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FRÉCHET INTERMEDIATE DIFFERENTIABILITY OF LIPSCHITZ FUNCTIONS ON ASPLUND SPACES

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Abstract

The deep Preiss theorem states that a Lipschitz function on a nonempty open subset of an Asplund space is densely Fréchet differentiable. However, the simpler Fabian–Preiss lemma implies that it is Fréchet intermediately differentiable on a dense subset and that for a large class of Lipschitz functions this dense subset is residual. Results are presented for Asplund generated spaces.

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A real-valued function ψ on a nonempty open subset G of a Banach space X is *locally Lipschitz* if for each $x \in G$ there exist $K(x) > 0$ and $\delta(x) > 0$ such that

$$|\psi(y) - \psi(z)| \leq K\|y - z\| \quad \text{for all } y, z \in B(x; \delta).$$

Recall that ψ is *Gâteaux differentiable* at $x \in G$ if there exists a continuous linear functional $\psi'(x)$ on X such that

$$\psi'(x)(h) = \lim_{\lambda \to 0} \frac{\psi(x + \lambda h) - \psi(x)}{\lambda} \quad \text{for each } h \in X;$$

furthermore, ψ is *Fréchet differentiable* at x if given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$|\psi(x + h) - \psi(x) - \psi'(x)(h)| \leq \varepsilon\|h\| \quad \text{for all } h \in X, \|h\| < \delta.$$

A Banach space X is an *Asplund space* if every continuous convex function on a nonempty open convex subset of X is Fréchet differentiable at the points of a residual subset of its domain. Preiss [8] proved the remarkable result that on an Asplund space every locally Lipschitz function on a nonempty open subset is Fréchet differentiable at the points of a dense subset of its domain. The proof is technical even in its later rewritten version presented with Lindenstrauss [5].

Giles and Sciffer [3] introduced a weaker form of Fréchet differentiability with the aim of exploring density properties in an elementary way. A locally Lipschitz function ψ on a nonempty open subset G of a Banach space X is said to be *Fréchet intermediately differentiable* at $x \in G$ if there exists a continuous linear functional ξ on X and given $\varepsilon > 0$ there exists $0 < \delta(\varepsilon) < \varepsilon$ such that

$$\left|\frac{\psi(x+\lambda h)-\psi(x)}{\lambda}-\xi(h)\right|<\varepsilon \quad \text{for all } \frac{\delta}{2}<\lambda<\delta \text{ and } h\in X,\ \|h\|=1.$$

Clearly a continuous convex function ϕ on a nonempty open convex subset G is Fréchet differentiable at $x \in G$ if and only if it is Fréchet intermediately differentiable at x. However, even for a Lipschitz function on the real line Fréchet intermediate differentiability does not necessarily imply Fréchet differentiability, [3, p. 358].

Without relying on the Preiss theorem, we aim to show that every locally Lipschitz function on a nonempty open subset of an Asplund space is densely Fréchet intermediately differentiable.

We recall that even on the real line there are Lipschitz functions where the set of points of differentiability is not residual. Nevertheless, using the Preiss theorem, Giles and Sciffer [3, p. 355] proved that a significant class of locally Lipschitz functions on an Asplund space are residually Fréchet intermediately differentiable. However, Fabian *et al.* [1, p. 701] have recently shown that this result can be proved without the Preiss theorem. For completeness of our study we include their proof.

We present our results in a slightly more general setting. A Banach space X is an *Asplund generated space* if there exist an Asplund space Y and a continuous linear mapping $T : Y \to X$ such that $X = \overline{T(Y)}$. Since T has dense range then the conjugate mapping $T^* : X^* \to Y^*$ is one-to-one. So $\|\cdot\|' : X^* \to \mathbb{R}$ defined by $\|\xi\|' = \|T^*\xi\|$ is a norm on X^*.

