Canad. Math. Bull. Vol. 64 (4), 2023, pp. 1296–1312
http://dx.doi.org/10.4153/S0008439523000371

Examples of exponential bases on union of intervals

Oleg Asipchuk and Vladyslav Drezels

Abstract. In this paper, we construct explicit exponential bases of unions of segments of total measure one. Our construction applies to finite or infinite unions of segments, with some conditions on the gaps between them. We also construct exponential bases on finite or infinite unions of cubes in $\mathbb{R}^d$ and prove a stability result for unions of segments that generalize Kadec's $\frac{1}{4}$-theorem.

1 Introduction

The main purpose of this paper is to construct explicit exponential bases on finite or infinite unions of intervals of the real line. We assume that our intervals have total measure one, until otherwise specified.

We recall that an exponential basis on a domain $D \subset \mathbb{R}^d$ is an unconditional Schäuder basis for $L^2(D)$ in the form of $\{e^{2\pi i \lambda_n \cdot x}\}_{n \in \mathbb{Z}^d}$ with $\lambda_n \in \mathbb{R}^d$. An important example of exponential basis is the Fourier basis $\mathcal{E} = \{e^{2\pi i n x}\}_{n \in \mathbb{Z}}$ for $L^2(0,1)$.

Non-orthonormal exponential bases on intervals of the real line are well studied and well understood in the context of nonharmonic Fourier series (see [13, 14, 18, 20] just to cite a few). Proving the existence of an exponential basis is in general a difficult problem, and constructing explicit bases can be even more difficult.

In [14], the author proved the existence of bases on finite unions of intervals under some conditions on the lengths of the intervals. It is proved in [11] that exponential bases on any finite union of intervals exist, but the construction of such bases is not explicit. It is not clear whether exponential bases on arbitrary infinite unions of intervals exist or not.

In [4], the author proves necessary and sufficient conditions for which sets in the form of $\{e^{2\pi i (n+\delta_j)x}\}_{n \in \mathbb{Z}, j \leq N}$ are exponential bases on unions of N intervals of a unit length separated by integer gaps and gave an explicit expression for the frame constants of these bases. Such a result can be used to construct explicit exponential bases on intervals with rational endpoints. However, the conditions involve evaluating the eigenvalues of $N \times N$ matrices, which can be a difficult task. Some of the results in [4] appear also in other papers, for example, in [10].

Received by the editors February 22, 2023; revised May 8, 2023; accepted May 8, 2023.
Published online on Cambridge Core May 15, 2023.
The research of this project started during the Summer 2022 REU program "AMRPU FIU" that took place at the Department of Mathematics and Statistics, Florida International University, which was supported by the NSA grant H982302210016 and the NSF (REU Site) grant DMS-2050971.
AMS subject classification: **42C15**, 42C30.
Keywords: Exponential bases, Riesz bases, Riesz constants, frames, Kadec stability theorem.

In [12], it is proved that if D is the union of two disjoint intervals of total length 1, then D has an orthonormal basis of exponentials if and only if it tiles $\mathbb{R}$ by translations. Recall that a measurable set D tiles $\mathbb{R}^d$ by translation if we can fill the space with translated copies of D without overlaps. Results for unions of three intervals that tile the real line are in [3].

In many applications, such as aircraft instrument communications, air traffic control simulation, or telemetry [7], one can consider the possibility of obtaining sampling expansion which involved sample values of a function and its derivatives. That translated into finding bases of the form of $\{x^k e^{2\pi i x\lambda_n}\}_{n\in\mathbb{Z}}$, with $k \in \mathbb{N} \cup \{0\}$. See [21].

It is worth mentioning the recent [19], where the authors partition the interval $[0,1]$ into intervals $I_1, \ldots, I_n$ and the set $\mathbb{Z}$ into $\Lambda_1, \ldots, \Lambda_n$ such that the complex exponential functions with frequencies in Λ_k form a Riesz basis for $L^2(I_k)$.

The existence of orthonormal bases on a domain of $\mathbb{R}^d$ is a difficult problem related to the tiling properties of the domain. It has been recently proved in [15] that convex sets tile $\mathbb{R}^d$ by translations if and only if they have an exponential basis. In [6], it is proved that the set $\mathcal{E} = \{e^{2\pi i n\cdot x}\}_{n\in\mathbb{Z}^d}$ is an exponential basis on a domain $D \subset \mathbb{R}^d$ of measure 1 if and only if D tiles $\mathbb{R}^d$. Furthermore, $\mathcal{E}$ is orthonormal for $L^2(D)$.

The aforementioned results in [6] are related to Theorem 1 in [9], where it is proved that if a set $\{e^{2\pi i x\lambda}\}_{\lambda\in\Lambda}$ is an orthonormal basis on a domain $D \subset \mathbb{R}$, then Λ is periodic, i.e., $\Lambda = T + \Lambda$ for some $T \in \mathbb{N}$.

1.1 Our results

Before we introduce our results, we need some more notations. By $m \in \mathbb{N} \cup \{\infty\}$, we mean that m is either a natural number or it is infinity. We let $I = [0,1)$. Given $0 = a_0 < a_1 < \cdots < a_{m-1} < \cdots < 1$ be a sequence for $m \in \mathbb{N} \cup \{\infty\}$, we let $I_j = [a_j, a_{j+1})$, so that $I = \bigcup I_j$.

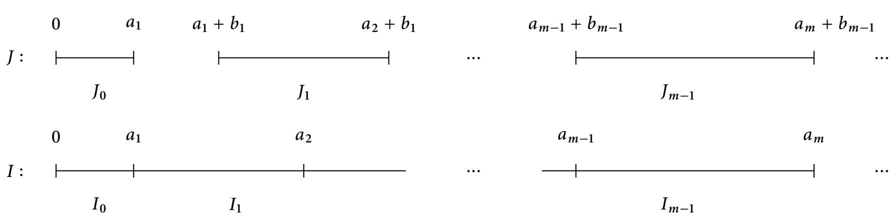
Given $0 = b_0 < b_1 < \cdots < b_{m-1} < \cdots$, we let

$$J = \bigcup_j J_j, \quad \text{with } J_j = [a_j + b_j, a_{j+1} + b_j). \tag{1.1}$$

We can also write

$$J = \bigcup_j [c_j - \gamma_j, c_j + \gamma_j) \quad \text{with } c_j = \frac{a_{j-1} + a_j}{2} + b_j, \quad \gamma_j = \frac{a_j - a_{j-1}}{2}. \tag{1.2}$$

Note that $\sum_j \gamma_j = 1$. So, on graph, we have



https://doi.org/10.4153/S0008439523000371 Published online by Cambridge University Press

Let $\{x\}$ be the decimal part of a real number x, let $\lfloor x \rfloor$ (the *floor function*) be the largest integer, that is, $\leq x$, and let $\lceil x \rceil$ (the *ceiling function*) be the smallest integer, that is, $\geq x$. If $E \subset \mathbb{R}$, we define the distance function for $x \in \mathbb{R}$ and $E \subset \mathbb{R}$ as $\operatorname{dist}(E, x) := \min_{y \in E}\{|y - x|\}$ or $\operatorname{dist}(E, F) := \min_{y \in E, x \in F}\{|y - x|\}$.

