Canad. Math. Bull. Vol. **47** (3), 2004 pp. 389–397

An Inversion Formula of the Radon Transform on the Heisenberg Group

Jianxun He

Abstract. In this paper we give an inversion formula of the Radon transform on the Heisenberg group by using the wavelets defined in [3]. In addition, we characterize a space such that the inversion formula of the Radon transform holds in the weak sense.

1 Introduction

Wavelet analysis has many applications in pure and applied mathematics. The theory of wavelet analysis on $\mathbf{R}^n$ is familiar to us. The concept of continuous wavelet transform is deeply related to the theory of square integrable group representations (see [1]). In [3, 4, 7] the authors extended the theory of wavelet analysis to the Heisenberg group. The Radon transform on $\mathbf{R}^n$ is a very useful analysis tool. New developments, in particular applications to partial differential equations, X-ray technology, and radio astronomy, have widened interest in this subject. For research about this see [5] and the references there. It is very useful to give the inversion formula of the Radon Transforms by using inverse wavelet transforms because we have a large choice of wavelets that we can use in the formula. The first result in this area is due to M. Holschneider who considered the classical Radon transform on the two-dimensional plane (see [6]). B. Rubin in [9, 10] extended the results in [6] to the k-dimensional Radon transform on $\mathbf{R}^n$ and totally geodesic Radon transforms on the sphere and hyperbolic space. R. S. Strichartz [12] discussed the Radon transform on the Heisenberg group. When one considers the problems of radial functions on the Heisenberg group, the fundamental manifold is the Laguerre hypergroup $K = [0, \infty) \times \mathbf{R}$. Using the generalized wavelets on K, M. M. Nessibi and K. Trimèche [8] gave an inversion formula of the Radon transform on K. In this paper we give inversion formulas of the Radon transform on the Heisenberg group by using the wavelets defined in [3]. Furthermore, we show that the inversion formula of the Radon transform on $R(\mathbf{H}_n)$ holds in the weak sense. The result of this paper is an extension of that of M. M. Nessibi and K. Trimèche [8].

Let $\mathbf{H}_n = \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ denote the $2n+1$-dimensional Heisenberg group with the multiplication law

$$(x, y, t)(x', y', t') = \big(x + x', y + y', t + t' + 2(y \cdot x' - x \cdot y')\big). \tag{1.1}$$

Received by the editors October 1, 2002.

The work for this paper was supported by the Foundation of the National Natural Science of China (Grants 10071039 and 10371087) and the Foundation of Education Commission of Jiangsu Province.

AMS subject classification: 43A85, 44A15.

Keywords: wavelet transform, Radon transform, Heisenberg group.

Let

$$\mathbf{P} = \{(x, y, t, \rho) : (x, y, t) \in \mathbf{H}_n, \rho > 0\}.$$

Then $\mathbf{P}$ is a locally compact nonunimodular group, the left and right Haar measures on $\mathbf{P}$ are, respectively, given by

$$d\mu_l(x, y, t, \rho) = \frac{dx\,dy\,dt\,d\rho}{\rho^{n+2}}, \quad d\mu_r(x, y, t, \rho) = \frac{dx\,dy\,dt\,d\rho}{\rho}$$

where dx and dy denote the Lebesgue measures on $\mathbf{R}^n$. The square integrable unitary representation of $\mathbf{P}$ on $L^2(\mathbf{H}_n)$ is defined by

(1.2)

$$U(x, y, t, \rho) f(x', y', t') = \rho^{-\frac{n+1}{2}} f\left(\frac{x - x'}{\sqrt{\rho}}, \frac{y' - y}{\sqrt{\rho}}, \frac{t' - t - 2(y \cdot x' - x \cdot y')}{\rho} \right).$$

Let $\mathbf{Z}^+ = \{0, 1, 2, \dots\}$, $k \in \mathbf{Z}^+$, $t \in \mathbf{R}$, the Hermite polynomials $H_k(t)$ are defined by

$$H_k(t) = \sum_{j=0}^{[k/2]} (-1)^j \frac{k!}{j!(k-2j)!} (2t)^{k-2j} = (-1)^k e^{t^2} \left(\frac{d^k}{dt^k} e^{-t^2} \right). \tag{1.3}$$

