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CLOSED INCOMPRESSIBLE SURFACES IN 2-GENERATOR HYPERBOLIC 3-MANIFOLDS WITH A SINGLE CUSP

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A knot K is said to have tunnel number 1 if there is an embedded arc A in S^3, with endpoints on K, whose interior is disjoint from K and such that the complement of a regular neighbourhood of $K \cup A$ is a genus 2 handlebody. In particular the fundamental group of the complement of a tunnel number one knot is 2-generator. There has been some interest in the question as to whether there exists a hyperbolic tunnel number one knot whose complement contains a closed essential surface. The aim of this paper is to prove the existence of infinitely many 2-generator hyperbolic 3-manifolds with a single cusp which contain a closed essential surface. One such example is a knot complement in $\mathbf{RP}^3$. The methods used are of interest as they include the possibility that one of our examples is a knot complement in S^3.

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1. Introduction

By a hyperbolic 3-manifold we shall always mean a complete orientable hyperbolic 3-manifold of finite volume. A hyperbolic 3-manifold M is called *n-generator* if the minimal number of elements required to generate $\pi_1(M)$ is n. It is the aim of this note to prove the following result. Recall a surface embedded and properly embedded in a compact orientable 3-manifold with non-empty boundary is called *essential* if it is incompressible, boundary incompressible and non-boundary parallel.

Theorem 1. *There exist infinitely many 2-generator hyperbolic 3-manifolds with a single cusp which contain a separating closed essential surface.*

Our interest in such a phenomena was motivated by a question of Cameron Gordon concerning tunnel number one knots, namely: does there exist a hyperbolic tunnel number one knot whose complement in S^3 contains a closed essential surface; such a surface necessarily separates. Recall that a knot K has tunnel number one if there is an embedded arc A in S^3, with endpoints on K, whose interior is disjoint from K and such that the complement of a regular neighbourhood of $K \cup A$ in S^3 is a genus 2 handlebody. In particular a tunnel number one knot complement is 2-generator. Our examples are certain knot complements in Lens Spaces, including the possibility of S^3. In fact our examples can be thought of as "tunnel number 1" knots in Lens Spaces, as it follows from their construction that they arise by attaching a 2-handle to a genus 2 handlebody.

With reference to the question stated above concerning tunnel number one knots; it is known that there exist tunnel number one knots whose complements in S^3 contain an essential torus, see [15]. However, these are of course non-hyperbolic. Indeed tunnel number one knots whose complements contain an essential torus are classified in [15]. In addition every 2-bridge knot has tunnel number one, so it follows from [13], that these tunnel number one knots cannot contain a closed essential surface in their complements. Now if there is an example of a hyperbolic tunnel number one knot with a closed essential surface in its complement, there are certain geometric restrictions forced on the surface by the existence of a hyperbolic structure. For example by [14], the surface cannot be totally geodesic. However, little else seems to be known.

The remainder of the paper is organized as follows. In Section 2 we gather some standard facts about 2-bridge links which will be required, together with some preliminary lemmas. In addition, we prove in Theorem 2 a result which limits the number of non-hyperbolic Dehn surgeries on one component of "most" hyperbolic 2-bridge link complements. In Section 3 we prove Theorem 1 and in the final section we give some explicit examples of manifolds as stated in Theorem 1, one of which is a knot complement in $\mathbf{RP}^3$.

2. Preliminaries on 2-bridge links

Our examples will be constructed by performing Dehn surgery on one component of 2-bridge link complements. We first recall some standard facts abour 2-bridge links. Further details can be found in [6, Chapter 12].

2.1. Every 2-bridge link has a presentation as a 4-plat which is constructed as follows. Let L be a 2-bridge link with normal form (α,β); here α and β are coprime integers with α positive and even, β odd and $-\alpha<\beta<\alpha$. We can expand β/α as a continued fraction, with m odd:

$$\beta/\alpha=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots a_{m-1}+\cfrac{1}{a_m}}}}.$$

We will denote this by $[a_1,a_2,\ldots,a_m]$. From this continued fraction expansion we can construct the 4-plat, $\sigma_2^{a_1}\sigma_1^{-a_2}\ldots\sigma_2^{a_m}$, using the standard braid group generators σ_1 and σ_2, where the a_i's are as above and m is odd; see Figure 1. It is well-known that this 4-plat is equivalent to the 2-bridge link L above, see [6]. Indeed, equivalences of 2-bridge knots and links can be transformed to an equivalence of 4-plats, which in turn is related to "equivalence" of continued fractions, see [6] for details.

