Adv. Appl. Prob. (SGSA) **37,** 884–907 (2005)
Printed in Northern Ireland

RANDOM DYNAMICS AND THERMODYNAMIC LIMITS FOR POLYGONAL MARKOV FIELDS IN THE PLANE

TOMASZ SCHREIBER,* *Nicolaus Copernicus University*

Abstract

We construct random dynamics for collections of nonintersecting planar contours, leaving invariant the distributions of length- and area-interacting polygonal Markov fields with V-shaped nodes. The first of these dynamics is based on the dynamic construction of consistent polygonal fields, as presented in the original articles by Arak (1983) and Arak and Surgailis (1989), (1991), and it provides an easy-to-implement Metropolis-type simulation algorithm. The second dynamics leads to a graphical construction in the spirit of Fernández *et al*. (1998), (2002) and yields a perfect simulation scheme in a finite window in the infinite-volume limit. This algorithm seems difficult to implement, yet its value lies in that it allows for theoretical analysis of the thermodynamic limit behaviour of length-interacting polygonal fields. The results thus obtained include, in the class of infinite-volume Gibbs measures without infinite contours, the uniqueness and exponential α-mixing of the thermodynamic limit of such fields in the low-temperature region. Outside this class, we conjecture the existence of an infinite number of extreme phases breaking both the translational and rotational symmetries.

Keywords: Polygonal Markov field; random dynamics; Metropolis simulation; perfect simulation; thermodynamic limit; phase transition

2000 Mathematics Subject Classification: Primary 60D05; 60K35; 82B21

1. Introduction

An example of a planar Markov field with polygonal realisations was first introduced in Arak (1983). The original Arak process in a bounded open convex set D is constructed as briefly sketched below. We define the family Γ_D of admissible polygonal configurations on D by taking all the finite planar graphs γ in $D \cup \partial D$, with straight line-segments as edges, such that

(P1) the edges of γ do not intersect,

(P2) all the interior vertices of γ (lying in D) are of degree 2,

(P3) all the boundary vertices of γ (lying in ∂D) are of degree 1, and

(P4) no two edges of γ are collinear.

In other words, γ consists of a finite number of disjoint polygons, possibly nested and truncated by the boundary. Furthermore, for a finite collection $(l) = (l_i)_{i=1}^{n}$ of straight lines intersecting

Received 17 August 2004; revision received 3 July 2005.
* Postal address: Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Toruń, 87-100, Poland. Email address: tomeks@mat.uni.torun.pl

https://doi.org/10.1239/aap/1134587745 Published online by Cambridge University Press

D, we write $\Gamma_D(l)$ for the family of admissible configurations γ with the additional properties that $\gamma \subseteq \bigcup_{i=1}^n l_i$ and $\gamma \cap l_i$ is a single interval of strictly positive length for each l_i, $i = 1, \dots, n$, possibly with some isolated points added.

Let Λ_D be the restriction to D of a homogeneous Poisson line process Λ with intensity measure given by the standard isometry-invariant Lebesgue measure μ on the space of straight lines in $\mathbb{R}^2$. One possible construction of μ is to identify a straight line l with the pair $(\phi, \rho) \in [0, \pi) \times \mathbb{R}$ (where $(\rho \sin(\phi), \rho \cos(\phi))$ is the vector orthogonal to l), join it to the origin, and then endow the parameter space $[0, \pi) \times \mathbb{R}$ with the usual Lebesgue measure. In the above notation, the polygonal Arak process $\mathcal{A}_D$ on D arises as the Gibbsian modification of the process induced on Γ_D by Λ_D, with Hamiltonian given by twice the total edge length, that is to say

$$\mathrm{P}(\mathcal{A}_D \in G) = \frac{\mathrm{E} \sum_{\gamma \in \Gamma_D(\Lambda_D) \cap G} \exp(-2 \operatorname{length}(\gamma))}{\mathrm{E} \sum_{\gamma \in \Gamma_D(\Lambda_D)} \exp(-2 \operatorname{length}(\gamma))} \tag{1}$$

for all sets $G \subseteq \Gamma_D$ Borel measurable with respect to, say, the usual Hausdorff distance topology; see Section 4 of Arak and Surgailis (1989). Recall that the Hausdorff distance between two compact sets C_1 and C_2 is given by

$$\varrho_{\mathrm{H}}(C_1, C_2) := \max\Big(\max_{x \in C_1} \operatorname{dist}(x, C_2), \max_{y \in C_2} \operatorname{dist}(y, C_1) \Big),$$

where $\operatorname{dist}(x, C) := \inf_{y \in C} \operatorname{dist}(x, y)$. The Arak process has a number of remarkable properties. It is exactly solvable (an explicit formula for the partition function is available), consistent ($\mathcal{A}_D$ coincides in distribution with the restriction of $\mathcal{A}_C$ to D, for $C \supseteq D$), and has a two-dimensional Markov property stating that the conditional behaviour of the process in an open bounded domain depends on the exterior configuration only through arbitrarily close neighbourhoods of the boundary; again see Section 4 of Arak and Surgailis (1989).

These nice features are shared by a much broader class of processes called consistent polygonal Markov fields, which were introduced and investigated in detail in Arak and Surgailis (1989), (1991). Arak *et al.* (1993) introduced an alternative, point-based, rather than line-based, representation of these models. Our description below specialises the standard Arak process $\mathcal{A}_D$.

For a given point configuration $\bar{x} = \{x_1, \dots, x_n\} \subseteq D \cup \partial D$, denote by $\Gamma_D(\bar{x})$ the family of admissible configurations γ whose vertex sets coincide with $\bar{x}$. Write Π_D for the Poisson point process in $D \cup \partial D$ with intensity measure given by the area element on D and by the length element on ∂D. By Theorem 1 of Arak *et al.* (1993) (see also Equation (2.6) thereof) the Arak process $\mathcal{A}_D$ coincides with the Gibbsian modification of the process on Γ_D induced by Π_D, with Hamiltonian

$$\Phi(\gamma) := 2 \operatorname{length}(\gamma) + \sum_{e \in \operatorname{Edges}(\gamma)} \log \operatorname{length}(e) - \sum_{x \in \operatorname{Vertices}(\gamma)} \log |\sin \phi_x|, \tag{2}$$

where $\operatorname{Edges}(\gamma)$ and $\operatorname{Vertices}(\gamma)$ are respectively the edge and vertex sets of γ while ϕ_x stands for the angle between the edges meeting at x, if $x \in D$, and for the angle between the edge and the tangent to ∂D at x, if $x \in \partial D$. This means that

$$\mathrm{P}(\mathcal{A}_D \in G) = \frac{\mathrm{E} \sum_{\gamma \in \Gamma_D(\Pi_D) \cap G} \exp(-\Phi(\gamma))}{\mathrm{E} \sum_{\gamma \in \Gamma_D(\Pi_D)} \exp(-\Phi(\gamma))} \tag{3}$$

for all Borel sets $G \subseteq \Gamma_D$.

The third equivalent description of polygonal Markov fields is available in terms of the equilibrium evolution of one-dimensional particle systems, tracing the polygonal realisations of the process in two-dimensional time–space. This description, usually referred to as the dynamic representation and introduced in the original Arak work of 1983, turned out to be very useful in establishing the essential properties of the models. Below, we discuss the dynamic representation for the Arak process; see Section 4 of Arak and Surgailis (1989). We interpret the open convex domain D as a set of *time–space* points $(t, y) \in D$, with t referred to as the *time* coordinate and y the *spatial* coordinate of a particle at time t. In this language, a straight line-segment in D represents a piece of the time–space trajectory of a freely moving particle. For a straight line l that is not parallel to the time axis and crosses the domain D, we define in the obvious way its entry and exit points to and from D, respectively $\mathrm{in}(l, D) \in \partial D$ and $\mathrm{out}(l, D) \in \partial D$.

We choose the time–space birth coordinates for the new particles according to the superposition of a homogeneous Poisson point process in D (interior birth sites) with intensity π and a Poisson point process on the boundary (boundary birth sites) with intensity measure

$$\kappa(B) = \mathrm{E}\,\mathrm{card}\{l \in \Lambda\colon\ \mathrm{in}(l, D) \in B\}, \qquad B \subseteq \partial D. \tag{4}$$

Each interior birth site emits two particles, which move with initial velocities v' and v'' chosen according to the joint distribution

$$\theta(\mathrm{d}v', \mathrm{d}v'') = \pi^{-1}|v' - v''|(1 + v'^2)^{-3/2}(1 + v''^2)^{-3/2}\,\mathrm{d}v'\,\mathrm{d}v''. \tag{5}$$

This can be shown to be equivalent to choosing the directions of the straight lines representing the time–space trajectories of the emitted particles according to the distribution of the *typical angle* between two lines of Λ; see Sections 3 and 4 of Arak and Surgailis (1989) and the references therein. Each boundary birth site $x \in \partial D$ yields one particle, with initial speed v determined according to the distribution $\theta_x(\mathrm{d}v)$, identified by requiring that the direction of the line entering D at x and representing the time–space trajectory of the emitted particle be chosen according to the distribution of a straight line $l \in \Lambda$ conditioned on the event $\{x = \mathrm{in}(l, D)\}$.

All the particles evolve independently in time according to the following rules.

Rule 1. *Between the critical moments listed below, each particle moves freely with constant velocity, meaning that* $\mathrm{d}y = v\,\mathrm{d}t$.

Rule 2. *When a particle touches the boundary* ∂D, *it dies.*

Rule 3. *In case of a collision of two particles (i.e. they have identical spatial coordinates* y *at some moment* t, *with* $(t, y) \in D$*), both of them die.*

Rule 4. *The time evolution of the velocity* v_t *of an individual particle is given by a pure-jump Markov process, meaning that*

$$\mathrm{P}(v_{t+\mathrm{d}t} \in \mathrm{d}u \mid v_t = v) = q(v, \mathrm{d}u)\,\mathrm{d}t$$

with transition kernel

$$q(v, \mathrm{d}u) := |u - v|(1 + u^2)^{-3/2}\,\mathrm{d}u.$$

It has been proved (see, e.g. Lemma 4.1 of Arak and Surgailis (1989)) that, with the above construction of the interacting particle system, the time–space trajectories traced by the evolving particles coincide in distribution with the Arak process $\mathcal{A}_D$. Moreover, a much broader class of consistent polygonal Markov fields admit analogous dynamic representations, possibly enhanced to allow vertices of higher degree (namely 3 and 4); again see Lemma 4.1 of Arak and Surgailis (1989). The problem of characterising the class of all polygonal Markov fields admitting dynamic representation is far from trivial; a description of this class was conjectured in Arak *et al*. (1993).

The above dynamic construction of the Arak process makes it very suitable for simulation. However, in this paper we focus our interest on the family of processes $\hat{\mathcal{A}}_D^{[\alpha,\beta]}$, $\alpha, \beta \in \mathbb{R}$, arising as the Ising-like length- and area-interacting Gibbsian modifications of $\mathcal{A}_D$. To this end we *colour* the original Arak process $\mathcal{A}_D$ as follows. We require that the polygonal contours of $\mathcal{A}_D$ stand for interfaces between black- and white-coloured regions in D, which leaves us almost surely with two possible ways of colouring D in black and white. These can be interchanged by a simple colour flip. We choose one of these colourings at random, with probability $\frac{1}{2}$, thus obtaining a coloured version of $\mathcal{A}_D$, denoted in the sequel by $\hat{\mathcal{A}}_D$.

The family of all admissible coloured polygonal configurations in D, carrying information not only about the planar contours it consists of, but also about the associated colourings, will be denoted by $\hat{\Gamma}_D$. With this notation and terminology, we define the (coloured) processes $\hat{\mathcal{A}}_D^{[\alpha,\beta]}$ by

$$\frac{\mathrm{d}\mathcal{L}(\hat{\mathcal{A}}_D^{[\alpha,\beta]})}{\mathrm{d}\mathcal{L}(\hat{\mathcal{A}}_D)}[\hat{\gamma}] = \frac{\exp(-\mathcal{H}_D^{[\alpha,\beta]}(\hat{\gamma}))}{\mathrm{E}\exp(-\mathcal{H}_D^{[\alpha,\beta]}(\hat{\mathcal{A}}_D))}, \qquad \hat{\gamma} \in \hat{\Gamma}_D, \tag{6}$$

with $\mathcal{L}(\cdot)$ denoting the law of the argument random object and

$$\mathcal{H}_D^{[\alpha,\beta]}(\hat{\gamma}) = \alpha A(\mathrm{black}[\hat{\gamma}]) + \beta\,\mathrm{length}(\hat{\gamma}), \tag{7}$$

where black$[\hat{\gamma}]$ is the black-coloured region in D for $\hat{\gamma}$ and $A(\cdot)$ stands for the area measure. Also, we write $\mathcal{A}_D^{[\alpha,\beta]}$ for the contour ensemble of $\hat{\mathcal{A}}_D^{[\alpha,\beta]}$, with the colours 'forgotten', and, likewise, write γ for the colourless version of $\hat{\gamma} \in \hat{\Gamma}_D$. Note that, by using the symmetry between black and white and possibly flipping the colours, whenever convenient we may assume without loss of generality that $\alpha \geq 0$ (and we do so in the proof of Theorem 3, below).