Thus the normalized Hermite functions are given by

$$h_k(t) = (2^k \sqrt{\pi} k!)^{-\frac{1}{2}} e^{-\frac{1}{2}t^2} H_k(t). \tag{1.4}$$

Now let $k = (k_1, k_2, \dots, k_n) \in (\mathbf{Z}^+)^n$, $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$. The higher dimensional Hermite functions denoted by Φ_k are then obtained by taking tensor products:

$$\Phi_k(x) = \prod_{j=1}^{n} h_{k_j}(x_j). \tag{1.5}$$

We know that the family $\{\Phi_k : k \in (\mathbf{Z}^+)^n\}$ is an orthonormal basis for $L^2(\mathbf{R}^n)$. For $\lambda \in \mathbf{R} \setminus \{0\} = \mathbf{R}^*$, let $\pi_\lambda(x, y, t)$ denote the Schrödinger representation of $\mathbf{H}_n$ which acts on $L^2(\mathbf{R}^n)$ by

$$\pi_\lambda(x, y, t)\Phi(\eta) = e^{i\lambda t - 2i\lambda x \cdot y + 4i\lambda \eta \cdot y} \Phi(\eta - x), \tag{1.6}$$

where $\Phi \in L^2(\mathbf{R}^n)$. The group Fourier transform of a function $f \in L^1(\mathbf{H}_n)$ is defined by

$$\widehat{f}(\lambda) = \int_{\mathbf{H}_n} f(x, y, t) \pi_\lambda(x, y, t)\, dx\, dy\, dt. \tag{1.7}$$

For $f, g \in L^2(\mathbf{H}_n)$, one has the following formula

$$\langle f, g \rangle_{L^2(\mathbf{H}_n)} = \frac{2^{n-1}}{\pi^{n+1}} \int_{-\infty}^{+\infty} \operatorname{tr}\left(\widehat{g}(\lambda)^* \widehat{f}(\lambda) \right) |\lambda|^n \, d\lambda, \tag{1.8}$$

where $\widehat{g}(\lambda)^*$ denotes the adjoint of $\widehat{g}(\lambda)$. Let $\mathcal{S}(\mathbf{H}_n)$ denote the Schwartz space on $\mathbf{H}_n$. From [13] we know that if $f \in \mathcal{S}(\mathbf{H}_n)$, then for all $(x, y, t) \in \mathbf{H}_n$, the inversion of the Fourier transform holds:

$$f(x, y, t) = \frac{2^{n-1}}{\pi^{n+1}} \int_{-\infty}^{\infty} \operatorname{tr}\left(\pi_\lambda(x, y, t)^* \widehat{f}(\lambda) \right) |\lambda|^n \, d\lambda. \tag{1.9}$$

The further details of harmonic analysis on $\mathbf{H}_n$ can be found in [2] and [13].

2 Wavelet Transform

Let $\mathbf{R}^+$ denote the set of all positive real numbers, $\mathbf{R}^- = -\mathbf{R}^+$. Let $h \in L^2(\mathbf{H}_n)$, if there exists a positive constant C_h such that for all $k \in (\mathbf{Z}^+)^n$ and $\sigma = +$ or $-$,

$$C_h = \left\langle \int_{\mathbf{R}^\sigma} \widehat{h}(\lambda)^* \widehat{h}(\lambda) \frac{d\lambda}{|\lambda|} \Phi_k, \Phi_k \right\rangle_{L^2(\mathbf{R}^n)}, \tag{2.1}$$

then we call h a wavelet on $\mathbf{H}_n$ (see [3]), and write $h \in AW$. Let $h \in AW$, $f \in L^2(\mathbf{H}_n)$, the continuous wavelet transform of f with respect to h is defined by

$$(W_h f)(x, y, t, \rho) = \langle f, U(x, y, t, \rho)h \rangle_{L^2(\mathbf{H}_n)}. \tag{2.2}$$

