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Counting of finite topologies and a dissection of Stirling numbers of the second kind

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Certain new combinatorial numbers which arise in the counting of finite topologies are introduced and formulae obtained. These numbers are used to obtain a known formula for t_n , the number of labelled topologies on n points in terms of the Stirling numbers $S(n, p)$ and d_n , the number of labelled T_0-topologies on n points. The numbers d_n are computed for $n \le 5$ with the help of a method of Comtet (1966) (which seems to have been missed by later authors), reinterpreted for transitive digraphs.

1. Introduction

Let $X = \{1, 2, \ldots, n\}$. Let t_n (and d_n) stand for the number of labelled topologies (and labelled T_0-topologies) respectively, on X . That

$$t_n = \sum_p S(n, p)d_p \tag{1}$$

is well known (*cf.* Evans, Harary and Lynn, [*2*], Comtet, [*1*], Gupta, [*3*]) and implicit in Shafaat, [*6*]. Comtet, [*1*], also derived a formula for the calculation of d_p and Shafaat, [*6*], has a similar formula.

Received 23 August 1974. Propositions 1 and 2 and Theorem 1 were done, a few years ago, jointly with M.S. Radhakrishnan and S. Raghunath. {Publication of this paper was delayed to enable the author to make a minor correction to Example 1. Editor}

In this paper we introduce certain combinatorial numbers, $\lambda(n : r : p)$, which arise in the counting of finite topologies on X. These numbers satisfy

(2) $$\sum_r \lambda(n : r : p) = S(n, p) .$$

We prove, independently of (1), that

(3) $$t_n = \sum_p \sum_r \lambda(n : r : p) d_p .$$

We then take up the calculation of d_p and provide, via transitive digraphs, what seems to be an easier version of Comtet's formula. Our computed values of $\lambda(n : r : p)$ and of d_n, $n \leq 5$, lead to known values of t_n as given in Comtet, [1], and Evans, Harary and Lynn, [2]. Shafaat's method [6], which is akin to that of Comtet, ends up in results for t_5 and t_6 that are wrong.

Unless otherwise mentioned all our topologies and graphs are labelled.

2. Combinatorial numbers $\lambda(n : r : p)$ and proof of (3)

We start with the concept of 'Borel equivalence' introduced by Rayburn [5]. Let $T(X)$ be the set of all topologies on X. Two topologies on X are said to be *Borel equivalent* if they generate the same Borel field; that is, a topology in which every open set is also closed, or, what Sharp, [7], calls, a symmetric topology. This equivalence partitions $T(X)$ into what are called Borel equivalence classes. 'How many topologies are there in each Borel equivalence class?' was a question posed by Rayburn.

Recall [2] that $T(X)$ is in one-to-one correspondence with transitive digraphs (shortly, transgraphs) in the following manner. Given $\tau \in T(X)$, denote by B_i the smallest τ-open set containing i. Construct the directed graph $G(\tau)$ on X by stipulating that, for $j \neq i$, $(i, j) \in G(\tau)$ if and only if $j \in B_i$. (Here, and throughout the paper, (i, j) means the directed edge leading from i to j.) The fact that this construction results in $G(\tau)$ being transitive and that the correspondence $\tau \to G(\tau)$ is bijective are proved in [2] and [4]. Under this correspondence, T_0-topologies and Borel fields show themselves up as two extremes in $T(X)$. Let us use the term 'dwicycle' to denote a directed cycle of length two.

Then T_0-topologies correspond to transgraphs which have no dwicycles (*cf.* [2] and [7]) and Borel fields correspond to transgraphs in which every edge is part of a dwicycle. These can be seen easily by noting that:

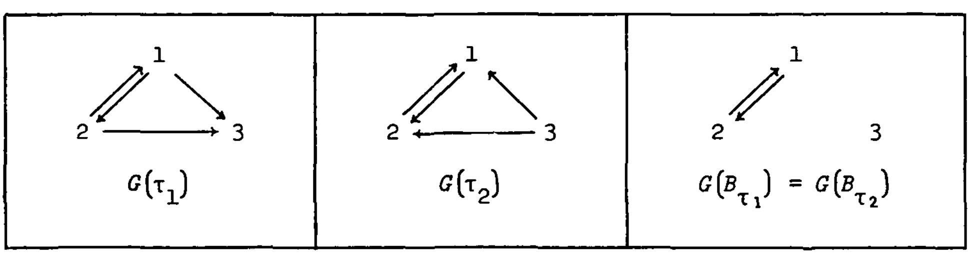
(1) τ is a T_0-topology if and only if $j \in B_i \Rightarrow i \notin B_j$; and

(2) τ is a Borel field if and only if $j \in B_i \Rightarrow i \in B_j$ (*cf.* Rayburn [5]).

