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# MOVING ERGODIC THEOREMS FOR SUPERADDITIVE PROCESSES

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ABSTRACT. Let $\tau = (\tau_u)_{u\in\mathbb{Z}_+^d}$ be a semigroup of measure preserving transformations on a measure space $(\Omega, \mathcal{F}, \mu)$. The main result of the paper is the proof of a.e. convergence for the moving averages

$$\frac{1}{\#I_n}F_{I_n}(\omega)$$

where $\{F_{I_n}\}$ is a superadditive process and $\{I_n\}$ is a sequence of cubes in $\mathbb{Z}_+^d$ satisfying the "cone-condition". The identification of the limit is given. A moving local theorem is also proved.

1. **Introduction.** In [7] A. Nagel and M. Stein developed a method to investigate the pointwise convergence for general approach regions in harmonic analysis which was later simplified by J. Sueiro [9]. These techniques have, in turn, been adapted to deal with convergence of "moving" averages in Ergodic Theory by A. Bellow, R. Jones and J. Rosenblatt [4]. We refer to [1] for a more up to date list of references in the applications of these methods to Ergodic Theory. The purpose of this article is to extend some of these results to the superadditive setting. More specifically, in Theorem 3.3 we prove pointwise convergence for multiparameter superadditive processes when the sequence is indexed by a family of cubes (in $\mathbb{Z}_+^d$) satisfying a condition which is equivalent to the "cone-condition". We also identify the limit function. This result generalises the main result in [5] where a.e. convergence was shown to hold for multiparameter additive processes. An important step of the proof is an improved maximal inequality, Theorem 3.1, which combined with techniques from [2] and [6] gives the a.e. convergence. The proof of Theorem 3.1 is related to the one of the central results in [8] (Theorem 1.7, p. 514). In fact, using the ideas contained in the proof of Theorem 3.1 it can be seen that there is a version of Theorem 1.7 ([8]) which holds for superadditive processes. At the moment it is not clear if such a version can be used as it has been done in [8] (*e.g.* Theorem 4.7 there, see also [10]). We remark that our results also generalise results in [2] where stronger conditions were imposed on the family of sets of integers. In a final section we prove that an alternative definition for the moving averages in the superadditive setting fails to give pointwise convergence. In that section we also prove a moving local theorem.

2. **Preliminaries.** Let $d \geq 1$ be a fixed integer and $S_1 = \mathbb{Z}_+^d$ be the additive semigroup of $d$-dimensional vectors with nonnegative integer coordinates. We denote by 0

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and $e$ the vectors with all coordinates equal to 0 and 1 respectively. For a fixed integer $m \geq 1$ we let:

$$S_m = (m\mathbb{Z}_+)^d = \{mu \mid u \in S_1\}.$$

If $a = (a_i)$ and $b = (b_i)$ are two vectors in $\mathbb{Z}^d$ then $[a, b)$ denotes the set $\{u \mid u = (u_i) \in \mathbb{Z}^d, a_i \leq u_i < b_i\}$ and $\mathfrak{F}_m = \{[a, b) \mid a, b \in S_m\}$. If $A \subseteq \mathbb{Z}^d$ is finite, the number of elements in $A$ is denoted by $\#A$. Also, let $J_n = [0, ne)$ for $n \in \mathbb{Z}_+$. To avoid misinterpretations the notation $[v]_k$ will be used occasionally to indicate the $k$-th component of $v \in S_1$.

Given a measure space $(\Omega, \mathcal{F}, \mu)$ we say that a set function

$$F: \mathfrak{F}_m \longrightarrow L_1(\Omega, \mathcal{F}, \mu)$$

is a *superadditive process* (with respect to $\tau$) on $\mathfrak{F}_m$ if there exists $\tau = (\tau_u)_{u \in S_m}$, a semigroup of measure preserving transformations on $(\Omega, \mathcal{F}, \mu)$, such that the following conditions are satisfied:

$F_I \circ \tau_u = F_{I+u}$ whenever $I \in \mathfrak{F}_m$ and $u \in S_m$.

If $I_1, \ldots, I_n$ are disjoint sets in $\mathfrak{F}_m$ and if $I = \bigcup_{i=1}^n I_i$ is also in $\mathfrak{F}_m$ then $\sum_{i=1}^n F_{I_i} \leq F_I$.

$\sup\{\frac{1}{\#I} \int F_I \, d\mu \mid I \in \mathfrak{F}_m, \#I > 0\} = \gamma(F) < \infty$.

If $-F$ is superadditive then $F$ is called *subadditive*. If both $F$ and $-F$ are superadditive then $F$ is called *additive*.

LEMMA 2.1. *If $F$ is a superadditive process on $\mathfrak{F}_m$, then*

$$\gamma(F) = \lim_{n \to \infty} \frac{1}{\#J_{nm}} \int F_{J_{nm}} \, d\mu.$$

For a proof see [2].

To make the connection with the cone-condition (see condition C) below) used in [4] and [5] we mention that: for a given sequence of cubes $\{I_n = [v_n, v_n + er_n)\}_{n=1,\ldots}$ in $\mathfrak{F}_1$ (*i.e.* $v_n \in S_1, r_n \in \mathbb{Z}_+, r_n \geq 1$) the conditions B) and C) below are equivalent.

B) There is a constant $B$ such that for any cube $I = [a, a + re), a \in \mathbb{Z}^d, r \in \mathbb{Z}_+, r > 0$:

$$\#\{u \in \mathbb{Z}^d \mid \exists n, u + I_n \subseteq I\} \leq B\#I.$$

C) There is a constant $C$ and $\alpha > 0$ such that for all $s > 0$

$$\#\{u \in \mathbb{Z}^d \mid \exists n, |u - v_n| \leq \alpha(s - r_n)\} \leq C s^d.$$

The proof of the equivalence goes as in [1]. In this paper only the condition B) restricted to $\mathbb{Z}_+^d$ will be used (see the definition of a $B$-sequence below).

**3.** In this section we prove the main result of the paper: Theorem 3.3. As it has been mentioned before, the key step is Theorem 3.1 below. For the case of dimension one ($d = 1$) it is possible to give a different proof of Theorem 3.3. This may be done by making use of the existence of exact dominants for these processes. However, this proof can not be extended to higher dimensions (see [2], Section 5.1).