Thus we can obtain the following

Theorem 1 *Let $h \in AW$, $f \in \mathcal{S}(\mathbf{H}_n)$. Then for all $(x', y', t') \in \mathbf{H}_n$, we have the following inversion:*

(2.3)
$$f(x', y', t') = \frac{1}{C_h} \int_0^\infty \int_{\mathbf{H}_n} (W_h f)(x, y, t, \rho) U(x, y, t, \rho) h(x', y', t') \frac{dx\, dy\, dt\, d\rho}{\rho^{n+2}}.$$

Proof It is not difficult to verify that

$$\int_{\mathbf{H}_n} U(x, y, t, \rho) h(x', y', t') \pi_\lambda(x', y', t')\, dx'\, dy'\, dt' = \rho^{\frac{n+1}{2}} \pi_\lambda(x, y, t) \widehat{h}(\rho\lambda) \tag{2.4}$$

and

$$\int_{\mathbf{H}_n} (W_h f)(x, y, t, \rho) \pi_\lambda(x, y, t)\, dx\, dy\, dt = \rho^{\frac{n+1}{2}} \widehat{f}(\lambda) \widehat{h}(\rho\lambda)^*. \tag{2.5}$$

Let P_k ($k \in (\mathbf{Z}^+)^n$) be the projection from $L^2(\mathbf{R}^n)$ to 1-dimensional subspace spanned by Φ_k, and let

$$H_k^\sigma = \{ f \in L^2(\mathbf{H}_n) : \widehat{f}(\lambda) = \widehat{f}(\lambda) P_k, \text{ and } \widehat{f}(\lambda) = 0 \text{ if } \lambda \notin \mathbf{R}^\sigma \}.$$

By Theorem 1 in [7],

$$L^2(\mathbf{H}_n) = \bigoplus_{k \in (\mathbf{Z}^+)^n} (H_k^+ \oplus H_k^-),$$

thus h and f can be expressed in the form $f = \sum_{k,\sigma} f_k^\sigma$, $h = \sum_{k,\sigma} h_k^\sigma$, where $f_k^\sigma, h_k^\sigma \in H_k^\sigma$. By (2.4) and (2.5) we have

$$\begin{aligned}
&\Big\langle \int_{\mathbf{H}_n} \Big(\frac{1}{C_h} \int_0^\infty \int_{\mathbf{H}_n} (W_h f)(x, y, t, \rho) U(x, y, t, \rho) h(x', y', t') \frac{dx\, dy\, dt\, d\rho}{\rho^{n+2}} \Big) \\
&\qquad \times \pi_\lambda(x', y', t') dx'\, dy'\, dt' \Phi_k, \Phi_k \Big\rangle_{L^2(\mathbf{R}^n)} \\
&\quad = \Big\langle \frac{1}{C_h} \int_0^\infty \widehat{f}(\lambda) \widehat{h}(\rho\lambda)^* \widehat{h}(\rho\lambda) \frac{d\rho}{\rho} \Phi_k, \Phi_k \Big\rangle_{L^2(\mathbf{R}^n)} \\
&\quad = \frac{1}{C_h} \Big\langle \int_0^\infty \widehat{h}_k^\sigma(\rho\lambda)^* \widehat{h}_k^\sigma(\rho\lambda) \frac{d\rho}{\rho} \Phi_k, \Phi_k \Big\rangle_{L^2(\mathbf{R}^n)} \langle \widehat{f}_k^\sigma(\lambda) \Phi_k, \Phi_k \rangle_{L^2(\mathbf{R}^n)} \\
&\quad = \langle \widehat{f}(\lambda) \Phi_k, \Phi_k \rangle_{L^2(\mathbf{R}^n)},
\end{aligned}$$

https://doi.org/10.4153/CMB-2004-038-9 Published online by Cambridge University Press

where $\sigma = +$ if $\lambda > 0$, otherwise $\sigma = -$. By (1.9) we complete the proof. ■

