The Differentiation of an Indefinite Integral.

By J. M. WHITTAKER.

(*Received 15th May 1925. Read 5th June 1925.*)

1. Theorem 1 needs very little explanation. It is the converse of the well known theorem * that the indefinite integral $F(x)$ of a function $f(x)$ possesses a derivate on the right at every point at which $f(x+0)$ exists. If $f(x+0)$ does not exist, nothing can be said as to the existence or otherwise of $F_+(x)$; but in a general way we might expect that the integral of a function which oscillates comparatively slowly, say $\sin(\log x)$ at $x=0$, would be more likely to possess a derivate than that of a function which oscillates more rapidly, say $\sin\frac{1}{x}$. It appears from Theorem 1 that this is not by any means the case. In fact the integral of $\sin(\log x)$ has not a definite derivate at $x=0$ while that of $\sin\frac{1}{x}$ has such a derivate.†

The theorem is very similar in character to one due to Mr Hardy concerning summable series.‡ The relation between $f(x+0)$ and $F_+(x)$ is indeed just the same as that between the sum and the sum $(C1)$ of an infinite series.

2. THEOREM 1. *Let $f(x)$ be continuous at all points of the interval $(0, a)$ except possibly the point $x=0$, and let the integral of $f(x)$ from ϵ to x, $(0<\epsilon<x\leq a)$, tend to a limit $F(x)$ as ϵ tends to zero.*

---

* HOBSON. *Functions of a Real Variable.* 2nd Edition, I. p. 454.

† *Cf.* LAKSHMI NARAYAN, *Bull. Calcutta Math. Soc.*, 8 (1916-17) p. 71.

‡ *Cf.* WHITTAKER and WATSON. *Modern Analysis*, 3rd Edition, p. 156.

Then $f(+0)$ *exists and* $= F_+(0)$ *if*

(i) $F_+(0)$ *exists.*

(ii) $(\alpha)\ \underline{D}_+ f(x) < \dfrac{K}{x}$ $\qquad (0 < x \leq x_0)$

or $(\beta)\ \overline{D}_+ f(x) > -\dfrac{K}{x}$ $\qquad$,,

$\log x \sin \dfrac{1}{x}$ is an example of an unbounded function which satisfies all the conditions except (ii).

Writing

$$g(x) = f(x) - F_+(0)$$

we have to prove that each of the hypotheses (ii), say (α), combined with

$$(2{\cdot}1) \qquad \int_0^x g(t)\,dt = xo(1) \qquad \text{as } x \to 0$$

implies

$$g(x) = o(1).$$

If this is not the case there must be a number h such that in every interval $0 \leq x \leq x_1$ there is a point ξ at which either

$$(a)\quad g(\xi) > h, \text{ or } (b)\quad g(\xi) < -h.$$

Take (a), and chose K so large that

$$\frac{h}{K} < 1\,.$$

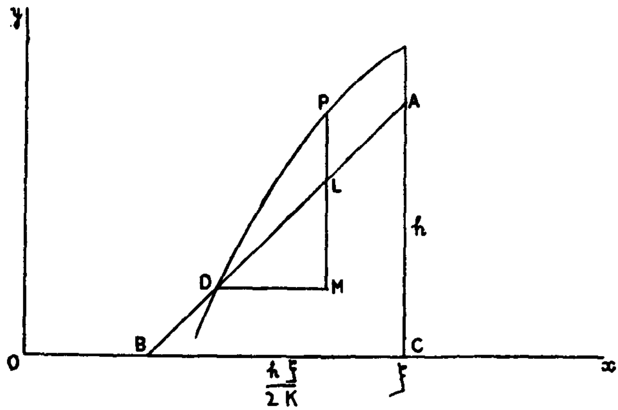

Fig. 1.

https://doi.org/10.1017/S0013091500036385 Published online by Cambridge University Press

We consider only points x_1 to the left of x_0.

From the point A (ξ, h) draw a line AB whose slope is $\frac{2K}{\xi}$. Then the curve $y = g(x)$ must lie above this line. For if not let η be the abscissa of D, the nearest point to A at which the curve cuts the line. The existence of a nearest point follows from the continuity of $g(x)$. Then if P be on the curve between D and A it is clear from Fig. 1 that

$$\frac{MP}{DM} > \frac{ML}{DM}$$

or

$$\frac{g(x) - g(\eta)}{x - \eta} > \frac{2K}{\xi} \qquad (\eta < x < \xi)$$

and taking bounds

$$\underline{D}_+ f(\eta) \geq \frac{2K}{\xi} > \frac{K}{\eta}$$

since

$$\eta \geq \xi - \frac{h\xi}{2K} > \xi - \tfrac{1}{2}\xi = \tfrac{1}{2}\xi.$$

This contradicts (α). Since therefore the curve must lie wholly above the line AB it is clear that

$$\int_{\xi_1}^{\xi} g(t)\,dt > \triangle ABC = \frac{h^2}{4K}\xi.$$

But by $(2 \cdot 1)$

$$\left| \int_{\xi_1}^{\xi} g(t)\,dt \right| \leq \left| \int_0^{\xi} g(t)\,dt \right| + \left| \int_0^{\xi_1} g(t)\,dt \right|$$
$$= \xi\, o(1) + \xi_1\, o(1)$$
$$= \xi\, o(1).$$