Observe that modifications of the type (6) only fall into the general setting considered by Arak and Surgailis (1989) for $\beta \geq 0$; see Corollary 4.1 thereof. However, we find it natural to admit negative βs also, since there is no obvious *infinite-temperature* noninteracting field available as the reference object for polygonal Markov fields. Consequently, in the sequel we will abuse the language by referring to large positive values of β as the low-temperature region, and to small, possibly negative values of β as the high-temperature regime. For $\beta < 0$, we must check that the partition function

$$\mathrm{E}\exp(-\mathcal{H}_D^{[\alpha,\beta]}(\hat{\mathcal{A}}_D))$$

is finite. In Corollary 2, we show that this is indeed the case and, consequently, that the definition (6) is correct for all $\beta \in \mathbb{R}$. Clearly there are no such problems for α, since the overall black or white area is deterministically bounded by $A(D)$. It should be emphasised, however, that we are at present able to establish the existence of the thermodynamic limit only for $\beta > 0$; see Theorem 3.

Models of type (6) have recently found interest in the physics literature; see Nicholls (2001). In particular, it has been argued that they exhibit a phase transition similar to that of the planar Ising model, with the low-temperature phase admitting only finite contour nesting (as rigorously shown in Nicholls (2001)) and with the high-temperature phase conjectured (although not yet proved) to exhibit infinite contour nesting.

Below, we shall also consider versions of the above models with empty boundary conditions, which arise when we condition the original model in the event of there being no vertices on the boundary, in which case we define

$$\mathcal{L}(\hat{\mathcal{A}}^{[\alpha,\beta]}_{D\,|\,\varnothing}) = \mathcal{L}(\hat{\mathcal{A}}^{[\alpha,\beta]}_D \mid \mathcal{A}^{[\alpha,\beta]}_D \cap \partial D = \varnothing). \tag{8}$$

In particular,

$$\hat{\mathcal{A}}_{D\,|\,\varnothing} := \hat{\mathcal{A}}^{[0,0]}_{D\,|\,\varnothing}.$$

Likewise, we shall consider versions of these models with black (or white) boundary conditions, given by

$$\mathcal{L}(\hat{\mathcal{A}}^{[\alpha,\beta]}_{D\,|\,\mathrm{bd}}) := \mathcal{L}(\hat{\mathcal{A}}^{[\alpha,\beta]}_D \mid \mathcal{A}^{[\alpha,\beta]}_D \cap \partial D = \varnothing,\ \partial D \text{ is bd}), \qquad \mathrm{bd} \in \{\text{black, white}\},$$

with

$$\hat{\mathcal{A}}_{D\,|\,\mathrm{bd}} := \hat{\mathcal{A}}^{[0,0]}_{D\,|\,\mathrm{bd}}, \qquad \mathrm{bd} \in \{\text{black, white}\}.$$

As a direct result of (6), we obtain

$$\frac{\mathrm{d}\mathcal{L}(\hat{\mathcal{A}}^{[\alpha,\beta]}_{D\,|\,\mathrm{bd}})}{\mathrm{d}\mathcal{L}(\hat{\mathcal{A}}_{D\,|\,\mathrm{bd}})}[\hat{\gamma}] = \frac{\exp(-\mathcal{H}^{[\alpha,\beta]}_D(\hat{\gamma}))}{\mathrm{E}\exp(-\mathcal{H}^{[\alpha,\beta]}_D(\hat{\mathcal{A}}_{D\,|\,\mathrm{bd}}))}, \qquad \hat{\gamma} \in \hat{\Gamma}_D,\ \gamma \cap \partial D = \varnothing, \tag{9}$$

for bd $\in \{\varnothing$, black, white$\}$. Observe that, unlike the unconditioned finite-volume fields $\hat{\mathcal{A}}^{[\alpha,\beta]}_D$, $\alpha \neq 0$, the conditioned fields with monochromatic boundary conditions are well defined also for nonconvex bounded open sets D with piecewise-smooth boundaries. Indeed, take any bounded open convex set D' containing D, and set $\hat{\mathcal{A}}^{[\alpha,\beta]}_{D\,|\,\mathrm{bd}}$, bd $\in \{$black, white$\}$, to coincide with $\hat{\mathcal{A}}^{[\alpha,\beta]}_{D'}$ conditioned on the event that no edge hits ∂D and that the colour of ∂D agrees with that specified by bd. The Markov property of polygonal fields (see Arak and Surgailis (1989)) implies that this construction does not depend on the choice of D'. Note that this argument does not apply for the empty boundary condition, bd $= \varnothing$, unless $\alpha = 0$.

The purpose of this paper is to construct, for $\alpha, \beta \in \mathbb{R}$, a family of random dynamics on $\hat{\Gamma}_D$ that leaves the distribution of $\hat{\mathcal{A}}^{[\alpha,\beta]}_D$ invariant. This yields simulation algorithms for $\hat{\mathcal{A}}^{[\alpha,\beta]}_D$ both of Metropolis type and of 'perfect' type in the spirit of Fernández *et al.* (1998), (2002). While the Metropolis algorithm is given for all $\alpha, \beta \in \mathbb{R}$ and can be readily implemented (which is a subject of the author's work in progress), the perfect scheme is applicable only for $\alpha = 0$ and seems to be more difficult to implement; its value lies mainly in that it provides important theoretical information about the thermodynamic limit behaviour of $\mathcal{A}^{[0,\beta]}$ in the low-temperature region (large β) and in that it can be used to simulate in finite windows directly in the thermodynamic limit. The finite-volume dynamics are discussed in Section 2. In Section 3, we discuss infinite-volume thermodynamic limits of polygonal fields and establish their existence.

For $\alpha = 0$ and β sufficiently large, one of our dynamics, constructed in Subsection 2.2, admits an infinite-volume extension and, as mentioned above, yields a perfect simulation scheme that enables us to show, in Section 4, that for $\mathcal{A}^{[0,\beta]}$ there exists exactly one thermodynamic limit without infinite chains (to be made specific below), and that this limit is isometry invariant as well as exponentially α-mixing. In particular, it follows that the class of infinite-volume measures without infinite chains contains exactly two extremal infinite-volume Gibbs measures for $\hat{\mathcal{A}}^{[0,\beta]}$, corresponding to the same contour distribution: the black-dominated phase and the white-dominated phase. In this context, it should be noted that this simple picture does not seem to extend to the whole simplex of infinite-volume Gibbs measures for $\mathcal{A}^{[0,\beta]}$: we conjecture the existence and sketch, in Section 3, a tentative construction of an infinite number of infinite-volume states admitting infinite chains and breaking both the translational and rotational symmetries.

As mentioned above, the implementation of the algorithms described in this paper is a subject of the author's current work in progress. It should be emphasised that an algorithm for simulating polygonal Markov fields, very different from ours, has already been given in the literature by Clifford and Nicholls (1994).

2. Finite-volume dynamics

Below, we construct two families of random dynamics that leave invariant the laws of the Gibbs-modified polygonal random fields $\hat{\mathcal{A}}_D^{[\alpha,\beta]}$ in a bounded open convex domain $D \subseteq \mathbb{R}^2$. The first of these dynamics, which leads to a practicable and easy-to-implement Metropolis-type simulation algorithm, is based on the dynamic representation of the Arak process. The second one relies mainly on the point- and line-based representation of general polygonal Markov fields and, after some additional work, leads to a graphical construction and a perfect algorithm, discussed in Section 4. We postpone the proof of the finiteness of the partition function in (6) to Corollary 2.

2.1. Disagreement loop birth-and-death dynamics

A concept important below will be that of a *disagreement loop*, borrowed from Section 2.2 of Schreiber (2004). This arises from the dynamic construction of the Arak process as provided by the evolution rules, Rules 1–4, with the corresponding birth rules; see (4) and (5).

Suppose that we observe a particular realisation $\gamma \in \Gamma_D$ of the colourless basic Arak process $\mathcal{A}_D$ and that we modify the configuration by adding an extra birth site x_0 to the existing collection of birth sites for γ, while keeping Rules 1–4 for all the particles, including the two newly added ones, if $x_0 \in D$, and the single newly added one, if $x_0 \in \partial D$. Denote the resulting new random (colourless) polygonal configuration by $\gamma \oplus x_0$. A simple yet crucial observation is that, for $x_0 \in D$, the symmetric difference $\gamma \triangle [\gamma \oplus x_0]$ is almost surely a single loop (a closed polygonal curve), possibly self-intersecting, and possibly truncated by the boundary. This can be seen as follows. The left-most point of the loop $\gamma \triangle [\gamma \oplus x_0]$ is of course x_0.

Each of the two *new* particles emitted from x_0, p_1 and p_2, move independently, according to Rules 1–4, each giving rise to a *disagreement path*. The initial segments of such a disagreement path correspond to the movement of a particle, say p_1, before its annihilation in the first collision. If this is a collision with the boundary, the disagreement path terminates there. If this is a collision with a segment of the original configuration γ corresponding to a certain *old* particle p_3, then the *new* particle p_1 dies but the disagreement path continues along the part of the trajectory of p_3 that is contained in γ but not in $\gamma \oplus x_0$. At some further moment, p_3 itself dies in γ, touching the boundary or killing another particle p_4 in γ. In the second case, however, this collision only happens for γ and not for $\gamma \oplus x_0$, meaning that the particle p_4 survives (for some time) in $\gamma \oplus x_0$, yielding a further connected portion of the disagreement path for p_1, which is contained in $\gamma \oplus x_0$ but not in γ.

A recursive continuation of this construction shows that the disagreement path initiated by p_1 consists of connected polygonal subpaths alternately contained in $[\gamma \oplus x_0] \setminus \gamma$ (call these *positive* parts) and in $\gamma \setminus [\gamma \oplus x_0]$ (call these *negative* parts). Note that this disagreement path is self-avoiding and, in fact, can be represented as the graph of some piecewise-linear function $t \mapsto y(t)$. Clearly, the same applies for the disagreement path initiated by p_2. An important observation is that whenever two positive or two negative segments of the two disagreement paths hit each other, both disagreement paths die at the point of collision and the disagreement loop closes (as opposed to intersections of segments of opposite sign, which do not have this effect). Obviously, if the disagreement loop does not close in the above way, it is eventually truncated by the boundary. We shall write $\Delta^{\oplus}[x_0; \gamma] = \gamma \triangle [\gamma \oplus x_0]$ to denote the (random) disagreement loop constructed above. It remains to consider the case $x_0 \in \partial D$, which is much simpler because there is only one emitted particle and, so, $\Delta^{\oplus}[x_0; \gamma] = \gamma \triangle [\gamma \oplus x_0]$ is a single self-avoiding polygonal path eventually truncated by the boundary. We abuse our notation by calling such a $\Delta^{\oplus}[x_0; \gamma]$ a (degenerate) disagreement loop as well.

Likewise, a disagreement loop arises if we *remove* one birth site x_0 from the collection of birth sites of an admissible polygonal configuration $\gamma \in \Gamma_D$, while keeping the evolution rules for all the remaining particles. We write $\gamma \ominus x_0$ for the configuration obtained from γ by removing x_0 from the list of birth sites, and the resulting random disagreement loop is denoted by $\Delta^{\ominus}[x_0; \gamma] = \gamma \triangle [\gamma \ominus x_0]$.

We point out that a formal proof of the fact that adding or removing a birth site to or from a polygonal configuration always results in a disagreement loop can be provided by noting first that all interior vertices of the planar graphs $\gamma \triangle [\gamma \oplus x_0]$ and $\gamma \triangle [\gamma \ominus x_0]$ are of order 2 and, likewise, all boundary vertices of these graphs are of order 1. This is easily seen by considering the following cases, where $[\gamma \odot x_0]$ denotes $[\gamma \oplus x_0]$ or $[\gamma \ominus x_0]$, as appropriate.

- An internal vertex arises from the added or the removed birth site.
- An internal vertex arises from the velocity update of a particle in γ or $[\gamma \odot x_0]$.
- An internal vertex arises from the collision of two particles in $\gamma \setminus [\gamma \odot x_0]$ or two particles in $[\gamma \odot x_0] \setminus \gamma$.
- An internal vertex arises from the collision of a part of one particle's trajectory absent in γ but present in $[\gamma \odot x_0]$ with another particle present in both γ and $[\gamma \odot x_0]$.
- An internal vertex arises from the collision of a part of one particle's trajectory absent in $[\gamma \odot x_0]$ but present in γ with another particle present in both γ and $[\gamma \odot x_0]$.
- A boundary vertex arises from the added or the removed birth site.
- A boundary vertex arises from the collision of a particle in either $\gamma \setminus [\gamma \odot x_0]$ or $[\gamma \odot x_0] \setminus \gamma$ with the boundary of the domain.

From these observations we conclude that $\gamma \triangle [\gamma \odot x_0]$ is a collection of loops, possibly truncated by the boundary and possibly degenerate to a single polygonal path starting and ending at a boundary point. It remains to show that this collection always consists of exactly one loop. However, this is easily seen by noting that $\gamma \triangle [\gamma \odot x_0]$ has exactly one extreme-left vertex, i.e. a vertex with no left-outgoing segment, which completes our argument.