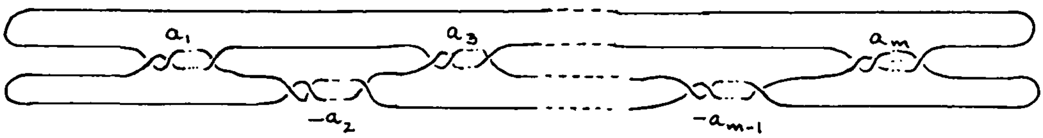

FIGURE 1

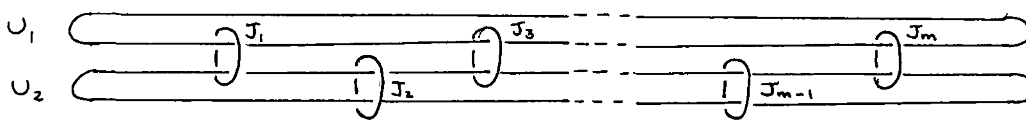

FIGURE 2

Notation. Let L be a 2-bridge link with normal form (α,β). We shall denote the complement of L in S^3 by $M(\alpha,\beta)$, and any continued fraction $[a_1,\ldots,a_m]$ for β/α as above we will refer to as being associated to $M(\alpha,\beta)$.

Using this description of 2-bridge links we have the following lemma which is implicit in Chapter 4 of [**1**].

Lemma 1. *Let $M(\alpha,\beta)$ be as above with associated continued fraction expansion $[a_1,a_2,\ldots,a_m]$, with m odd. Then $M(\alpha,\beta)$ is obtained by performing $(1,(-1)^{1+i}a_i)$-Dehn surgery on the component J_i of the hyperbolic link L_m shown in Figure 2.*

Proof. Consider the 4-plat presentation of L defined by the continued fraction $[a_1,a_2,\ldots,a_m]$. Evidently as the components J_i are unknotted $M(\alpha,\beta)$ is obtained by the sequence of Dehn surgeries described. That L_m is a hyperbolic link is proved in [**1**, Lemma 4.2]. Briefly, it is shown in [**1**, Lemma 4.2] that $S^3\backslash L_m$ can be built up by a succession of cut-and-paste operations called *belted sums* and these preserve hyperbolicity, see [**1**] and [**2**]. This operation involves cutting link complements along certain embedded twice punctured discs (which are necessarily totally geodesic by [**2**]) and then pasting (see [**1**] and [**2**] for further details). □

Remark. In fact it is shown in [**1**, Lemma 4.2] that $S^3\backslash L_m$ is obtained from the belted sum of $(m-1)$ copies of the complement of the link V shown in Figure 3, and this link is hyperbolic, since it is simply a different projection of the Borromean rings.

Theorem 1 will be derived from our next result, which is of independent interest.

Theorem 2. *There exists an integer m_0, such that for all m odd $>m_0$, and sufficiently large positive integers $a_1,\ldots,a_m$, all but $(1,0)$-Dehn surgery on one component of $M(\alpha,\beta)$ with associated continued fraction $[a_1,\ldots,a_m]$ is a hyperbolic 3-manifold.*

The proof of Theorem 2 occupies most of Section 2.2. The idea of the proof of

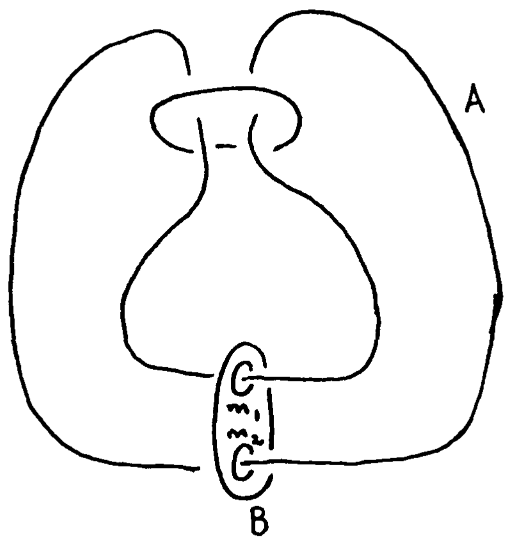

FIGURE 3

Theorem 2 is to get a description of the "shape" of the cusps of $S^3\backslash L_m$ which remain true cusps under the sequence of Dehn surgeries given above and apply the 2π-Theorem of Gromov and Thurston, see [11] or [5]. We therefore need to describe in more detail the geometry of these cusps of $S^3\backslash L_m$. This is intimately related to the geometry of the cusp C_A of $S^3\backslash V$ corresponding to the component labelled A in Figure 3. Let us briefly recall some salient points.