DEFINITION. We say that $\{I_n\}_{n=1,\dots}$ is a *B-sequence* of cubes in $\mathfrak{F}_m$ if: $I_n = [v_n, v_n + r_n e)$ where $r_n \in (m\mathbb{Z}_+)$, $r_n > 0$ and $v_n \in S_m$ and for every cube $I \in \mathfrak{F}_m$,

$$\#\{u \in S_m \mid \exists n, u + I_n \subseteq I\} \leq B\#I$$

where $B$ is a constant independent of $I$.

Examples of $B$-sequences can be found in [4] and [8].

THEOREM 3.1. *Let $\{I_n\}_{n=1,\dots}$ be a B-sequence of cubes in $\mathfrak{F}_m$ and $F$ a nonnegative superadditive process on $\mathfrak{F}_m$ and let $\alpha > 0$. If*

$$E = \left\{\omega \in \Omega \,\Big|\, \sup_{n\geq 1} \frac{1}{\#I_n} F_{I_n}(\omega) > \alpha\right\}$$

*then*

$$\mu(E) \leq \frac{3^d B}{\alpha} m^d \gamma(F).$$

PROOF. If

$$E_N \equiv \left\{\omega \in \Omega \,\Big|\, \max_{1\leq n\leq N} \frac{1}{\#I_n} F_{I_n}(\omega) > \alpha\right\}$$

we just need to show that $\mu(E_N) \leq \frac{3^d B}{\alpha} m^d \gamma(F)$ for every $N \geq 1$. Let $K > N$ and denote $V \equiv J_{m(K-N)}$, $L \equiv J_{m(K+b_N)}$ where

$$b_N \equiv \max_{\substack{1\leq n\leq N \\ 1\leq k\leq d}} [er_n + v_n]_k$$

and

$$I_n = [v_n, v_n + r_n e)$$

Therefore $V + I_n = \{v + t \mid v \in V, t \in I_n\} \subseteq L$ for $n = 1, \dots, N$.

For a given $\omega \in \Omega$ define $A_1(\omega) = \{u \in V \cap S_m \mid \tau_u\omega \in E_N\}$. Let $n: A_1(\omega) \longrightarrow [1, N]$ denote the multiple valued function $n: u \longmapsto n(u)$ where $n(u)$ satisfies

$$\frac{1}{\#I_{n(u)}} F_{I_{n(u)}}(\tau_u\omega) > \alpha.$$

Now choose $\left(u^1, n(u^1)\right)$ in such a way that:

$$r_{n(u^1)} = \max_{u\in A_1(\omega)} r_{n(u)}.$$

Define

$$C_1 = \{u \in A_1(\omega) \mid u + I_{n(u)} \cap u^1 + I_{n(u^1)} \neq \emptyset\}$$
$$D_1 = [\gamma_1, \gamma_1 + 3r_{n(u^1)}e)$$

where $[\gamma_1]_k \equiv \max([u^1 + v_{n(u^1)} - r_{n(u^1)}e]_k, 0)$ for $k = 1, \dots, d$. Therefore

$$C_1 \subseteq \{u \in A_1(\omega) \mid u + I_{n(u)} \subseteq D_1\}$$

then

$$\#C_1 \leq \{u \in A_1(\omega) \mid u + I_{n(u)} \subseteq D_1\} \leq B\#D_1 = B3^d\#I_{n(u^1)}.$$

Continue recursively as follows: $A_2(\omega) = A_1(\omega) \setminus C_1$ and choose $u^2 \in A_2(\omega)$ in such a way that

$$r_{n(u^2)} = \max_{u \in A_2(\omega)} r_{n(u)}$$

and then define $C_2$ and $D_2$ similarly as above.

Because the set $A_1(\omega)$ is finite we can see that there exists an integer $r \geq 1$ which satisfies

- $A_1(\omega) = \bigcup_{i=1}^r C_i$
- $\#C_i \leq B3^d\#I_{n(u^i)}$ for $1 \leq i \leq r$
- $u^i + I_{n(u^i)} \cap u^j + I_{n(u^j)} = \emptyset$ if $1 \leq i \neq j \leq r$.

Making use of these properties and the fact that $F$ is a nonnegative superadditive process we obtain the following inequalities

$$\begin{aligned} \alpha\#A_1(\omega) &\leq \alpha \sum_{i=1}^r \#C_i \leq \alpha 3^d B \sum_{i=1}^r \#I_{n(u^i)} \\ &\leq 3^d B \sum_{i=1}^r F_{u^i + I_{n(u^i)}}(\omega) \leq 3^d B F_L(\omega). \end{aligned}$$

We also notice that

$$\begin{aligned} \int_\Omega \#A_1(\omega)\, d\mu(\omega) &= \sum_{u \in (V \cap S_m)} \int_\Omega \psi_{\tau_u^{-1} E_N}(\omega)\, d\mu(\omega) \\ &= \sum_{u \in (V \cap S_m)} \mu(\tau_u^{-1} E_N) = \#(V \cap S_m)\mu(E_N) \end{aligned}$$

where $\psi_E$ is the indicator function of the set $E \subseteq \Omega$.

Combining these two results we obtain:

$$\#(V \cap S_m)\mu(E_N)\alpha \leq \alpha \int_\Omega \#A_1(\omega) \leq 3^d B \int_\Omega F_L.$$

So

$$\mu(E_N) \leq \frac{3^d}{\alpha} B \frac{1}{\#(V \cap S_m)} \int_\Omega F_L = \frac{3^d B}{\alpha} \frac{m^d (K + b_N)^d}{(K - N)^d} \frac{1}{\#L} \int_\Omega F_L \leq \frac{3^d B}{\alpha} \frac{m^d (K + b_N)^d}{(K - N)^d} \gamma(F).$$

Taking $K \longrightarrow \infty$ gives the desired inequality. ■

REMARK. We will now prove the a.e. convergence for multidimensional additive moving averages. This result is Theorem 2.1 in [5], therefore we only sketch a proof and add the identification of the limit function, which we will need for the proof of Theorem 3.4.