Given a nonempty bounded subset E of the dual X^* of a Banach space X, for $e \in X$ and $\alpha > 0$, a nonempty subset of the form

$$S\ell(E, e, \alpha) \equiv \{\xi \in E \mid \xi(e) > \sup E(e) - \alpha\}$$

is called a *weak* slice* of E.

We rely on the following property of Asplund generated spaces which is an extension of that for Asplund spaces, [6, p. 83].

LEMMA 1 [3, p. 359]. *An Asplund generated space $X = \overline{T(Y)}$ has the property that every weak* slice of a nonempty bounded subset E of X^* determined by Ty for some $y \in Y$ contains a weak* slice determined by Ty' for some $y' \in Y$ with arbitrarily small $\|\cdot\|'$-diameter.*

PROOF. Consider $S\ell(E, Ty, \alpha)$ a weak* slice in X^* determined by Ty where $y \in Y$ and $\alpha > 0$. Now $S\ell(E, Ty, \alpha) = (T^*)^{-1} S\ell(T^*E, y, \alpha)$. Since Y is Asplund, given

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$\varepsilon > 0$ there exist $y' \in Y$ and $\alpha' > 0$ such that $S\ell(T^*E, y, \alpha) \supseteq S\ell(T^*E, y', \alpha')$ with diameter less than ε. Now

$$S\ell(E, Ty, \alpha) = (T^*)^{-1} S\ell(T^*E, y, \alpha) \supseteq (T^*)^{-1} S\ell(T^*E, y', \alpha') = S\ell(E, Ty', \alpha').$$

But also $\|\cdot\|'$-diam $S\ell(E, Ty', \alpha') = \text{diam } S\ell(T^*E, y', \alpha') < \varepsilon$. □

The appropriate generalization of Fréchet intermediate differentiability is as follows. A locally Lipschitz function ψ on a nonempty open subset G of an Asplund generated space $X = \overline{T(Y)}$ is said to be (T, Y)-*Fréchet intermediately differentiable* at $x \in G$ if there exists a continuous linear functional ξ on X and given $\varepsilon > 0$ there exists $0 < \delta(\varepsilon) < \varepsilon$ such that

$$\left| \frac{\psi(x + \lambda Ty) - \psi(x)}{\lambda} - \xi(Ty) \right| < \varepsilon \quad \text{for all } \frac{\delta}{2} < \lambda < \delta \text{ and } y \in X, \|y\| = 1.$$

For the study of the differentiability of a locally Lipschitz function ψ on a nonempty open subset G of a Banach space X an essential tool is the *Clarke subdifferential* of ψ at $x \in G$, which is the nonempty weak* compact convex set

$$\partial\psi(x) \equiv \left\{ \xi \in X^* \mid \xi(y) \le \limsup_{\substack{\lambda \to 0^+ \\ z \to x}} \frac{\psi(z + \lambda y) - \psi(z)}{\lambda} \quad \text{for all } y \in X \right\}.$$

Our principal tool is the Fabian–Preiss lemma developed to establish residual intermediate differentiability for Lipschitz functions on subspaces of Asplund generated spaces [2]. Fabian *et al.* [1] showed that several differentiability properties derived from the Preiss theorem could in fact be deduced from the simpler Fabian–Preiss lemma.

LEMMA 2 [2, p. 375]. *Consider a Lipschitz 1 function ψ on a nonempty open subset G of an Asplund generated space $X = \overline{T(Y)}$. If for $e \in X$ and $\alpha > 0$ the weak* slice $S\ell(\partial\psi(G), e, \alpha)$ has $\|\cdot\|'$-diameter less than $d > 0$, then there exist $x \in G$, $\xi \in \partial\psi(x) \cap S\ell(\partial\psi(G), e, \alpha)$ and $\delta > 0$ such that*

$$|\psi(x + \lambda Ty) - \psi(x) - \xi(\lambda Ty)| \le 3\lambda d \quad \textit{for } 0 < \lambda < \delta \textit{ and all } y \in X, \|y\| = 1.$$