2.2. Let M be hyperbolic 3-manifold which is diffeomorphic to the complement of a link in S^3 and C a cusp of M. The *homological cusp shape* of C is defined as follows, cf. [3]. Let ∞ be the centre of a horoball covering C, and let T_μ and T_λ be the two covering transformations which fix ∞ which are lifts of the meridian and longitude of the link component corresponding to C. Then T_μ and T_λ act by translation on the complex plane by complex numbers $\mathscr{M}$ and l respectively. The ratio $l/\mathscr{M}$ is a complex number which describes the parallelogram which is a fundamental domain for the action of the group $\langle T_\mu, T_\lambda\rangle$ acting on the complex plane. In particular the magnitude of $l/\mathscr{M}$ describes the ratio of the lengths of two adjacent sides and the argument the angle between two adjacent sides. This ratio is the homological cusp shape of C. In the sequel we will refer to this as simply the *shape of the cusp*. In [3] it is the ratio $\mathscr{M}/l$ that is defined to be the homological cusp shape.

Returning to the consideration of the geometry of the cusps of $S^3\backslash L_m$. It is known that (cf. [3]) the shape of each cusp of the complement of the Borromean rings is $2i$. In particular the shape of the fundamental domain (as above) for each cusp is rectangular. With this we can describe the shapes of the cusps of $S^3\backslash L_m$ corresponding to the components labelled U_1 and U_2 in Figure 2. In the sequel we will use $U_1(m)$ and $U_2(m)$

to denote the cusps corresponding to the link components labelled U_1 and U_2 of L_m in Figure 2.

Lemma 2. *Denote the cusp shape of $U_i(m)$ by the complex number $r_i(m)$. Then $|r_i(m)|$ gets arbitrarily large as $m \to \infty$.*

Proof. Let us first consider L_3. In this case $S^3 \backslash L_3$ is obtained from the belted sum of two copies of $S^3 \backslash V$. That is, we cut open $S^3 \backslash V$ along the twice punctured disc S bounded by the component of V labelled B in Figure 3 and then paste two copies of the result together along S—see Figure 4. Indeed, the manifold $S^3 \backslash L_3$ is simply the double cover of $S^3 \backslash V$ which is obtained as the double branched cover of S^3 branched over the component B of V. In particular the other two cusps of $S^3 \backslash V$ are each covered by two tori whose cusp shapes are those of C_A. From this we deduce the shape of the cusps $U_i(3)$ is $2i$ for $i = 1, 2$.

We can also analyze the effect of the cut-and-paste as follows, this will be useful in subsequent discussions. Let m_1 and m_2 be the meridian curves shown in Figure 3. The cut-and-paste operation described above that results in $S^3 \backslash L_3$ cuts a torus corresponding to a horospherical cross-section of the cusp C_A into a pair of annuli as shown in Figure 5.

The effect on a fundamental domain for the group of translations, as described above, corresponding to the cusp C_A of this pasting is described in Figure 6.

Let us now consider the case of $S^3 \backslash L_5$; this is obtained as the belted sum of two copies of $S^3 \backslash L_3$ along the twice punctured disc labelled S' in Figure 4. Repeating the analysis of the cusps in this case we see that fundamental domains for the cusps $U_1(5)$ (respectively $U_2(5)$) will consist of two copies of the rectangle which is the fundamental domain for $U_1(3)$ (respectively $U_2(3)$). Also note that the meridional translations remain the same. In particular the shape of the cusps in this case is $4i$.

Repeating this procedure for arbitrary large m it is clear that fundamental domains for the cusps $U_1(m)$ and $U_2(m)$ will continue to be rectangles, and it can be checked in analogous manner to the above that $r_i(m) = (m-1)i$. It is clear from this that $|r_i(m)|$ will get arbitrarily large with m. □

We are now in a position to prove Theorem 2.