Next, we introduce the sequence δ_n^*. Let $\beta > 0$ be fixed; for every $n \in \mathbb{Z}$, we let

$$\delta_n^* = \begin{cases} \frac{\lfloor \beta n \rfloor + 1}{\beta} - n, & \text{if } \{\beta n\} \geq \frac{1}{2}, \\ \frac{\lfloor \beta n \rfloor}{\beta} - n, & \text{if } \{\beta n\} < \frac{1}{2}. \end{cases} \tag{1.3}$$

We can see at once that

$$\delta_n^* = \frac{\xi_n}{\beta} \min\{\{\beta n\}, 1 - \{\beta n\}\} = \frac{\xi_n}{\beta} \operatorname{dist}(\mathbb{Z}, \beta n),$$

where

$$\xi_n = \begin{cases} 1, & \text{if } \{\beta n\} \geq \frac{1}{2}, \\ -1, & \text{if } \{\beta n\} < \frac{1}{2}. \end{cases}$$

Let

$$\mathcal{B}^* = \{e^{2\pi i x (n + \delta_n^*)}\}_{n \in \mathbb{Z}}, \tag{1.4}$$

where the δ_n^* are defined as in (1.3). Our main results are the following.

Theorem 1.1 *Let J be as in (1.1). If there exists $\beta \geq 1$ such that $\frac{b_k}{\beta} \in \mathbb{Z}$ for all k, then the set $\mathcal{B}^*$ defined in (1.4) is an exponential basis for $L^2(J)$.*

Theorem 1.2 *Let J and β be as in Theorem 1.1. Let $m \in \mathbb{N}$, and let $\Lambda = \{n + \delta_n^*\}$, where δ^* is as in (1.3). If $\frac{b_k}{\beta} \in \mathbb{Z}$ for all $k = 1, \ldots, m - 1$, then the set $\mathcal{E}(\frac{1}{\Delta}\mathbb{Z} \backslash \Lambda)$ is an exponential basis for $L^2([0, \Delta] \backslash J)$, where*

$$\Delta = \left\lceil \frac{1 + b_{m-1}}{\beta} \right\rceil \beta.$$

We will prove in Lemma 3.1 that $\mathcal{B}^*$ is also a basis in $L^2(0,1)$; our proof of Theorem 1.1 shows that $\mathcal{B}^*$ has the same frame constants in $L^2(J)$ and $L^2(0,1)$.

In our Theorem 1.1, the gaps b_k are integer multiples of β and so the set J is unbounded when $m = \infty$. To the best of our knowledge, there are very few examples of exponential bases on unbounded sets in the literature. The existence of exponential frames on unbounded sets of finite measure has been recently proved in [16].

We also observe that in Theorem 1.2, we cannot consider $m = \infty$ because J is unbounded, and in the proof, we need to consider a finite interval $[0, \Delta]$ that contains J.

Our paper is organized as follows: in Section 2, we recall some preliminaries and we prove some important lemmas. In Section 3, we prove our main results. In Section 4, we prove the result for unions of cubes in $\mathbb{R}^d$ and a stability result.

2 Preliminaries

We have used the excellent textbooks [8, 24] for the definitions and some of the results presented in this section.

Let H be a separable Hilbert space with inner product $\langle\ ,\ \rangle$ and norm $\|\ \| = \sqrt{\langle\ ,\ \rangle}$. We will mostly work with $L^2(D)$, where $D \subset \mathbb{R}^d$. So, the norm will be $\|f\|_2^2 = \int_D |f(x)|^2 dx$. We denote the characteristic function on D by χ_D.

A sequence of vectors $\mathcal{V} = \{v_j\}_{j\in\mathbb{Z}}$ in H is a *Riesz basis* if there exist constants A, $B > 0$ such that, for any $w \in H$ and for all finite sequences $\{a_j\}_{j\in J} \subset \mathbb{C}$, the following inequalities hold:

$$A\sum_{j\in J}|a_j|^2 \leq \Big\|\sum_{j\in J} a_j v_j\Big\|^2 \leq B\sum_{j\in J}|a_j|^2, \tag{2.1}$$

$$A\|w\|^2 \leq \sum_{j=1}^{\infty}|\langle w, v_j\rangle|^2 \leq B\|w\|^2. \tag{2.2}$$

The constants A and B are called *frame constants* of the basis. The left inequality in (2.1) implies that $\mathcal{V}$ is linearly independent, and the left inequality in (2.2) implies that $\mathcal{V}$ is complete. If the condition (2.1) holds, we call $\mathcal{V}$ a Riesz sequence. We call $\mathcal{V}$ a frame if the condition (2.2) holds. If the condition (2.2) holds and $A = B$, then we call $\mathcal{V}$ a tight frame. If $A = B = 1$, then we have a Parseval frame. The following lemma is well known, but for the reader's convenience, we will prove it.

Lemma 2.1 *If a sequence of vectors $\mathcal{V} = \{v_j\}_{j\in\mathbb{Z}}$ is a frame with upper constant B, then the right inequality in (2.1) holds, i.e., for all finite sequences $\{a_j\}_{j\in J} \subset \mathbb{C}$,*

$$\Big\|\sum_{j\in J} a_j v_j\Big\|^2 \leq B\sum_{j\in J}|a_j|^2.$$

Proof $\mathcal{V}$ is a frame, so for all finite sequences $\{a_j\}_{j\in J} \subset \mathbb{C}$, there is $f \in H$ such that $f = \sum_{j\in J} a_j v_j$. So,

$$\begin{aligned}\|f\|^2 &= \langle \sum_{j\in J} a_j v_j, f\rangle = \sum_{j\in J} a_j \langle v_j, f\rangle \\ &\leq \sqrt{\sum_{j\in J}|a_j|^2}\sqrt{\sum_{j\in J}|\langle v_j, f\rangle|^2} \leq \sqrt{\sum_{j\in J}|a_j|^2}\sqrt{B}\|f\|.\end{aligned}$$

Therefore,

$$\Big\|\sum_{j\in J} a_j v_j\Big\| \leq \sqrt{\sum_{j\in J}|a_j|^2}\sqrt{B}.$$

■

One more lemma that makes a connection between frames and bases is the following.

Lemma 2.2 *If $E(\Lambda)$ is basis for $L^2(D)$ and $D' \subset D$, then $E(\Lambda)$ is a frame for $L^2(D')$ with at least the same frame constants. In particular, if $E(\Lambda)$ is an orthogonal basis for $L^2(D)$, then it is a tight frame for $L^2(D')$.*

https://doi.org/10.4153/S0008439523000371 Published online by Cambridge University Press

Proof Let $E(\Lambda)$ is a basis for $L^2(D)$, then for all $w \in L^2(D)$,

$$A\|w\|^2 \le \sum_{j=1}^{\infty} |\langle w, v_j\rangle|^2 \le B\|w\|^2.$$

Also, for any $f \in L^2(D')$, there is $w \in L^2(D)$ such that $f = w\chi_{D'}$. So, the frame inequalities hold for any $f \in L^2(D')$. Therefore, $E(\Lambda)$ is a frame for $L^2(D')$ with at least the same frame constants. ∎

An important characterization of Riesz bases is that they are bounded and unconditional Schäuder bases. See, e.g., [8].

Let $\vec{v} \in \mathbb{R}^d$ and $\rho > 0$; we denote by $d_\rho D = \{\rho x : x \in D\}$ and by $t_{\vec{v}} D = \{x + \vec{v} : x \in D\}$ the dilation and translation of D. Sometimes, we will write $\vec{v} + D$ instead of $t_{\vec{v}} D$ when there is no risk of confusion.

The following lemma can easily be proved with a change of variables in (2.1) and (2.2).

Lemma 2.3 *Let $\vec{v} \in \mathbb{R}^d$ and $\rho > 0$. The set $\mathcal{V} = \{e^{2\pi i\langle x, \lambda_n\rangle}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(D)$ with constants A and B if and only if the set $\{e^{2\pi i\langle x, \frac{1}{\rho}\lambda_n\rangle}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(t_{\vec{v}}(d_\rho D))$ with constants $A\rho^d$ and $B\rho^d$.*

2.1 Paley–Wiener and Kadec stability theorem

Bases in Banach spaces are stable, in the sense that small perturbations of a basis still produce bases.