Thus for a set of values of ξ tending to zero

$$\xi\, o(1) > \frac{h^2}{4K}\xi.$$

The contradiction implies that hypothesis (a) is untenable.

Consider now the consequences of (b). There must exist a point $\bar{x}$ such that if $0 < x_1 \leq \bar{x}$ there is at least one point ξ between

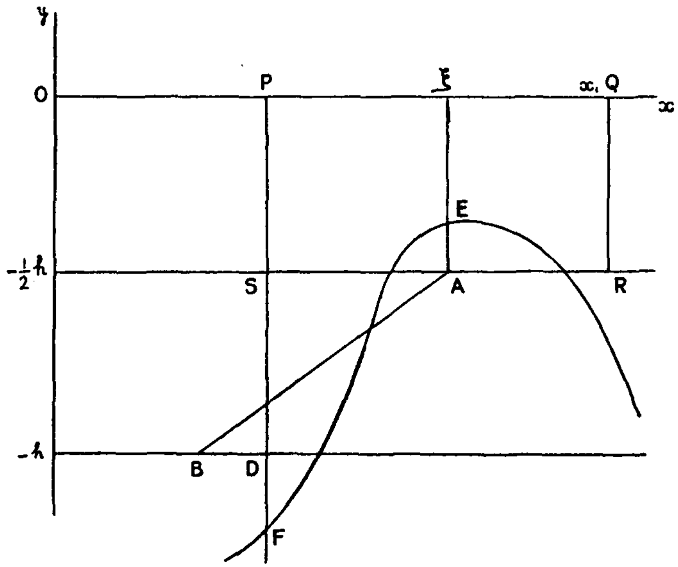

Fig. 2.

$\left(1-\frac{h}{4K}\right)x_1$ and x_1 at which the curve rises above the line $y=-\frac{h}{2}$. For if this is not so it is obvious from the diagram (Fig. 2) that

$$\int_{\left(1-\frac{h}{4K}\right)x_1}^{x_1} g(t)\,dt < \text{area of rectangle } PQRS = -\frac{h^2}{8K}x_1$$

and we are led to a contradiction, as above.

Let x_1 be a point to the left both of x_0 and of $\bar{x}$ such that at $x_1\left(1-\frac{h}{4K}\right)$ the curve is below the line $y=-h$. Then we have seen that there is a point ξ, $x_1\left(1-\frac{h}{4K}\right) < \xi < x_1$, at which the curve is above the line $y = -\frac{h}{2}$. From $A\left(\xi, -\frac{h}{2}\right)$ draw a line AB of slope $\frac{2K}{\xi}$ cutting $y = -h$ in B. The abscissa of B is

$$\xi - \frac{h}{2} \cdot \frac{\xi}{2K} = \xi\left(1 - \frac{h}{4K}\right) < x_1\left(1 - \frac{h}{4K}\right)$$

so that B lies to the left of D.

Thus, as is clear from the diagram, the curve must cut AB in order to get from E to F. This, as we have seen, contradicts (α). Thus hypothesis (b) is untenable and the theorem is proved.

The argument is similar if we assume (β) in place of (α).

Added 8th August 1925. 3. As an example consider the power series $\sum_0^\infty a_n x^n \equiv f(x)$, the coefficients a_n being real, supposed convergent for $|x| < 1$. As in Theorem 1, let $f(x)$ possess an improper integral in (0, 1). Then we deduce.

THEOREM 2

$$f(x) \to A \qquad as\ x \to 1-0$$

if (i) $\frac{1}{1-x}\int_x^1 f(t)\,dt \to A$,,

(ii) $n a_n < K$ (Ck).

For, by (ii)

$$f'(x) = \sum_1^\infty n a_n x^{n-1} = (1-x)^k \sum_1^\infty S_n^k x^{n-1}$$

$$< (1-x)^k \sum_1^\infty K A_n^k x^{n-1} = K(1-x)^k (1-x)^{-k-1}$$

$$= \frac{K}{1-x}.$$

https://doi.org/10.1017/S0013091500036385 Published online by Cambridge University Press