Proof of Theorem 2. By Thurston's theory of hyperbolic Dehn surgery [**16**], for a_i $(i = 1, \ldots, m)$ sufficiently large the geometry of the manifolds obtained by the sequence of Dehn surgeries on the cusps corresponding to the components $J_1, \ldots, J_m$ of L_m stated in Lemma 1 approximates that of $S^3 \backslash L_m$. In particular for sufficiently large a_i's the shape of the cusps of $M(\alpha, \beta)$ is approximately that of $U_1(m)$ and $U_2(m)$. Thus the fundamental domains of the cusps of $M(\alpha, \beta)$ are approximately rectangular and by Lemma 2 the shape of these cusps can be made arbitrarily large by choosing m large.

In the remainder of the proof let us fix attention on the cusp of $M(\alpha, \beta)$ which approximates $U_1(m)$. Let us denote this cusp by C. An analogous argument applies to the other cusp.

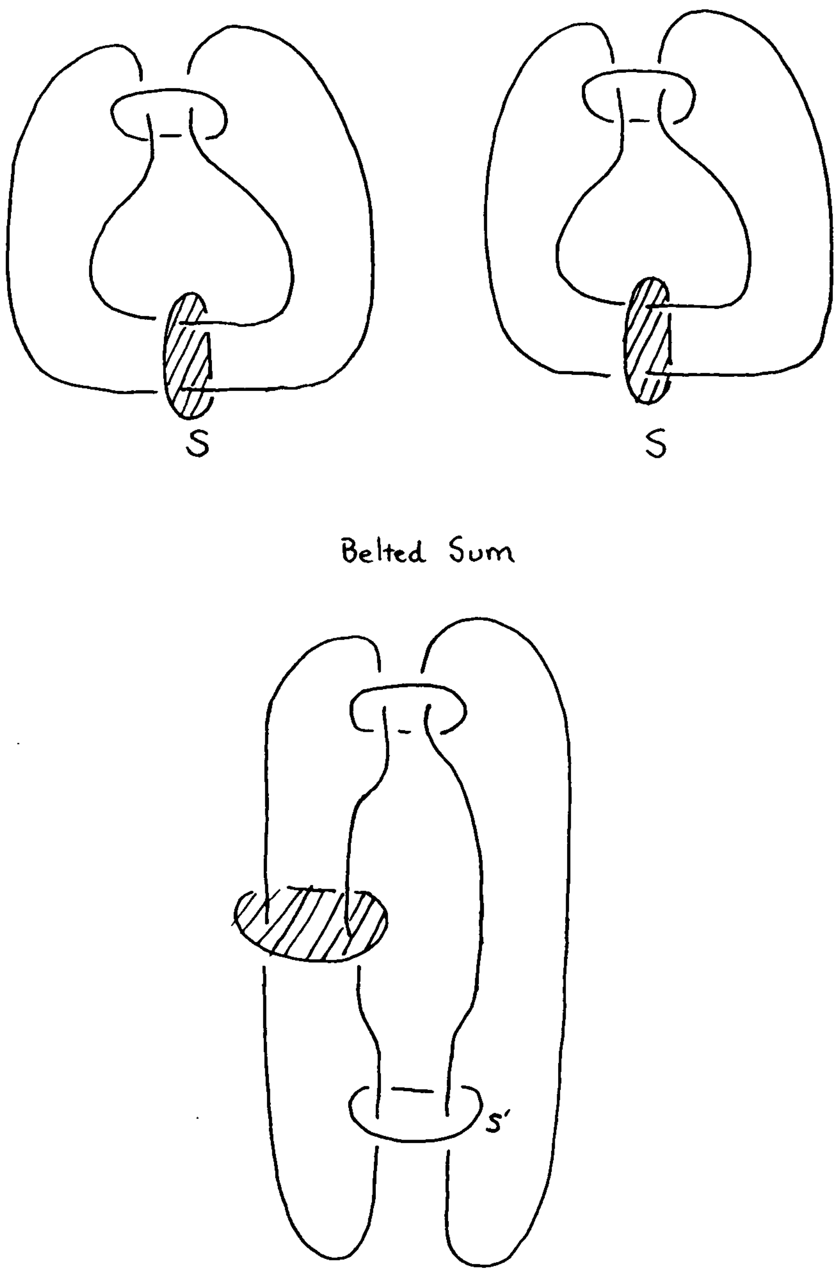

FIGURE 4

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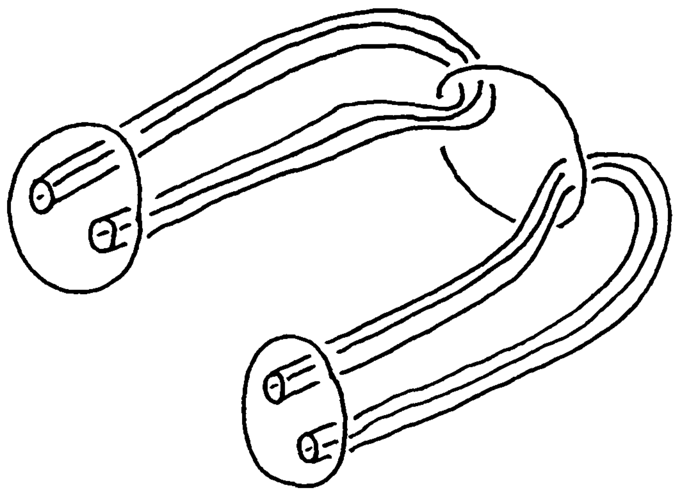

FIGURE 5

Lifting C to $\mathbf{H}^3$, choose a maximal cusp corresponding to C and arrange so that one of the horoballs is centred at ∞. In addition, we can arrange it so that the horosphere, which we shall denote by $\mathscr{H}$, bounding this horoball is at Euclidean height one above the complex plane. Let T_μ and T_λ be the lifts of the meridian and longitude which generate $\pi_1(C)$ acting by translation on $\mathscr{H}$. Since the fundamental domain for $G=\langle T_\mu, T_\lambda\rangle$ acting on $\mathscr{H}$ is approximately rectangular the shortest translation in G is simply that given by T_μ—choosing the integers a_i suitably large—and this (by choice of $\mathscr{H}$) is at least 1.

