THE MAXIMUM TERM AND THE RANK OF AN ENTIRE FUNCTION

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1. For an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, let $M(r,f)$, $\mu(r,f)$, and $\nu(r,f)$ denote the maximum modulus, the maximum term, and the rank for $|z| = r$, respectively. Also, let

$$M_2(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta\right)^{1/2},$$

$$L(f) = \limsup_{n\to\infty} \left|\frac{a_n^{\,2}}{a_{n-1}a_{n+1}}\right|, \quad R_n = \left|\frac{a_{n-1}}{a_n}\right|,$$

and $\lambda(r)$ the lower proximate order relative to $\log M(r,f)$. For the properties of the lower proximate order, we refer the reader to the paper by Shah (**1**).

2. We prove the following theorems.

THEOREM 1. *For an entire function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$,

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\mu(r,f)}{M(r,f)} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\mu(r,f^1)}{M(r,f^1)},$$

where $\mu(r,f^1)$ *and* $M(r,f^1)$ *correspond to* $f^1(z)$, *the derivative of* $f(z)$, *provided* $(n+1)R_n < nR_{n+1}$, *when* $L(f) > 1$.

It is well known that $M_2(r,f) \leqq M(r,f)$. We now obtain an inequality in the opposite direction.

THEOREM 2. *Let* $\epsilon > 0$. *For an entire function* $f(z)$ *of non-null and finite order* ρ,

$$(M_2(r,f))^2(2r^{\rho+\epsilon}+1) + O(W(r,f)) \geqq (W(r,f))^2 \geqq (M(r,f))^2,$$

where $W(r,f) = \sum_{p=0}^{\infty} |a_p| r^p$.

THEOREM 3. *Let* $G(z) = G_1(z)G_2(z)$, *where* $G_1(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $G_2(z) = \sum_{n=0}^{\infty} b_n z^n$ *are two entire functions, such that* $M(r,G) = O(M(r,G_1)M(r,G_2))$. *If* $|a_{n-1}/a_n|$ *is a strictly increasing function of* n, *and* $L(G_1) = 1$, *then*

$$\lim_{r\to\infty} \frac{\mu(r,G)}{M(r,G)} = 0.$$

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THEOREM 4. (i) *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is an entire function of lower order* $\lambda = 0$, *then*

$$\liminf_{r\to\infty} \frac{\nu(r,f)}{r^{\lambda(r)}} = 0.$$

(ii) *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is an entire function such that*

$$\liminf_{r\to\infty} \frac{\nu(r,f)}{\log M(r,f)} = \infty,$$

then $\rho = \infty$.

3. Proof of Theorem 1. *Case* 1. $L(f) = 1$. The maximum term $\mu(r,f) = \operatorname{Sup}_n |a_n| r^n$. Let $n_1, n_2, \ldots, n_k$ be the values assumed by $\nu(r,f)$. Hence, if n_k denotes the rank of the maximum term for $|z| = r$, then it is obvious that

$$g_m - m \log r \geqq g_{n_k} - n_k \log r,$$

where $g_m = -\log |a_m|$, $m \neq n_k$. Hence, on letting $m = n_k - 1$, we have that $R_{nk} \leqq r$. Since the term which has the greatest rank is usually called the maximum term (even when there is more than one term which is equal to it), we obtain $R_{n_k+1} > r$ by letting $m = n_k + 1$. Hence, for $R_{n_k} \leqq r < R_{n_k+1}$, we have that $\mu(r,f) = |a_{nk}| r^{n_k}$ and $\nu(r,f) = n_k$. Clearly,

$$M_2(r,f) = \left(\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \right)^{1/2}.$$

Thus, for $R_{n_k} \leqq r < R_{n_k+1}$, we have that

$$\left(\frac{M_2(r,f)}{\mu(r,f)}\right)^2 > \left(\frac{r}{R_{nk+1}}\right)^2 + \ldots + \left(\frac{r^p}{R_{nk+p}}\right)^2$$
$$> \left(\frac{r}{R_{nk+1}}\right)^2 + \ldots + \left(\frac{r^p}{(R_{nk+1})^p L_1^{p-1}}\right)^2$$
$$> \frac{p}{2},$$

since $R_{n+1} < L_1 R_n$ for $n \geqq n_0$, where $L_1 > 1$. Since L_1 can be chosen as close to 1 as possible and p arbitrarily large, we have that

$$\lim_{r\to\infty} \frac{M_2(r,f)}{\mu(r,f)} = \infty.$$

Hence, *a fortiori*

$$\lim_{r\to\infty} \frac{M(r,f)}{\mu(r,f)} = \infty.$$

Furthermore, $L(f^1) = 1$. Hence, proceeding as above, we have that

$$\lim_{r\to\infty} \frac{M(r,f^1)}{\mu(r,f^1)} = \infty.$$

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Thus, the result is true when $L(f) = 1$.