One of the fundamental stability criteria, and historically the first, is due to Paley and Wiener in [17].

Theorem 2.4 (Paley–Wiener theorem) *Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in a Banach space X. Let λ be a real number $(0 < \lambda < 1)$ such that*

$$\left\|\sum_n a_n(x_n - y_n)\right\| \le \lambda \left\|\sum_n a_n x_n\right\|$$

holds for any arbitrary finite set of scalars $\{a_n\} \subset \mathbb{C}$. Then, if $\{x_n\}$ is a basis, so is $\{y_n\}$. Moreover, if $\{x_n\}$ has Riesz constants A and B, then

$$(1-\lambda)A\sum |a_n|^2 \le \left\|\sum a_n y_n\right\|^2 \le (1+\lambda)B\sum |a_n|^2.$$

We will use the following important observation: if $\mathcal{B} = \{x_n\}_{n\in\mathbb{N}}$ is a bounded and unconditional basis in a Banach space $(X, \|\,\|)$, and $\|\,\|_*$ is a norm equivalent to $\|\,\|$, then $\mathcal{B}$ is also a bounded and unconditional basis in $(X, \|\,\|_*)$. Note that two norms $\|\,\|_*$ and $\|\,\|$ are equivalent if there are two constants c and C such that for all elements of the space, $c\|x\|_* \le \|x\| \le C\|x\|_*$.

Let a set $\{f_n\}_{n\in\mathbb{N}}$ be a Riesz basis for $L^2(D)$ with norm $\|\,\|_2$ if $\|\,\|_*$ is equivalent to $\|\,\|_2$, then $\{f_n\}_{n\in\mathbb{N}}$ is a bounded and unconditional basis of $(L^2(D), \|\,\|_*)$. So, if a sequence $\{g_n\}_{n\in\mathbb{N}} \subset L^2(D)$ satisfies the conditions of the Paley–Wiener theorem with respect to the norm $\| \|_*$, i.e.,

$$\left\| \sum_n c_n(f_n - g_n) \right\|_* \le \lambda \left\| \sum_n c_n f_n \right\|_*$$

with $0 < \lambda < 1$, then $\{g_n\}_{n\in\mathbb{N}}$ is a bounded and unconditional basis of $(L^2(D), \| \|_*)$ and hence also of $(L^2(D), \| \|)$. Thus, $\{g_n\}_{n\in\mathbb{N}}$ is a Riesz basis in $(L^2(D), \| \|)$. This observation proves the following lemma, which will be useful later on.

Lemma 2.5 *Let $m \in \mathbb{N}$ and $D = \cup_{j=1}^m D_j \subset \mathbb{R}^d$ where $D_j \cap D_k = \varnothing$. Let $\{g_n\}_{n\in N}$ be a Riesz basis for $L^2(D)$. Let $\{h_n\}_{n\in\mathbb{Z}} \subset L^2(D)$ be such that for every finite sequence $\{a_n\} \in \mathbb{C}$,*

$$\sup_{j\le m} \left\| \sum_n a_n(g_n - h_n) \right\|_{L^2(D_j)} \le \alpha \sup_j \left\| \sum_n a_n g_n \right\|_{L^2(D_j)}.$$

Then, the set $\{h_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(D)$.

The celebrated Kadec stability theorem (also called Kadec's $\frac{1}{4}$-theorem) gives an optimal measure of how the standard orthonormal basis $\mathcal{E} = \{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ on the unit interval $[0,1]$ can be perturbed to still obtain an exponential basis.

Theorem 2.6 *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{R}$ for which*

$$|\lambda_n - n| \le L < \frac{1}{4}$$

whenever $n \in \mathbb{Z}$. Then, $E(\Lambda) = \left\{e^{2\pi i\lambda_n x}\right\}_{\lambda\in\Lambda}$ is an exponential basis for $L^2(0,1)$ with frame constants $A = \cos(\pi L) - \sin(\pi L)$ and $B = 2 - \cos(\pi L) + \sin(\pi L)$. The constant $\frac{1}{4}$ cannot be replaced by any larger constant.

The theorem is proved using the Paley–Wiener theorem and a clever Fourier series expansion of the function $1 - e^{2\pi i\delta x}$. The quantity

$$D(L) = 1 - \cos(\pi L) + \sin(\pi L) \tag{2.3}$$

plays an important part in the proof of the theorem, as well as in other generalizations. So, we can rewrite the frame constants of $E(\Lambda)$ as

$$A = 1 - D(L) = \cos(\pi L) - \sin(\pi L),$$
$$B = 1 + D(L) = 2 - \cos(\pi L) + \sin(\pi L).$$

Kadec's theorem has been generalized to prove the stability of general exponential frames. See Theorem 1 in [2].

An important generalization of Kadec's theorem is due to Avdonin [1].

Theorem 2.7 (Special version of Avdonin's theorem) *Let $\lambda_n = n + \delta_n$ and suppose $\{\lambda_n\}_{n\in\mathbb{Z}}$ is separated, i.e., $\inf_{n\ne k} |\lambda_n - \lambda_k| > 0$. If there exist a positive integer N and a* *positive real number* $\varepsilon < \frac{1}{4}$ *such that*

$$\left|\sum_{n=mN+1}^{(m+1)N} \delta_n\right| \le \varepsilon N \tag{2.4}$$

for all integers m, *then the system* $\{e^{2\pi i x\lambda_n}\}_{n\in\mathbb{Z}}$ *is a Riesz basis for* $L^2([0,1])$.

This special version of Avdonin's theorem can be found in [20]. The condition (2.4) is hard to prove in the case when the sequence is not periodic. But, in our case, the following lemma will help.

Lemma 2.8 *When* g *is Riemann integrable in* $[0,1]$, *and periodic of period* 1, *and* β *is irrational, then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g(n\beta) = \int_0^1 g(x)dx.$$

This lemma is Corollary 2.3 on page 110 in [22].

2.2 Bases on disconnected domains

Let $E(\Lambda)$ be an exponential basis on a domain $D \subset \mathbb{R}^d$. If D is partitioned into disjoint sets $D_1, \dots, D_m, \dots$, which are then translated with translations $\tau_1, \dots \tau_m, \dots$ in such a way that the translated pieces do not intersect, then in general $E(\Lambda)$ is not a basis on the "broken domain" $\tilde{D} = D_1 + \tau_1 \cup \cdots \cup D_m + \tau_m \cup \cdots$. The following lemma shows how a basis $E(\Lambda)$ for $L^2(D)$ can be transformed into a basis for $L^2(\tilde{D})$.

Lemma 2.9 *Let* $D \subset \mathbb{R}^d$ *be measurable, with* $|D| < \infty$; *let for* $m \in \mathbb{N} \cup \{\infty\}$, *we have* $D = \bigcup_{j=0}^m D_j$, *with* $|D_j| > 0$ *for all* j *and* k $D_k \cap D_j = \varnothing$ *when* $k \ne j$. *Let* τ_j *be translations such that* $\tau_j(D_j) \cap \tau_k(D_k) = \varnothing$ *when* $k \ne j$. *Let* $\tilde{D} = \bigcup_{j=1}^m \tau_j(D_j)$. *If* $\mathcal{B} = \{\psi_n(x)\}_{n\in\mathbb{N}} \subset L^2(D)$ *is a Riesz basis for* $L^2(D)$, *then*

$$\tilde{\mathcal{B}} = \left\{\sum_{j=1}^m \chi_{\tau_j(D_j)}\psi_n(\tau_j^{-1}x)\right\}_{n\in\mathbb{N}}$$

is a Riesz basis for $L^2(\tilde{D})$ *with the same frame constants.*

Proof With some abuse of notation, we will let $\tau_j(x) = x + \tau_j$ and $\tau_j(D_j) = D'_j$. Thus, $\tilde{D} = \bigcup_{j=1}^m D'_j$.