With the above terminology, we are in a position to describe a random dynamics on the coloured configuration space $\hat{\Gamma}_D$ that leaves invariant the law of the basic Arak process $\hat{\mathcal{A}}_D$. Particular care is needed, however, to distinguish between the notion of time considered both in the dynamic representation of the Arak process and throughout the above construction of the disagreement loops, and the notion of time to be introduced for the random dynamics

https://doi.org/10.1239/aap/1134587745 Published online by Cambridge University Press

on $\hat{\Gamma}_D$, constructed below. To make this distinction clear, we shall refer to the former as the *representation time* (or r-time), t, while the latter will be called the *simulation time* (or s-time), s.

Consider the following rules for pure-jump birth-and-death-type Markovian dynamics on $\hat{\Gamma}_D$.

Rule 5. (DL:birth.) *With intensity* $[\pi \, \mathrm{d}x + \kappa(\mathrm{d}x)] \, \mathrm{d}s$*, with* κ *as in (4), set* $\gamma_{s+\mathrm{d}s} = \gamma_s \oplus x$ *and then construct* $\hat{\gamma}_{s+\mathrm{d}s}$ *by randomly choosing, with probability* $\frac{1}{2}$*, either of the two possible colourings for* $\gamma_{s+\mathrm{d}s}$.

Rule 6. (DL:death.) *For each birth site* x *in* γ_s*, with intensity* 1 *set* $\gamma_{s+\mathrm{d}s} = \gamma_s \ominus x$ *and then construct* $\hat{\gamma}_{s+\mathrm{d}s}$ *by randomly choosing, with probability* $\frac{1}{2}$*, either of the two possible colourings for* $\gamma_{s+\mathrm{d}s}$.

If neither of the above updates occurs we set $\hat{\gamma}_{s+\mathrm{d}s} = \hat{\gamma}_s$. It is convenient to picture the above dynamics as generating random disagreement loops λ and setting $\gamma_{s+\mathrm{d}s} = \gamma_s \triangle \lambda$, with the loops of type $\Delta^{\oplus}[\cdot;\cdot]$ corresponding to the DL:birth rule and those of type $\Delta^{\ominus}[\cdot;\cdot]$ to the DL:death rule.

As an direct consequence of the dynamic representation of the Arak process $\hat{\mathcal{A}}_D$, we make the following proposition.

Proposition 1. *The distribution of the Arak process* $\hat{\mathcal{A}}_D$ *is the unique invariant law of the dynamics given by the DL:birth rule and the DL:death rule. The resulting stationary process is reversible. Moreover, for any initial distribution of* $\hat{\gamma}_0$ *the laws of the random polygonal fields* $\hat{\gamma}_s$ *converge in variational distance to the law of* $\hat{\mathcal{A}}_D$ *as* $s \to \infty$.

The uniqueness and convergence statements in the above proposition require a short justification. They both follow from the observation that, in a finite volume and regardless of the initial state, the process $\hat{\gamma}_s$ spends a non-null fraction of time in the state 'black' (in which there are no contours and the whole domain D is coloured black). By a standard coupling argument, e.g. along the lines of the proof of Theorem 1.2 of Liggett (1985, pp. 65–67), this observation implies the required uniqueness and convergence.

Below, we show that the laws of the Gibbs-modified polygonal fields $\hat{\mathcal{A}}_D^{[\alpha,\beta]}$ arise as the unique invariant distributions of appropriate modifications of the reference dynamics DL:birth and DL:death. The main change is that the birth and death updates are no longer performed unconditionally; they must pass an *acceptance test* instead and are accepted with certain state-dependent probabilities. Upon failure of the acceptance test, the update is discarded. For an $a \geq 0$ and a $b \geq 0$ such that $\alpha + a \geq 0$ and $\beta + b \geq 0$, consider the following dynamics.

Rule 7. (DL:birth$[\alpha, \beta; a, b]$.) *With intensity* $[\pi \, \mathrm{d}x + \kappa(\mathrm{d}x)] \, \mathrm{d}s$*,*

- *set* $\delta = \gamma_s \oplus x$*,*
- *construct* $\hat{\delta}$ *by randomly choosing, with probability* $\frac{1}{2}$*, either of the two possible colourings for* δ*,*
- *accept* $\hat{\delta}$ *with probability*

$$p[\hat{\delta}; \hat{\gamma}_s] = \exp(-\alpha A(\mathrm{black}[\hat{\delta}] \setminus \mathrm{black}[\hat{\gamma}_s]) - \beta \, \mathrm{length}(\delta \setminus \gamma_s)) \\ \times \exp(-a A(\mathrm{black}[\hat{\delta}] \triangle \mathrm{black}[\hat{\gamma}_s]) - b \, \mathrm{length}(\delta \triangle \gamma_s)), \tag{10}$$

- *if* $\hat{\delta}$ *is accepted, set* $\hat{\gamma}_{s+\mathrm{d}s} = \hat{\delta}$*; otherwise, set* $\hat{\gamma}_{s+\mathrm{d}s} = \hat{\gamma}_s$.

https://doi.org/10.1239/aap/1134587745 Published online by Cambridge University Press

Rule 8. (DL:death[$\alpha, \beta; a, b$].) *For each birth site x in γ_s, with intensity* 1

- *set $\delta = \gamma_s \ominus x$,*
- *construct $\hat{\delta}$ by randomly choosing, with probability $\frac{1}{2}$, either of the two possible colourings for δ,*
- *accept $\hat{\delta}$ with probability $p[\hat{\delta}; \hat{\gamma}_s]$ as given in (10),*
- *if $\hat{\delta}$ is accepted, set $\hat{\gamma}_{s+\mathrm{d}s} = \hat{\delta}$; otherwise, set $\hat{\gamma}_{s+\mathrm{d}s} = \hat{\gamma}_s$.*

In analogy with their original reference forms DL:birth and DL:death, the above dynamics should be thought of as generating random disagreement loops λ and setting $\gamma_{s+\mathrm{d}s} := \gamma \triangle \lambda$ provided that λ passes the acceptance test. It should be emphasised that the random disagreement loops above are generated according to the dynamic representation of the original Arak process $\mathcal{A}_D$. The following theorem justifies the above construction.

Theorem 1. *For each $a \geq 0, b \geq 0$, α, and β such that $\alpha + a \geq 0$ and $\beta + b \geq 0$, the law of the Gibbs-modified Arak process $\hat{\mathcal{A}}_D^{[\alpha,\beta]}$ is the unique invariant distribution of the dynamics* DL:birth[$\alpha, \beta; a, b$] *and* DL:death[$\alpha, \beta; a, b$]*. The resulting stationary process is reversible. For any initial distribution of $\hat{\gamma}_0$, the laws of the random polygonal fields $\hat{\gamma}_s$ converge in variational distance to the law of $\hat{\mathcal{A}}_D^{[\alpha,\beta]}$ as $s \to \infty$.*

Theorem 1 follows easily from Proposition 1 by a straightforward check of the detailed balance conditions. We chose, however, to provide below a geometric proof of this result for the case $\alpha, \beta \geq 0$, so revealing, in the author's opinion, the geometric intuition underlying the dynamics (a similar proof can be provided for $\alpha < 0$ or $\beta < 0$, as well). Note that the role of the first factor in the acceptance probability $p[\cdot; \cdot]$, as given in (10), is to ensure the detailed balance for the dynamics, while the second factors, involving the additional parameters a and b, exponentially suppress the flow between configurations exhibiting strong differences (measured by $\mathrm{length}(\delta \setminus \gamma)$ and $A(\mathrm{black}[\hat{\delta}] \triangle \mathrm{black}[\hat{\gamma}])$) while keeping the detailed balance unperturbed. The rate of this suppression is controlled by a and b. In fact, the reason for introducing the parameters a and b with the possibility that $a > 0, b > 0, \alpha + a > 0$, and $\beta + b > 0$ was to gain direct control over the diameter of the region affected by a single update, which exhibits exponentially decaying tails in the current dynamics. The control of the diameter of the affected region is a sine qua non for infinite-volume extensions of the dynamics in Rules 7 and 8, which are the subject of the author's current work in progress and will be discussed in a future paper. Clearly, we could also have chosen another standard set of acceptance probabilities conforming to the detailed balance conditions, e.g. we could accept a transition $\hat{\gamma}_s \mapsto \hat{\gamma}_{s+\mathrm{d}s} = \hat{\delta}$ with probability

$$\min(1, \exp(\mathcal{H}_D^{[\alpha,\beta]}(\hat{\gamma}_s) - \mathcal{H}_D^{[\alpha,\beta]}(\hat{\delta}))),$$

and a direct check of the detailed balance conditions, based on Proposition 1, would show that the law of $\hat{\mathcal{A}}_D^{[\alpha,\beta]}$ is invariant with respect to such a dynamics. However, in this dynamics, in general we cannot efficiently control the size of the region affected in a single update.

We can easily construct versions of the disagreement loop birth-and-death dynamics that leave invariant the distributions of the polygonal fields $\hat{\mathcal{A}}_{D\,|\,\varnothing}^{[\alpha,\beta]}$, $\hat{\mathcal{A}}_{D\,|\,\mathrm{black}}^{[\alpha,\beta]}$, and $\hat{\mathcal{A}}_{D\,|\,\mathrm{white}}^{[\alpha,\beta]}$ with empty, black, and white boundary conditions, respectively. To this end, we modify the dynamics DL:birth[$\alpha, \beta; a, b$] and DL:death[$\alpha, \beta; a, b$] accordingly, by discarding all the updates that cause the contour collection γ_s to hit the boundary. In addition, for the monochromatic black or white boundary conditions, upon an update we do not pick the colouring at random but rather choose the unique one compatible with the boundary condition. By denoting the dynamics so constructed by adding a subscript indicating the boundary conditions, we immediately obtain the following corollary from Theorem 1.

Corollary 1. *For each $a \geq 0$, $b \geq 0$, α, and β such that $\alpha + a \geq 0$ and $\beta + b \geq 0$, the laws of the Gibbs-modified Arak processes $\hat{\mathcal{A}}^{[\alpha,\beta]}_{D\,|\,\varnothing}$, $\hat{\mathcal{A}}^{[\alpha,\beta]}_{D\,|\,\text{black}}$, and $\hat{\mathcal{A}}^{[\alpha,\beta]}_{D\,|\,\text{white}}$ are respectively the unique invariant distributions of the dynamics*

- $\mathrm{DL}_{\varnothing}$:birth$[\alpha, \beta; a, b]$ *and* $\mathrm{DL}_{\varnothing}$:death$[\alpha, \beta; a, b]$,
- $\mathrm{DL}_{\text{black}}$:birth$[\alpha, \beta; a, b]$ *and* $\mathrm{DL}_{\text{black}}$:death$[\alpha, \beta; a, b]$,
- $\mathrm{DL}_{\text{white}}$:birth$[\alpha, \beta; a, b]$ *and* $\mathrm{DL}_{\text{white}}$:death$[\alpha, \beta; a, b]$.

The resulting stationary processes are reversible. For any initial distribution of $\hat{\gamma}_0$, the laws of the random polygonal fields $\hat{\gamma}_s$ converge in variational distance to the laws of $\hat{\mathcal{A}}^{[\alpha,\beta]}_{D\,|\,\varnothing}$, $\hat{\mathcal{A}}^{[\alpha,\beta]}_{D\,|\,\text{black}}$, and $\hat{\mathcal{A}}^{[\alpha,\beta]}_{D\,|\,\text{white}}$, as appropriate, as $s \to \infty$.

The author believes that a very similar dynamics could be used to simulate length- and area-interacting modifications of more general consistent polygonal Markov fields admitting the dynamic representation discussed in Arak and Surgailis (1989), (1991) and Arak *et al*. (1993). The only change would be an appropriate redefinition of the operations $\Delta^{\oplus}[\cdot;\cdot]$ and $\Delta^{\ominus}[\cdot;\cdot]$, and the resulting disagreement field would no longer be a single loop.

2.2. Contour birth-and-death dynamics

As already mentioned, unlike the previous dynamics, that discussed in this subsection is constructed in a much narrower setting, restricted to colourless contour configurations that do not hit the boundary, and is meant to leave invariant the distributions of $\mathcal{A}^{[0,\beta]}_{D\,|\,\varnothing}$. Recall, from the discussion following (9), that in this setting we can take D to be an arbitrary bounded open set in $\mathbb{R}^d$ with a piecewise-smooth boundary, and do not need convexity. The approach developed in this section leads to a simulation algorithm, discussed in Section 4 below, that, though perfect, seems to be impracticable due to the nonconstructive description of the intensity measure of contour births. However, its value lies in the fact that its infinite-volume extension provides important theoretical information about the thermodynamic limit $\mathcal{A}^{[0,\beta]}$, yielding, in particular, the uniqueness of the thermodynamic limit for sufficiently large values of β. Observe that the dynamics constructed in this section could in principle also be used directly for Metropolis sampling; however, the previous disagreement loop dynamics seems much better suited for this purpose.