By choosing m large enough, it follows from Lemma 2 that the Euclidean length of a simple closed curve $pT_\mu+qT_\lambda$ on the torus boundary component corresponding to C will be at least 2π whenever $q\neq 0$. Thus by the 2π-Theorem of Gromov and Thurston, cf. [**11**] and [**5**], the manifold obtained by any Dehn surgery on C other than (1,0)-Dehn surgery will carry a metric of negative curvature. If a manifold obtained by such a Dehn surgery is not the solid torus (which of course admits a metric of negative curvature) then the manifold is irreducible by the Cartan–Hadamard Theorem (see [**4**, Chapter 2] for example) and any embedded incompressible torus is boundary parallel, see [**4**, Chapter 10] for example. Thus the manifold is atoroidal and Haken. Hence by Thurston's Geometrization Theorem for Haken manifolds, the manifold constructed is hyperbolic.

It remains to rule out the possibility that we could obtain the solid torus by Dehn surgery on one component of $M(\alpha,\beta)$. Let us denote the boundary components of $M(\alpha,\beta)$ by T_1 and T_2, where the cusp corresponding to T_i approximates that of the component $U_i(m)$ of L_m. Suppose that (p,q) Dehn surgery on T_1 yields a solid torus. Then every Dehn surgery on T_2 will produce a Lens Space. However, by the argument just used we know that large Dehn surgeries on T_2 have a negligible effect on the shape