Define the operator $T : L^2(D) \to L^2(\tilde{D})$ by $T^{-1}(f)(x) = \sum_{k=1}^m f(x - \tau_k)\chi_{D_k}$. This is a linear transformation. We can also check that T is invertible, and its inverse is the operator $T^{-1} : L^2(\tilde{D}) \to L^2(D)$ defined as $T^{-1}(f)(x) = \sum_{k=1}^m f(x + \tau_k)\chi_{D'_k}$. Let us show that T (and also T^{-1}) are isometry. Indeed, for every $f \in L^2(D)$,

$$\|T(f)\|^2_{L^2(\tilde{D})} = \sum_{k=1}^{m} \|T(f)\|^2_{L^2(D'_k)} = \sum_{k=1}^{m} \int_{\tau_k + D_k} |f(x-\tau_k)|^2 dx$$
$$= \sum_{k=1}^{m} \int_{D_k} |f(x)|^2 dx = \|f\|^2_{L^2(D)},$$

where the third equality comes from a change of variables in the integrals. An invertible isometry maps bases into bases, and the frame constants are the same. Since $\tilde{\mathcal{B}} = T(\mathcal{B})$, we have proved that $\tilde{\mathcal{B}}$ is a basis for $L^2(\tilde{D})$. ∎

Remark 2.10 Let $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}^d$, $\{b_k\}_{k=0}^{m-1} \subset \mathbb{R}^d$, with $m \in \mathbb{N} \cup \{\infty\}$, and D and $\tilde{D}$, as in Lemma 2.9. Also, let $w_n = \sum_{k=0}^{m-1} e^{2\pi i b_k \lambda_n} \chi_{\tilde{D}_k}$. The set $\{e^{2\pi i x \lambda_n}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(\tilde{D})$ if and only if the set $\{w_n e^{2\pi i x \lambda_n}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(D)$. Moreover, those two bases have the same Riesz constants. We can also see that the set $\{\overline{w_n} e^{2\pi i x \lambda_n}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(D)$ if and only if the set $\{e^{2\pi i x \lambda_n}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(\tilde{D})$. Moreover, those two bases have the same Riesz constants.

If we replace D by J, when

$$J = \bigcup J_j, \quad \text{with } J_j = [a_j + b_j, a_{j+1} + b_j),$$

with $J_j = [a_j + b_j, a_{j+1} + b_j)$, as in (1.1), then we obtain a special case of Lemma 2.9.

Lemma 2.11 *For m finite or infinite, the sequence* $\{g_n\}_{n\in\mathbb{Z}}$, *where*

$$g_n = \sum_{k=0}^{m} e^{2\pi i x \lambda_n} e^{2\pi i b_k \lambda_n} \chi_{J_k},$$

is a Riesz basis for $L^2(J)$ *if and only if* $\mathcal{B} = \{e^{2\pi i x \lambda_n}\}_{n\in\mathbb{Z}}$ *is a Riesz basis for* $L^2(I)$. *Moreover, two bases* $\mathcal{B}$ *and* $\{g_n\}$ *have the same Riesz constants. Conversely, the set* $\{\tilde{g}_n\}_{n\in\mathbb{Z}}$, *where*

$$\tilde{g}_n = \sum_{k=0}^{m} e^{2\pi i x \lambda_n} e^{-2\pi i b_k \lambda_n} \chi_{I_k},$$

is a Riesz basis for $L^2(I)$ *if and only if* $\mathcal{B}$ *is a Riesz basis for* $L^2(J)$. *Moreover, two bases* $\mathcal{B}$ *and* $\{\tilde{g}_n\}$ *have the same Riesz constants.*

A version of Lemma 2.11 is also in [5].

3 Proofs of the main results

In this section, we will prove our main results. But first, we remind the reader that $\mathcal{B}^* = \{e^{2\pi i x (n+\delta_n^*)}\}_{n\in\mathbb{Z}}$, where

$$\delta_n^* = \begin{cases} \frac{\lfloor \beta n \rfloor + 1}{\beta} - n, & \text{if } \{\beta n\} \geq \frac{1}{2}, \\ \frac{\lfloor \beta n \rfloor}{\beta} - n, & \text{if } \{\beta n\} < \frac{1}{2}, \end{cases}$$

for some $\beta > 0$ (see (1.4) and (4.5)).

https://doi.org/10.4153/S0008439523000371 Published online by Cambridge University Press

3.1 A useful lemma

Lemma 3.1 *Let $\beta \geq 1$. Then $\mathcal{B}^*$ is an exponential basis for $L^2([0,1])$.*

Proof First, we let $\beta > 2$. Then

$$\sup_{\beta \in \mathbb{Z}} |\delta_n^*| = \sup_{\beta \in \mathbb{Z}} \left| \frac{\operatorname{dist}(\mathbb{Z}, \beta n)}{\beta} \right| \leq \frac{1}{2\beta} < \frac{1}{4}.$$

So, by Theorem 2.6, $\left\{e^{2\pi i x(n+\delta_n^*)}\right\}_{n\in\mathbb{Z}}$ is an exponential basis for $L^2([0,1])$.

Next, let $1 \leq \beta \leq 2$ and $\beta \in \mathbb{Q}$. First, trivial case with $\beta = 1$ or $\beta = 2$. In this case, $\delta_n^* = 0$ for all n. So, $\mathcal{B}^* = \left\{e^{2\pi i x n}\right\}_{n\in\mathbb{Z}}$, the standard basis for $L^2([0,1])$. Now, let $1 < \beta < 2$ and $\beta \in \mathbb{Q}$, so there are two integers p and q such that $\beta = \frac{p}{q}$. We are going to use Theorem 2.7, so we need to check if $\{\lambda_n\}_{n\in\mathbb{Z}} = \{n + \delta_n^*\}_{n\in\mathbb{Z}}$ is separated and if there exist a positive integer N and a positive real number $\varepsilon < \frac{1}{4}$ such that

$$\left| \sum_{n=mN+1}^{(m+1)N} \delta_n^* \right| \leq \varepsilon N$$

for all integers m (see (2.4)). For all $n \in \mathbb{Z}$, we compare λ_n and λ_{n+1}

$$\lambda_n = n + \begin{cases} \frac{\lfloor \beta n \rfloor + 1}{\beta} - n, & \text{if } \{\beta n\} \geq \frac{1}{2} \\ \frac{\lfloor \beta n \rfloor}{\beta} - n, & \text{if } \{\beta n\} < \frac{1}{2} \end{cases} \leq n + \frac{1}{2b},$$

$$\lambda_{n+1} = n + 1 + \begin{cases} \frac{\lfloor \beta(n+1) \rfloor + 1}{\beta} - n - 1, & \text{if } \{\beta(n+1)\} \geq \frac{1}{2} \\ \frac{\lfloor \beta(n+1) \rfloor}{\beta} - n - 1, & \text{if } \{\beta(n+1)\} < \frac{1}{2} \end{cases} \geq n + 1 - \frac{1}{2\beta} > \lambda_n.$$

So, the sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$ is increasing. Moreover,

$$\sup_{n\in\mathbb{Z}} |\lambda_{n+1} - \lambda_n| \geq n + 1 - \frac{1}{2\beta} - \left(n + \frac{1}{2\beta}\right) = 1 - \frac{1}{\beta} > 0.$$

Therefore, $\{\lambda_n\}_{n\in\mathbb{Z}}$ is separated.