To proceed, we consider the space $\mathcal{C}_D$ consisting of all closed polygonal contours in D that do not touch the boundary ∂D. For a given point configuration $\bar{x} = \{x_1, \dots, x_n\}$, we denote by $\mathcal{C}_D(\bar{x})$ the family of those polygonal contours in $\mathcal{C}_D$ that belong to $\Gamma_D(\bar{x})$, i.e. whose vertex sets coincide with $\bar{x}$. We construct the so-called free contour measure Θ_D on $\mathcal{C}_D$ by

$$\Theta_D(C) = \int_{\mathrm{Fin}(D)} \sum_{\theta \in C \cap \mathcal{C}_D(\bar{x})} \exp(-\Phi(\theta))\nu^*(\mathrm{d}\bar{x}), \tag{11}$$

where $C \subseteq \mathcal{C}_D$ is a set measurable with respect to, say, the Borel σ-field generated by the Hausdorff distance topology, the Hamiltonian Φ is as in (2), $\mathrm{Fin}(D)$ stands for the family of finite point configurations in D, and ν^* is the measure on $\mathrm{Fin}(D)$ given by $\mathrm{d}\nu^*(\bar{x}) = \mathrm{d}x_1 \cdots \mathrm{d}x_n$. In order to provide an alternative line- rather than point-based expression for Θ_D, for a given finite configuration $(l) = (l_1, \dots, l_n)$ of straight lines intersecting D denote by $\mathcal{C}_D(l)$ the family of those polygonal contours in $\mathcal{C}_D$ that belong to $\Gamma_D(l)$. We then have (see, e.g. Equation (3.8) of Arak *et al*. (1993))

$$\Theta_D(C) = \int_{\mathrm{Fin}(L[D])} \sum_{\theta \in C \cap \mathcal{C}_D(l)} \exp(-2\,\mathrm{length}(\theta))\,\mathrm{d}\mu^*((l)), \tag{12}$$

where $\mathrm{Fin}(L[D])$ stands for the family of finite line configurations intersecting D and μ^* is the measure on $\mathrm{Fin}(L[D])$ given by $\mathrm{d}\mu^*((l_1, \dots, l_n)) = \mathrm{d}\mu(l_1) \cdots \mathrm{d}\mu(l_n)$, with μ as defined in the discussion preceding (1).

For $\beta \in \mathbb{R}$, we consider the exponential modification $\Theta_D^{[\beta]}$ of the free measure Θ_D given by

$$\Theta_D^{[\beta]}(\mathrm{d}\theta) = \exp(-\beta\,\mathrm{length}(\theta))\Theta_D(\mathrm{d}\theta). \tag{13}$$

It is easily seen that the total mass $\Theta_D^{[\beta]}(\mathcal{C}_D)$ is always finite. Indeed, by using (12), taking into account that the length of a line-segment in D can be at most $\mathrm{diam}(D)$ (the diameter of D), and recalling that, by standard integral geometry, $M := \mu(\{l : l \cap D \neq \varnothing\}) \leq \mathrm{length}(\partial\,\mathrm{conv}(D))$, we conclude that

$$\Theta_D^{[\beta]}(\mathcal{C}_D) \leq \sum_{k=0}^{\infty} \frac{M^k \exp(k|\beta|\,\mathrm{diam}(D))}{k!} \leq \exp(\mathrm{length}(\partial\,\mathrm{conv}(D)) \exp(|\beta|\,\mathrm{diam}(D))) < \infty. \tag{14}$$

Let $\mathcal{P}_{\Theta_D^{[\beta]}}$ be the Poisson point process on $\mathcal{C}_D$ with intensity measure $\Theta_D^{[\beta]}$. It then follows directly from (11), the point-based representation (3), and (8) that, for all $\beta \in \mathbb{R}$ for which the partition function $\mathrm{E}\exp(-\mathcal{H}_D^{[\alpha,\beta]}(\hat{\mathcal{A}}_{D\,|\,\varnothing}))$ in (9) is finite (we show that this in fact holds for all $\beta \in \mathbb{R}$ in Corollary 2), the polygonal field $\mathcal{A}_{D\,|\,\varnothing}^{[0,\beta]}$ coincides in distribution with the union of contours in $\mathcal{P}_{\Theta_D^{[\beta]}}$ conditioned on the event that they are disjoint, i.e.

$$\mathcal{L}(\mathcal{A}_{D\,|\,\varnothing}^{[0,\beta]}) = \mathcal{L}\biggl(\bigcup_{\theta \in \mathcal{P}_{\Theta_D^{[\beta]}}} \theta \;\bigg|\; \theta \cap \theta' = \varnothing \text{ for all } \theta, \theta' \in \mathcal{P}_{\Theta_D^{[\beta]}},\ \theta \neq \theta'\biggr), \tag{15}$$

where the conditioning is well defined in view of (14). In particular, taking into account (1) and (12), for all β such that (9) makes sense we have

$$\mathrm{P}(\theta \cap \theta' = \varnothing \text{ for all } \theta, \theta' \in \mathcal{P}_{\Theta_D^{[\beta]}},\ \theta \neq \theta') = \mathrm{E} \sum_{\delta \in \Gamma_{D\,|\,\varnothing}(\Lambda_D)} \exp(-[2+\beta]\,\mathrm{length}(\delta)),$$

where $\Gamma_{D\,|\,\varnothing}$ stands for the family of admissible polygonal configurations in D that do not touch ∂D. It follows easily that the law of $\mathcal{A}_{D\,|\,\varnothing}^{[0,\beta]}$ is invariant and reversible with respect to the following contour birth-and-death dynamics for $(\gamma_s)_{s \geq 0}$ on $\Gamma_{D\,|\,\varnothing}$.

Rule 9. (C:birth[β].) *With intensity* $\Theta_D^{[\beta]}(\mathrm{d}\theta)\,\mathrm{d}s$,

- *choose a new contour* θ,
- *if* $\theta \cap \gamma_s = \varnothing$, *accept* θ *and set* $\gamma_{s+\mathrm{d}s} = \gamma_s \cup \theta$,
- *otherwise, reject* θ *and set* $\gamma_{s+\mathrm{d}s} = \gamma_s$.

Rule 10. (C:death[β].) *With intensity* 1, *for each contour* $\theta \in \gamma_s$, *remove* θ *from* γ_s, *setting* $\gamma_{s+\mathrm{d}s} = \gamma_s \setminus \theta$.

It is worth noting that, should we accept all the new contours without performing the disjointness test in the above dynamics, we would obtain the Poisson contour process $\mathcal{P}_{\Theta_D^{[\beta]}}$ as the stationary state.

Observing that the process γ_s constructed above spends a non-null fraction of time in the state $\varnothing$, and using a standard coupling argument (cf. the proof of Theorem 1.2 of Liggett (1985, pp. 65–67)), we are led to our next theorem.

Theorem 2. *The law of the Gibbs-modified Arak process* $\mathcal{A}_{D\,|\,\varnothing}^{[0,\beta]}$ *is the unique invariant distribution of the dynamics* C:birth[β] *and* C:death[β]. *The resulting stationary process is reversible. For any initial distribution of* γ_0 *the laws of random polygonal fields* γ_s *converge in variational distance to the law of* $\mathcal{A}_{D\,|\,\varnothing}^{[0,\beta]}$ *as* $s \to \infty$.

All our results in this section are conditional on the partition function in (9) being finite. We claim here that this holds for all $\beta \in \mathbb{R}$. Indeed, since $\beta = 0$ clearly satisfies this condition, as this choice corresponds to the basic Arak process $\mathcal{A}_{D\,|\,\varnothing}$, Theorem 2 can be used for $\beta = 0$. The dynamics in Rules 9 and 10 implies that the empty-boundary Arak process $\mathcal{A}_{D\,|\,\varnothing}$ is stochastically dominated (in the sense of inclusion) by the union of contours in $\mathcal{P}_{\Theta_D}$; see Corollary 5. In particular, by (14), for all $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}
\mathrm{E}\exp(-\mathcal{H}_D^{[\alpha,\beta]}(\hat{\mathcal{A}}_{D\,|\,\varnothing})) &\le \exp(|\alpha|A(D))\,\mathrm{E}\exp\Bigg(|\beta|\sum_{\theta\in\mathcal{P}_{\Theta_D}}\mathrm{length}(\theta)\Bigg)\\
&= \exp(|\alpha|A(D))\exp(-\Theta_D(\mathcal{C}_D))\\
&\quad\times\sum_{k=0}^{\infty}\frac{[\Theta_D(\mathcal{C})]^k}{k!}\bigg(\int_{\mathcal{C}_D}\exp(|\beta|\,\mathrm{length}(\theta))\Theta_D(d\theta)\bigg)^k\\
&= \exp(|\alpha|A(D))\exp(\Theta_D(\mathcal{C}_D)[\Theta_D^{[|\beta|]}(\mathcal{C}_D)-1]) < \infty.
\end{aligned}$$

By an appropriate redefinition of Θ_D that admits edges truncated by the boundary, the same argument can be repeated with $\mathcal{A}_{D\,|\,\varnothing}$ replaced by $\mathcal{A}_D$. Thus, we have proved the following corollary.

Corollary 2. *For each bounded open domain* $D \subseteq \mathbb{R}^2$, *both the partition functions*

$$\mathrm{E}\exp(-\mathcal{H}_D^{[\alpha,\beta]}(\hat{\mathcal{A}}_D))$$

in (6) and

$$\mathrm{E}\exp(-\mathcal{H}_D^{[\alpha,\beta]}(\hat{\mathcal{A}}_{D\,|\,\varnothing}))$$

in (9) are finite for all $\alpha, \beta \in \mathbb{R}$.

3. Thermodynamic limit

The purpose of this section is to define the notion, and establish the existence (see Surgailis (1991)), of a thermodynamic limit for the polygonal fields under consideration.

For a smooth closed simple (nonintersecting) curve c in D, by the *trace* of a polygonal configuration $\hat{\gamma}$ on c, denoted in the sequel by $\hat{\gamma} \wedge c$, we mean the knowledge of

- the intersection points and intersection directions of $\hat{\gamma}$ with c, and
- the colouring of the points of c.

https://doi.org/10.1239/aap/1134587745 Published online by Cambridge University Press

Although this concept can be formalised in various compatible ways, we keep the above informal definition in the hope that it does not lead to any ambiguities, while allowing us to avoid unnecessary technicalities. For convenience, we assume that no edge of $\hat{\gamma}$ is tangent to c, which can be ensured with probability 1 in view of the smoothness of c.

Fix $\alpha, \beta \in \mathbb{R}$. In view of the Gibbsian representations (1), (3), and (6), we can easily check that, for each curve c as above and a trace $\hat{\theta}$ on c, there exists a stochastic kernel $\hat{\mathcal{A}}^{[\alpha,\beta]}_{\operatorname{int} c}(\cdot \mid \hat{\theta})$ with the property that

$$\mathcal{L}_{\operatorname{int} c}(\hat{\mathcal{A}}^{[\alpha,\beta]}_{D} \mid \hat{\mathcal{A}}^{[\alpha,\beta]}_{D} \wedge c = \hat{\theta}) = \mathcal{L}_{\operatorname{int} c}(\hat{\mathcal{A}}^{[\alpha,\beta]}_{D\mid \mathrm{bd}} \mid \hat{\mathcal{A}}^{[\alpha,\beta]}_{D\mid \mathrm{bd}} \wedge c = \hat{\theta}) = \hat{\mathcal{A}}^{[\alpha,\beta]}_{\operatorname{int} c}(\cdot \mid \hat{\theta})$$

for all bounded open and convex sets $D \supseteq \operatorname{int} c$ and for bd $\in$ {black, white}, where $\mathcal{L}_{\operatorname{int} c}$ denotes the law of the argument random element restricted to $\operatorname{int} c$ (the interior of c).

Consider the family $\Gamma_{\mathbb{R}^2}$ of whole-plane admissible polygonal configurations, determined by (P1), (P2), and (P4) ((P3) is meaningless in this context) and by the requirement of local finiteness (any bounded set is hit by at most a finite number of edges). Let $\hat{\Gamma}_{\mathbb{R}^2}$ be the corresponding collection of black-and-white whole-plane admissible polygonal configurations (note that there are exactly two elements of $\hat{\Gamma}_{\mathbb{R}^2}$ corresponding to a given one in $\Gamma_{\mathbb{R}^2}$, representing two possible colouring strategies: setting the origin to be either black or white). It is natural to define the family $\mathcal{G}(\hat{\mathcal{A}}^{[\alpha,\beta]})$ of infinite-volume Gibbs measures (thermodynamic limits) for $\hat{\mathcal{A}}^{[\alpha,\beta]}$ as the collection of all probability measures on $\hat{\Gamma}_{\mathbb{R}^2}$ with the accordingly distributed random element $\hat{\mathcal{A}}$ satisfying

$$\mathcal{L}_{\operatorname{int} c}(\hat{\mathcal{A}} \mid \hat{\mathcal{A}} \wedge c = \hat{\theta}) = \hat{\mathcal{A}}^{[\alpha,\beta]}_{\operatorname{int} c}(\cdot \mid \hat{\theta}).$$

In addition, we shall consider the family $\mathcal{G}_{\tau}(\hat{\mathcal{A}}^{[\alpha,\beta]})$ of isometry-invariant measures in $\mathcal{G}(\hat{\mathcal{A}}^{[\alpha,\beta]})$. Using an appropriate relative-compactness argument, much along the same lines as Schreiber (2004), we will readily ascertain the existence of at least one isometry-invariant thermodynamic limit for each $\beta > 0$.