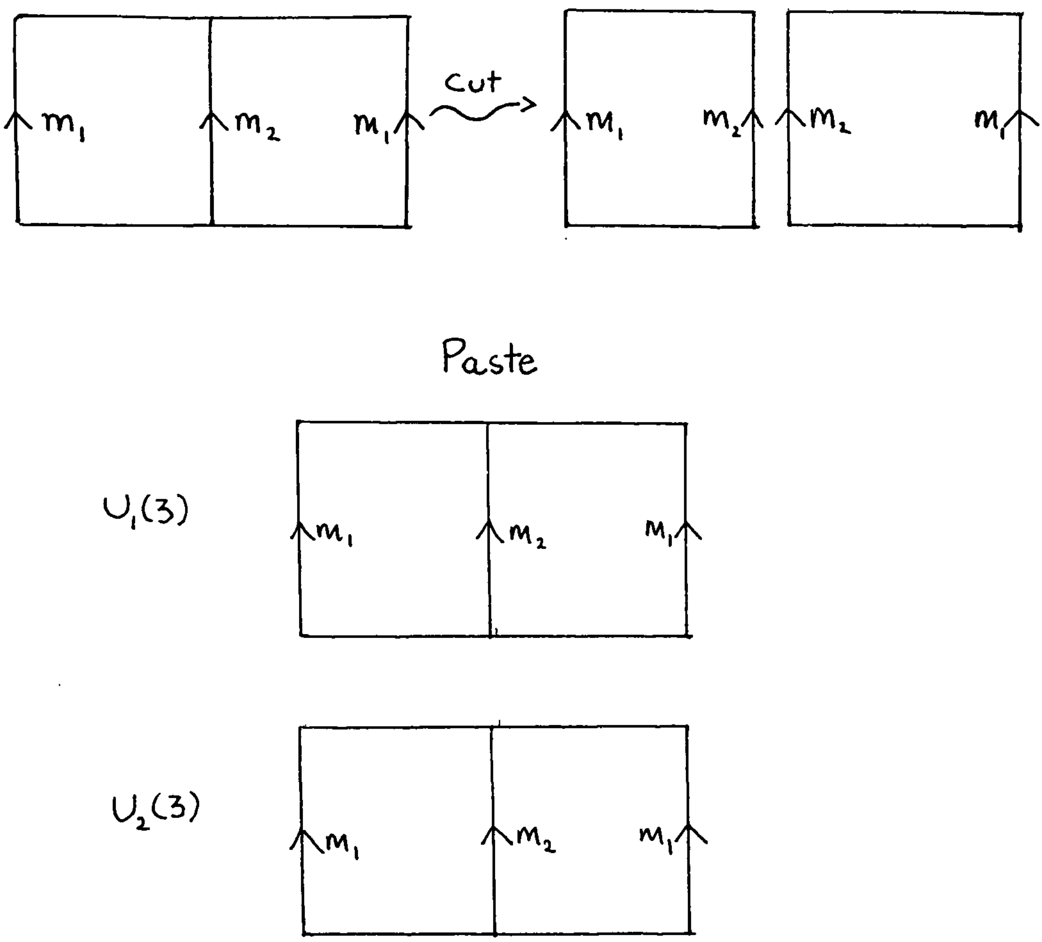

FIGURE 6

of the cusp corresponding to T_1. Hence for large enough Dehn surgeries on T_2, (p,q) Dehn surgery on T_1 will admit a metric of negative curvature. In particular, the universal cover will be $\mathbf{R}^3$ by the Cartan–Hadamard Theorem, thus excluding the possibility of a Lens Space. This complete the proof of Theorem 2. □

3. Proof of Theorem 1

We require the following Lemma which is simply a special case of Theorem 2 of **[10]**. We shall give a proof for the sake of completeness.

Lemma 3. *Let L be a hyperbolic 2-bridge link with 2-bridge normal form* (α,β). *Then*

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$M(\alpha,\beta)$ contains an embedded properly embedded incompressible surface S whose boundary meets only one boundary component of $M(\alpha,\beta)$. Moreover, we may choose S to be separating.

Proof. Let X denote the component of the character variety of $M(\alpha,\beta)$ containing the complete structure. The complex dimension of X is at least two, see [**16**, Chapter 5]; indeed as $M(\alpha,\beta)$ has no closed essential surface it is exactly 2, cf. [**9**, Proposition 3.2.3]. Let the two torus boundary components of $M(\alpha,\beta)$ be labelled T_1 and T_2, and let p be a non-trivial element of $\pi_1(T_1)$. Under the complete representation p becomes parabolic. Let us assume that under the complete representation the trace of p is 2 as opposed to -2—an analogous argument works for T_2.

As in [**9**] let $I_p: X \to \mathbf{C}$ be the map defined by $I_p(\chi)=\chi(p)$. By standard results in algebraic geometry the pre-image of every point is an affine algebraic set all of whose components have dimension at least one.

Let C be the component of the pre-image of 2 under I_p which contains the character corresponding to the complete representation. One can check that C has dimension exactly one. Thus C is a complex curve of characters all of which satisfy $I_p(\chi)=2$. In brief the trace of p under the representation associated to these χ's remains constant with value 2. Furthermore, if x is any other element of $\pi_1(T_1)$, this also must satisfy $\chi(x)=2$, for any $\chi \in C$, since there is a Zariski open subset of C for which this is true, namely characters associated to representations induced by hyperbolic Dehn surgery on the component T_2.