Next, we observe that for all $n \in \mathbb{Z}$,

$$\delta_{n+q}^* = \begin{cases} \frac{\lfloor \frac{p}{q}(n+q) \rfloor + 1}{\frac{p}{q}} - n - q = \frac{\lfloor \frac{p}{q} n \rfloor + p + 1}{\frac{p}{q}} - n - q, & \text{if } \left\{\frac{p}{q}(n-q)\right\} \geq \frac{1}{2}, \\ \frac{\lfloor \frac{p}{q}(n+q) \rfloor}{\frac{p}{q}} - n - q = \frac{\lfloor \frac{p}{q} n \rfloor + p}{\frac{p}{q}} - n - q, & \text{if } \left\{\frac{p}{q}(n-q)\right\} < \frac{1}{2}, \end{cases}$$

$$= \begin{cases} \frac{\lfloor \frac{p}{q} n \rfloor + 1}{\frac{p}{q}} - n, & \text{if } \left\{\frac{p}{q} n\right\} \geq \frac{1}{2}, \\ \frac{\lfloor \frac{p}{q} n \rfloor}{\frac{p}{q}} - n, & \text{if } \left\{\frac{p}{q} n\right\} < \frac{1}{2}. \end{cases}$$

https://doi.org/10.4153/S0008439523000371 Published online by Cambridge University Press

So, $\delta_n^* = \delta_{n+q}^*$ and we also observe that $\delta_q^* = 0$. Thus, in order to apply (2.4), it is enough to consider $\left|\sum_{n=1}^{q-1} \delta_n^*\right|$. Now, for all $n = 1, \dots, \lfloor \frac{q}{2} \rfloor$,

$$\delta_{q-n}^* = \begin{cases} \frac{\lfloor \frac{p}{q}(q-n) \rfloor + 1}{\frac{p}{q}} - q + n = \frac{\lfloor -\frac{p}{q} n \rfloor + p + 1}{\frac{p}{q}} - q + n, & \text{if } \left\{\frac{p}{q}(q-n)\right\} \geq \frac{1}{2}, \\ \frac{\lfloor \frac{p}{q}(q-n) \rfloor}{\frac{p}{q}} - q + n = \frac{\lfloor -\frac{p}{q} n \rfloor + p}{\frac{p}{q}} - q + n, & \text{if } \left\{\frac{p}{q}(q-n)\right\} < \frac{1}{2}, \end{cases}$$

$$= \begin{cases} -\frac{\lfloor \frac{p}{q} n \rfloor + 1}{\frac{p}{q}} + n, & \text{if } \left\{\frac{p}{q} n\right\} \geq \frac{1}{2}, \\ -\frac{\lfloor \frac{p}{q} n \rfloor}{\frac{p}{q}} + n, & \text{if } \left\{\frac{p}{q} n\right\} < \frac{1}{2}. \end{cases}$$

It means that $\delta_n^* + \delta_{q-n}^* = 0$. Moreover, if q is even, then $\delta_{\frac{q}{2}}^* + \delta_{\frac{q}{2}}^* = 0$, and then $\delta_{\frac{q}{2}}^* = 0$. Thus,

$$\left|\sum_{n=1}^{q-1} \delta_n^*\right| = 0.$$

If q is odd, then

$$\left|\sum_{n=1}^{q} \delta_n^*\right| = \left|\delta_{\frac{q+1}{2}}^*\right| = \left|\begin{cases} \frac{\lfloor \frac{p}{2} \rfloor + 1}{\frac{p}{q}} - \frac{q}{2} & \text{if } \left\{\frac{p}{2}\right\} \geq \frac{1}{2} \\ \frac{\lfloor \frac{p}{2} \rfloor}{\frac{p}{q}} - \frac{q}{2} & \text{if } \left\{\frac{p}{2}\right\} < \frac{1}{2} \end{cases}\right| = \begin{cases} 0, & \text{if } p \text{ is even} \\ \frac{1}{2b}, & \text{if } p \text{ is odd} \end{cases} < \frac{1}{4} < \frac{q}{4},$$

because $q > 1$. Therefore, using Theorem 2.7, we conclude that $\left\{e^{2\pi x(n+\delta_n^*)}\right\}_{n\in\mathbb{Z}}$ is an exponential basis for $L^2([0,1])$.

Next, we consider the case when $1 < \beta < 2$ is irrational. We can rewrite our sequence in the form

$$\delta_n^* = \frac{g(n\beta)}{\beta},$$

where

$$g(x) = \begin{cases} 1 - \{x\}, & \text{if } \{x\} \geq \frac{1}{2}, \\ -\{x\}, & \text{if } \{x\} < \frac{1}{2}. \end{cases}$$

g is Riemann integrable in $[0,1]$, and periodic of period 1. Moreover,

$$\int_0^1 g(x)dx = 0.$$

So, by Lemma 2.8,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(n\beta) = 0.$$

Moreover, by simple translation, we can get that for all $m \in \mathbb{Z}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=mN+1}^{(m+1)N} \frac{g(n\beta)}{\beta} = 0.$$

It means that for any $\varepsilon < \frac{1}{4}$, there is an N_0 such that for all $m \in \mathbb{Z}$ and for all $N > N_0$,

$$\left| \frac{1}{N} \sum_{n=mN+1}^{(m+1)N} \frac{g(n\beta)}{\beta} \right| = \left| \frac{1}{N} \sum_{n=mN+1}^{(m+1)N} \delta_n^* \right| < \varepsilon.$$

Therefore, using Theorem 2.7, we conclude that $\left\{ e^{2\pi x(n+\delta_n^*)} \right\}_{n\in\mathbb{Z}}$ is an exponential basis for $L^2([0,1])$. ∎

Remark 3.2 Since we proved that δ_n^* is periodic when $1 \le \beta \le 2$ and $\beta \in \mathbb{Q}$, we could also have used Corollary 3.1 from [4] to conclude that $\mathcal{B}^*$ is an exponential basis for $L^2([0,1])$.

Remark 3.3 If $\beta > 2$, then Theorem 2.6 shows that the frame constants of the basis are $A = \cos(\pi L) - \sin(\pi L)$ and $B = 2 - \cos(\pi L) + \sin(\pi L)$, where $L = \sup_{n\in\mathbb{Z}} |\delta_n|$.

3.2 Proof of Theorem 1.1

Proof Let m be infinite or finite. By Lemma 3.1, $\mathcal{B}^*$ is an exponential basis for $L^2(I)$ with the Riesz constants A and B. We can obtain a Riesz basis $\{g_n\}_{n\in\mathbb{Z}}$ for $L^2(J)$ using Lemma 2.11, where $g_n = \sum_{k=0}^{m-1} e^{2\pi i(x-b_k)(n+\delta_n^*)} \chi_{J_k}$. Next, we use the Paley–Wiener theorem to show that $\mathcal{B}^*$ is a basis for $L^2(J)$. So, we need to show that there is $0 \le \alpha < 1$ such that for all sequence $\{a_n\}$ with the property $\sum |a_n|^2 = 1$,

$$\|\sum a_n (g_n - e^{2\pi i x(n+\delta_n^*)})\|^2_{L^2(J)} \le \alpha \|\sum a_n g_n\|^2_{L^2(J)}. \tag{3.1}$$

Using a simple substitution and the Riesz constants of the basis $\mathcal{B}^*$ we, can estimate the right-hand side of the inequality (3.1)

$$0 < A \le \|\sum a_n e^{2\pi i x(n+\delta_n^*)}\|^2_{L^2(I)} = \|\sum a_n g_n\|^2_{L^2(J)}.$$