Theorem 3. *For all $\alpha \in \mathbb{R}$ and $\beta > 0$, the family $\mathcal{G}_{\tau}(\hat{\mathcal{A}}^{[\alpha,\beta]})$ is nonempty.*

Note that, for $\alpha = 0$ and β sufficiently large, this statement also follows from the theorem of Surgailis (1991).

In the sequel, we will establish certain uniqueness results for the thermodynamic limit in the low-temperature region within a particular class of infinite-volume measures without infinite contours. However, we do conjecture that, for $\alpha = 0$, outside this class there exist an infinite number of extreme infinite-volume phases breaking both the rotational and translational symmetries. We briefly and informally sketch their tentative construction. For the increasing sequence of squares $(-n, n)^2$, $n = 1, 2, \ldots$, we consider a sequence of boundary conditions arising by requiring that a large number $C(n)$ of edges hit the left-hand side of $(-n, n)^2$ (with the intersection points located more or less uniformly over the edge), the same number of edges intersect the opposite right-hand side, but no edges hit the upper or lower sides. We believe that, by choosing an appropriate growth rate for $C(n)$, we can ensure that the resulting sequence of polygonal fields on $(-n, n)^2$ is uniformly tight (e.g. in the topology discussed in the proof of Theorem 3) and that the accumulation points of this sequence are thermodynamic limits for $\mathcal{A}^{[0,\beta]}$ with an infinite number of infinite *left-to-right* polygonal chains. Moreover, the expected number of such chains hitting a disk of radius 1 should exhibit untempered growth to infinity with the distance of the centre of the disk from the origin.

https://doi.org/10.1239/aap/1134587745 Published online by Cambridge University Press

We conjecture that such untempered thermodynamic limits should exist even for $\beta = 0$, in which case, in the language of the dynamic time–space construction of the basic Arak process, we could, roughly speaking, have an infinite-density cloud of particles born at time $-\infty$. Such constructions are possible because, under very rapid edge density growth with the distance from the origin, we can enforce a situation in which the influence of the boundary conditions on $\partial(-n, n)^2$ is of equal importance or even dominates the stabilising bulk effects within $(-n, n)^2$. Clearly, such phenomena cannot occur in the stationary regime; see Schreiber (2004) for a discussion.

4. Perfect simulation from the thermodynamic limit and exponential mixing

The purpose of this section is to study the contour birth-and-death dynamics described in Subsection 2.2 in the context of the perfect, infinite-volume simulation scheme developed by Fernández *et al*. (1998), (2002). This approach is valid only for sufficiently large β. It yields a perfect algorithm for simulating thermodynamic limits in finite windows and certain uniqueness and mixing results for the thermodynamic limit in the low-temperature regime.

To this end, we observe first that, for all bounded open sets D with piecewise-smooth boundaries, the free contour measures Θ_D, defined in (11), arise as the respective restrictions to $\mathcal{C}_D$ of the measure $\Theta \equiv \Theta_{\mathbb{R}^2}$ on $\mathcal{C} := \bigcup_{n=1}^{\infty} \mathcal{C}_{(-n,n)^2}$, which is referred to in the sequel as the infinite-volume free contour measure. Indeed, this follows easily from the observation that Θ_{D_1} restricted to $\mathcal{C}_{D_2}$ coincides with Θ_{D_2}, for $D_2 \subseteq D_1$. We construct the infinite-volume exponentially modified measures $\Theta^{[\beta]} = \Theta^{[\beta]}_{\mathbb{R}^2}$ in the same way. The next result, which is related to the lemma in the appendix of Nicholls (2001), gives the exponential decay of the measure $\Theta^{[\beta]}$ with respect to the contour size, and is crucial for what follows.

Lemma 1. *For $\beta \geq 2$, we have*

$$\Theta^{[\beta]}(\{\theta \colon \mathrm{d}x \in \mathrm{Vertices}(\theta),\ \mathrm{length}(\theta) > R\}) \leq 8\pi \exp(-[\beta - 2]R)\, \mathrm{d}x. \tag{16}$$

Moreover, there exists a constant $\varepsilon > 0$ such that, for $\beta \geq 2$,

$$\Theta^{[\beta]}(\{\theta \colon 0 \in \mathrm{int}\, \theta,\ \mathrm{length}(\theta) > R\}) \leq \exp(-[\beta - 2 + \varepsilon/2]R + o(R)). \tag{17}$$

We note that, in view of (15), a standard Peierls-type argument can be applied to conclude from Lemma 1 that there is no infinite contour nesting for $\mathcal{A}^{[0,\beta]}$ whenever $\beta \geq 2$.

The approach of Fernández *et al*. (1998), (2002), specialised for our purposes, relies on the following graphical construction, briefly sketched below; see the original articles for further details. Choose a sufficiently large $\beta \geq 2$, as specified below. Define $\mathcal{F}(\mathcal{C})$ to be the space of countable and locally finite collections of contours from $\mathcal{C}$, with the local finiteness requirement meaning that at most a finite number of contours can hit a bounded subset of $\mathbb{R}^2$. Write $\mathcal{F}^{\Gamma}(\mathcal{C})$ for the family of contour collections in $\mathcal{F}(\mathcal{C})$ that correspond to admissible configurations in $\Gamma_{\mathbb{R}^2}$; in particular, intersections between contours are forbidden in $\mathcal{F}^{\Gamma}(\mathcal{C})$. Observe that $\mathcal{F}^{\Gamma}(\mathcal{C})$ is a proper subset of both $\mathcal{F}(\mathcal{C})$ (in which contours are allowed to intersect) and $\Gamma_{\mathbb{R}^2}$ (in which infinite polygonal chains are admitted, whereas $\mathcal{F}(\mathcal{C})$ contains bounded contours only). On the s-time–space $\mathbb{R} \times \mathcal{F}(\mathcal{C})$, we construct the stationary *unconstrained* (*free*) contour birth-and-death process $(\varrho_s)_{s \in \mathbb{R}}$, with the birth intensity measure given by $\Theta^{[\beta]}$ and with death intensity 1. Note that *unconstrained*, or *free*, means here that every new contour is accepted regardless of whether it hits the union of already-existing contours or not; moreover, we admit negative time here, letting s range through $\mathbb{R}$ rather than $\mathbb{R}_+$. Observe also that the birth measure $\Theta^{[\beta]}$ must

https://doi.org/10.1239/aap/1134587745 Published online by Cambridge University Press

be finite on the set $\{\theta \in \mathcal{C} : \theta \cap A \neq \varnothing\}$, for all bounded Borel sets $A \subseteq \mathbb{R}^2$, in order that the process $(\varrho_s)_{s\in\mathbb{R}}$ be well defined on $\mathbb{R} \times \mathcal{F}(\mathcal{C})$. By Lemma 1, this is ensured when $\beta \geq 2$. It is easily seen that, for each $s \in \mathbb{R}$, ϱ_s coincides in distribution with the whole-plane Poisson contour process $\mathcal{P}_{\Theta^{[\beta]}}$.

To proceed, for the free process $(\varrho_s)_{s\in\mathbb{R}}$ we perform the following *trimming* procedure. We make a directed connection between each s-time–space instance of a contour appearing in $(\varrho_s)_{s\in\mathbb{R}}$, denoted by $\theta \times [s_0, s_1)$ with θ standing for the contour and $[s_0, s_1)$ for its lifespan, and all s-time–space contour instances $\theta' \times [s_0', s_1')$ with $\theta' \cap \theta \neq \varnothing$, $s_0' \leq s_0$, and $s_1' > s_0$. In other words, we connect $\theta \times [s_0, s_1)$ to all those contour instances that may have affected the acceptance status of $\theta \times [s_0, s_1)$ in the *constrained* contour birth-and-death dynamics of Rules 9 and 10. These connections yield directed chains of s-time–space contour instances; we call them the *ancestor chains* in the sequel. Following Fernández *et al*. (2002), we refer to the union of all ancestor chains stemming from a given contour instance as its *clan of ancestors*. By using Lemma 1 combined with a general technique of stochastic domination by subcritical multitype branching processes (as discussed in detail in Fernández *et al*. (1998), (2002)), for all sufficiently large β we can ensure that all such clans of ancestors are almost surely (a.s.) finite and that a single clan size has an exponentially decaying tail (i.e. the probability that the clan size exceeds R is $O(\exp(-cR))$ for some $c > 0$). In this case, we can uniquely determine the acceptance status of all the clan members: contour instances with no ancestors are a.s. accepted, which automatically and uniquely determines the acceptance status of all the remaining members of the clan by recursive application of the intercontour exclusion rule. Discarding the unaccepted contour instances leaves us with an s-time–space representation of a stationary evolution $(\gamma_s)_{s\in\mathbb{R}}$ on $\mathcal{F}^\Gamma(\mathcal{C})$. The graphical construction and the arguments of Fernández *et al*. (1998), (2002), specialised to our setting, yield our next result.

Theorem 4. *Choose a $\beta \geq 2$ sufficiently large that all the ancestor clans in the above graphical construction are a.s. finite and a single clan size exhibits an exponentially decaying tail. The following statements then hold.*

(i) *The $\mathcal{F}^\Gamma(\mathcal{C})$-valued process $(\gamma_s)_{s\geq 0}$ given above is well defined, stationary, and reversible.*

(ii) *The stationary distribution $\mathcal{L}(\gamma_0)$ on $\mathcal{F}^\Gamma(\mathcal{C})$ is isometry invariant and belongs to $\mathcal{G}_\tau(\mathcal{A}^{[0,\beta]})$.*

(iii) *The dynamics of $(\gamma_s)_{s\in\mathbb{R}}$ is an infinite-volume extension of the contour birth-and-death dynamics of Rules 9 and 10, i.e. $(\gamma_s)_{s\in\mathbb{R}}$ is a Markov process on $\mathcal{F}^\Gamma(\mathcal{C})$ with infinitesimal generator*

$$[L^{[\beta]}F](\eta) = \int_{\mathcal{C}} [F(\eta \cup \{\theta\}) - F(\eta)]\, \mathbf{1}_{\{\theta\cap\delta=\varnothing \text{ for all } \delta\in\eta\}}\, \mathrm{d}\Theta^{[\beta]}(\theta) + \sum_{\theta\in\eta} [F(\eta \setminus \{\theta\}) - F(\eta)] \tag{18}$$

for $\eta \in \mathcal{F}^\Gamma(\mathcal{C})$ and bounded functions $F : \mathcal{F}^\Gamma(\mathcal{C}) \to \mathbb{R}$ such that $F(\eta)$ depends only on $\eta \cap D$, for some bounded convex set D.

(iv) *$(\gamma_s)_{s\in\mathbb{R}}$ exhibits exponential s-time–space α-mixing, in that there exists a $c > 0$ such that*

$$\sup_{\substack{\mathcal{E}_1\in \mathrm{Im}_{B(x,1)\times[s_0,s_1]} \\ \mathcal{E}_2\in \mathrm{Im}_{B(y,1)\times[s_0',s_1']}}} |\mathrm{P}(\mathcal{E}_1\cap\mathcal{E}_2) - \mathrm{P}(\mathcal{E}_1)\,\mathrm{P}(\mathcal{E}_2)| \leq \exp(-c[\mathrm{dist}(x, y) + \mathrm{dist}([s_0, s_1], [s_0', s_1'])])$$

https://doi.org/10.1239/aap/1134587745 Published online by Cambridge University Press

when $\operatorname{dist}(x, y)$ *is sufficiently large, with* $\operatorname{Im}_{B(x,1)\times[s_0,s_1]}$ *standing for the* σ*-field generated by the restriction of* $(\gamma_s)_{s\in\mathbb{R}}$ *to the space–s-time region* $B(x, 1) \times [s_0, s_1]$, *where* $B(x, 1)$ *is the disk of radius* 1 *centred at* $x \in \mathbb{R}^2$.

(v) *Consequently, the stationary distribution* $\mathcal{L}(\gamma_0)$ *exhibits exponential spatial* α*-mixing.*

It is worth noting that, even if β is not large enough to ensure the a.s. finiteness of ancestor clans, a weaker version of the above graphical construction can be provided if the birth intensity measure $\Theta^{[\beta]}$ is finite on $\{\theta \in \mathcal{C} : \theta \cap A \neq \varnothing\}$ for all bounded $A \subseteq \mathbb{R}^2$, which, by Lemma 1, is the case when $\beta \geq 2$. To this end, we restrict the s-time to $\mathbb{R}_+$ and choose an initial condition that is a $\mathcal{F}^{\Gamma}(\mathcal{C})$-valued random element independent of the free birth-and-death process of the graphical construction. The birth-and-death process here is also restricted to positive times, in that no contours are born or alive before the s-time 0. In other words, the birth-and-death process starts with the initial state $\varnothing$ at s-time 0; consequently, it is no longer stationary. In this context, the local finiteness of $\Theta^{[\beta]}$ allows us to conclude that, for each contour instance $\theta \times [s_0, s_1),\ s_0, s_1 > 0$, the expected cardinality of its ancestor clan extending down to s-time 0 is finite and, consequently, the clan is a.s. finite (note that, in the original graphical construction, it might extend in negative s-time to an infinite clan). Thus, with the initial state given, the acceptance status of each contour instance is uniquely determined by the intercontour exclusion rule. This motivates the following corollary.