THEOREM 3.2. *Let $H\colon \mathfrak{F}_1 \longrightarrow L_1(\Omega, \mathcal{F}, \mu)$ be an additive process on $\mathfrak{F}_1$ and $\{I_n\}$ a B-sequence of cubes in $\mathfrak{F}_1$ with $\#I_n \longrightarrow \infty$, then*

$$\frac{1}{\#I_n} H_{I_n}(\omega) \quad \text{converges a.e.}$$

*to a function $h \in L_1(\Omega, \mathcal{F}, \mu)$ which is invariant under $\tau$ (where $\tau$ is the semigroup associated to the additive process H). If $\mu(\Omega) < \infty$, $\frac{1}{\#I_n} H_{I_n}$ converges also in $L_1$-norm to h. Moreover, if $\mathcal{I}$ denotes the $\sigma$-algebra of $\tau$-invariant subsets of $\Omega$ then*

$$\int_A h = \int_A H_{[0,e)} \quad \forall A \in \mathcal{I} \text{ and } \mu(A) < \infty$$

PROOF. Let $f(\omega) = H_{[0,e)}(\omega)$ therefore

$$H_I(\omega) = \sum_{u \in I} f(\tau_u \omega) \quad \forall I \in \mathfrak{F}_1.$$

Define

$$\rho f(\omega) \equiv \limsup_{n \longrightarrow \infty} \frac{1}{\#I_n} \sum_{u \in I_n} f(\tau_u \omega) - \liminf \frac{1}{\#I_n} \sum_{u \in I_n} f(\tau_u \omega).$$

Fix $\varepsilon > 0$, it can be seen that

$$f = \sum_{i=1}^{d} (g_i - g_i \circ \tau_{e_i}) + f_1 + f_2 + f_3$$

where

$$g_i \in L_\infty,\ f_1 \in L_2 \text{ and } f_1 \circ \tau_u = f_1 \quad \forall u \in S_1,$$
$$\|f_2\|_\infty < \varepsilon \text{ and } \|f_3\|_1 < \varepsilon^2.$$

We also used the notation $e_j = (\delta^j_i)_{i=1,\dots,d}, \delta^j_i = \begin{cases} 1 & j = i \\ 0 & j \neq i. \end{cases}$ Using

$$\lim_{n \longrightarrow \infty} \frac{\#(I_n \Delta v + I_n)}{\#I_n} = 0$$

we obtain

$$\rho(g_i - g_i \circ \tau_{e_i})(\omega) = 0.$$

Clearly $\rho f_1(\omega) = 0$ and $\rho f_2(\omega) \leq 2\|f_2\|_\infty < 2\varepsilon$.

Now notice that

$$\rho f_3(\omega) \leq 2 \sup_{n \geq 1} \frac{1}{\#I_n} \sum_{u \in I_n} |f_3(\tau_u \omega)|.$$

Therefore for a given $\alpha > 0$

$$\mu(\rho f_3 > \alpha) = \mu\Big(\Big\{\omega \mid \sup_{n \geq 1} \frac{1}{\#I_n} \sum_{u \in I_n} |f_3(\tau_u \omega)| > \frac{\alpha}{2}\Big\}\Big) \leq \frac{2\, 3^d B}{\alpha} \|f_3\|_1 \leq \frac{2\, 3^d B \varepsilon^2}{\alpha}.$$

Since

$$\rho f(\omega) \leq \sum_{i=1}^{d} \rho\Big((g_i - g \circ \tau_{e_i}(\omega)\Big) + \sum_{i=1}^{3} \rho f_i(\omega)$$

by making use of the above inequalities and choosing $\varepsilon$ appropriately we can make $\mu(\{\omega \mid \rho f(\omega) > \alpha\})$ arbitrarily small. This ends the proof of a.e. convergence.

To prove the other statements in Theorem 3.2 it is enough to consider $f \geq 0$. The integrability of $h(\omega) = \lim_{n\to\infty} \frac{1}{\#I_n} H_{I_n}(\omega)$ follows from Fatou's lemma. Consider $\nu \in S_1$,

$$h(\tau_\nu\omega) = \lim_{n\to\infty} \frac{1}{\#I_n} \sum_{u\in I_n} f(\tau_{u+\nu}\omega) = \lim_{n\to\infty} \left[\frac{1}{\#I_n} \sum_{u\in I_n} f(\tau_u\omega) + \theta_n(\omega)\right]$$

where

$$\theta_n(\omega) = \frac{1}{\#I_n} \sum_{u\in(\nu+I_n\setminus I_n)} f(\tau_u\omega) - \frac{1}{\#I_n} \sum_{u\in(I_n\setminus \nu+I_n)} f(\tau_u\omega)$$

Now

$$\begin{aligned} 0 &\leq \int \lim_{n\to\infty} |\theta_n(\omega)| \leq \liminf_{n\to\infty} \int |\theta_n(\omega)| \\ &\leq \liminf_{n\to\infty} \frac{1}{\#I_n}[\#(\nu + I_n \setminus I_n) + \#(I_n \setminus \nu + I_n)] \int_\Omega f = 0. \end{aligned}$$

This proves the $\tau$-invariance of the limit function. To prove $\int_A h = \int_A f \; \forall A \in I$ and $\mu(A) < \infty$ it is enough to prove $\int_\Omega h = \int_\Omega f$ when $\mu(\Omega) < \infty$. If $\mu(\Omega) < \infty$ it is easy to see that the sequence $\frac{1}{\#I_n} H_{I_n}$ is uniformly integrable and so we have $L_1$-norm convergence to, necessarily, $h$. Then $\int_\Omega f = \int_\Omega h$ follows from $\int_\Omega \frac{1}{\#I_n} H_{I_n} = \int_\Omega f$. For details see [6] p. 10. ∎

REMARK. The same results are true when $H$ is an additive process on $\mathfrak{F}_m$ and the sets $I_n$ are replaced by the sets $I_n \cap S_m$. This follows by using the natural bijection between $S_m$ and $S_1$. When $\mu(\Omega) < \infty$, it follows from Theorem 3.2 that the limit function $h$ is independent of the $B$-sequence of cubes $\{I_n\}$ with $\#I_n \longrightarrow \infty$.

THEOREM 3.3. *Let $\{I_n\}_{n=1,\ldots}$ be a B-sequence of cubes in $\mathfrak{F}_1$ and $\#I_n \longrightarrow \infty$, $F\colon \mathfrak{F}_1 \longrightarrow L_1(\Omega, \mathcal{F}, \mu)$ a superadditive process on $\mathfrak{F}_1$, with semigroup $\tau = \{\tau_\nu\}_{\nu\in S_1}$, then:*

$$\lim_{n\to\infty} \frac{1}{\#I_n} F_{I_n}(\omega) = \bar{F}(\omega) \quad \textit{exists a.e.},$$

*where $\bar{F}$ is invariant under $\tau$ and integrable. If $\mu(\Omega) < \infty$, $\frac{1}{\#I_n} F_{I_n}$ converges also in $L_1$-norm to $\bar{F}$. Moreover if $I$ denotes the $\sigma$-algebra of $\tau$-invariant subsets of $\Omega$ then*

$$\int_A \bar{F} = \lim_{n\to\infty} \frac{1}{\#J_n} \int_A F_{J_n}$$

*$\forall A \in I$ and $\mu(A) < \infty$.*

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PROOF. Define

$$G_I(\omega) = \sum_{u \in I} F_{J_1}(\tau_u \omega) \quad \forall I \in \mathfrak{F}_1$$

then $G$ is an additive process on $\mathfrak{F}_1$ and $F - G$ a nonnegative superadditive process on $\mathfrak{F}_1$. Then by Theorem 3.2 it is enough to assume, for the a.e. convergence, that $F$ is nonnegative.