3 Inversion of the Radon Transform

The Radon transform R on $\mathbf{H}_n$ is defined by

$$R(f)(x,y,t) = \int_{\mathbf{R}^n\times\mathbf{R}^n} f\big((x,y,t)(u,v,0)\big)\,du\,dv$$

$$= \int_{\mathbf{R}^n\times\mathbf{R}^n} f\big(u,v,t+2(y\cdot u - x\cdot v)\big)\,du\,dv \tag{3.1}$$

(see [12]). Write

$$\mathcal{F}_3(f)(u,v,\lambda) = \int_{\mathbf{R}} f(u,v,t)e^{i\lambda t}\,dt.$$

Then we can get

$$\big(\widehat{R(f)}(\lambda)\Phi_k\big)(\eta) = \int_{\mathbf{H}_n} R(f)(x,y,t)e^{i\lambda t-2i\lambda x\cdot y+4i\lambda\eta\cdot y}\Phi_k(\eta-x)\,dx\,dy\,dt$$

$$= \int_{\mathbf{R}^n\times\mathbf{R}^n} \mathcal{F}_3(f)(u,v,\lambda)\Big(\int_{\mathbf{R}^n\times\mathbf{R}^n} e^{-2i\lambda x\cdot y+4i\lambda\eta\cdot y-2i\lambda(y\cdot u-x\cdot v)}\Phi_k(\eta-x)\,dx\,dy\Big)\,du\,dv.$$

Let $\widehat{\Phi_k}$ denote the ordinary Fourier transform of Φ_k on $\mathbf{R}^n$. Then we have

$$\begin{aligned}
&\int_{\mathbf{R}^n\times\mathbf{R}^n} e^{-2i\lambda x\cdot y+4i\lambda\eta\cdot y-2i\lambda(y\cdot u-x\cdot v)}\Phi_k(\eta-x)\,dx\,dy\\
&\quad= \int_{\mathbf{R}^n} e^{2i\lambda\eta\cdot v-2i\lambda y\cdot u+2i\lambda\eta\cdot y}\Big(\int_{\mathbf{R}^n}\Phi_k(\eta-x)e^{-2i\lambda(\eta-x)\cdot(v-y)}\,dx\Big)\,dy\\
&\quad= e^{2i\lambda\eta\cdot v-2i\lambda v\cdot(u-\eta)}\int_{\mathbf{R}^n}\widehat{\Phi_k}\big(2\lambda(v-y)\big)\,e^{2i\lambda(v-y)\cdot(u-\eta)}\,dy\\
&\quad= (2|\lambda|)^{-n}(2\pi)^n e^{-2i\lambda v\cdot u+4i\lambda\eta\cdot v}\Phi_k(u-\eta).
\end{aligned}$$

On the other hand, by the recursion formula of Hermite polynomials (see [11, pp. 198–199]) we can obtain

$$\Phi_k(-\eta) = (-1)^{|k|}\Phi_k(\eta)$$

where $|k| = \sum_{j=1}^n k_j$. Hence

$$\big(\widehat{R(f)}(\lambda)\Phi_k\big)(\eta) = (-1)^{|k|}\pi^n|\lambda|^{-n}\big(\widehat{f}(\lambda)\Phi_k\big)(\eta). \tag{3.2}$$

Thus we can get

Theorem 2 *Let $f \in L^2(\mathbf{H}_n)$. Then*

$$|\lambda|^n\big(\widehat{R(f)}(\lambda)\Phi_k\big)(\eta) = (-1)^{|k|}\pi^n\big(\widehat{f}(\lambda)\Phi_k\big)(\eta). \tag{3.3}$$

https://doi.org/10.4153/CMB-2004-038-9 Published online by Cambridge University Press

Let $L = (\frac{\partial}{\partial t})^n$, $h \in \mathcal{S}(\mathbf{H}_n)$. Then

$$\int_{\mathbf{R}} L(h)(x, y, t) e^{i\lambda t}\, dt = (-i\lambda)^n \mathcal{F}_3(h)(x, y, \lambda).$$