THEOREM 3. *A locally Lipschitz function ψ on a nonempty open subset G of an Asplund generated space $X = \overline{T(Y)}$ is (T, Y)-Fréchet intermediately differentiable at the points of a dense subset of G.*

PROOF. We may assume that ψ is Lipschitz on G with Lipschitz constant 1. Consider $x_0 \in G$ and $0 < r_0 < 1$ such that $B[x_0; r_0] \subset G$. Since $x = \overline{T(Y)}$ is Asplund generated there exists a weak* slice $S\ell(\partial\psi(B(x_0; r_0)), Te_0, \alpha_0)$ where $e_0 \in Y$ and $\alpha_0 > 0$ with $\|\cdot\|'$-diameter less than 1. By the Fabian–Preiss lemma, there exist $x_1 \in B(x_0; r_0)$, $\xi \in \partial\psi(x_1) \cap S\ell(\partial\psi(B(x_0; r_0)), Te_0, \alpha_0)$ and $0 < \delta_0 < 1$ such that

$$|\psi(x_1 + \lambda Ty) - \psi(x_1) - \xi_1(\lambda Ty)| \le 3\lambda \quad \text{for } 0 < \lambda < \delta_1 \text{ and all } y \in Y, \|y\| = 1.$$

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Choose $0 < r_1 < \delta_1$ and such that $B(x_1; r_1) \subset B(x_0; r_0)$. Since $X = \overline{T(Y)}$ is Asplund generated there exists a weak* slice

$$S\ell(\partial\psi(B(x_1; r_1)), Te_1, \alpha_1) \subset S\ell(\partial\psi(B(x_0); r_0), Te_0, \alpha_0)$$

where $e_1 \in Y$ and $\alpha_1 > 0$ with $\|\cdot\|'$-diameter less than $\frac{1}{2}$.

At the nth stage of the induction we have for each $k \in \{1, 2, \ldots, n\}$, $x_k \in B(x_0; r_0)$ and $\xi_k \in \partial\psi(x_k)$ and there exists $0 < \delta_k < 1/k$ such that

$$|\psi(x_k + \lambda Ty) - \psi(x_k) - \xi_k(\lambda Ty)| \leq \frac{3\lambda}{k} \quad \text{for } 0 < \lambda < \delta_k \text{ and all } y \in Y, \ \|y\| = 1.$$

We have a nested sequence of balls where we have chosen $0 < r_k < \delta_k/k$ and such that $B(x_k; r_k) \subset B(x_{k-1}; r_{k-1})$ and a nested sequence of weak* slices

$$S\ell(\partial\psi(B(x_k; r_k)), Te_k, \alpha_k) \subset S\ell(\partial\psi(B(x_{k-1}; r_{k-1})), Te_{k-1}, \alpha_{k-1})$$

where $e_k \in Y$ and $\alpha_k > 0$ with $\|\cdot\|'$-diameter less than $1/(k+1)$.

By the Fabian–Preiss lemma there exist $x_{n+1} \in B(x_n; r_n)$,

$$\xi_{n+1} \in \partial\psi(x_{n+1}) \cap S\ell(\partial\psi(B(x_n; r_n)), Te_n, \alpha_0) \quad \text{and} \quad 0 < \delta_{n+1} < \frac{1}{n+1}$$

such that

$$|\psi(x_{n+1} + \lambda Ty) - \psi(x_{n+1}) - \xi_{n+1}(\lambda Ty)| \leq \frac{3\lambda}{n+1}$$

for $0 < \lambda < \delta_{n+1}$ and all $y \in Y$, $\|y\| = 1$.
Choose $0 < r_{n+1} < \delta_{n+1}/(n+1)$ and such that $B(x_{n+1}; r_{n+1}) \subset B(x_n; r_n)$. Since $X = \overline{T(Y)}$ is Asplund generated there exists a weak* slice

$$S\ell(\partial\psi(B(x_{n+1}; r_{n+1})), Te_{n+1}, \alpha_{n+1}) \subset S\ell(\partial\psi(B(x_n; r_n)), Te_n, \alpha_n)$$

where $e_{n+1} \in Y$ and $\alpha_{n+1} > 0$ with $\|\cdot\|'$-diameter less than $1/(n+2)$.

