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Ratio tests for the convergence of integrals

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1. Ratio tests for the convergence or divergence of infinite series of positive terms are well known; they are used in and out of season. On the other hand, ratio tests for infinite integrals are never used. What is the reason for this disparity between series and integrals?

The answer to our query is simple. The ratio tests for integrals are not hard to find, are very easy to prove, and are quite useless. For example, the test corresponding to the familiar u_n/u_{n+1} test says no more than this:—

(i) the integral of $f(x)$ will diverge as $x \to \infty$ if $f(x)$ is positive and increases as x increases,

(ii) the integral of $f(x)$ will converge as $x \to \infty$ if $f(x)$ is comparable with $\exp(-kx)$, where k is a positive constant.

Yet the tests may be worth giving since they show more simply than do their series analogues the simplicity of the underlying ideas.

2. "Kummer's" ratio tests for integrals

THEOREM 1. *Let $f(x)$, $\phi(x)$ be positive, differentiable functions of x. Let there be a number X such that*

$$\frac{d}{dx}\{f(x)\,\phi(x)\} \geqq 0 \ \textit{when}\ x \geqq X; \tag{1}$$

then $\int^{\infty} f(x)\,dx$ *diverges if* $\int^{\infty} \{1/\phi(x)\}\,dx$ *diverges.*

Proof. When $x \geqq X$, we have $f(x)\,\phi(x) \geqq f(X)\,\phi(X)$, and so

$$\int_X^{X_1} f(x)\,dx \geqq f(X)\,\phi(X) \int_X^{X_1} \frac{dx}{\phi(x)}.$$

Hence, if the integral of $\{1/\phi(x)\}$ diverges so does the integral of $f(x)$.

COROLLARY. As a particular case, if

$$\frac{d}{dx}\{f(x)\,\phi(x)\} \to L > 0 \text{ as } x \to \infty, \tag{2}$$

then there is a number X for which (1) is satisfied.

THEOREM 2. *Let $f(x)$, $\phi(x)$ be positive differentiable functions, and let their differential coefficients be continuous in any finite interval*[1]. *Let there be positive numbers X, k such that*

$$\frac{d}{dx}\{f(x)\,\phi(x)\} \leqq -k f(x) \text{ when } x \geqq X. \tag{3}$$

Then the integral of $f(x)$, as x tends to infinity, is convergent.

Proof. By hypothesis, when $x > X$,

$$\begin{aligned}\int_X^x f(t)\,dt &\leqq -\frac{1}{k}\int_X^x \frac{d}{dt}\{f(t)\,\phi(t)\}\,dt \\ &= \frac{1}{k}\{f(X)\,\phi(X) - f(x)\,\phi(x)\} \\ &< f(X)\,\phi(X)/k.\end{aligned}$$

Hence if $F(t)$ is the integral of $f(t)$, $F(x)$ is bounded above as $x \to \infty$. Moreover, since $f(x)$ is positive, $F(x)$ is monotonic increasing. Hence $F(x)$ converges to a finite limit as $x \to \infty$.

COROLLARY. As a particular case, if

$$\frac{1}{f(x)} \cdot \frac{d}{dx}\{f(x)\,\phi(x)\} \to L < 0 \text{ as } x \to \infty, \tag{4}$$

then there are positive numbers k (for example, $-\tfrac{1}{2}L$) and X for which (3) is satisfied.

3. Why the tests are never used

D'Alembert's test. Take $\phi(x) = 1$, so that we have the integral analogue of d'Alembert's test for series. Suppose that (3) holds; that is,

$$\frac{d}{dx}\{f(x)\,.\,1\} \leqq -k f(x) \text{ when } x > X. \tag{5}$$

[1] We need *some* condition to ensure that $f(x)\,\phi(x)$ is the integral of its differential coefficient; we have chosen the simplest.

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Then, since $f(x)$ is positive,

$$\frac{f'(x)}{f(x)} \leqq -k, \quad -\frac{f'(x)}{f(x)} \geqq k,$$

and so

$$\log\{1/f(x)\} \geqq kx + \text{constant}.$$

That is to say,

$$f(x) \leqq B \exp(-kx), \tag{6}$$

where B, k are positive constants.

In any possible application of Theorem 2 it will be easier to see whether (6) is satisfied than it will be to see whether (5) is satisfied.

Raabe's test. The analogue of Raabe's test is obtained by putting $\phi(x) = x$. If we suppose that, for some positive k and X,

$$\frac{d}{dx}\{f(x) \,.\, x\} \leqq -kf(x) \text{ when } x > X, \tag{7}$$

then, as a little calculation shows,

$$f(x) \leqq Ax^{-1-k}, \tag{8}$$

where A is a positive constant. No one would prefer (7) to (8) as a criterion of convergence and (8), like (6), is a well-known test for the convergence of infinite integrals.

The next test, in the usual order, is given by taking $\phi(x) = x \log x$ in Theorems 1 and 2. That the test is useless may be seen from the fact (mildly interesting in its proof) that

$$\frac{d}{dx}\{(x \log x) f(x)\} \leqq -kf(x) \text{ when } x > X$$

implies

$$f(x) \leqq A/x(\log x)^{1+k}.$$

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A further note on differentials

By E. G. PHILLIPS.

Since the publication of my article[1] on "The advantage of differentials in the technique of differentiation" both Dr H. A. Hayden and Prof. A. Oppenheim have kindly pointed out to me that

[1] *Math. Notes* 30, May 1937.