(\lambda)'}$. Eliminate between these the C's, excepting C_0, which *do not occur in the first row* of $C_{(\lambda)}$, namely C_1, C_3, C_4. The eliminant has the form

$$0 = H_{(\lambda)'} - C_2H_{(\mu)} - C_5H_{(\nu)} - C_6H'_{(\lambda)'} \quad \dots\dots\dots\dots\dots (1)$$

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But Wronski's relations remain valid when the suffixes of C's are all diminished by the same integer. Hence we annex to (1) two other equations corresponding to the remaining rows of C,

$$0 = 0 - C_0 H_{(\mu)} - C_3 H_{(\nu)} - C_4 H'_{(\lambda)'} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$0 = 0 - \quad 0 \quad - C_1 H_{(\nu)} - C_2 H_{(\lambda)'} \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Solving now from (1), (2), (3), we have

$$C_{(\lambda)} : C'_{(\lambda)} = H_{(\lambda)'} : H'_{(\lambda)'}.$$

Hence the identity of $C_{(\lambda)}$ and $H_{(\lambda)'}$ depends on that of $C'_{(\lambda)}$ and $H'_{(\lambda)'}$, *always provided the latter are not zero.* (We have seen that they are *conjugate*.) This identity in its turn depends, with the same provision, on the identity of $C''_{(\lambda)}$ and $H''_{(\lambda)'}$, and so on. Thus the validity of the theorem depends ultimately on the case when one of the minors reduces to a single element, such as H_3. But then it is a well-known result, *e.g.*

$$H_3 = \begin{vmatrix} C_1 & C_2 & C_3 \\ C_0 & C_1 & C_2 \\ 0 & C_0 & C_1 \end{vmatrix},$$

or say

$$H\{* \;\; * \;\; *\} = C\left\{\begin{matrix} * \\ * \\ * \end{matrix}\right\}.$$

In the case excepted, where $C_{(\lambda)}$ is such that $C'_{(\lambda)}$ is zero, inspection shows that then both $C_{(\lambda)}$ and $H_{(\lambda)'}$ are factorizable into two or more determinants, the diagrams of which, when juxtaposed, constitute the complete diagram. The separate factors are conjugate in pairs, and their identity follows as before.

Thus the theorem is established.

§ 4. Determinant Factors.

A trivial example of factorization has just been noticed, sufficiently indicated by such a diagram as

$$C\left\{\begin{matrix} * & & & & \\ * & * & * & & \\ & & & * & * \end{matrix}\right\} = C\left\{\begin{matrix} * & & \\ * & * & * \end{matrix}\right\}. C\{* \;\; *\}.$$

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It simply involves a block of zero elements *immediately below the principal diagonal.*

In a more important case, considered for H-determinants by Segar[1], Nanson[2] and Muir[3], the possibility of factorization depends on the number of variables $a, b, c, \ldots k$ entering into the symmetric functions. For example, if there are m variables, Segar's theorem is essentially that when the order of $H_{(\lambda)}$ equals or exceeds m, then $H_{(\lambda)}$ is composed of simpler bi-alternant factors.[4] Now this becomes almost obvious if we examine the conjugate determinant $C_{(\lambda)'}$.

Consider for example a determinantal symmetric function of three variables, a, b, c,

$$\begin{vmatrix} H_4 & H_5 & H_7 \\ H_2 & H_3 & H_5 \\ H_0 & H_1 & H_3 \end{vmatrix} = \begin{vmatrix} C_1 & 0 & 0 & 0 & 0 \\ C_0 & C_3 & 0 & 0 & 0 \\ 0 & C_2 & C_3 & 0 & 0 \\ 0 & C_0 & C_1 & C_2 & C_3 \\ 0 & 0 & 0 & C_0 & C_1 \end{vmatrix}.$$

Since there are only three variables, $C_r = 0$ for $r > 3$, and so the C-determinant has blocks of zeros above the principal diagonal, causing it to break up into the factors

$$\begin{vmatrix} C_2 & C_3 \\ C_0 & C_1 \end{vmatrix} . C_3^2 . C_1, \text{ or } \begin{vmatrix} H_2 & H_3 \\ H_0 & H_1 \end{vmatrix} \begin{vmatrix} H_1 & H_2 & H_3 \\ H_0 & H_1 & H_2 \\ 0 & H_0 & H_1 \end{vmatrix}^2 . H_1 .$$

From this point of view, which is quite general, Segar's theorem is a simple consequence of the vanishing of the higher C's.

The diagrammatic interpretation of the factorization of $H_{(\lambda)}$ may be of interest. In the example before us it is

$$H \left\{ \begin{matrix} * & * & * & * & \\ & * & * & * & \\ & & * & * & * \end{matrix} \right\} = H \left\{ \begin{matrix} * & * \\ & * \end{matrix} \right\} H \left\{ \begin{matrix} * \\ * \\ * \end{matrix} \right\} H \left\{ \begin{matrix} * \\ * \\ * \end{matrix} \right\} H \{ * \}.$$

[1] On a determinantal theorem due to Jacobi. *Messenger of Math.* 21 (1892), pp. 148, 150.

[2] On a theorem of Segar's. *Messenger of Math.* 36 (1906), pp. 77–78.

[3] Note on a determinant whose elements are aleph functions. *Messenger of Math.* 46 (1916), pp. 108-110.

[4] In this form the theorem had really been given by Naegelsbach, *op. cit.*, in 1871. *Cf.* Muir's *History*, Vol. III, p. 147.

The rule is easily seen to be as follows: *if the diagram can be dissected by drawing vertical lines which just cover all the asterisks on either side of them and have a vertical span equal to or exceeding the space of m asterisks, then $H_{(\lambda)}$ has factors, one for each dissected part.*

Thus in the example given $m = 3$, and we make dissections

$$\begin{array}{cc|c|c|c} \ast & \ast & \ast & \ast & \\ & \ast & \ast & \ast & \\ & & \ast & \ast & \ast \end{array}.$$

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