Now there exist $x \in G$ and $\xi \in X^*$ where, for each $n \in \mathbb{N}$, $\|x_n - x\| \leq r_n < \delta_n/n$ and $\|\xi - \xi_n\|' \leq 1/(n+1)$. So for $\lambda > 0$,

$$\begin{aligned}
&\left|\frac{\psi(x + \lambda Ty) - \psi(x)}{\lambda} - \xi(Ty)\right| \\
&\quad \leq \frac{2\|x_n - x\|}{\lambda} + \left|\frac{\psi(x_n + \lambda Ty) - \psi(x_n)}{\lambda} - \xi_n(Ty)\right| + \|\xi_n - \xi\|' \\
&\quad \leq 2\frac{\delta_n}{n}\frac{2}{\delta_n} + \frac{3}{n} + \frac{1}{n} = \frac{8}{n} \quad \text{when } \frac{\delta_n}{2} < \lambda < \delta_n \text{ and all } y \in Y, \ \|y\| = 1.
\end{aligned}$$

Then we conclude that every open subset G of $X = \overline{T(Y)}$ contains a point x where ψ is (T, Y)-Fréchet intermediately differentiable. □

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While Theorem 3 establishes the density of Fréchet intermediate differentiability, the proof suggests that we can show that certain Lipschitz functions are residually Fréchet intermediately differentiable.

Given a locally Lipschitz function ψ on a nonempty open subset G of a Banach space X, its subdifferential mapping $x \mapsto \partial\psi(x)$ is said to be *quasi weak* hyperplane lower semicontinuous* at $x_0 \in G$ if for any weak* open half space W of X^* where $\partial\psi(x_0) \cap W \neq \emptyset$ there exists a nonempty open subset U of G such that $x_0 \in \overline{U}$ and $\partial\psi(x) \cap W \neq \emptyset$ for each $x \in U$. A continuous convex function ϕ on a nonempty open convex subset G of X always has its subdifferential mapping $x \mapsto \partial\phi(x)$ quasi weak* hyperplane lower semicontinuous on G. However, not all Lipschitz functions have this property, [3, p. 357].

THEOREM 4. *A locally Lipschitz function ψ on a nonempty open subset G of an Asplund generated space $X = \overline{T(Y)}$ with subdifferential mapping $x \mapsto \partial\psi(x)$ quasi weak* hyperplane lower semicontinuous on G is (T, Y)-Fréchet intermediately differentiable at the points of a residual subset of G.*

PROOF. We may assume that ψ is Lipschitz on G with Lipschitz constant 1 and that the subdifferential mapping $x \mapsto \partial\psi(x)$ is bounded on G. We play a Banach–Mazur game between two players A and B on G. A *play* is a decreasing sequence of nonempty open subsets of G, $U_1 \supset V_1 \supset U_2 \supset V_2 \supset \cdots$ chosen alternately by players A and B with player A making the first move U_1. We show that no matter how player A moves, player B can move with a strategy to ensure that $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$ and consists of points where ψ is (T, Y)-Fréchet intermediately differentiable. By the Banach–Mazur Theorem [7, p. 69], this will imply that the set of points where ψ is (T, Y)-Fréchet intermediately differentiable is residual in G.