Let $\varepsilon > 0$, using Lemma 2.1 find $m \in \mathbb{Z}_+$ such that

$$\gamma(F) < \frac{1}{\#J_{\tilde{m}}} \int F_{J_{\tilde{m}}} + \varepsilon \quad \forall \tilde{m} \geq m,\ \tilde{m} \in \mathbb{Z}_+.$$

Define

$$F_I^m(\omega) = F_I(\omega) - H_I^m(\omega) \quad \forall I \in \mathfrak{F}_m$$

where

$$H_I^m(\omega) = \sum_{u \in I \cap S_m} F_{J_m}(\tau_u \omega) \quad \forall I \in \mathfrak{F}_m.$$

Notice that $H^m$ and $F^m$ are respectively nonnegative additive and superadditive processes on $\mathfrak{F}_m$. Moreover

$$\begin{aligned}\lim_{n \to \infty} \frac{1}{\#J_{mn}} \int F_{J_{mn}}^m &= \lim_{n \to \infty} \Big[ \frac{1}{\#J_{mn}} \int F_{J_{mn}} - \frac{1}{\#J_{mn}} \#(J_{mn} \cap S_m) \int F_{J_m} \Big] \\ &= \gamma(F) - \frac{1}{\#J_m} \int F_{J_m} < \varepsilon\end{aligned}$$

so $\gamma(F^m) < \varepsilon$ by Lemma 2.1.

Define:

$\tilde{I}_n^m$ = largest possible cube in $\mathfrak{F}_m$ such that $\tilde{I}_n^m \subseteq I_n$.

$I_n^m$ = smallest possible cube in $\mathfrak{F}_m$ such that $I_n \subseteq I_n^m$.

We will now estimate the $B$-constant associated to the sequence $\{I_n^m\}_{n=1\ldots}$ (considered as a sequence in $\mathfrak{F}_m$). Let $I = [a, a + re)$ be a cube in $\mathfrak{F}_m$ and $I^* \equiv [a, a + 2re)$. Then

$$\begin{aligned}\#\{u \in S_m \mid \exists n, u + I_n^m \subseteq I\} &\leq \frac{1}{m^d} \#\{u \in S_1 \mid \exists n, u + I_n^m \subseteq I^*\} \\ &\leq \frac{1}{m^d} \#\{u \in S_1 \mid \exists n, u + I_n \subseteq I^*\} \\ &\leq \frac{1}{m^d} B \# I^* = \frac{2^d B}{m^d} \# I,\end{aligned}$$

where the first inequality follows easily once we notice that if $u \in S_m$ and $u + I_n^m \subseteq I$ then

$$[u, u + me) \subseteq \{u \in S_1 \mid \exists n, u + I_n^m \subseteq I^*\}.$$

Hence the $B$-constant associated with $\{I_n^m\}_{n=1,\ldots}$ is $B_m \equiv \frac{2^d B}{m^d}$.

As regards the sequence $\{\tilde{I}_n^m\}_{n=1,\ldots}$ notice that $\tilde{I}_n^m \neq \emptyset$ for $n$ large enough. Moreover if $I \neq \emptyset$ is a cube in $\mathfrak{F}_1$ there exists $I^1$, a cube in $\mathfrak{F}_1$, such that

$$\# I^1 = (r^1)^d \leq (r + 2m)^d$$

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where $\#I = r^d$ and

$$\{u \in S_m \mid \exists n, u + \tilde{I}_n^m \subseteq I\} \subseteq \{u \in S_m \mid \exists n, u + I_n \subseteq I^1\}.$$

This shows that we can consider $\{\tilde{I}_n^m\}_{n=1,\ldots}$ as a $B$-sequence of cubes in $\mathfrak{F}_m$ with constant $\tilde{B}_m \equiv B(1+2m)^d$.

For later use we notice that

$$\lim_{n\to\infty} \frac{\#I_n^m}{\#I_n} = \lim_{n\to\infty} \frac{\#\tilde{I}_n^m}{\#I_n} = 1$$

and

$$\lim_{n\to\infty} \frac{\#I_n^m \cap S_m}{\#I_n^m} = \lim_{n\to\infty} \frac{\#\tilde{I}_n^m \cap S_m}{\#\tilde{I}_n^m} = \frac{1}{m^d}.$$

Due to the fact that $\{\tilde{I}_n^m\}$ and $\{I_n^m\}$ are $B$-sequences, an application of Theorem 3.2 gives the existence of both limits below

$$\lim_{n\to\infty} \frac{1}{\#(\tilde{I}_n^m \cap S_m)} H_{\tilde{I}_n^m}^m \tag{3.4}$$

$$\lim_{n\to\infty} \frac{1}{\#(I_n^m \cap S_m)} H_{I_n^m}^m \tag{3.5}$$

Using

$$\lim_{n\to\infty} \frac{\#[(I_n^m \cap S_m) \setminus (I_n \cap S_m)]}{\#(I_n \cap S_m)} = 0$$

(similarly for the sets $\tilde{I}_n^m$) we conclude that both limits, (3.4) and (3.5), are equal to

$$h^m(\omega) = \lim_{n\to\infty} \frac{1}{\#[I_n \cap S_m]} H_{I_n}^m(\omega)$$

Define

$$\bar{f}(\omega) = \limsup_{n\to\infty} \frac{1}{\#I_n} F_{I_n}(\omega)$$

$$\underline{f}(\omega) = \liminf_{n\to\infty} \frac{1}{\#I_n} F_{I_n}(\omega)$$

both functions are finite a.e. by an application of Theorem 3.1, then

$$\begin{aligned}
\bar{f}(\omega) - \underline{f}(\omega) &\leq \limsup_{n\to\infty} \frac{1}{\#I_n} F_{I_n^m}(\omega) - \underline{f}(\omega) \\
&\leq \limsup_{n\to\infty} \frac{1}{\#I_n^m} F_{I_n^m}^m(\omega) + \limsup_{n\to\infty} \frac{1}{\#\tilde{I}_n^m} H_{\tilde{I}_n^m}^m(\omega) - \underline{f}(\omega) \\
&\leq \sup_{n\geq 1} \frac{1}{\#I_n^m} F_{I_n^m}^m(\omega) + \frac{1}{m^d} \lim_{n\to\infty} \frac{1}{[\tilde{I}_n^m \cap S_m]} H_{\tilde{I}_n^m}^m(\omega) - \liminf_{n\to\infty} \frac{1}{\#I_n} H_{\tilde{I}_n^m}^m(\omega) \\
&= \sup_{n\geq 1} \frac{1}{\#I_n^m} F_{I_n^m}^m(\omega) + \frac{1}{m^d} h^m(\omega) - \frac{1}{m^d} h^m(\omega).
\end{aligned}$$