Case 2. $L(f) > 1$. Clearly, for $E_k = R_{n_k} \leqq r < R_{n_k+1}$,

$$\mu(r, f^1) \geqq \frac{\nu(r,f)\mu(r,f)}{r}. \tag{3.1}$$

Also, for $F_k = ((n_k - 1)/n_k)R_{n_k} \leqq r < (n_k/(n_k + 1))R_{n_k+1}$,

$$\mu(r, f^1) \leqq \frac{\mu(r,f)\nu(r,f^1)}{r}. \tag{3.2}$$

Since $L(f) > 1$, t_k is contained in $E_k F_k$, where

$$t_k = R_{nk} \leqq r < \frac{n_k}{n_k + 1} R_{nk+1}.$$

Hence, for all points r in t_k, $\nu(r, f^1) = \nu(r, f) = n_k$. Thus, from (3.2) we have that

$$\mu(r, f^1) \leqq \frac{\mu(r,f)\nu(r,f)}{r}. \tag{3.3}$$

Let s_k be the segment in which the variation of $\log r$ is less than $K\nu(R/k, f)^{-1/12}$ $(r > R)$. Also, let $S = \sum_{k=1}^{\infty} s_k$, and CS be the complement of S. For points r in CS (**2**, p. 105),

$$rM(r, f^1) \sim M(r,f)\nu(r,f). \tag{3.4}$$

Clearly, the total variation of $\log r$ in t_n tends to infinity with n. Let $T = \sum_{k=1}^{\infty} t_k$. Thus, for all points r in TCS, (3.1), (3.3), and (3.4) hold. Therefore, the result follows from (3.1), (3.3), and (3.4). Hence, this completes the proof.

Proof of Theorem 2. From a well-known inequality of Cauchy we obtain

$$\left(\sum_{p=0}^{n} |a_p|^2 r^{2p}\right)(n + 1) \geqq \left(\sum_{p=0}^{n} |a_p| r^p\right)^2. \tag{3.5}$$

Therefore,

$$\left((M_2(r,f))^2 - \sum_{p=n+1}^{\infty} |a_p|^2 r^{2p}\right)(n + 1) \geqq \left(W(r,f) - \sum_{p=n+1}^{\infty} |a_p| r^p\right)^2.$$

This yields

$$(W(r,f))^2 \leqq (M_2(r,f))^2(n + 1) - (n + 1)\left(\sum_{p=n+1}^{\infty} |a_p|^2 r^{2p}\right) + 2\left(\sum_{p=0}^{n} |a_p| r^p \sum_{p=n+1}^{\infty} |a_p| r^p\right) + \left(\sum_{p=n+1}^{\infty} |a_p| r^p\right)^2.$$

Using (3.5) we have that

$$(W(r,f))^2 \leqq (M_2(r,f))^2(n + 1) + \left(\sum_{p=n+1}^{\infty} |a_p| r^p\right)\left(\left(2\sum_{p=0}^{n} |a_p| r^p\right) - n \sum_{p=n+1}^{\infty} |a_p| r^p\right). \tag{3.6}$$

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Obviously, $|a_p|r^p < p^{-p/(\rho+\epsilon)}r^p$, since

$$\limsup_{n\to\infty} \frac{n \log n}{\log 1/|a_n|} = \rho.$$

Hence,

$$\sum_{p=n+1}^{\infty} |a_p|r^p < \sum_{p=n+1}^{\infty} p^{-p/(\rho+\epsilon)}r^p = \sum_{p=n+1}^{\infty} \left(\frac{r^{\rho+\epsilon}}{p}\right)^{p/(\rho+\epsilon)} \leqq \sum_{p=n+1}^{\infty} \left(\frac{r^{\rho+\epsilon}}{n}\right)^{p/(\rho+\epsilon)}.$$

Let $n = 2[r^{\rho+\epsilon}]$, where $[r^{\rho+\epsilon}]$ denotes the integral part of $r^{\rho+\epsilon}$. Thus, we have that

$$\sum_{p=n+1}^{\infty} |a_p|r^p < \sum_{p=n+1}^{\infty} 2^{-p/(\rho+\epsilon)} = O(1). \tag{3.7}$$

From (3.6) and (3.7) we obtain the required result.