Player A begins by choosing a nonempty open subset U_1 of G. Since $X = \overline{T(Y)}$ is an Asplund generated space there exists a weak* slice $S\ell(\partial\psi(U_1), Te_1, \alpha_1)$ where $e_1 \in Y$ and $\alpha_1 > 0$ with $\|\cdot\|'$-diameter less than 1. By the Fabian–Preiss lemma there exist $x_1 \in U_1$, $\xi_1 \in \partial\psi(x_1) \cap S\ell(\partial\psi(U_1), Te_1, \alpha_1)$ and $0 < \delta_1 < 1$ such that

$$|\psi(x_1 + \lambda Ty) - \psi(x_1) - \xi_1(\lambda Ty)| \leq 3\lambda \quad \text{for } 0 < \lambda < \delta_1 \text{ and all } y \in Y, \|y\| = 1.$$

Since the subdifferential mapping $x \mapsto \partial\psi(x)$ is quasi weak* hyperplane lower semicontinuous at x_1, player B chooses $0 < r_1 < \delta_1$ and $V_1 \subset B(x_1; r_1) \subset B[x_1; r_1] \subset U_1$ such that $x_1 \in \overline{V_1}$ and $\partial\psi(x) \cap S\ell(\partial\psi(U_1), Te_1, \alpha_1) \neq \emptyset$ for each $x \in V_1$.

At the nth stage of play we assume that $U_1 \supset V_1 \supset U_2 \supset V_2 \supset \cdots \supset V_{n-1}$ and player A chooses $U_n \subset V_{n-1}$. Since $X = \overline{T(Y)}$ is Asplund generated there exists a weak* slice

$$S\ell(\partial\psi(U_n), Te_n, \alpha_n) \subset S\ell(\partial\psi(U_{n-1}), Te_{n-1}, \alpha_{n-1})$$

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where $e_n, e_{n-1} \in Y$ and $\alpha_n, \alpha_{n-1} > 0$ with $\|\cdot\|'$-diameter less than $1/n$. By the Fabian–Preiss lemma there exist $x_n \in U_n$, $\xi_n \in \partial\psi(x_n) \cap S\ell(\partial\psi(x_n), Te_n, \alpha_n)$ and $0 < \delta_n < 1/n$ such that

$$|\psi(x_n + \lambda Ty) - \psi(x_n) - \xi_n(\lambda Ty)| \le \frac{3\lambda}{n} \quad \text{for all } 0 < \lambda < \delta_n \text{ and all } y \in Y,\ \|y\| = 1.$$

Since the subdifferential mapping $x \mapsto \partial\psi(x)$ is quasi weak* hyperplane lower semicontinuous at x_n, player B chooses $0 < r_n < \delta_n/n$ and $V_n \subset B(x_n; r_n) \subset B[x_n; r_n] \subset U_n$ such that $x_n \in \overline{V_n}$ and $\partial\psi(x) \cap S\ell(\partial\psi(U_n), Te_n, \alpha_n) \ne \emptyset$ for each $x \in V_n$.

Clearly the sequence $\{x_n\}$ is norm convergent to $x \in \bigcap_{n=1}^{\infty} \overline{V_n} = \bigcap_{n=1}^{\infty} V_n$ and the sequence $\{\xi_n\}$ is $\|\cdot\|'$-convergent to $\xi \in X^*$ where

$$\xi \in \bigcap_{n=1}^{\infty} \overline{S\ell(\partial\psi(U_n), Te_n, \alpha_n)}.$$

We show that ψ is Fréchet intermediately differentiable at x. Given $n \in \mathbb{N}$, we have $\|x_n - x\| \le r_n < \delta_n/n$ and $\|\xi_n - \xi\| \le 1/n$. For $\lambda > 0$,

$$\begin{aligned} &\left|\frac{\psi(x + \lambda Ty) - \psi(x)}{\lambda} - \xi(Ty)\right| \\ &\quad \le \frac{2\|x_n - x\|}{\lambda} + \left|\frac{\psi(x_n + \lambda Ty) - \psi(x_n)}{\lambda} - \xi_n(Ty)\right| + \|\xi_n - \xi\|' \\ &\quad \le 2\frac{\delta_n}{n}\frac{2}{\delta_n} + \frac{3}{n} + \frac{1}{n} = \frac{8}{n} \quad \text{when } \frac{\delta_n}{2} < \lambda < \delta_n \text{ and all } y \in Y,\ \|y\| = 1. \end{aligned}$$

It follows from the Banach–Mazur theorem that ψ is Fréchet intermediately differentiable at the points of a residual subset of G. □