Now let $\tilde{C}$ be the smooth model of the projective completion of C, cf. [**9**]. From above it follows that on degenerating to an ideal point of $\tilde{C}$ the characters remain constant with value 2. By [**9**], we deduce a splitting of the group for which $\pi_1(T_1)$ lies in a vertex stabilizer. Hence by standard 3-manifold techniques, see [**9**, Proposition 2.3.1], we can find an incompressible, boundary incompressible surface which is disjoint from T_1. Moreover, as a 2-bridge link complement cannot contain a closed essential surface (see for instance [**13**]) the boundary of the surface constructed is non-empty. Let us denote the surface by S_1 and assume its boundary curves are all homotopic to the simple closed non-contractible curve g_1 on T_2. That the surface is separating follows by an elementary homology calculation given in [**10**]. For completeness we sketch the argument.

First of all notice that there are at least two incompressible boundary incompressible surfaces meeting only the torus T_2. To see this observe that if $q \in \pi_1(T_2)$ then $I_q: X \to \mathbf{C}$ defined by $I_q(\chi)=\chi(q)$ is non-constant on C. For if it were the case that I_q were constant on C, then since at the character corresponding to the complete structure q is parabolic it would follow that I_q would take the value -2 or 2 on C. However this yields degenerations which remain constant with values 2 on T_1 and ± 2 on T_2. Thus by the results of [**9**] it would follow that $M(\alpha,\beta)$ would contain a closed essential surface, which as we noted above is false. It follows from this discussion that there exists another way of degenerating to an ideal point of $\tilde{C}$ to give a distinct boundary slope on T_2.

Let S_2 be the surface that corresponds to this second boundary slope constructed above. Let boundary components of S_2 be parallel to g_2. Assume that both of S_1 and S_2 are non-separating. In what follows all coefficients for homology are rational. As $H_2(T_2)$ surjects onto $H_2(M(\alpha,\beta))$ it follows that the map $H_2(M(\alpha,\beta),T_2)\to H_1(T_2)$ is injective. Since both of the surfaces are non-separating, it follows that the homology class of the boundary of S_i, ∂S_i in ∂T_2 is k_ig_i for non-zero integers k_i, $i=1$ and 2. However, a standard intersection argument shows that for surfaces meeting only one boundary component, there is at most one non-zero class in $\operatorname{Ker}\{i_*:H_1(T_2)\to H_1(M)\}$ and this is a contradiction. Whence one of the surfaces must be separating. □

To complete the proof of Theorem 1, we argue as follows. We shall be using the contents of [**8**, Chapter 2] and the reader is referred there for all terminology.

Observe that by Lemma 3 any hyperbolic 2-bridge link complement $M(\alpha,\beta)$ (with the two torus boundary components of $M(\alpha,\beta)$ labelled by T_1 and T_2) contains a properly embedded incompressible surface S with non-empty boundary meeting only one boundary component of $M(\alpha,\beta)$. Let the boundary slope of S be p/q, and assume that ∂S lies entirely on T_2. We remark that we will always assume that S has been chosen so that it is minimal with respect to the action, i.e., it has minimal number of boundary components.

Claim 1. There are infinitely many Dehn surgeries on T_1 for which S remains incompressible.

Denote the manifold obtained by doubling $M(\alpha,\beta)$ along T_2 by $D(\alpha,\beta)$. Then the double of S along its boundary yields a closed surface $DS\subset D(\alpha,\beta)$. By the minimality of S, DS is incompressible. Let us fix attention on one copy of T_1, say T, in the boundary of $D(\alpha,\beta)$. To prove the claim we use the contents of Section 2.4 of [**8**]. The argument splits into two cases depending on the existence of certain annuli as we now describe.

If there is no annulus in $D(\alpha,\beta)$ with one boundary component on DS and the other on T, then [**8**, Theorem 2.4.2] implies that if $r_1=p_1/q_1$ and $r_2=p_2/q_2$ are slopes on T with $\triangle(r_1,r_2)>2$ (where $\triangle(x,y)$ denotes the geometric intersection number of a pair of slopes x and y) DS remains incompressible in at least one of (p_1,q_1) or (p_2,q_2) Dehn surgery on T. If there is an annulus of the type described above then in this case, denoting the slope of the boundary curve of the annulus on T by $r_0=p_0/q_0$, we can apply [**8**, Theorem 2.4.3] to deduce that for any slope $r=p/q$ with $\triangle(r_0,r)>1$, DS remains incompressible in the manifold obtained by (p,q)-Dehn surgery on T. Note that the case of $T^2\times I$ of Theorem 2.4.3 of [**8**] is excluded, as $D(\alpha,\beta)$ is the double of a hyperbolic manifold.