Also note that τ_1 is finer than τ_2 if and only if $G(\tau_1)$ is a subgraph of $G(\tau_2)$. This tells us that, to generate the Borel field B_τ containing τ , we have only to look at the subgraph of $G(\tau)$ and pick the largest subgraph G_0 which has nothing but dwicycles in it. This G_0 will be $G(B_\tau)$. We have thus proved

PROPOSITION 1. *If* $\tau \in T(X)$ *and* B_τ *is the Borel field generated by* τ *then* $G(B_\tau)$ *can be obtained from* $G(\tau)$ *by deleting all the edges in the latter which are not part of dwicycles.*

As an illustration, note the following transgraphs on three points:



If τ is in the Borel equivalence class $B(B)$ determined by B then $G(\tau)$ and $G(B)$ differ only in the single lines which do not form part of dwicycles. To construct $G(\tau)$ from $G(B)$, we have, therefore, only to add other lines to $G(B)$ in such a way that

(i) the resulting graph is a transgraph, and

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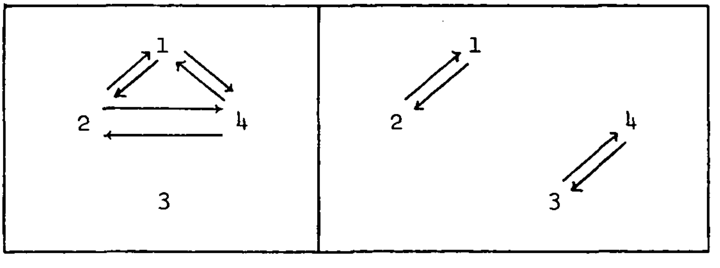
(ii) no new dwicycles are introduced.

Note that, in this construction, if (i, j) and (j, i) form a dwicycle and k is any other vertex, either we add both (i, k) and (j, k) or not at all and similarly, either we add both (k, i) and (k, j) or not at all. So, for the purpose of this construction, we can identify pairs of points which are connected by a dwicycle. Consider the resulting smaller set X_0 of points and construct transgraphs on X_0 without dwicycles. For each such transgraph on X_0 (which is now a T_0-topology on X_0) we can recover a topology on X which belongs to $\mathcal{B}(B)$. This is done by recovering all the identified points and the dwicycles connecting them. Conversely every T_0-topology on X_0 in the same way gives rise to a topology on X which belongs to $\mathcal{B}(B)$. Thus the number of topologies in $\mathcal{B}(B)$ is the number of T_0-topologies on the set X_0 as obtained above. Hence we have proved the following

PROPOSITION 2. *Let B be a Borel field and $C(G(B))$ be the graph obtained by identifying pairs of points in $G(B)$ which are connected by dwicycles. Let p be the number of vertices in $C(G(B))$. Then the number of topologies in the Borel equivalence class determined by B is d_p.*

Now in order to count $|T(X)|$ we have only to list the various Borel equivalence classes there are and sum up d_p for the various values of p that arise. But it happens that the same p may arise from distinct Borel fields, as can be seen from the *two* transgraphs on four points shown below.



So it is necessary to take into account the number r of dwicycles that occur in $G(B)$. Each unlabelled Borel field B is determined by two parameters r and p. So we make the following definition.

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DEFINITION 1. Given integers n, r, p such that $n \geq 2$, $0 \leq r \leq \binom{n}{2}$ and $1 \leq p \leq n$, $\lambda(n : r : p)$ denotes the number of labelled Borel fields B on a set of n points, with r dwicycles and with $p = |C(G(B))|$. If there is no such Borel field for a pair (r_0, p_0), $\lambda(n : r_0 : p_0) = 0$.

Putting aside the calculation of $\lambda(n : r : p)$ for a while, we first note that Proposition 2 and the discussion following it gives us the following

THEOREM 1. $t_n = \sum_p \sum_r \lambda(n : r : p) d_p$.

Recall (*cf.* Sharp [7]) that $T(X)$ is in bijective correspondence with the set of quasiorders (reflexive and transitive relations) on X, by the rule

$$j \in B_i \Leftrightarrow iRj .$$

Under this correspondence T_0-topologies correspond to partial orders and Borel fields correspond to equivalence relations. Given p, the problem of constructing all the $\sum_r \lambda(n : r : p)$ labelled Borel fields is the problem of distributing n distinct objects (the vertices $1, 2, \ldots, n$ in this case) into p distinct cells (the vertices of $C(G(B))$, in this case). Hence

$$\sum_r \lambda(n : r : p) = S(n, p) .$$

This observation completes the promised independent proof of (1).