Proof of Theorem 3. Let $G(z) = \sum_{n=0}^{\infty} c_n z^n$. Then

$$c_n = b_0 a_n + b_1 a_{n-1} + \ldots + b_n a_0.$$

Now,

$$\begin{aligned} |c_n|r^n &\leqq |b_0|\,|a_n|r^n + |b_1|r|a_{n-1}|r^{n-1} + \ldots + |b_n|r^n|a_0| \\ &= \sum_{s=0}^{n} |b_s|r^s|a_{n-s}|r^{n-s} \\ &\leqq \left(\sum_{s=0}^{n} |b_s|^2 r^{2s}\right)^{1/2} \left(\sum_{s=0}^{n} |a_s|^2 r^{2s}\right)^{1/2} \\ &< M_2(r, G_1)M_2(r, G_2). \end{aligned}$$

This is true for all n. Hence,

$$\mu(r, G) < M_2(r, G_1)M_2(r, G_2). \tag{3.8}$$

Since $R_n = |a_{n-1}/a_n|$ is a strictly increasing function of n, $G_1(z) = P(z) + A\phi_1(z)$, where A is a constant, $P(z)$ a polynomial, and

$$\phi_1(z) = \sum_{n=1}^{\infty} z^n e^{i\theta_n}/R_1R_2 \ldots R_n.$$

Thus,

$$M(r, G_1) \sim M(r, \phi_1)|A| \quad \text{and} \quad M_2(r, G_1) \sim M_2(r, \phi_1)|A|. \tag{3.9}$$

Let $\phi(z) = \sum_{n=1}^{\infty} z^n/R_1R_2 \ldots R_n$. For this function,

$$M(r, \phi) = \sum_{n=1}^{\infty} r^n/R_1R_2 \ldots R_n \quad \text{and} \quad M_2(r, \phi) = \left(\sum_{n=1}^{\infty} r^{2n}/(R_1R_2 \ldots R_n)^2\right)^{1/2}.$$

Hence,

$$\lim_{r\to\infty} \frac{M_2(r, \phi)}{M(r, \phi)} = 0, \tag{3.10}$$

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since $R_n \sim R_{n+1}$. Also, we observe that $(M_2(r, \phi_1))^2 = M_2(r^2, h)$ and $(M(r, \phi_1))^2 \geqq M(r^2, h)$, if $h(z) = \sum_{n=1}^{\infty} z^n/(R_1R_2 \ldots R_n)^2$. Hence,

$$\frac{M_2(r, \phi_1)}{M(r, \phi_1)} \leqq \left(\frac{M_2(r^2, h)}{M(r^2, h)}\right)^{1/2}. \tag{3.11}$$

From (3.9), (3.10), and (3.11), we have that

$$\lim_{r\to\infty} \frac{M_2(r, G_1)}{M(r, G_1)} = 0. \tag{3.12}$$

Therefore, from (3.8), (3.12), and the hypothesis, we have that

$$\lim_{r\to\infty} \frac{\mu(r, G)}{M(r, G)} = 0.$$

This is the required result.

Proof of Theorem 4. (i) Let us suppose that

$$\liminf_{r\to\infty} \nu(r,f)/r^{\lambda(r)} = A > 0.$$

Thus, $\nu(r,f) > (A - \epsilon)r^{\lambda(r)}$ for $r \geqq r_0$. This yields $\lim_{r\to\infty} \log \mu(r,f)/r^{\lambda(r)} \geqq A/\lambda$, since

$$\log \mu(r,f) = \int_0^r \frac{\nu(x,f)}{x}\, dx.$$

Now,

$$\liminf_{r\to\infty} \log \mu(r,f)/r^{\lambda(r)} \leqq \liminf_{r\to\infty} \log M(r,f)/r^{\lambda(r)} = 1.$$

Thus, $\lambda > 0$. However, from the hypothesis, we have that $\lambda = 0$. Hence the result is proved.

(ii) Let us suppose that $\rho < \infty$. We can choose a positive number $\alpha > \rho$ such that $\int_{r_0}^{r} \nu(x,f)/x^\alpha\, dx$ is convergent. Also, from the hypothesis, we have that $\nu(r,f) > \sigma(r) \log M(r,f)$ for $r \geqq r_0$, where $\sigma(r) \to \infty$, as $r \to \infty$. Integrating from r_0 to r, with respect to r, we obtain, after dividing by r, $M/\sigma(r) > \log M(r_0,f)/(\alpha - 1)r_0^{\alpha-1}$, where r_0 is large but suitably fixed. Clearly, $\alpha \geqq 1$; otherwise, $\int_{r_0}^{\infty} \nu(x,f)/x^\alpha\, dx$ is divergent. Letting $r \to \infty$ we obtain a contradiction, and hence the result is proved.

References

1. S. M. Shah, *A note on lower proximate orders*, J. Indian Math. Soc. (N.S.) *12* (1948), 31–32.
2. G. Valiron, *Lectures on the general theory of integral functions*, Translated by E. F. Collingwood (Imprimerie et Librairie, Edouere Privat, Toulouse, 1923).

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