In either case we can conclude that there are infinitely many Dehn surgeries on T that keep DS incompressible. In particular splitting the manifold open again, we deduce infinitely many Dehn surgeries on T_1 for which S remains incompressible which justifies the claim.

Claim 2. Choose L so that a continued fraction $[a_1,\ldots,a_m]$ associated to $M(\alpha,\beta)$ satisfies the hypothesis of Theorem 2. Then the surface S is non-planar.

This follows from a similar analysis to that above. Briefly, observe that by performing a sufficiently large Dehn surgery on T_1 we may arrange that the shape of the cusp corresponding to T_2 is not changed much. Therefore, any subsequent nontrivial surgery on T_1 gives a manifold of negative curvature, which has as its universal cover Euclidean space; in particular, this violates all the possibilities of 2.3.1 in [**8**] which establishes the claim.

The claims established above have shown the existence of (infinitely many) 2-bridge link complements containing an incompressible, boundary incompressible surface meeting only one boundary component (in the convention established, it is the component T_2) of the link complement which remains incompressible upon infinitely many Dehn surgeries on T_1.

By Hatcher's theorem on boundary slopes [**12**], there can be at most finitely many incompressible, boundary incompressible embedded surfaces meeting only the boundary component T_1 of $M(\alpha, \beta)$. Combining this observation, Claim 1 and Thurston's Hyperbolic Dehn Surgery Theorem [**16**], it follows that there is some Dehn surgery on T_1, which is a hyperbolic 3-manifold containing no essential closed surfaces and for which S remains incompressible. By Claim 2 S is non-planar, therefore by [**8**, Proposition 2.2.1] Dehn surgery on the boundary slope p/q of S yields a closed manifold with an essential surface $\hat{S}$. If we now remove the surgery torus from T_1, $\hat{S}$ remains incompressible.

To summarize; we have constructed a compact, orientable 3-manifold, say M with torus boundary containing a closed essential surface. To complete the proof we need to exclude the possibility that the boundary slope p/q (as above) is infinity. For then by a judicious choice of L, (as in Claim 2) Theorem 2 shows that M admits a metric of negative curvature and is therefore hyperbolic. As a 2-bridge link complement is 2-generator, the manifold obtained by Dehn surgery is also 2-generator. In addition as L is 2-bridge, both components of L are unknotted, thus the result of (p, q) Dehn surgery on one component yields a knot complement in a Lens Space—including the possibility of S^3.

To prove that infinity is not a boundary slope of a surface meeting only one boundary component of $M(\alpha, \beta)$ we argue as follows. As above assume the boundary of the surface (denoted as above by S) lies entirely on T_2. Doing 1/0-Dehn surgery on T_2 will produce a solid torus, so that a subsequent Dehn surgery on T_1 will produce a Lens Space. Arguing as above we can perform p_0/q_0-Dehn surgery on T_1 such that the resultant manifold is hyperbolic, does not contain a closed essential surface and the surface S remains incompressible. However, as noted above 1/0-Dehn surgery will then produce a Lens Space. This contradicts the possible outcomes of surgery on a boundary slope exhibited by Theorem 2.0.3 of [**8**]. □

4. An example

The statement of Theorem 2 might suggest that one needs to take rather a complicated 2-bridge, link in order to obtain an example. However in practice using Snappea (see [**3**] for a discussion of this program), one can find 2-bridge link complements in the link tables whose cusp shapes make them amenable to the methods indicated in Section 2 and Section 3. For example the link 9^2_9 yields examples. We briefly sketch the calculation. Full details of the ideas that are the basis of this method of calculating boundary slopes can be found in [7].

Let L denote the link 9^2_9, and let x and y be a pair of meridional generators for the fundamental group of the complement of L. We seek to find a curve of representations for which the trace of y remains bounded. As in the proof of Lemma 3, we see that to do this it suffices to consider deformations of the complete structure keeping the element y parabolic. As discussed in [7], standard computations using elimination theory show that the slopes of incompressible boundary incompressible surfaces which arise from surfaces meeting only the boundary component whose meridian is x are 0, -2 and ± 4. Using Snappea one may compute the shape of this cusp and one finds that all the manifolds obtained by doing Dehn surgeries on the boundary slopes are hyperbolic manifolds. As observed in the proof of Theorem 1, these manifolds are knot complements in Lens Spaces. The boundary slope -2 gives an example of a knot complement in $\mathbf{RP}^3$.

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