In the study of the differentiability of Lipschitz functions, Fréchet intermediate differentiability appears to be an interesting generalization of Fréchet differentiability. However, there are serious restrictions on its usefulness, [3, p. 357]. A Lipschitz function intermediately differentiable at a point does not necessarily have a unique Fréchet intermediate subgradient at that point. Of course, if the function is also Gâteaux differentiable at the point then its Fréchet intermediate derivative is unique; but uniqueness of the Fréchet intermediate derivative does not necessarily imply that the Lipschitz function is Gâteaux differentiable at the point. If the Lipschitz function is Gâteaux differentiable at a point and Fréchet intermediately differentiable at the point then it is not necessarily Fréchet differentiable at the point.

A major problem remaining is to determine whether the set of points of Fréchet intermediate differentiability of a Lipschitz function on an Asplund space is residual in general. Further to this problem we should note that there exists a Lipschitz function on an Asplund space where the set of points where it is Fréchet intermediately differentiable but not Fréchet differentiable is residual [3, p. 358].

But we should also note the relation of Fréchet intermediate differentiability to a weaker intermediate differentiability property.

A locally Lipschitz function ψ on a nonempty open subset G of a Banach space X is said to be *uniformly intermediately differentiable* at $x \in G$ if given $n \in \mathbb{N}$ there exist $0 < \delta_n < 1/n$ and $\xi_n \in X^*$ such that

$$\left| \frac{\psi(x+\lambda y) - \psi(x)}{\lambda} - \xi_n(y) \right| < \frac{1}{n} \quad \text{for all } \frac{\delta_n}{2} < \lambda < \delta_n \text{ and all } y \in X, \ \|y\| = 1.$$

If X is an Asplund space and $\{\xi_n\}$ is bounded then $\{\xi_n\}$ has a subsequence $\{\xi_{n_k}\}$ which is weak* convergent to $\xi \in X^*$. Then for any $\delta_{n_k}/2 < \lambda < \delta_{n_k}$,

$$\xi(y) = \lim_{k \to \infty} \frac{\psi(x+\lambda_k y) - \psi(x)}{\lambda_k} \quad \text{for all } y \in X, \ \|y\| = 1.$$

Giles and Sciffer [4, p. 840] prove that on an Asplund space X, ψ is uniformly intermediately differentiable at the points of a residual subset of G. The proof in that paper used the Preiss theorem, but Fabian *et al.* [1, p. 700] proved it using the Fabian–Preiss lemma.

It is clear that on a finite-dimensional space, ψ is Fréchet intermediately differentiable if it is uniformly intermediately differentiable. So we have that on a finite-dimensional space ψ is Fréchet intermediately differentiable at the points of a residual subset of G. However, in general Fréchet intermediate differentiability and uniform intermediate differentiability are not the same, as is shown by the following example due to Scott Sciffer.

EXAMPLE. Consider the function $f : [0, \infty) \mapsto \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2}, \\ 2x - 1 & \text{for } \frac{1}{2} \leq x \leq 1, \\ 2 - x & \text{for } 1 < x \leq 2, \\ 0 & \text{for } x > 2 \end{cases}$$

and the function $\rho : (0, 1] \to \ell_2^*$ defined by

$$\rho(r) = \sum_{n=1}^{\infty} f(2^{n-1} r) e_n^*.$$

Given $n \in \mathbb{N}$ and $1/2^n < r < 1/2^{n-1}$, we obtain $\frac{1}{2} < 2^{n-1} r < 1$ and $1 < 2^n r < 2$, and writing ρ_n for ρ restricted to $(1/2^n, 1/2^{n-1})$

$$\rho(r) = \sum_{n=1}^{\infty} \rho_n(r) \quad \text{where } \rho_n(r) = (2^n r - 1) e_n^* + (2 - 2^n r) e_{n+1}^*.$$

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Define $\psi : \ell_2 \to \mathbb{R}$ by

$$\psi(x) = \begin{cases} \rho(\|x\|)(x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