For the left-hand side, using the definition of g_n and Minkowski's inequality,

$$\begin{aligned} \|\sum a_n (g_n - e^{2\pi i x(n+\delta_n^*)})\|^2_{L^2(J)} &\le \sum_{k=1}^{m-1} \|\sum a_n (e^{2\pi i(x-b_k)(n+\delta_n^*)} - e^{2\pi i x(n+\delta_n^*)})\|^2_{L^2(J_k)} \\ &= \sum_{k=1}^{m-1} \|\sum a_n (e^{-2\pi i b_k(n+\delta_n^*)} - 1) e^{2\pi i x(n+\delta_n^*)}\|^2_{L^2(J_k)}. \end{aligned}$$

Next, we recall that

$$\delta_n^* = \begin{cases} \frac{\lfloor \beta n \rfloor + 1}{\beta} - n, & \text{if } \{\beta n\} \ge \frac{1}{2}, \\ \frac{\lfloor \beta n \rfloor}{\beta} - n, & \text{if } \{\beta n\} < \frac{1}{2}. \end{cases}$$

It means that for each $n \in \mathbb{Z}$, we can find $M_n \in \mathbb{Z}$ such that $\delta_n^* = \frac{M_n}{\beta} - n$. Thus, for all $k = 1, \dots, m-1$,

$$e^{-2\pi i b_k(n+\delta_n^*)} - 1 = e^{-2\pi i \frac{M_n b_k}{\beta}} - 1 = 0,$$

https://doi.org/10.4153/S0008439523000371 Published online by Cambridge University Press

because $\frac{b_k}{\beta} \in \mathbb{Z}$ for $k = 1, \dots, m-1$. So,

$$\| \sum a_n (g_n - e^{2\pi i x(n+\delta_n^*)}) \|_{L^2(J)}^2 = 0,$$

and the inequality (3.1) holds. Therefore, $\mathcal{B}^*$ is an exponential basis for $L^2(J)$ with Riesz constants A and B. ∎

3.3 Proof of Theorem 1.2

In [19], the reader can find the following result.

Theorem 3.4 *Let $\Delta > 0$ and $S \subset [0, \Delta]$. Suppose that for some $\Lambda \subset \frac{1}{\Delta}\mathbb{Z}$, $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2(S)$. Then $\mathcal{E}(\frac{1}{\Delta}\mathbb{Z} \backslash \Lambda)$ is a Riesz basis for $L^2([0, \Delta] \backslash S)$.*

Proof of Theorem 1.2 Let β and b_k be real numbers as in Theorem 1.2. First, we introduce the interval $[0, \Delta]$, where $\Delta = \lceil \frac{1 + b_{m-1}}{\beta} \rceil \beta$ and $\lceil x \rceil$ is a ceiling function. The set J defined as (1.1) will be a subset of $[0, \Delta]$. Let $\Lambda = \{n + \delta_n^*\}_{n \in \mathbb{Z}} = \{\lambda_n\}_{n \in \mathbb{Z}}$, where

$$\lambda_n = \begin{cases} \frac{\lfloor \beta n \rfloor + 1}{\beta}, & \text{if } \{\beta n\} \geq \frac{1}{2}, \\ \frac{\lfloor \beta n \rfloor}{\beta}, & \text{if } \{\beta n\} < \frac{1}{2}. \end{cases}$$

In view of the definition of β, we have that $\Lambda \subset \frac{1}{\Delta}\mathbb{Z}$. By Theorem 1.1, $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2(J)$. So, by Theorem 3.4, $\mathcal{E}(\frac{1}{\Delta}\mathbb{Z} \backslash \Lambda)$ is a Riesz basis for $L^2([0, \Delta] \backslash J)$. ∎

4 Extensions and generalizations

4.1 A stability theorem

Theorem 4.1 *(Stability theorem for m-segments). Let $m \in \mathbb{N} \backslash \{1\}$ and $b_j \in \mathbb{N}$ for all $j = 1, \dots, m-1$ and $L := \sup_n |\delta_n| < \frac{1}{4}$. Then the system $\{e^{2\pi i(n+\delta_n)x}\}$ is a Riesz basis for $L^2(J)$ if for all $j = 1, \dots, m-1$ and $n \in \mathbb{Z}$,*

$$\max_{j=1,\dots,m-1} \{B_{\gamma_j} \sup_n |\sin(\pi \operatorname{dist}(\mathbb{Z}, \delta_n b_j))|\} < \frac{A}{2\sqrt{m}}, \tag{4.1}$$

where

$$\begin{aligned} A &= A(L) = \cos(\pi L) - \sin(\pi L), \\ B_{\gamma_j} &= B_{\gamma_j}(L) = 2 - \cos(\pi \gamma_j L) + \sin(\pi \gamma_j L). \end{aligned} \tag{4.2}$$

Proof By the $\frac{1}{4}$-Kadec theorem, if $L := \sup_n |\delta_n| < \frac{1}{4}$, then $e^{2\pi i x(n+\delta_n)}$ is the Riesz basis for $L^2(I)$ with constants $A = \cos(\pi L) - \sin(\pi L)$ and $B = 2 - \cos(\pi L) + \sin(\pi L)$. Then, using Lemma 2.11, we can obtain a Riesz basis $\{g_n\}_{n \in \mathbb{Z}}$ for $L^2(J)$, where

$$g_n = \sum_{k=0}^{m} e^{2\pi i x \lambda_n} e^{2\pi i b_k \lambda_n} \chi_{J_k}.$$

https://doi.org/10.4153/S0008439523000371 Published online by Cambridge University Press

Next, we are going to use Theorem 2.4 with the norm

$$\|\cdot\|^2_{L^{2,\infty}(J)} = \sup_{k=0,\dots,m-1} \|\cdot\|^2_{L^2(J_k)}.$$

We can estimate the right-hand side of the inequality using substitution and the Riesz sequence definition as

$$\begin{aligned}\|\sum a_n g_n\|_{L^{2,\infty}(J)} &\geq \frac{1}{\sqrt{m}}\|\sum a_n g_n\|_{L^2(J)} \\ &= \frac{1}{\sqrt{m}}\|\sum a_n e^{i(n+\delta_n)x}\|_{L^2(I)}.\end{aligned}$$

By the elementary inequality $\sum\{|a_1|,\dots,|a_m|\} \geq \frac{1}{\sqrt{(m)}}\sqrt{\sum_n |a_n|^2}$, we have

$$\frac{1}{\sqrt{m}}\|\sum a_n e^{i(n+\delta_n)x}\|_{L^2(I)} \geq \frac{A}{\sqrt{m}}\sqrt{\sum |a_n|^2} = \frac{A}{\sqrt{m}}.$$

For the left-hand side, we have

$$\begin{aligned}\|\sum a_n\left(g_n - e^{2\pi i(n+\delta_n)x}\right)\|_{L^{2,\infty}(J)} &\leq \max_{j=1,\dots,m} \|\sum a_n e^{2\pi i(n+\delta_n)x}\left(e^{-2\pi i(n+\delta_n)b_j} - 1\right)\|_{L^{2,\infty}(J_j)} \\ &\leq \max_{j=1,\dots,m}\left\{B_{\gamma_j}\sqrt{\sum\left|a_n\left(e^{-2\pi i(n+\delta_n)b_j} - 1\right)\right|^2}\right\} \\ &\leq 2\max_{j=1,\dots,m}\{B_j \sup_n |\sin(\pi(n+\delta_n)b_j)|\} \\ &= 2\max_{j=1,\dots,m}\{B_{\gamma_j}\sup_n |\sin(\pi\delta_n b_j)|\},\end{aligned}$$

where $B_{\gamma_j} = 2 - \cos(\pi\gamma_j L) + \sin(\pi\gamma_j L)$ for $j = 1,\dots,m-1$. So, we need

$$\max_{j=1,\dots,m-1}\{B_{\gamma_j}\sup_n |\sin(\pi\delta_n b_j)|\} < \frac{A}{2\sqrt{m}}$$

or

$$\max_{j=1,\dots,m-1}\{B_{\gamma_j}\sup_n |\sin(\min\{\{\delta_n b_j\}, 1 - \{\delta_n b_j\}\}\pi)|\} < \frac{A}{2\sqrt{m}}.$$

■

Remark 4.2 If $m = 1$, we only have one interval, so Kadec's theorem holds.