Furthermore,

$$\widehat{L(h)}(\lambda) = (-i\lambda)^n \widehat{h}(\lambda). \tag{3.4}$$

By Theorem 2 we have

$$\left(\widehat{LR(h)}(\lambda)\Phi_k\right)(\eta) = \begin{cases} i^n(-1)^{n+|k|}\pi^n\left(\widehat{h}(\lambda)\Phi_k\right)(\eta), & \text{if } \lambda > 0,\\ i^n(-1)^{|k|}\pi^n\left(\widehat{h}(\lambda)\Phi_k\right)(\eta), & \text{if } \lambda < 0. \end{cases} \tag{3.5}$$

It is easy to see that if $\widehat{(LR)^2(h)}(\lambda) = (-1)^n\pi^{2n}\widehat{h}(\lambda)$, then $R^{-1} = (-1)^n\pi^{-2n}LRL$.

Let $h_\rho(x, y, t) = h(\frac{x}{\sqrt{\rho}}, \frac{y}{\sqrt{\rho}}, \frac{t}{\rho})$. Then

$$(\widehat{h_\rho})(\lambda) = \rho^{n+1}\widehat{h}(\rho\lambda). \tag{3.6}$$

Define the operator

$$(\widetilde{W}_h f)(x, y, t) = \int_{\mathbf{H}_n} f(x', y', t')\bar{h}\left((x', y', t')^{-1}(x, y, t)\right)\, dx'\, dy'\, dt'.$$

Then

$$(\widehat{\widetilde{W}_h f})(\lambda) = \widehat{f}(\lambda)\widehat{h}(\lambda)^*. \tag{3.7}$$

By the relations (3.4) and (3.6) together with (3.7), we deduce

$$\begin{aligned}(\widehat{\widetilde{W}_{L(h)_\rho} f})(\lambda) &= \widehat{f}(\lambda)\widehat{L(h)_\rho}(\lambda)^*\\ &= i^n\rho^{2n+1}\lambda^n\widehat{f}(\lambda)\widehat{h}(\rho\lambda)^*.\end{aligned}$$

Similarly, we have

$$\begin{aligned}(\widehat{\widetilde{W}_{h_\rho} L(f)})(\lambda) &= \widehat{L(f)}(\lambda)\widehat{h}_\rho(\lambda)^*\\ &= (-1)^n i^n\rho^{n+1}\lambda^n\widehat{f}(\lambda)\widehat{h}(\rho\lambda)^*.\end{aligned}$$

Then we obtain

$$(\widetilde{W}_{L(h)_\rho} f)(x, y, t) = (-1)^n\rho^n\left(\widetilde{W}_{h_\rho}L(f)\right)(x, y, t). \tag{3.8}$$

Theorem 3 *Let $h \in \mathcal{S}(\mathbf{H}_n) \cap AW$, $f \in \mathcal{S}(\mathbf{H}_n)$. Then*

$$\left(\widetilde{W}_{LRL(h)_\rho}R(f)\right)(x, y, t) = \pi^{2n}\rho^{\frac{3n+1}{2}}(W_h f)(x, y, t, \rho). \tag{3.9}$$

https://doi.org/10.4153/CMB-2004-038-9 Published online by Cambridge University Press

Proof It is easy to verify the commutative relation of L and R, *i.e.* $LR = RL$. Thus by (3.8) we have

$$\big(\widetilde{W}_{LRL(h)_\rho}R(f)\big)(x,y,t) = (-1)^n\rho^n\big(\widetilde{W}_{LR(h)_\rho}LR(f)\big)(x,y,t).$$

Using the identities (3.5) (3.6) and (3.7), we obtain

$$\begin{aligned}\big(\widehat{\widetilde{W}_{LRL(h)_\rho}R(f)}\big)(\lambda) &= (-1)^n\rho^n\big(\widehat{\widetilde{W}_{LR(h)_\rho}LR(f)}\big)(\lambda)\\ &= \pi^{2n}\rho^{2n+1}\widehat{f}(\lambda)\widehat{h}(\rho\lambda)^*.\end{aligned}$$