Corollary 3. *With* $\beta \geq 2$, *for each* $\mathcal{F}^{\Gamma}(\mathcal{C})$*-valued initial condition* γ_0 *there exists a Markov process* $(\gamma_s)_{s\geq 0}$ *on* $\mathcal{F}^{\Gamma}(\mathcal{C})$ *with infinitesimal generator given by (18).*

In the remainder of the present section, we will not use Corollary 3; rather, unless otherwise stated, we shall assume that β stays within the region of validity of the original graphical construction, described immediately above Theorem 4. We denote by $\mu^{[\beta]}$ the infinite-volume stationary distribution $\mathcal{L}(\gamma_0)$ arising in this graphical construction. That $\mu^{[\beta]}$ is concentrated on $\mathcal{F}(\mathcal{C})$ means that it contains no infinite polygonal chains; all the contours are bounded and closed. Below we show that, under the assumptions of Theorem 4, $\mu^{[\beta]}$ is in fact the unique element of $\mathcal{G}(\mathcal{A}^{[0,\beta]})$ concentrated on $\mathcal{F}(\mathcal{C})$ (and, hence, on $\mathcal{F}^{\Gamma}(\mathcal{C})$), although we conjecture that $\mathcal{G}(\mathcal{A}^{[0,\beta]})$ is infinite, as argued in Section 3. To proceed with our argument, we consider finite-volume versions of the above graphical construction, with the infinite-volume birth intensity measure $\Theta^{[\beta]}$ replaced by its finite-volume restrictions $\Theta_D^{[\beta]}$ for bounded and open sets D with piecewise-smooth boundaries. Clearly, the graphical construction then yields a version of the finite-volume contour birth-and-death dynamics of Rules 9 and 10. For each set D, denote the resulting finite-volume stationary process on $\mathcal{F}(\mathcal{C}_D)$ by $(\gamma_s^D)_{s\in\mathbb{R}}$. Also, write (ϱ_s^D) for the corresponding *free* contour birth-and-death process. Note that this finite-volume construction is valid for all $\beta \in \mathbb{R}$, even though in this section it is only used for values of β satisfying the hypothesis of Theorem 4. In view of Theorem 2, we see that γ_s^D coincides in distribution with $\mathcal{A}_{D\,|\,\varnothing}^{[0,\beta]}$ for all $s \in \mathbb{R}$. Moreover, it is easily seen that ϱ_s^D coincides in distribution with $\mathcal{P}_{\Theta_D^{[\beta]}}$ for all $s \in \mathbb{R}$. From the construction, Lemma 1, and the general theory developed in Fernández *et al*. (1998), (2002), we have the following proposition.

Proposition 2. *With* β *as in Theorem 4, the finite-volume graphical constructions for different sets* $D \subseteq \mathbb{R}^2$ *and the infinite-volume graphical construction can be coupled on a common probability space such that there exists a* $c > 0$ *with*

$$\mathrm{P}(\gamma_s^{D_1} \cap B(x, 1) \neq \gamma_s^{D_2} \cap B(x, 1)) \leq \exp(-c \min(\operatorname{dist}(x, \partial D_1), \operatorname{dist}(x, \partial D_2)))$$

https://doi.org/10.1239/aap/1134587745 Published online by Cambridge University Press

for bounded sets D_1, $D_2 \subseteq \mathbb{R}^2$, x sufficiently far from ∂D_1 and ∂D_2, and all $s \in \mathbb{R}$. Moreover,

$$\mathrm{P}(\gamma_s^D \cap B(x, 1) \neq \gamma_s \cap B(x, 1)) \leq \exp(-c \operatorname{dist}(x, \partial D))$$

for bounded sets $D \subseteq \mathbb{R}^2$, x sufficiently far from ∂D, and all $s \in \mathbb{R}$.

To proceed, observe that, for each contour collection in $\mathcal{F}^\Gamma(\mathcal{C})$, every bounded region can be surrounded by a smooth curve that does not hit any of the contours. Consequently, by recalling that γ_s^D coincides in distribution with $\mathcal{A}_{D\,|\,\varnothing}^{[0,\beta]}$ we can use the Markov property of the considered polygonal fields, combined with Proposition 2, to conclude the next result.

Corollary 4. *For β as in Theorem 4, the measure $\mu^{[\beta]}$ is the only element of $\mathcal{G}(\mathcal{A}^{[0,\beta]})$ concentrated on $\mathcal{F}(\mathcal{C})$.*

For β as in Theorem 4, by using Lemma 1 we easily conclude that the number of contours in ϱ_0 surrounding a given point is a.s. finite. Consequently, the number of contours surrounding a given point in γ_0 is also a.s. finite, whence there is no infinite contour nesting. Thus, we observe a unique infinite connected region surrounding finitely nested contour collections. Colouring this region black or white gives rise to two distinct, black- or white-dominated phases. There are no other extreme phases without infinite chains in the coloured model, because their corresponding colourless contour ensembles must coincide with $\mu^{[\beta]}$.

The last important conclusion of the graphical construction, based on the above observations that $\gamma_s \subseteq \varrho_s$ and $\gamma_s^D \subseteq \varrho_s^D$ a.s., and

$$\gamma_s \stackrel{\mathrm{D}}{=} \mathcal{A}^{[0,\beta]}, \qquad \gamma_s^D \stackrel{\mathrm{D}}{=} \mathcal{A}_{D\,|\,\varnothing}^{[0,\beta]}, \qquad \varrho_s \stackrel{\mathrm{D}}{=} \mathcal{P}_{\Theta^{[\beta]}}, \quad \text{and} \quad \varrho_s^D \stackrel{\mathrm{D}}{=} \mathcal{P}_{\Theta_D^{[\beta]}}$$

(where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution), is the following statement of stochastic domination.

Corollary 5. *The Poisson contour process $\mathcal{P}_{\Theta^{[\beta]}}$ stochastically dominates the polygonal field $\mathcal{A}^{[0,\beta]}$ (in the sense of inclusion of contour collections). Likewise, for each bounded set D with a piecewise-smooth boundary, the Poisson process $\mathcal{P}_{\Theta_D^{[\beta]}}$ stochastically dominates the finite-volume polygonal field $\mathcal{A}_D^{[0,\beta]}$.*

5. Proofs

5.1. Proof of Theorem 1

In order to provide a geometrical and intuitive proof of the theorem, we construct an auxiliary model. For $r > 0$, define $\hat{\mathcal{A}}_D^{[\alpha,\beta;r]}$ to be the Gibbsian modification of $\hat{\mathcal{A}}_D$ with the Hamiltonian

$$\mathcal{H}_D^{[\alpha,\beta;r]}(\hat{\gamma}) = r^{-1}\beta A((\gamma +_{\mathrm{M}} B(r)) \cap D) + \alpha A((\mathrm{black}[\hat{\gamma}] +_{\mathrm{M}} B(r)) \cap D),$$

with '$+_{\mathrm{M}}$' standing for the usual Minkowski addition, i.e.

$$C_1 +_{\mathrm{M}} C_2 = \{x + y,\ x \in C_1,\ y \in C_2\},$$

and with $B(r)$ denoting the disk of radius r centred at 0 in $\mathbb{R}^2$. It is easily seen that, for each $\hat{\gamma} \in \hat{\Gamma}_D$,

$$\lim_{r \to 0} \mathcal{H}_D^{[\alpha,\beta;r]}(\hat{\gamma}) = \mathcal{H}_D^{[\alpha,\beta]}(\hat{\gamma}), \tag{19}$$

meaning that $\mathcal{H}_D^{[\alpha,\beta;r]}$ is an approximation of $\mathcal{H}_D^{[\alpha,\beta]}$ for small r. Take $\Pi^{[\alpha+a]}$, $\Pi^{[r^{-1}(\beta+b)]}$, $\Pi^{[a]}$, and $\Pi^{[r^{-1}b]}$ to be independent homogeneous Poisson point processes on D, jointly

https://doi.org/10.1239/aap/1134587745 Published online by Cambridge University Press

independent of $\hat{\mathcal{A}}_D$, with respective intensities $\alpha + a, r^{-1}(\beta + b), a$, and $r^{-1}b$. We claim that $\hat{\mathcal{A}}_D^{[\alpha,\beta;r]}$ coincides in distribution with $\hat{\mathcal{A}}_D$ conditioned, jointly with $\Pi^{[\alpha+a]}$, $\Pi^{[r^{-1}(\beta+b)]}$, $\Pi^{[a]}$, and $\Pi^{[r^{-1}b]}$, on the event, $\mathcal{E}[\alpha, \beta; a, b; r]$, that the following conditions are simultaneously satisfied:

- $\Pi^{[r^{-1}(\beta+b)]} \cap [\gamma +_{\mathrm{M}} B(r)] = \varnothing$,
- $\Pi^{[r^{-1}b]} \subseteq [\gamma +_{\mathrm{M}} B(r)]$,
- $\Pi^{[\alpha+a]} \cap [\mathrm{black}[\hat{\gamma}] +_{\mathrm{M}} B(r)] = \varnothing$,
- $\Pi^{[a]} \subseteq [\mathrm{black}[\hat{\gamma}] +_{\mathrm{M}} B(r)] = \varnothing$.

Then

$$\mathcal{L}(\hat{\mathcal{A}}_D^{[\alpha,\beta;r]}) = \mathcal{L}(\hat{\mathcal{A}}_D \mid \mathcal{E}[\alpha, \beta; a, b; r]). \tag{20}$$

Indeed, for a given $\hat{\gamma} \in \hat{\Gamma}_D$, the probability of the event $\mathcal{E}[\alpha, \beta; a, b; r]$ is

$$\begin{aligned} \mathrm{P}(\mathcal{E}[\alpha, \beta; a, b; r] \mid \hat{\gamma}) &= \exp(-r^{-1}[\beta + b]A([\gamma +_{\mathrm{M}} B(r)] \cap D)) \\ &\quad \times \exp(-r^{-1}b[A(D) - A([\gamma +_{\mathrm{M}} B(r)] \cap D)]) \\ &\quad \times \exp(-[\alpha + a]A([\mathrm{black}[\hat{\gamma}] +_{\mathrm{M}} B(r)] \cap D)) \\ &\quad \times \exp(-a[A(D) - A([\mathrm{black}[\hat{\gamma}] +_{\mathrm{M}} B(r)] \cap D)]) \\ &= \exp(-\mathcal{H}_D^{[\alpha,\beta;r]}(\hat{\gamma})) \exp(-[a + r^{-1}b]A(D)), \end{aligned}$$

which yields (20) by definition of $\hat{\mathcal{A}}_D^{[\alpha,\beta;r]}$.

To proceed, we construct an auxiliary Markovian dynamics that leaves invariant the joint distribution of $\hat{\mathcal{A}}_D$, $\Pi^{[r^{-1}(\beta+b)]}$, $\Pi^{[\alpha+a]}$, $\Pi^{[r^{-1}b]}$, and $\Pi^{[a]}$ and makes the resulting stationary process reversible. To this end, let

$$\hat{\gamma}_0 = \hat{\mathcal{A}}_D, \qquad \pi_0^{\alpha} = \Pi^{[\alpha+a]}, \qquad \pi_0^{\beta} = \Pi^{[\beta+b]}, \qquad \pi_0^{a} = \Pi^{[a]}, \qquad \pi_0^{b} = \Pi^{[b]},$$

and let the quintuple $(\hat{\gamma}_s, \pi_s^{\alpha}, \pi_s^{\beta}, \pi_s^{a}, \pi_s^{b})_{s \geq 0}$ evolve according to the following rules, applied independently to each component.

Rule 11. (Aux1.) *The graph* $\hat{\gamma}_s$ *evolves according to* DL:birth *and* DL:death.

Rule 12. (Aux2.) *The processes* $\pi_s^{\alpha}, \pi_s^{\beta}, \pi_s^{a}$, *and* π_s^{b} *evolve according to birth-and-death processes with death intensity* 1 *and birth intensities* $\alpha+a$, $r^{-1}(\beta+b)$, a, *and* $r^{-1}b$, *respectively.*

These invariance and reversibility statements follow as direct consequences of Proposition 1. Thus, we conclude that the joint distribution of $(\hat{\mathcal{A}}_D, \Pi^{[r^{-1}(\beta+b)]}, \Pi^{[\alpha+a]}, \Pi^{[r^{-1}b]}, \Pi^{[a]})$ conditioned on the event $\mathcal{E}[\alpha, \beta; a, b; r]$ is invariant and reversible with respect to the following Markovian dynamics, which arise from Rules 11 and 12 by adding an appropriate acceptance test to be passed only by admissible updates.