Remark 4.3 Observe that $A(L)$ defined as (4.2) is a concave down function of L on the interval $\left[0, \frac{1}{4}\right)$ and $B_{\gamma_j}(L)$ defined as (4.2) is a concave up function of L on the interval $\left[0, \frac{1}{4\gamma_j}\right)$ for $j = 1,\dots,m-1$. Also, we can use the fact that $\sin(\pi \mathrm{dist}(\delta_n b_j, \mathbb{Z})) \leq \pi \mathrm{dist}(\delta_n b_j, \mathbb{Z})$. So, the condition

$$\max_{j=1,\dots,m-1}\{(1 + 4L\gamma_j)\,\mathrm{dist}(\mathbb{Z}, \delta_n b_j)\} \leq \frac{(1-4L)}{2\sqrt{m}\pi},$$

for $j = 1,\dots,m-1$, guarantees that (4.1) holds.

https://doi.org/10.4153/S0008439523000371 Published online by Cambridge University Press

4.2 *d*-dimensions

We use the arguments developed in previous sections to find bases on "split cubes." Let $Q = [0,1)^d$, and let $0 = a_{0,k} < a_{1,k} < \cdots < a_{m_k-1,k} < \cdots < 1$, where $m_k \in \mathbb{N} \cup \{\infty\}$, for all $k = 1, \ldots, d$. Also, we let $I_{j,k} = [a_{j,k}, a_{j+1,k})$ and

$$R_{\vec{j}} = R_{j_1,j_2,\ldots,j_d} = I_{j_1,1} \times I_{j_2,2} \times \cdots \times I_{j_k,k},$$

so

$$Q = \bigcup_{j_1,j_2,\ldots,j_d} R_{j_1,j_2,\ldots,j_d}.$$

Given that for all $k = 1, \ldots, d$, we have $0 = b_{0,k} < b_{1,k} < \cdots < b_{m_k-1,k}$, and we let $\vec{\beta}_{\vec{j}} = (b_{j_1,1}, b_{j_2,2}, \ldots, b_{j_d,d})$. Now, we can define a "split cube" as

$$\tilde{Q} = \bigcup_{\vec{j}} \left(\tau_{\vec{\beta}_{\vec{j}}} + R_{\vec{j}} \right). \tag{4.3}$$

Let $\beta_k > 0$ be fixed for $k = 1, \ldots, d$; for every $n_k \in \mathbb{Z}$, we let

$$\delta^*_{n_k} = \begin{cases} \frac{\lfloor \beta_k n_k \rfloor + 1}{\beta_k} - n_k, & \text{if } \{\beta_k n_k\} \geq \frac{1}{2}, \\ \frac{\lfloor \beta_k n_k \rfloor}{\beta_k} - n_k, & \text{if } \{\beta_k n_k\} < \frac{1}{2}. \end{cases} \tag{4.4}$$

We let $\vec{\delta}^*_n = (\delta^*_{n_1}, \ldots, \delta^*_{n_d})$. Let

$$\begin{aligned} \mathcal{B}^*_k &= \{e^{2\pi i x_k (n_k + \delta^*_{n_k})}\}_{n_k \in \mathbb{Z}}, \\ \mathcal{B}^* &= \{e^{2\pi i x \cdot (n + \vec{\delta}^*_n)}\}_{n \in \mathbb{Z}^d} \quad \text{with } x \in \mathbb{R}^d. \end{aligned} \tag{4.5}$$

Lemma 2.1 in [23] can be generalized in the following way.

Lemma 4.4 *Let each the set $\mathcal{U}_j = \{e^{2\pi i x \lambda_j(n)}\}_{n \in \mathbb{Z}}$ be a basis on a domain $D_j \subset \mathbb{R}$, with constants A_j and B_j, then the set $\{e^{2\pi i (\lambda_1(n_1)x_1 + \cdots + \lambda_d(n_d)x_d}\}_{n_1,\ldots,n_d \in \mathbb{Z}}$ is a basis on $L^2(D_1 \times \cdots \times D_d)$, with the constants $A = A_1 \cdot \ldots \cdot A_d$ and $B = B_1 \cdot \ldots \cdot B_d$.*

Proof We consider only the case $d = 2$. If $d > 2$, the proof is similar. Let $\mathcal{U}_j = \{e^{2\pi i x (\lambda_j(n))}\}_{n \in \mathbb{Z}}$ be a basis on a domain $D_j \subset \mathbb{R}$, with constants A_j and B_j, $j = 1, 2$. Also, to simplify formulas, we use the following notations:

$$v_{j,n_j} = e^{2\pi i x \lambda_j(n_j)} \text{ and } \tilde{v}_{n_1,n_2} = v_{1,n_1} \cdot v_{2,n_2}.$$

For any $f \in L^2(D_1 \times D_2)$, we have

$$\begin{aligned} \sum_{n_1,n_2 \in \mathbb{Z}} |\langle f, \tilde{v}_{n_1,n_2} \rangle_{L^2(D_1 \times D_2)}|^2 &= \sum_{n_2} \sum_{n_1} \left| \int_{D_1} \left(\int_{D_2} f(x_1,x_2) v_{2,n_2} dx_2 \right) v_{1,n_1} dx_1 \right|^2 \\ &\leq B_1 \int_{D_1} \sum_{n_2} \left| \int_{D_2} f(x_1,x_2) v_{2,n_2} dx_2 \right|^2 dx_1 \\ &\leq B_1 B_2 \, \|f\|^2_{L^2(D_1 \times D_2)}. \end{aligned}$$

A similar argument shows that

$$\sum_{n_1,n_2\in\mathbb{Z}} |\langle f, \tilde{v}_{n_1,n_2}\rangle_{L^2(D_1\times D_2)}|^2 \geq A_1A_2 \|f\|^2_{L^2(D_1\times D_2)}.$$

Thus, $\{e^{2\pi i(\lambda_1(n_1)x_1+\lambda_2(n_2)x_2}\}_{n_1,n_2\in\mathbb{Z}}$ is a frame for $L^2(D_1 \times D_2)$. Next, for any finite sequence of complex numbers c_{n_1,n_2}, we have

$$\begin{aligned}\Big\|\sum_{n_1}\sum_{n_2} c_{n_1,n_2}\tilde{v}_{n_1,n_2}\Big\|^2 &= \int_{D_2}\int_{D_1}\left|\sum_{n_1}\left(\sum_{n_2} c_{n_1,n_2}v_{2,n_2}\right)v_{1,n_1}\right|^2 dx_1dx_2\\ &\leq B_1\sum_{n_1}\int_{D_2}\left|\sum_{n_2} c_{n_1,n_2}v_{2,n_2}\right|^2 dx_2\\ &\leq B_1B_2\sum_{n_1}\sum_{n_2}|c_{n_1,n_2}|^2.\end{aligned}$$

A similar argument shows that

$$\Big\|\sum_{n_1}\sum_{n_2} c_{n_1,n_2}\tilde{v}_{n_1,n_2}\Big\|^2 \geq A_1A_2\sum_{n_1}\sum_{n_2}|c_{n_1,n_2}|^2.$$

Therefore, $\{e^{2\pi i(\lambda_1(n_1)x_1+\lambda_2(n_2)x_2}\}_{n_1,n_2\in\mathbb{Z}}$ is a basis for $L^2(D_1\times D_2)$. Moreover, $A = A_1\cdot A_2$ and $B = B_1\cdot B_2$ are the Riesz constants. ■

Now, we can use Lemma 4.4 to generalize some results from Section 3 in d-dimensions.