On the other hand we can compute

$$(\widehat{W_h f})(\lambda) = \rho^{\frac{n+1}{2}}\widehat{f}(\lambda)\widehat{h}(\rho\lambda)^*.$$

This completes the proof of the formula (3.9). ■

From Theorem 1 we obtain

Theorem 4 *Let $h \in \mathcal{S}(\mathbf{H}_n) \cap AW$, $f \in \mathcal{S}(\mathbf{H}_n)$. Then for all $(x,y,t) \in \mathbf{H}_n$ we have*

$$(3.10)\quad f(x,y,t) = \frac{1}{\pi^{2n}C_h}\int_{\mathbf{H}_n\times\mathbf{R}^+}\widetilde{W}_{LRL(h)_\rho}R(f)(x',y',t')U(x',y',t',\rho)h(x,y,t)\frac{dx'\,dy'\,dt'\,d\rho}{\rho^{\frac{5(n+1)}{2}}}$$

and

$$(3.11)\quad R^{-1}(f)(x,y,t) = \frac{1}{\pi^{2n}C_h}\int_{\mathbf{H}_n\times\mathbf{R}^+}\widetilde{W}_{LRL(h)_\rho}f(x',y',t')U(x',y',t',\rho)h(x,y,t)\frac{dx'\,dy'\,dt'\,d\rho}{\rho^{\frac{5(n+1)}{2}}}.$$

Let $h = h_k^\sigma \in \mathcal{S}(\mathbf{H}_n) \cap H_k^\sigma$, if it satisfies the condition

$$0 < C_{h_k^\sigma} = \int_{\mathbf{R}^\sigma}\|\widehat{h}(\lambda)\|_{HS}^2\frac{d\lambda}{|\lambda|} < \infty$$

(see [7]), then for all $f_k^\sigma \in \mathcal{S}(\mathbf{H}_n)\cap H_k^\sigma$, we have a simple inversion formula as follows:

$$(3.12)\quad R^{-1}(f_k^\sigma)(x,y,t) = \frac{1}{\pi^{2n}C_{h_k^\sigma}}\int_{\mathbf{H}_n\times\mathbf{R}^+}\widetilde{W}_{LRL(h_k^\sigma)_\rho}f_k^\sigma(x',y',t')U(x',y',t',\rho)h_k^\sigma(x,y,t)\frac{dx'\,dy'\,dt'\,d\rho}{\rho^{\frac{5(n+1)}{2}}}.$$

Here the condition on h is weaker than that in Theorem 4. From Theorem 1 in [4], we know that

$$L^2(\mathbf{H}_n) = \bigoplus_{l\in\mathbf{Z}^+}(H_l^+ \oplus H_l^-),$$

where

$$H_l^\sigma = \{f \in L^2(\mathbf{H}_n) : \widehat{f}(\lambda) = \widehat{f}(\lambda)P_l, \text{ and } \widehat{f}(\lambda) = 0 \text{ if } \lambda \notin \mathbf{R}^\sigma\},$$

and P_l is the orthogonal projection from $L^2(\mathbf{R}^n)$ to subspace spanned by $\{\Phi_k : |k| = l\}$. Let $h = h_l^\sigma = \sum_{|k|=l} h_k^\sigma \in \mathcal{S}(\mathbf{H}_n) \cap H_l^\sigma$, if $C_{h_l^\sigma}$ is a positive number such that

$$\begin{aligned} C_{h_l^\sigma} &= \left\langle \int_{\mathbf{R}^\sigma} \widehat{h_k^\sigma}(\lambda)^* \widehat{h_k^\sigma}(\lambda) \frac{d\lambda}{|\lambda|} \Phi_k, \Phi_k \right\rangle_{L^2(\mathbf{R}^n)} \\ &= \left\langle \int_{\mathbf{R}^\sigma} \widehat{h_{k'}^\sigma}(\lambda)^* \widehat{h_{k'}^\sigma}(\lambda) \frac{d\lambda}{|\lambda|} \Phi_{k'}, \Phi_{k'} \right\rangle_{L^2(\mathbf{R}^n)} \end{aligned}$$