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Therefore

$$\mu(\{\omega \mid \bar{f}(\omega) - \underline{f}(\omega) > \alpha\}) \leq \mu\Big(\Big\{\omega \mid \sup_{n\geq 1} \frac{1}{\#I_n^m} F_{I_n^m}^m(\omega) > \alpha\Big\}\Big)$$
$$\leq \frac{3^d\, m^d B_m}{\alpha}\gamma(F^m) = \frac{3^d\, 2^d B}{\alpha}\gamma(F^m) < \frac{3^d\, 2^d B\varepsilon}{\alpha}.$$

Because $\alpha$ and $\varepsilon$ are arbitrary, we have proved $\bar{f}(\omega) = \underline{f}(\omega)$ a.e.

To prove the rest of the statements of the theorem it is enough to consider that $F$ is nonnegative. We use the notation $\bar{F}(\omega) = \lim_{n\to\infty} \frac{1}{\#I_n} F_{I_n}(\omega)$. We will follow the ideas in [6] p. 37–38.

To prove $\bar{F} = \bar{F} \circ \tau_u\ \forall u \in S_1$ it is enough to prove $\bar{F} = \bar{F} \circ \tau_{e_j}$ for $j = 1, \ldots, d$, where $e_j = (\delta_i^j)_{i=1,\ldots,d}, \delta_i^j = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$.

Let $\alpha, \varepsilon > 0$ and find $m \in \mathbb{Z}_+$ as above, *i.e.* $\gamma(F) < \frac{1}{\#J_m} \int F_{J_m} + \varepsilon$. We obtain the following inequalities:

$$0 \leq \lim_{n\to\infty} \frac{1}{\#I_n} F_{I_n}(\omega) - \lim_{n\to\infty} \frac{1}{\#I_n} H_{I_n^m}^m(\omega)$$
$$\leq \lim_{n\to\infty} \frac{1}{\#I_n^m} F_{I_n^m}(\omega) - \lim_{n\to\infty} \frac{1}{\#I_n^m} H_{I_n^m}^m(\omega)$$
$$= \lim_{n\to\infty} \frac{1}{\#I_n^m} F_{I_n^m}^m(\omega) \leq \sup_{n\geq 1} \frac{1}{\#I_n^m} F_{I_n^m}^m(\omega)$$

Let $v_m \in S_m$, and $\tilde{h}^m(\omega) \equiv \frac{1}{m^d} h^m(\omega)$, then:

$$\mu\Big(\Big\{\omega \,\Big|\, |\bar{F}(\omega) - \bar{F}(\tau_{v_m}\omega)| > 2\alpha\Big\}\Big) \leq \mu\Big(\Big\{\omega \,\Big|\, |\bar{F}(\omega) - \tilde{h}^m(\omega)| > \alpha\Big\}\Big)$$
$$+\mu\Big(\Big\{\omega \,\Big|\, |\tilde{h}^m(\omega) - \bar{F}(\tau_{v_m}\omega)| > \alpha\Big\}\Big)$$
$$= 2\mu\Big(\Big\{\omega \,\Big|\, |\bar{F}(\omega) - \tilde{h}^m(\omega)| > \alpha\Big\}\Big)$$
$$\leq 2\mu\Big(\Big\{\omega \mid \sup_n \frac{1}{\#I_n^m} F_{I_n^m}^m(\omega) > \alpha\Big\}\Big) \leq \frac{2^{d+1}\, 3^d B\varepsilon}{\alpha},$$

where we made use of the inequalities above and the fact that $\tilde{h}^m = \tilde{h}^m \circ \tau_u\ \forall u \in S_m$, a.e.

Take $v_m = me_j$, then we obtain

$$\mu\Big(\Big\{\omega \,\Big|\, |\bar{F}(\omega) - \bar{F}(\tau_{me_j}\omega)| > 2\alpha\Big\}\Big) \leq 2^{d+1}\frac{3^d B\varepsilon}{\alpha}. \tag{3.6}$$

Inequality (3.6), by an application of Lemma 2.1, is also valid for $m + 1$. Therefore (3.6) for $m$ and $m + 1$ implies the following inequality:

$$\mu\Big(\Big\{\omega \,\Big|\, |\bar{F}(\omega) - \bar{F}(\tau_{e_j}\omega)| > 4\alpha\Big\}\Big) = \mu\Big(\{\omega \mid |\bar{F}(\tau_{me_j}\omega) - \bar{F}(\tau_{(m+1)e_j}\omega)| > 4\alpha\}\Big)$$
$$\leq \mu\Big(\Big\{\omega \,\Big|\, |\bar{F}(\omega) - \bar{F}(\tau_{me_j}\omega)| > 2\alpha\Big\}\Big)$$
$$+\mu\Big(\Big\{\omega \,\Big|\, |\bar{F}(\omega) - \bar{F}(\tau_{(m+1)e_j}\omega)| > 2\alpha\Big\}\Big)$$
$$\leq \frac{2^{d+2} B 3^d \varepsilon}{\alpha}.$$

Now, $\alpha$ and $\varepsilon$ being arbitrary, this proves the $\tau$-invariance of $\bar{F}$.