Lemma 4.5 *For all $k = 1,\dots,d$, let $\beta_k \geq 1$. Then $\mathcal{B}^*$ is an exponential basis for $L^2([0,1]^d)$, where $\mathcal{B}^*$ is defined as in (4.5).*

Proof From Lemma 3.1, we have that $\mathcal{B}^*_k$, defined as in (4.5), is a basis for $L^2([0,1])$. Therefore, by Lemma 4.5, $\mathcal{B}^*$ is an exponential basis for $L^2([0,1]^d)$. ■

Theorem 4.6 *Let $\vec{m} = (m_1,\dots,m_d)$, when $m_k \in \mathbb{N}\cup\{\infty\}$. For all $k = 1,\dots,d$, let $\beta_k \geq 1$. If for all $k = 1,\dots,d$ $\frac{b_{j,k}}{\beta_k} \in \mathbb{Z}$ for all j, then the set $\mathcal{B}^*$ defined in (4.5) is an exponential basis for $L^2(\tilde{Q})$, where $\mathcal{B}^*$ is defined as in (4.5). Moreover, $\mathcal{B}^*$ has the same frame constants for $L^2(\tilde{Q})$ and $L^2(Q)$.*

Proof From Theorem 1.1, we have that $\mathcal{B}^*_k$, defined as in (4.5), is a basis for $L^2(D_k)$, where D_k is a projection of $\tilde{Q}$ on kth coordinate. Therefore, by Lemma 4.5, $\mathcal{B}^*$ is an exponential basis for $L^2(\tilde{Q})$. ■

5 Remarks and open problems

Theorem 1.1 provides explicit exponential bases for split intervals under conditions on b_k. In Remark 3.3, we have observed that we can obtain the frame constants for the basis when $b \geq 2$. The problem of finding explicit exponential bases for general split intervals, and explicit frame constants for these bases, is still waiting for a solution. The same situation occurs with exponential bases on split cubes in $\mathbb{R}^d$.

We have provided explicit exponential bases on certain infinite unions of intervals of total finite measure. We would like to generalize our results and prove the existence of exponential bases on arbitrary infinite unions of intervals or rectangles.

Our Theorem 4.1 reduces to Kadec's theorem when the interval is not split, but in the other cases, we obtain stability bounds that depend on the gaps between the intervals. We believe that this result can be improved, and we hope to do so in another paper.

Acknowledgment Foremost, we would like to express our sincere gratitude to our mentor Professor Laura De Carli for her continuous support of our research. We acknowledge the collaboration of Mark Leal and Izzy Morell in the initial stage of this project and their participation in the technical report (available at go.fiu.edu/amrpu).

References

[1] S. A. Avdonin, *On the question of Riesz bases of exponential functions in L^2*. Vestn. Leningr. Univ. No. 13 Mat. Meh. Astron. Vyp. **3** (1974), 5–12, 154.

[2] R. Balan, *Stability theorems for Fourier frames and wavelet Riesz bases. Dedicated to the memory of Richard J. Duffin*. J. Fourier Anal. Appl. **3**(1997), no. 5, 499–504.

[3] D. Bose, C. P. A. Kumar, R. Krishnan, and S. Madan, *On Fuglede's conjecture for three intervals*. Online J. Anal. Comb. 5(2010), Article no. 1, 24 pp

[4] L. De Carli, *Concerning exponential bases on multi-rectangles of R^d*. In: Topics in classical and modern analysis: in memory of Yingkang Hu, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, Cham, 2019, pp. 65–85.

[5] L. De Carli and A. Kumar, *Exponential bases on two dimensional trapezoids*. Proc. Amer. Math. Soc. **143**(2015), no. 7, 2893–2903.

[6] L. De Carli, A. Mizrahi, and A. Tepper, *Three problems on exponential bases*. Can. Math. Bull. **62**(2019), no. 1, 55–70.

[7] L. J. Fogel, *A note on the sampling theorem*. IRE Trans. Inform. Theory **1**(1955), 47–48.

[8] C. Heil, *A basis theory primer*, expanded edition, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York, 2011, xxvi + 534 pp.

[9] A. Iosevich and M. N. Kolountzakis, *Periodicity of the spectrum in dimension one*. Anal. PDE **6**(2013), no. 4, 819–827.

[10] M. N. Kolountzakis, *The study of translational tiling with Fourier analysis*. In: Fourier analysis and convexity, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, MA, 2004, pp. 131–187.

[11] G. Kozma and S. Nitzan, *Combining Riesz bases in R^d*. Invent. Math. **199**(2015), no. 1, 267–285.

[12] I. Laba, *Fuglede's conjecture for a union of two intervals*. Proc. Amer. Math. Soc. **129**(2001), no. 10, 2965–2972.

[13] H. J. Landau, *Necessary density conditions for sampling and interpolation of certain entire functions*. Acta Math. **117**(1967), 37–52.

[14] N. Lev, *Riesz bases of exponentials on multiband spectra*. Proc. Amer. Math. Soc. **140**(2012), no. 9, 3127–3132.

[15] N. Lev and M. Matolcsi, *The Fuglede conjecture for convex domains is true in all dimensions*. Acta Math. **228**(2022), no. 2, 385–420.

[16] S. Nitzan, A. Olevskii, and A. Ulanovskii, *Exponential frames on unbounded sets*. Proc. Amer. Math. Soc. **144**(2016), no. 1, 109–118.

[17] R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, American Mathematical Society Colloquium Publications, 19, American Mathematical Society, Providence, RI, 1987, x + 184 pp. Reprint of the 1934 original

[18] B. S. Pavlov, *The basis property of a system of exponentials and the condition of Muckenhoupt*. Dokl. Akad. Nauk SSSR **247**(1979), no. 1, 37–40 (in Russian).

[19] G. Pfander, R. Shauna, and D. Walnut, *Exponential bases for partitions of intervals*. Preprint, 2021. arXiv:2109.04441

[20] K. Seip, On the connection between exponential bases and certain related sequences in $L^2(-\pi, \pi)$. J. Funct. Anal. **130**(1995), no. 1, 131–160.
[21] A. A. Selvan and R. Ghosh, *Sampling with derivatives in periodic shift-invariant spaces*. Numer. Funct. Anal. Optim. **43**(2022), no. 13, 1591–1615.
[22] E. M. Stein and R. Shakarchi, *Fourier analysis*, Princeton University Press, Princeton, NJ, 2003.
[23] W. Sun and X. Zhou, *On the stability of multivariate trigonometric systems*. J. Math. Anal. Appl. **235**(1999), no. 1, 159–167.
[24] R. M. Young, *An introduction to nonharmonic Fourier series*, Pure and Applied Mathematics, 93, Academic Press (Harcourt Brace Jovanovich), New York–London, 1980, x + 246 pp.

Department of Mathematics and Statistics, Florida International University, Miami, FL 33199, USA
e-mail: aasip001@fiu.edu vdrez001@fiu.edu

https://doi.org/10.4153/S0008439523000371 Published online by Cambridge University Press