Rule 13. • *Choose an update* $(\hat{\delta}, \theta^{\alpha}, \theta^{\beta}, \theta^{a}, \theta^{b})$ *for* $(\hat{\gamma}_{s+\mathrm{d}s}, \pi_{s+\mathrm{d}s}^{\alpha}, \pi_{s+\mathrm{d}s}^{\beta}, \pi_{s+\mathrm{d}s}^{a}, \pi_{s+\mathrm{d}s}^{b})$ *according to Rules 11 and 12.*

• *Accept the update, setting*

$$(\hat{\gamma}_{s+\mathrm{d}s}, \pi_{s+\mathrm{d}s}^{\alpha}, \pi_{s+\mathrm{d}s}^{\beta}, \pi_{s+\mathrm{d}s}^{a}, \pi_{s+\mathrm{d}s}^{b}) = (\hat{\delta}, \theta^{\alpha}, \theta^{\beta}, \theta^{a}, \theta^{b}),$$

https://doi.org/10.1239/aap/1134587745 Published online by Cambridge University Press

provided that the following conditions are satisfied:

$$2\theta^{\beta} \cap [\delta +_{\mathrm{M}} B(r)] = \varnothing, \qquad \theta^{b} \subseteq [\delta +_{\mathrm{M}} B(r)],$$
$$\theta^{\alpha} \cap [\mathrm{black}[\hat{\delta}] +_{\mathrm{M}} B(r)] = \varnothing, \qquad \theta^{a} \subseteq [\mathrm{black}[\hat{\delta}] +_{\mathrm{M}} B(r)].$$

- *Otherwise, discard the update, setting*

$$(\hat{\gamma}_{s+\mathrm{d}s}, \pi^{\alpha}_{s+\mathrm{d}s}, \pi^{\beta}_{s+\mathrm{d}s}, \pi^{a}_{s+\mathrm{d}s}, \pi^{b}_{s+\mathrm{d}s}) = (\hat{\gamma}_s, \pi^{\alpha}_s, \pi^{\beta}_s, \pi^{a}_s, \pi^{b}_s).$$

Consequently, in view of (20), under the stationary dynamics of Rule 13, and with the distribution at $s = 0$ given by the joint law of $(\hat{\mathcal{A}}_D, \Pi^{[r^{-1}(\beta+b)]}, \Pi^{[\alpha+a]}, \Pi^{[r^{-1}b]}, \Pi^{[a]})$ conditioned on the event $\mathcal{E}[\alpha, \beta; a, b; r]$, the first component $\hat{\gamma}_s$ coincides in distribution with $\hat{\mathcal{A}}_D^{[\alpha,\beta;r]}$ for all $s \in \mathbb{R}_+$. Moreover, the conditional distributions of the remaining components, given $\hat{\gamma}_s$, are also readily determined. Indeed, π^{α}_s is a homogeneous Poisson point process on $D \setminus [\mathrm{black}[\hat{\gamma}_s] +_{\mathrm{M}} B(r)]$ with intensity $\alpha + a$, π^{β}_s is a homogeneous Poisson point process on $D \setminus [\gamma +_{\mathrm{M}} B(r)]$ with intensity $r^{-1}(\beta + b)$, π^{a}_s is a homogeneous Poisson point process on $\mathrm{black}[\hat{\gamma}_s] +_{\mathrm{M}} B(r)$ with intensity a, while π^{b}_s is a homogeneous Poisson point process on $\gamma_s +_{\mathrm{M}} B(r)$ with intensity $r^{-1}b$. All four components, π^{α}_s, π^{β}_s, π^{a}_s, and π^{b}_s, are jointly independent, given $\hat{\gamma}_s$. Consequently, we observe that if we integrate out the Poisson components $\pi^{\alpha}, \pi^{\beta}, \pi^{a}$, and π^{b}, the polygonal field component $\hat{\gamma}_s$ turns out to evolve according to the following dynamics (see Subsection 2.1 for the notation).

Rule 14. (DL:birth$[\alpha, \beta; a, b; r]$.) *With intensity* $[\pi\, \mathrm{d}x + \kappa(\mathrm{d}x)]\, \mathrm{d}s$,

- *set* $\delta = \gamma_s \oplus x$,
- *construct* $\hat{\delta}$ *by randomly choosing, with probability* $\frac{1}{2}$, *either of the two possible colourings for* δ,
- *accept* $\hat{\delta}$ *with probability*

$$\begin{aligned}
&\exp(-[\alpha + a]A([\mathrm{black}[\hat{\delta}] +_{\mathrm{M}} B(r)] \setminus [\mathrm{black}[\hat{\gamma}_s] +_{\mathrm{M}} B(r)]))\\
&\times \exp(-r^{-1}[\beta + b]A([\delta +_{\mathrm{M}} B(r)] \setminus [\gamma_s +_{\mathrm{M}} B(r)]))\\
&\times \exp(-aA([\mathrm{black}[\hat{\gamma}_s] +_{\mathrm{M}} B(r)] \setminus [\mathrm{black}[\hat{\delta}] +_{\mathrm{M}} B(r)]))\\
&\times \exp(-r^{-1}bA([\gamma_s +_{\mathrm{M}} B(r)] \setminus [\delta +_{\mathrm{M}} B(r)]))\\
&\quad = \exp(-\alpha A([\mathrm{black}[\hat{\delta}] +_{\mathrm{M}} B(r)] \setminus [\mathrm{black}[\hat{\gamma}_s] +_{\mathrm{M}} B(r)]))\\
&\qquad \times \exp(-\beta r^{-1} A([\delta +_{\mathrm{M}} B(r)] \setminus [\gamma_s +_{\mathrm{M}} B(r)]))\\
&\qquad \times \exp(-aA([\mathrm{black}[\hat{\delta}] +_{\mathrm{M}} B(r)] \triangle [\mathrm{black}[\hat{\gamma}_s] +_{\mathrm{M}} B(r)]))\\
&\qquad \times \exp(-br^{-1} A([\delta +_{\mathrm{M}} B(r)] \triangle [\gamma_s +_{\mathrm{M}} B(r)])),
\end{aligned}$$

- *if* $\hat{\delta}$ *is accepted, set* $\hat{\gamma}_{s+\mathrm{d}s} = \hat{\delta}$; *otherwise, set* $\hat{\gamma}_{s+\mathrm{d}s} = \hat{\gamma}_s$.

Rule 15. (DL:death$[\alpha, \beta; a, b; r]$.) *For each birth site* x *in* γ_s, *with intensity* 1

- *set* $\delta = \gamma_s \ominus x$,
- *construct* $\hat{\delta}$ *by randomly choosing, with probability* $\frac{1}{2}$, *either of the two possible colourings for* δ,

- *accept* $\hat{\delta}$ *with probability*

$$\begin{aligned}&\exp(-\alpha A([\text{black}[\hat{\delta}] +_{\mathrm{M}} B(r)] \setminus [\text{black}[\hat{\gamma}_s] +_{\mathrm{M}} B(r)]))\\&\quad\times \exp(-\beta r^{-1} A([\delta +_{\mathrm{M}} B(r)] \setminus [\gamma_s +_{\mathrm{M}} B(r)]))\\&\quad\times \exp(-a A([\text{black}[\hat{\delta}] +_{\mathrm{M}} B(r)] \triangle [\text{black}[\hat{\gamma}_s] +_{\mathrm{M}} B(r)]))\\&\quad\times \exp(-b r^{-1} A([\delta +_{\mathrm{M}} B(r)] \triangle [\gamma_s +_{\mathrm{M}} B(r)])),\end{aligned}$$

- *if* $\hat{\delta}$ *is accepted, set* $\hat{\gamma}_{s+\mathrm{d}s} = \hat{\delta}$*; otherwise, set* $\hat{\gamma}_{s+\mathrm{d}s} = \hat{\gamma}_s$.

Thus, the distribution of $\hat{\mathcal{A}}_D^{[\alpha,\beta;r]}$ is invariant and reversible with respect to the above dynamics. Moreover, it is easily seen that the acceptance probabilities in Rules 14 and 15 converge to those in Rules 7 and 8 as $r \to 0$. By taking (19) into account and letting $r \to 0$, by a standard continuity argument (namely that the convergence of densities combined with the convergence of acceptance probabilities implies the weak convergence of stationary laws and ensures the detailed balance conditions in the limit) we find that $\hat{\mathcal{A}}_D^{[\alpha,\beta]}$ is invariant and reversible with respect to the dynamics DL:birth$[\alpha, \beta; a, b]$ and DL:death$[\alpha, \beta; a, b]$ of Rules 7 and 8.

To complete the proof of Theorem 1, it suffices now to establish the remaining uniqueness and convergence statements. However, these follow along the same lines as Proposition 1, by the observation that, in finite volume and regardless of the initial state, the process $\hat{\gamma}_s$ spends a non-null fraction of time in the state 'black' (with no contours and the whole domain D coloured black), and by a standard application of the coupling argument; see the proof of Theorem 1.2 of Liggett (1985, pp. 65–67). The proof is thus complete.

5.2. Proof of Theorem 3

Following the ideas of Schreiber (2004), it is convenient to consider the family $\Gamma_{\mathbb{R}^2}$ of admissible configurations in the plane embedded into the space $G_{\mathbb{R}^2}$ of locally finite, nonnegative Borel measures on $\mathbb{R}^2$, by identifying a configuration $\hat{\gamma} \in \Gamma_{\mathbb{R}^2}$ with the measure

$$M_{\hat{\gamma}}(U) = \text{length}(\gamma \cap U) + A(\text{black}[\hat{\gamma}] \cap U) + N(\gamma \cap U)$$

for Borel sets $U \subseteq \mathbb{R}^2$, where $N(\gamma \cap U)$ stands for the number of vertices of γ falling into U. Endow the space $G_{\mathbb{R}^2}$ with the vague topology defined as the weakest one to make continuous the mappings $\mu \mapsto \int f \,\mathrm{d}\mu$ for all continuous functions f with bounded support. Observe that, in general, $\Gamma_D \not\subseteq \Gamma_{\mathbb{R}^2}$ for $D \subset \mathbb{R}^2$, due to the presence of edges truncated by the boundary. Therefore, in order for our embedding also to be defined for finite-volume configurations, we set $M_{\hat{\gamma}}(D^{\mathrm{c}}) = 0$ for all $\hat{\gamma} \in \Gamma_D$. Note that only the internal vertices of finite-volume configurations are counted in $N(\cdot)$.

Consider the sequence $((-n, n)^2)_{n=1}^{\infty}$ of growing open squares in $\mathbb{R}^2$. By the properties of the basic Arak process (see Section 4 of Arak and Surgailis (1989) and Section 2.1 of Schreiber (2004)), it immediately follows that there exists a finite constant C with

$$\mathrm{E}\, M_{\hat{\mathcal{A}}^{[0,0]}_{(-n,n)^2}}((-n, n)^2) \leq C A((-n, n)^2) \tag{21}$$

for all $n \geq 1$. We will show that the above conclusion can be extended to arbitrary $\alpha \in \mathbb{R}$ and $\beta > 0$, in that there exists a $C^{[\alpha,\beta]} < \infty$ with

$$\mathrm{E}\, M_{\hat{\mathcal{A}}^{[\alpha,\beta]}_{(-n,n)^2}}((-n, n)^2) \leq C^{[\alpha,\beta]} A((-n, n)^2). \tag{22}$$

https://doi.org/10.1239/aap/1134587745 Published online by Cambridge University Press

Below, without loss of generality we assume that $\alpha \geq 0$, which can be done in view of the colour-flip symmetry. Observe first that, from (6),

$$\frac{\partial}{\partial h} \mathrm{E}\, \mathcal{H}^{[\alpha,\beta]}_{(-n,n)^2}(\hat{\mathcal{A}}^{[h\alpha,h\beta]}) = -\operatorname{var}(\mathcal{H}^{[\alpha,\beta]}_{(-n,n)^2}(\hat{\mathcal{A}}^{[h\alpha,h\beta]}_{(-n,n)^2})) < 0,$$

with Hamiltonian $\mathcal{H}^{[\alpha,\beta]}_{(-n,n)^2}$ as in (7). Consequently, taking into account that the area term in the Hamiltonian is bounded by $\alpha A((-n,n)^2)$ and that the Hamiltonian is always positive, we conclude from (21) that the expectation of the edge length term in the Hamiltonian admits an area-order upper bound. It remains to show that this is also the case for the number of vertices. Here, we sketch the argument, omitting standard technical details. Let $\mathrm{Po}(\tau)$ stand for a Poisson-distributed random variable with mean τ. We use the dynamic representation (as discussed in the introduction of this paper and in Section 4 of Arak and Surgailis (1989)) to conclude that, for the basic Arak process $\mathcal{A}_{(-n,n)^2}$, the number of internal extreme-left vertices (with the corresponding sharp angle lying to the right of the vertex) is $\mathrm{Po}(\pi A((-n,n)^2))$. The same applies for the number of internal extreme-right, extreme-upper, and extreme-lower vertices (recall that we do not count the boundary vertices here).