To prove $\int_A \bar{F} = \lim_{n\to\infty} \frac{1}{\#J_n} \int_A F_{J_n}$ $\forall A \in I$ and $\mu(A) < \infty$ it is enough to consider the case $A = \Omega$ and $\mu(\Omega) < \infty$. So assuming $\mu(\Omega) < \infty$, we will show below that $\frac{1}{\#I_n} F_{I_n}$ converges in $L_1$ to $h^\infty$, necessarily $h^\infty = \bar{F}$. Therefore, if $I_n = [v_n, v_n + er_n)$

$$\|\bar{F}\|_1 = \|h^\infty\|_1 = \lim_{n\to\infty} \frac{1}{\#I_n} \|F_{I_n}\|_1 = \lim_{n\to\infty} \frac{1}{\#J_{r_n}} \int_\Omega F_{[0,r_n e)} = \lim_{n\to\infty} \frac{1}{\#J_n} \int_\Omega F_{J_n}$$

We will now define $h^\infty$. By Theorem 3.2 and the remark that follows it we know that $\frac{1}{\#\tilde{I}_n^m} H_{\tilde{I}_n^m}$ converges in the $L_1$-norm to some function $\tilde{h}^m$ *i.e.*

$$\tilde{h}^m = L_1 - \lim_{n\to\infty} \frac{1}{\#\tilde{I}_n^m} H_{\tilde{I}_n^m}^m$$

and that the limit is independent of the $B$-sequence chosen, hence

$$\tilde{h}^m = L_1 - \lim_{n\to\infty} \frac{1}{\#J_{nm}} H_{J_{nm}}^m$$

Using this last expression for $\tilde{h}^m$ it is easy to obtain (using superadditivity of $F$) the following inequality

$$\tilde{h}^m \leq \tilde{h}^{2m}$$

so $h^\infty = L_1 - \lim_{i\to\infty} \tilde{h}^{2^i}$ exists.

Given $\eta > 0$ fix $m = 2^i$ such that

$$\|h^\infty - \tilde{h}^m\|_1 < \eta \quad \text{and} \quad \gamma(F) - \frac{1}{\#J_m} \int_\Omega F_{J_m} < \eta.$$

We remark that

$$\tilde{r}_n^m \leq r_n,\ r_n \leq (\tilde{r}_n^m + 2m) \quad \text{where } \#\tilde{I}_n^m = (\tilde{r}_n^m)^d.$$

Hence

$$\begin{aligned}\int_\Omega (F_{I_n} - H_{\tilde{I}_n^m}^m) &= \int_\Omega F_{[0,r_n e)} - \#[\tilde{I}_n^m \cap S_m] \int_\Omega F_{J_m}\\ &\leq (r_n)^d \gamma(F) - \left(\frac{\tilde{r}_n^m}{m}\right)^d \int_\Omega F_{J_m}\\ &\leq (\tilde{r}_n^m)^d \eta + a(n,m,d)\gamma(F)\end{aligned}$$

where $a(n,m,d) \equiv (\tilde{r}_n^m + 2m)^d - (\tilde{r}_n^m)^d$ and so $\lim_{n\to\infty} \frac{a(n,m,d)}{r_n^d} = 0$

Now:

$$\begin{aligned}\lim_{n\to\infty} \left\| \frac{1}{\#I_n} F_{I_n} - h^\infty \right\|_1 &\leq \lim_{n\to\infty} \left[ \frac{1}{\#I_n} \|F_{I_n} - H_{\tilde{I}_n^m}^m\|_1 + \left\| \frac{1}{\#I_n} H_{\tilde{I}_n^m}^m - \tilde{h}^m \right\|_1 + \|\tilde{h}^m - h^\infty\|_1 \right]\\ &\leq \lim_{n\to\infty} \frac{1}{\#I_n} \int_\Omega (F_{I_n} - H_{\tilde{I}_n^m}^m) + \eta\\ &\leq 2\eta \quad \text{where we used } \lim_{n\to\infty} \frac{r_n}{\tilde{r}_n^m} = 1 \text{ and } F_{I_n} - H_{\tilde{I}_n^m}^m \geq 0.\end{aligned}$$

Because $\eta$ was arbitrary this gives the $L_1$-norm convergence of $\frac{1}{\#I_n} F_{I_n}$ to $h^\infty$, and this ends the proof. ∎

**4.** In this final section we present a counterexample related to the material in Section 3. We also state a local theorem.

The following counterexample is related to an alternative definition for the moving averages in the superadditive setting. In [3] and [6], for example, the following type of superadditive process is considered:

$\{S_n\}_{n=0,1,2,\ldots}$, $S_0 = 0$

$S_n: \Omega \longrightarrow L_1(\Omega, \mathcal{F}, \mu)$ and

$\tau: \Omega \longrightarrow \Omega$ a measure preserving transformation such that:

- $S_n \circ \tau^m \leq S_{m+n} - S_m \; \forall m, n \geq 0$
- $\sup_{n\geq 1} \frac{1}{n} \int_\Omega S_n < \infty$

The connection between this one parameter type of superadditive process and the two-parameter processes considered in Section 2 (taking $d = 1$ in that definition) is the following:

$$F_{[m,n)}(\omega) = S_{n-m}(\tau^m \omega) \quad \text{for } n > m \geq 0.$$

Defining $f_n = S_{n+1} - S_n$, $n \geq 0$ we can write $S_n = f_0 + \cdots + f_{n-1}, f_n \in L_1$. In the additive case $f_n(\omega) = f_0(\tau^n \omega)$. The moving averages in the additive case can be written, for a given $B$-sequence $[v_k, v_k + r_k)$ $v_k, r_k \in \mathbb{Z}_+$, in two equivalent ways, namely ($f_j = f_0 \circ \tau^j$):

$$\frac{1}{r_k} \sum_{j=0}^{r_k-1} f_j(\tau^{v_k} \omega) = \frac{1}{r_k} \sum_{j=v_k}^{v_k+r_k-1} f_j(\omega) = \Big[ \sum_{j=0}^{v_k+r_k-1} f_j(\omega) - \sum_{j=0}^{v_k-1} f_j(\omega) \Big]. \tag{4.1}$$

In the superadditive version, the left-hand side of (4.1) can be written as follows:

$$\frac{1}{r_k} S_{r_k}(\tau^{v_k}\omega) = \frac{1}{r_k} F_{[v_k, v_k+r_k)}(\omega)$$

which by Theorem 3.3 it converges a.e. if $r_k \longrightarrow \infty$. The superadditive analog of the right-hand side of (4.1) is:

$$\frac{1}{r_k}\big(S_{v_k+r_k}(\omega) - S_{v_k}(\omega)\big).$$

We show below that this quantity does not converge in general.