(i) Now ψ is Lipschitz since, for $x = \sum_{k=1}^{\infty} \alpha_k e_k$ and $y = \sum_{k=1}^{\infty} \beta_k e_k$,

$$\begin{aligned} |\psi(x) - \psi(y)| &= |\rho(\|x\|)(x) - \rho(\|y\|)(y)| \\ &\leq \sum_{k=1}^{\infty} |\alpha_k - \beta_k| \leq \sqrt{2} \sqrt{\sum_{k=1}^{\infty} (\alpha_k - \beta_k)^2} = \sqrt{2}\|x - y\|. \end{aligned}$$

(ii) Also ψ is Gâteaux differentiable at 0 since, for $\lambda > 0$,

$$\frac{\psi(0 + \lambda y) - \psi(0)}{\lambda} = \frac{\psi(\lambda y)}{\lambda} = \rho(|\lambda|)(y) \quad \text{for } \|y\| = 1,$$

so when $1/2^n < \lambda < 1/2^{n-1}$ then

$$\rho(|\lambda|)(y) = \beta_n(2^n\lambda - 1) + \beta_{n+1}(2 - 2^n\lambda)$$

and as $\lambda \to 0$ then $n \to \infty$ and $\beta_n, \beta_{n+1} \to 0$; we have Gâteaux derivative 0.

(iii) We have ψ is uniformly intermediately differentiable at 0 since

$$\left| \frac{\psi(0 + \lambda y) - \psi(0)}{\lambda} - \rho_n(\lambda)(y) \right| = 0 \quad \text{for } \frac{1}{2^n} < \lambda < \frac{1}{2^{n-1}} \text{ and all } y \in \ell_2, \ \|y\| = 1.$$

(iv) But ψ is not Fréchet intermediately differentiable at 0, since although ρ_n converges weak* to 0, we have

$$\|\rho_n(\lambda)\| = \sqrt{(2^n\lambda - 1)^2 + (2 - 2^n\lambda)^2} \geq \frac{1}{\sqrt{2}} \quad \text{for all } n \in \mathbb{N}.$$ □

The proof strategy of Theorems 3 and 4 is based on the Fabian–Preiss lemma. The strategy of Lindenstrauss and Preiss for proving dense Fréchet differentiability of Lipschitz functions on an Asplund space is based on a similar but stronger property.

THEOREM 5 [5, p. 212]. *Given a Lipschitz function ψ on a nonempty open subset G of an Asplund space X, if there exists a sequence $\{x_k\}$ convergent to x in G where for each $j \in \mathbb{N}$ there are numbers $\omega_j > 0$ and $\delta_j > 0$ where $\omega_j \to 0$ and $\xi_j \in \partial\psi(x_j)$, and, for every $k > j$,*

$$|\psi(x_k + h) - \psi(x_k) - \xi_j(h)| \leq \omega_j \|h\| \quad \textit{for all } h \in X, \ \|h\| < \delta_j,$$

then ψ is Fréchet differentiable at x.

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PROOF. For j_1 and j_2 we have $\|\xi_{j_1} - \xi_{j_2}\| \le \omega_{j_1} + \omega_{j_2}$ so $\{\xi_j\}$ is convergent to some $\xi \in X^*$ and $\|\xi - \xi_j\| \le \omega_j$. Then

$$|\psi(x_k + h) - \psi(x_k) - \xi(h)| \le 2\omega_j \|h\| \quad \text{for all } h \in X,\ \|h\| < \delta_j \text{ and } k > j.$$

Keeping j fixed and letting $k \to \infty$,

$$|\psi(x + h) - \psi(x) - \xi(h)| \le 2\omega_j \|h\| \quad \text{for all } h \in X,\ \|h\| < \delta_j.$$

So ψ is Fréchet differentiable at x and $\xi = \psi'(x)$. □

The technicality of their result lies in constructing a suitable sequence $\{x_k\}$ in G.

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