for all $|k| = |k'| = l$, then for all $f_l^\sigma \in \mathcal{S}(\mathbf{H}_n) \cap H_l^\sigma$, we also have the inversion formula

$$\begin{aligned} &(3.13) \quad R^{-1}(f_l^\sigma)(x, y, t) \\ &= \frac{1}{\pi^{2n} C_{h_l^\sigma}} \int_{\mathbf{H}_n \times \mathbf{R}^+} \widetilde{W}_{LRL(h_l^\sigma)_\rho} f_l^\sigma(x', y', t') U(x', y', t', \rho) h_l^\sigma(x, y, t) \frac{dx'\, dy'\, dt'\, d\rho}{\rho^{\frac{5(n+1)}{2}}}. \end{aligned}$$

Generally, for $f \in L^2(\mathbf{H}_n)$, $R(f) \notin L^2(\mathbf{H}_n)$ (see [12]). We naturally hope to find a space such that the inversion formula of the Radon transform holds in the weak sense. Let $\mathbf{Z}$ be the set of all integers. We define $R(\mathbf{H}_n)$ by

$$R(\mathbf{H}_n) = \left\{ f \in L^2(\mathbf{H}_n) : \int_{\mathbf{R}} \|\widehat{f}(\lambda)\|_{HS}^2 |\lambda|^{(2m+1)n} d\lambda < \infty, m \in \mathbf{Z} \right\}.$$

Obviously, if $f \in R(\mathbf{H}_n)$, then $R(f) \in L^2(\mathbf{H}_n)$. From (3.2) it is easy to verify that if $f_1, f_2 \in R(\mathbf{H}_n)$, $f_1 \neq f_2$, then $R(f_1) \neq R(f_2)$. Furthermore, for any $g \in R(\mathbf{H}_n)$, we can find $f \in R(\mathbf{H}_n)$, such that $g = R(f)$. In fact, we take f satisfying $\widehat{f}(\lambda) = (-1)^{|k|} \pi^{-n} |\lambda|^n \widehat{g}(\lambda)$. Since

$$\int_{\mathbf{R}} \|\widehat{g}(\lambda)\|_{HS}^2 |\lambda|^{(2m+1)n} d\lambda = \pi^{2n} \int_{\mathbf{R}} \|\widehat{f}(\lambda)\|_{HS}^2 |\lambda|^{(2m-1)n} d\lambda,$$

we can see that $f \in R(\mathbf{H}_n)$. Thus the Radon transform R is a bijection from $R(\mathbf{H}_n)$ onto itself. On the other hand, we can see that (3.9) is also valid if $h \in \mathcal{S}(\mathbf{H}_n)$, $f \in R(\mathbf{H}_n)$. In fact, we only need to show

$$\left(\widehat{\widetilde{W}_{LRL(h)_\rho} R(f)} \right)(\lambda) = \pi^{2n} \rho^{2n+1} \widehat{f}(\lambda) \widehat{h}(\rho\lambda)^*.$$

https://doi.org/10.4153/CMB-2004-038-9 Published online by Cambridge University Press

By (3.4) and (3.5) we have

$$\begin{aligned}\left(\widehat{LRL(h)_\rho}\right)(\lambda) &= \rho^{n+1}(\widehat{LRL(h)})(\rho\lambda)\\ &= \begin{cases} i^n(-1)^{n+|k|}\pi^n\rho^{n+1}\widehat{L(h)}(\rho\lambda), & \text{if } \lambda > 0,\\ i^n(-1)^{|k|}\pi^n\rho^{n+1}\widehat{L(h)}(\rho\lambda), & \text{if } \lambda < 0\end{cases}\\ &= (-1)^{|k|}\pi^n\rho^{2n+1}|\lambda|^n\widehat{h}(\rho\lambda).\end{aligned}$$

It follows that

$$\left(\widehat{\widetilde{W}_{LRL(h)_\rho}R(f)}\right)(\lambda) = \widehat{R(f)}(\lambda)\widehat{LRL(h)_\rho}^{\,*}(\lambda) = \pi^{2n}\rho^{2n+1}\widehat{f}(\lambda)\widehat{h}(\rho\lambda)^*.$$

From the Calderón reproducing formula of continuous wavelet transform we have the following inversion formula of Radon transform in the weak sense.