Pick a $B$-sequence of intervals $\{[v_k, v_k + r_k)\}_{k=1,\ldots}$ satisfying

$$\sum_{k=1}^{\infty} \frac{r_k}{v_k + r_k} < \frac{1}{4} \tag{4.2}$$

and

$$0 < v_1 < v_2 < v_3 < \cdots$$
$$0 < r_1 < r_2 < r_3 < \cdots.$$

Set $\Omega = \{x_0\}$, $\mu(x_0) = 1$. We will define a new sequence $\{(v_k', r_k')\}_{k=1} \subseteq \{v_k, r_k)\}_{k=1,\ldots}$ as follows:

$$(v_1', r_1') = (v_1, r_1)$$
$$v_2' = v_{n_2},\ v_{n_2} \quad \text{such that } v_{n_2} > v_1' + r_1'$$
$$r_2' = r_{n_2},\ r_3' = r_{n_3}, r_{n_3} \quad \text{such that } \frac{v_2' + r_2'}{r_{n_3}} < \frac{1}{3} \text{ and } v_3' = v_{n_3}.$$

In general

$$v'_{2k} = v_{n_{2k}} \quad \text{and } v_{n_{2k}} > v'_{2k-1} + r'_{2k-1}$$

and $r'_{2k} = r_{n_{2k}}$ for $k = 1, 2, \ldots$ and

$$r'_{2k+1} = r_{n_{2k+1}} \quad \text{where } \frac{v'_{2k} + r'_{2k}}{r_{n_{2k+1}}} < \frac{1}{3} \text{ and } v'_{2k+1} = v_{n_{2k+1}} \quad \text{for } k = 1, 2, \ldots$$

Define $X^{2k} = (X_j^{2k})_{j=1,\ldots}$ where

$$X_j^{2k} = \begin{cases} 1 & \text{if } j \in \left[nv'_{2k} + (n-1)r'_{2k}, nv'_{2k} + nr'_{2k}\right) \text{ for some } n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

for $k = 1, 2, \ldots$ and

$$X^{2k+1} = (0)_{j=1,\ldots} \quad k = 0, 1, 2, \ldots$$

$$Y_r = \sum_{i=1}^{\infty} X_r^i \text{ and } S_n = \sum_{r=1}^{n} Y_r \quad S_0 = 0,\ Y_0 = 0.$$

Using (4.2) we obtain

$$\sup_{n \geq 1} \frac{S_n}{n} < \frac{1}{3}.$$

By making use of the definitions it can be seen that

$$S_m \leq S_{m+n} - S_n \quad \forall m, n \geq 0$$

and

$$\frac{S_{v'_{2k}+r'_{2k}} - S_{v'_{2k}}}{r'_{2k}} \geq 1 \quad \text{for } k = 1, 2, \ldots$$

and

$$\frac{S_{v'_{2k+1}+r'_{2k+1}} - S_{v'_{2k+1}}}{r'_{2k+1}} < \frac{1}{3} \quad \text{for } k = 0, 1, 2, \ldots.$$

Therefore

$$\lim_{n \to \infty} \frac{S_{v'_n + r'_n} - S_{v'_n}}{r'_n}$$

does not exist.

To state the moving local theorem we need to consider the previous concepts in $R^d_+$, the additive semigroup of nonnegative real numbers. From now on we will consider intervals in $R^d_+$, *i.e.* if $a = (a_i)$ and $b = (b_i)$ are two vectors in $R^d_+$, $[a, b)$ will now denote $[a, b) = \{x \mid x = (x_i) \in R^d_+,\ a_i \leq x_i < b_i, a, b, \in R^d_+\}$ and $\mathcal{R} = \{[a, b) \mid a, b \in R^d_+\}$. The definition of a continuous superadditive process, *i.e.* on $\mathcal{R}$, is similar to the one in Section 2. More precisely, replace $\mathfrak{F}_m$ by $\mathcal{R}$ and $\tau = (\tau_u)_{u \in S_m}$ by $\tau = (\tau_u)_{u \in R^d_+}$ in the definition of Section 2. Moreover we need to assume that $\tau = (\tau_u)_{u \in R^d_+}$ is a measurable semigroup of measure preserving transformations (see [6] p. 223). We say that $\{I_q\}$, where $q$ ranges over a subset of the positive rational numbers, is a *continuous B-sequence* of cubes in $R^d_+$ if $I_q = [v_q, v_q + er_q) \in \mathcal{R}$, $r_q > 0$, for every $q$ and:

$$\left|\{p \in R^d_+ \mid \exists q, p + I_q \subseteq I\}\right| \leq B|I| \quad \forall I \in \mathcal{R}.$$

where $B$ is a constant independent of $I$ and $|A|$ denotes the Lebesgue measure of the measurable set $A \subseteq R^d_+$. For the sake of simplicity we will drop the adjective "continuous" from now on. We now prove the maximal inequality that we will need for the local theorem.

THEOREM 4.3. *Let $\{I_n = [v_n, v_n + er_n)\}_{n=1,\ldots,N}$ be a finite B-sequence of cubes in $\mathcal{R}$ and F a nonnegative superadditive process on $\mathcal{R}$ and let $\alpha > 0$. If*

$$E_N = \left\{\omega \in \Omega \;\middle|\; \max_{1\leq n\leq N} \frac{1}{|I_n|} F_{I_n}(\omega) > \alpha\right\}$$

*and V and L are sets in $\mathcal{R}$ such that*

$$V + I_n \equiv \{u + t \mid u \in V, t \in I_n\} \subseteq L \quad \textit{for } n = 1, \ldots, N$$

*then*

$$\big(\mu(E_N) - \mu(\Omega \setminus R)\big) \leq \frac{3^d B}{\alpha} \frac{1}{|V|} \int_R F_L$$

*holds for any $R \in \mathcal{F}$.*

PROOF. The proof is similar to the proof of Theorem 3.1. For a given $\omega \in \Omega$ define

$$A_1(\omega) = \{u \in V \mid \tau_u \omega \in E_N\}$$

For $u \in A_1(\omega)$ we let n(u) to denote all the n's in [1, N] such that:

$$\frac{1}{|I_{n(u)}|} F_{I_n(u)}(\tau_u \omega) > \alpha.$$

Choose $(u^1, r_{n(u^1)})$ as in Theorem 3.1 and define $D_1$ as in that theorem too. Let

$$G_1 \equiv \{u \in R^d_+ \mid \exists n = 1, \ldots, N; u + I_n \subseteq D_1\}$$

Moreover, due to $\{I_n\}_{n=1,\ldots,N}$ is a $B$-sequence with constant $B$ we also have:

$$|G_1| \leq B|D_1| \leq 3^d B |I_{n(u^1)}|.$$

Let $A_2(\omega) = A_1(\omega) \setminus G_1$. Choose $(u^2, r_{n(u^2)})$, with $u^2 \in A_2(\omega)$, similarly as above. Continuing recursively in this way we obtain:

$$u^i + I_{n(u^i)} \cap u^j + I_{n(u^j)} = \emptyset \quad \text{if } i \neq j$$
$$|G_i| \leq 3^d B |I_{n(u^i)}$$