Consequently, the overall number of internal vertices $N(\mathcal{A}_{(-n,n)^2})$ is stochastically bounded by $4\,\mathrm{Po}(4\pi n^2)$ and has mean, of area order, not greater than $16\pi n^2$. In view of the representation (6), and taking into account that the Hamiltonian $\mathcal{H}^{[\alpha,\beta]}_{(-n,n)^2}$ is always positive, since $\alpha \geq 0$, we conclude that

$$\mathrm{P}(N(\hat{\mathcal{A}}^{[\alpha,\beta]}_{(-n,n)^2}) > 4K) \leq \frac{\mathrm{P}(\mathrm{Po}(4\pi n^2) > K)}{\mathrm{E}\exp(-\mathcal{H}^{[\alpha,\beta]}_{(-n,n)^2}(\hat{\mathcal{A}}^{[\alpha,\beta]}_{(-n,n)^2}))} \tag{23}$$

for all $K > 0$. Recall that Poisson distributions exhibit superexponentially decaying tails:

$$\mathrm{P}(\mathrm{Po}(4\pi n^2) > K) \leq \exp\left(-\frac{K}{4}\log\left(\frac{K}{8\pi n^2}\right)\right), \qquad K \geq 64\pi n^2;$$

see Shorack and Wellner (1986, p. 485). Moreover, the negative logarithm of the denominator in (23) exhibits at most area-order growth, which is due to the (easily verified) finiteness of the free energy density for $\mathcal{H}^{[\alpha,\beta]}$:

$$\liminf_{n\to\infty} \frac{1}{(2n)^2} \log \mathrm{E}\exp(-\mathcal{H}^{[\alpha,\beta]}_{(-n,n)^2}(\hat{\mathcal{A}}_{(-n,n)^2})) > -\infty.$$

Consequently, the required area-order bound for $\mathrm{E}\, N(\hat{\mathcal{A}}^{[\alpha,\beta]}_{(-n,n)^2})$ follows from (23) by a direct calculation. This completes the verification of (22).

To proceed with the proof of the theorem, consider the sequence $(M_n^{[\alpha,\beta]})_{n=1}^{\infty}$ of $G_{\mathbb{R}^2}$-valued random elements with laws given by

$$\begin{aligned}\mathcal{L}(M_n^{[\alpha,\beta]}) &= \frac{1}{4\pi(2n)^2}\int_{[0,2\pi)}\int_{(-n,n)^2} \mathcal{L}([\tau_x \circ R_\phi] M_{\hat{\mathcal{A}}^{[\alpha,\beta]}_{(-n,n)^2}})\,\mathrm{d}x\,\mathrm{d}\phi \\ &\quad + \frac{1}{4\pi(2n)^2}\int_{[0,2\pi)}\int_{(-n,n)^2} \mathcal{L}([\Sigma \circ \tau_x \circ R_\phi] M_{\hat{\mathcal{A}}^{[\alpha,\beta]}_{(-n,n)^2}})\,\mathrm{d}x\,\mathrm{d}\phi,\end{aligned} \tag{24}$$

where τ_x stands for the standard translation operator, such that $\tau_x\mu(U) = \mu(U+x)$, while R_ϕ, $\phi \in [0, 2\pi)$, is the rotation by angle ϕ around 0 and Σ is the reflection with respect to some fixed axis passing through the origin. From (22), it follows that

$$\mathrm{E}\, M_n^{[\alpha,\beta]}(U) < \infty$$

https://doi.org/10.1239/aap/1134587745 Published online by Cambridge University Press

for all bounded sets $U \subseteq \mathbb{R}^2$. By applying Corollary A2.6.V of Daley and Vere-Jones (1988, Corollary A2.6.V, p. 632) we conclude that the sequence of random measures $(M_n^{[\alpha,\beta]})_{n=1}^{\infty}$ is uniformly tight in $G_{\mathbb{R}^2}$ and, consequently, contains a subsequence converging in law to some M_∞ corresponding to a whole-plane polygonal field $\hat{\mathcal{A}}_\infty^{[\alpha,\beta]}$. In view of (24), it is clear that

$$\mathcal{L}(\hat{\mathcal{A}}_\infty^{[\alpha,\beta]}) \in \mathcal{G}_\tau(\hat{\mathcal{A}}^{[\alpha,\beta]}),$$

which completes the proof of the theorem.

5.3. Proof of Lemma 1

By definition (13) of the β-tilted contour measure $\Theta^{[\beta]}$, it is enough to establish the assertion of Lemma 1 for the case $\beta = 2$. Thus, we henceforth assume that $\beta = 2$. In order to establish (16), define the continuous-time random walk $(Z_t)_{t \geq 0}$ in $\mathbb{R}^2$ with the following transition mechanism:

- between critical events specified below, move in a constant direction with speed 1;
- with intensity given by four times the covered length element, update the movement direction, choosing the angle $\phi \in (0, 2\pi)$ between the old and new directions according to the density $|\sin(\phi)|/4$.

We start the random walk Z_t at time 0 at a given point x and with a given velocity vector. Moreover, we choose the *loop-closing angle* $\phi^* \in (0, 2\pi)$ according to the density $|\sin(\phi)|/4$ and draw an infinite *loop-closing line* l^* that starts at x and forms the angle ϕ^* with the initial velocity vector. Let $\hat{Z}_t$ be the random walk Z_t killed whenever it hits either its past trajectory or the loop-closing line l^*. The directed nature of the random walk trajectories constructed above requires us to consider, for each contour θ, two oriented instances: $\theta^{\rightarrow}$ (clockwise) and $\theta^{\leftarrow}$ (anticlockwise). We claim that, for $x \in \mathbb{R}^2$ and $\theta \in \mathcal{C}$ with $x \in \text{Vertices}(\theta)$,

$$16\pi \, \mathrm{d}x \mathrm{e}^{-4\,\text{length}(e^*)} \, \mathrm{P}(\hat{Z}_t \text{ reaches } l^* \text{ and the resulting contour falls into } \mathrm{d}\theta^{\rightarrow}) = \Theta^{[2]}(\mathrm{d}\theta), \tag{25}$$

where e^* stands for the last segment of $\theta^{\rightarrow}$, counting x as the initial vertex, which is to coincide with the segment of the loop-closing line l^* joining its intersection point with $\hat{Z}_t$ to x. Clearly, the same relation then holds for $\theta^{\leftarrow}$, and addition of the two versions of (25) (one for $\theta^{\rightarrow}$ and one for $\theta^{\leftarrow}$), which amounts to taking into account the two possible directions in which the random walk can move along θ, will yield $2\Theta^{[\beta]}(\mathrm{d}\theta)$ on the right-hand side of the equation. Relation (16) will then easily follow from the trivial upper bound 1 for the probability on the left-hand side of (25).

To establish (25), we observe that the probability element

$$\mathrm{P}(\hat{Z}_t \text{ reaches } l^* \text{ and the resulting contour falls into } \mathrm{d}\theta^{\rightarrow})$$

is exactly

$$\frac{1}{4[\mu \times \mu](\{(l, l^*) \colon l \cap l^* \in \mathrm{d}x\})} \exp(-4\,\text{length}(\theta \setminus e^*)) \prod_{i=1}^{k} \mathrm{d}\mu(l[e_i]), \tag{26}$$

where $e_1, \ldots, e_k$ are segments of θ, inclusive of e^*, and $l[e_i]$ stands for the straight line determined by e_i. Indeed, the prefactor $(4[\mu \times \mu](\{(l, l^*) \colon l \cap l^* \in \mathrm{d}x\}))^{-1}$ comes from the choice of the lines respectively containing the initial segment of $\theta^{\rightarrow}$ (counting from x) and

l^*, as well as from the choice between two equiprobable directions on each of these lines. For the remaining segments, we use the fact that, for any given straight line l_0, we have

$$\mu(\{l : l \cap l_0 \in \mathrm{d}\ell,\ \angle(l, l_0) \in \mathrm{d}\phi\}) = |\sin\phi|\mathrm{d}\ell\,\mathrm{d}\phi,$$

where $\mathrm{d}\ell$ stands for the length element on l_0 and $\angle(l_0, l)$ denotes the angle between l and l_0; see both Proposition 3.1 and the argument justifying the dynamic representation in Section 4 of Arak and Surgailis (1989). Note that the direction update intensity was set to 4 to coincide with $\int_0^{2\pi} |\sin\phi|\,\mathrm{d}\phi$.

To recover (25), it is now enough to use (26), recall the definitions of Θ and $\Theta^{[2]}$, and observe that $[\mu \times \mu](\{(l, l^*) : l \cap l^* \in \mathrm{d}x\}) = 4\pi\,\mathrm{d}x$ (as follows by standard integral geometry). This completes the proof of (16).

We now let $\tilde{Z}_t$ be the random walk Z_t killed whenever it hits its past trajectory (but not when it hits the loop-closing line l^*). Define

$$\varepsilon = -\lim_{T\to\infty} \frac{1}{T} \log \mathrm{P}(\tilde{\tau} > T), \tag{27}$$

where $\tilde{\tau}$ is the lifetime of $\tilde{Z}_t$ or, in other words, the first moment Z_t hits its past trajectory. The existence of the limit in (27) follows from a standard superadditivity argument (see Section 1.2 of Madras and Slade (1993)), and ε can in fact be regarded as the connective constant for the self-avoiding version of the random walk Z_t (again see Section 1.2 of Madras and Slade (1993)). It is easily checked that $\varepsilon > 0$, since, during each unit of time of its evolution the walk Z_t has a certain positive probability of hitting its past trajectory that is always uniformly bounded away from 0. Indeed, the probability that such a collision happens during the time period [0, 1] is clearly positive and can only increase in later unit time periods because of the growth of the past trajectory. To establish (17), observe that, as in the argument above,

$$\begin{aligned}\Theta^{[2]}(\{\theta : \mathrm{d}x \in \mathrm{Vertices}(\theta),\ \mathrm{length}(\theta) > R\}) \\ &\le 8\pi\,\mathrm{d}x\,\mathrm{P}(\tilde{Z}_t \text{ survives up to time } R/2) \\ &\le 8\pi\,\mathrm{d}x\,\mathrm{P}(\tilde{\tau} > R/2).\end{aligned} \tag{28}$$

Relation (17) then follows from (28), (27), and the observation that

$$\begin{aligned}&\Theta^{[2]}(\{\theta : 0 \in \mathrm{int}\,\theta,\ \mathrm{length}(\theta) > R\}) \\ &\quad \le \sum_{k=0}^{\infty} \Theta^{[2]}(\{\theta \mid \mathrm{Vertices}(\theta) \cap [B(0, k+1) \setminus B(0, k)] \ne \varnothing,\ \mathrm{length}(\theta) > \max(R, k)\}).\end{aligned}$$

This completes the proof of Lemma 1.

Acknowledgements

The author gratefully acknowledges the support of both the Foundation for Polish Science (FNP) and the Polish Minister of Scientific Research and Information Technology, grant 1 P03A 018 28 (2005-2007). Special thanks are also due to an anonymous referee whose suggestions were very helpful in improving the paper and whose comments will be useful in the author's further research.

References

ARAK, T. (1983). On Markovian random fields with finite numbers of values. In *Proc. 4th USSR–Japan Symp. Prob. Theory Math. Statist.* (Tbilisi, USSR, 1982), eds K. Ito and Yu. V. Prokhorov, Springer, New York.

ARAK, T. AND SURGAILIS, D. (1989). Markov fields with polygonal realisations. *Prob. Theory Relat. Fields* **80,** 543–579.

ARAK, T. AND SURGAILIS, D. (1991). Consistent polygonal fields. *Prob. Theory Relat. Fields* **89,** 319–346.

ARAK, T., CLIFFORD, P. AND SURGAILIS, D. (1993). Point-based polygonal models for random graphs. *Adv. Appl. Prob.* **25,** 348–372.

CLIFFORD, P. AND NICHOLLS, G. (1994). A Metropolis sampler for polygonal image reconstruction. Preprint, available at http://www.stats.ox.ac.uk/~clifford/papers/met_poly.html.

DALEY, D. J. AND VERE-JONES, D. (1988). *An Introduction to the Theory of Point Processes*. Springer, New York.

FERNÁNDEZ, R., FERRARI, P. AND GARCIA, N. (1998). Measures on contour, polymer or animal models. A probabilistic approach. *Markov Process. Relat. Fields* **4,** 479–497.

FERNÁNDEZ, R., FERRARI, P. AND GARCIA, N. (2002). Perfect simulation for interacting point processes, loss networks and Ising models. *Stoch. Process. Appl.* **102,** 63–88.

LIGGETT, T. (1985). *Interacting Particle Systems*. Springer, New York.

MADRAS, N. AND SLADE, G. (1993). *The Self-Avoiding Walk*. Birkhäuser, Boston, MA.

NICHOLLS, G. K. (2001). Spontaneous magnetisation in the plane. *J. Statist. Phys.* **102,** 1229–1251.

SCHREIBER, T. (2004). Mixing properties for polygonal Markov fields in the plane. Submitted.

SHORACK, G. R. AND WELLNER, J. A. (1986). *Empirical Processes with Applications to Statistics*. John Wiley, New York.

SURGAILIS, D. (1991). Thermodynamic limit of polygonal models. *Acta Appl. Math.* **22,** 77–102.

https://doi.org/10.1239/aap/1134587745 Published online by Cambridge University Press