Theorem 5 *Let $h \in \mathcal{S}(\mathbf{H}_n) \cap AW$, $f \in R(\mathbf{H}_n)$. Then the following formulas hold in the weak sense:*

(3.14)

$$f(x,y,t) = \frac{1}{\pi^{2n}C_h}\int_{\mathbf{H}_n\times\mathbf{R}^+}\widetilde{W}_{LRL(h)_\rho}R(f)(x',y',t')U(x',y',t',\rho)h(x,y,t)\frac{dx'\,dy'\,dt'\,d\rho}{\rho^{\frac{5(n+1)}{2}}}$$

and

(3.15)

$$R^{-1}(f)(x,y,t) = \frac{1}{\pi^{2n}C_h}\int_{\mathbf{H}_n\times\mathbf{R}^+}\widetilde{W}_{LRL(h)_\rho}f(x',y',t')U(x',y',t',\rho)h(x,y,t)\frac{dx'\,dy'\,dt'\,d\rho}{\rho^{\frac{5(n+1)}{2}}}.$$

Acknowledgment The author would like to thank the referee for his apt suggestions for improvement of this paper.

References

[1] H. G. Feichtinger and K. H. Gröchenig, *Banach Spaces Related to Integrable Group Representations and Their Atomic Decompositions.* J. Funct. Anal. **86**(1989), 307–340.

[2] D. Geller, *Fourier Analysis on the Heisenberg Group I.* J. Funct. Anal. **36**(1980), 205–254.

[3] J. He, *Continuous Multiscale Analysis on the Heisenberg Group*. Bull. Korean Math. Soc. **38**(2001), 517–526.

[4] J. He and H. Liu, *Admissible Wavelets Associated with the Affine Automorphism Group of the Siegel Upper Half-Plane*. J. Math. Anal. Appl. **208**(1997), 58–70.

[5] S. Helgason, *The Radon Transform.* Second Edition, Birkhäuser, Boston Basel Berlin, 1999.

[6] M. Holschneider, *Inverse Radon Transforms Through Inverse Wavelet Transforms.* Inverse Problems **7**(1991), 853–861.

[7] H. Liu and L. Peng, *Admissible Wavelets Associated with the Heisenberg Group*. Pacific Math. J. **180**(1997), 101–123.

[8] M. M. Nessibi and K. Trimèche, *Inversion of the Radon Transform on the Laguerre Hypergroup by Using Generalized Wavelets*. J. Math. Anal. Appl. **208**(1997), 337–363.

[9] B. Rubin, *The Calderón Reproducing Formula, Windowed X-ray Transforms and Radon Transforms in L^p-spaces.* J. Fourier Anal. Appl. **4**(1998), 175–197.

[10] ———, *Fractional Calculus and Wavelet Transforms in Integral Geometry*. Fract. Calculus Appl. Anal. **1**(1998), 193–219.
[11] J. B. Seaborn, *Hypergeometric Functions and Their Applications*. Springer-Verlag, New York, Berlin, Heidelberg, London, 1991.
[12] R. S. Strichartz, *L^p Harmonic Analysis and Radon Transforms on the Heisenberg Group*. J. Funct. Anal. **96**(1991), 350–406.
[13] S. Thangavelu, *Harmonic Analysis on the Heisenberg Group*. Birkhäuser, Boston, Basel, Berlin, 1998.

Department of Mathematics
College of Sciences
Guihuagang Campus
Guangzhou University
Guangzhou 510405
People's Republic of China
e-mail: h_jianxun@hotmail.com