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and

$$\frac{1}{|I_{n(u^i)}|} F_{I_{n(u^i)}}(\tau_{u^i}\omega) > \alpha.$$

We show below that, a.e. in $\omega$, there exists $r+1$ $\big(r = r(\omega)\big)$ an integer such that $A_{r+1}(\omega) = \emptyset$. Suppose we have done $t$-steps in the recursion. Let $\delta \equiv \min_{j=1,\dots,N} |I_j|$ then $|G_i| \geq \delta > 0$ for $i = 1, \dots, t$. Therefore

$$t\delta \leq \sum_{i=1}^{t} |G_i| \leq 3^d B \sum_{i=1}^{t} |I_{n(u^i)}| \leq \frac{3^d B}{\alpha} \sum_{i=1}^{t} F_{I_{n(u^i)}}(\tau_u \omega) \leq \frac{3^d B}{\alpha} F_L(\omega).$$

So

$$t \leq \frac{3^d B}{\alpha\delta} F_L(\omega)$$

because $F_L \in L_1(\Omega, \mathcal{F}, \mu)$, we conclude that $t$ is finite a.e. Then $A_1(\omega) \subseteq \bigcup_{i=1}^{r} G_i$ for some integer $r = r(\omega)$.

Take $R \in \mathcal{F}$, then:

$$\begin{aligned}\alpha|V|[\mu(E_N) - \mu(\Omega \setminus R)] &= \alpha \int_V [\mu(\tau_u^{-1} E_N) - \mu(\Omega \setminus R)]\, du \leq \alpha \int_V \mu(\tau_u^{-1} E_N \cap R)\, du \\ &= \alpha \int_R |A_1(\omega)|\, d\mu(\omega) \leq B3^d \int_R F_L(\omega)\, d\mu(\omega)\end{aligned}$$

where we used $\alpha|A_1(\omega)| \leq B3^d F_L(\omega)$ a.e. ■

We conclude with the sketch of a derivation of the moving local theorem. We only present a sketch because the proof is similar to the proof of Theorem 2.9 in [2]; the difference is that the maximal inequality used there (Theorem 4.2 in that paper) is replaced by our more general Theorem 4.3.

We first introduce some notation. We define $\lim_{q \to 0} I_q = 0$ by: $\forall \alpha > 0$ there exists $q_0 > 0$ such that $I_q \subseteq J_\alpha = [0, e\alpha) \in \mathcal{R}$ for all $q < q_0$. The notation $q-\lim$ indicates that the the limit is taken along the rational numbers (see [6] p. 230). We will call a process $F$ a bounded process if it satisfies

$$\sup \frac{1}{|I|} \int |F_I| < \infty$$

where the supremum is taken over all $I \in \mathcal{R}$ with $|I| > 0$.

THEOREM 4.4. *Let $F$ be a bounded superadditive process on $\mathcal{R}$. Let $\{I_q\}$ be a continuous B-sequence of cubes in $\mathcal{R}$ with* $\lim_{q \to 0} I_q = 0$ *then*

$$q - \lim_{q \to 0} \frac{1}{|I_q|} F_{I_q}(\omega) \quad \textit{exists a.e.}$$

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PROOF. By making use of Lemma 4.7 in [2] we can write $F = G + F'$ where $F'$ is a nonnegative superadditive process on $\mathcal{R}$ satisfying:

$$\lim_{r\to 0} \frac{1}{|J_r|} \int F_{J_r}(\omega)\, d\mu(\omega) = 0$$

and $G$ is a bounded additive process on $\mathcal{R}$. As mentioned in [2], p. 62, $G$ is always the difference of two bounded nonnegative processes. So we can assume $G \geq 0$. We will first prove $q - \lim_{q\to 0} \frac{1}{|I_q|} F_{I_q}(\omega) = 0$ a.e.

Let $E = \{\omega \in \Omega \mid \bar{f}(\omega) \equiv q \limsup_{q\to 0} \frac{1}{|I_q|} F_{I_q}(\omega) > \alpha\}$ with $\alpha > 0$. It is enough to show $\mu(E) = 0$. Let $\varepsilon > 0$ and choose $t > 0$ such that $\frac{1}{|J_t|} \int F_{J_t} < \varepsilon$. We also choose $\bar{r} > 0$ satisfying the following properties

- $I_q \subseteq J_{t/2}$ whenever $0 < q < \bar{r}$
- there exist positive rationals $q_i$ satisfying $0 < q_1 < q_2 < \cdots < q_N < \bar{r}$ moreover if $E' \equiv \{\omega \mid \sup_{1\leq n\leq N} \frac{1}{|I_{q_n}|} F_{I_{q_n}}(\omega) > \alpha\}$ then $\mu(E) \leq 2\mu(E')$.

We will apply Theorem 4.3 to the finite $B$-sequence $\{I_{q_1}, \ldots, I_{q_N}\}$ and the process $F'$ with $V = J_{t/2}$, $R = \Omega$ and $L = J_t$. Hence:

$$\mu(E) \leq 2\mu(E') \leq \frac{23^d B}{\alpha} \frac{1}{|J_{t/2}|} \int F_{J_t} \leq \frac{2^{d+1} 3^d B\varepsilon}{\alpha}.$$

To prove the a.e. convergence of $q - \lim_{r\to 0} \frac{1}{|I_q|} G_{I_q}(\omega)$ we write $G = G' + G''$ (for this decomposition see [2], p. 63) where

- $G'$ is absolutely continuous; *i.e.* there is a nonnegative integrable function $g$ on $\Omega$ and
$$G'_I(\omega) = \int_I g(\tau_u \omega)\, du, \qquad \text{for all } I \in \mathcal{R}.$$
- $G''$ is singular; *i.e.* for each $\varepsilon > 0$ there is a number $t > 0$ and a set $R \in \mathcal{F}$ such that $\mu(\Omega \setminus R) < \varepsilon$ and such that $\int_R G''_I\, d\mu \leq 3|I|$ whenever $I \in \mathcal{R}$ and $I \subseteq J_t$.

The proof of a.e. convergence for $G'$ and $G''$ is done as in Lemmas 4.11 and 4.12 in [2] and making use of Theorem 4.3 as we did above for the process $F'$. ∎

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