3. Calculation of $\lambda(n : r : p)$

When $r = 0$, $p = n$, and clearly $\lambda(n : 0 : n) = 1$. We shall suppose $r > 0$ in the rest of this section until we come to Theorem 2. The number r arises as follows. First, note that, as a consequence of transitivity, no dwicycle can exist in a transgraph except as part of a complete sub-transgraph. The number r will therefore be the sum of the numbers of dwicycles in the *complete* subtransgraphs of $G(B)$. But the number of dwicycles in a complete subtransgraph is $\binom{k}{2}$ where $k \geq 2$ is the number of vertices in the complete subtransgraph. So,

$$r = \binom{k_1}{2} + \binom{k_2}{2} + \dots ,$$

with $k_i \geq 2$, and $k_1 + k_2 + \dots = n$. The number p is the number of such complete subtransgraphs in $G(B)$. Thus, given the parameters n, r, p we arrive at a unique unordered partition of the integer n into p parts such that $n = k_1 + k_2 + \dots + k_p$ and $r = \sum_{i=1}^{p} \binom{k_i}{2}$.

Conversely, given an unordered partition of the integer n which has at least one part greater than 1, the parameters r and p are determined uniquely.

Thus this correspondence between unordered partitions with at least one part greater than 1 and the triads of parameters n, r, p for which $\lambda(n : r : p) > 0$ is bijective. So, to determine $\lambda(n : r : p)$, we take the corresponding partition

$$n = k_1 + k_2 + \dots + k_p$$

and regroup the integers $k_1, k_2, \dots, k_p$ into

α_1 integers each equal to p_1,

α_2 integers each equal to p_2, and so on.

(Note that we must have $\sum \alpha_i p_i = n$ and at least one $p_i \geq 2$.) The corresponding transgraph will consist of

α_1 disjoint complete transgraphs each on p_1 points;

α_2 disjoint complete transgraphs each on p_2 points; and so on;

subject to the understanding that wherever $p_j = 1$, the component corresponding to that reduces to a single point. "In how many ways can such a configuration arise, with n, r, p given?" is the question. The choice of α_1 subsets of p_1 vertices each can be made in

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$$\frac{\binom{n}{p_1}\binom{n-p_1}{p_1}\binom{n-2p_1}{p_1}\cdots\binom{n-(\alpha_1-1)p_1}{p_1}}{\alpha_1!}$$

ways. Having made this choice, the choice of α_2 subsets of p_2 vertices each can be made in

$$\frac{\binom{n-\alpha_1p_1}{p_2}\binom{n-\alpha_1p_1-p_2}{p_2}\cdots\binom{n-\alpha_1p_1-(\alpha_2-1)p_2}{p_2}}{\alpha_2!}$$

ways; and so on.

This completes the proof of the following

THEOREM 2. *Let* n *be any integer greater than or equal to* 2, $0 \le r \le \binom{n}{2}$ *and* $1 \le p \le n$ *such that* $n = k_1 + k_2 + \ldots + k_p$ *and* $r = \sum_i \binom{k_i}{2}$ *where* $\binom{k_i}{2} = 0$ *if* $k_i = 1$. *Then*

$$\lambda(n : r : p) = \frac{\binom{n}{p_1}\binom{n-p_1}{p_1}\binom{n-2p_1}{p_1}\cdots\binom{n-(\alpha_1-1)p_1}{p_1}}{\alpha_1!} \times \frac{\binom{n-\alpha_1p_1}{p_2}\binom{n-\alpha_1p_1-p_2}{p_2}\cdots\binom{n-\alpha_1p_1-(\alpha_2-1)p_2}{p_2}}{\alpha_2!} \times \ldots ,$$

where the integers $k_1, k_2, \ldots, k_p$ *have*

α_1 *integers each equal to* p_1,

α_2 *integers each equal to* p_2, *and so on.*

4. Calculation of d_p

It remains to calculate d_p for every $p > 0$. Clearly $d_1 = 1$ and $d_2 = 3$. To arrive at a general formula for d_p, we proceed by the method of Comtet but now use the concept of transgraphs intensively. Evans,

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Harary and Lynn [2] have done a similar computation but ours is different.

Let Γ_n be the set of transgraphs on n points without dwicycles. Let γ stand for an arbitrary element of Γ_n . We shall associate with each γ a unique ordered vector of non-empty subsets of X as follows. Count the outdegrees of each vertex of γ . (The outdegree of a vertex is the number of directed edges leaving it.) We claim that at least one of these outdegrees must be zero. To see this, start with any vertex $i \in \gamma$. If $j \in B_i$, then $(i, j) \in \gamma$ but $(j, i) \notin \gamma$. Now look at B_j . If $k \in B_j$ then k can be connected only to points other than i and j ; this follows easily from the transitivity of γ and the fact that it has no dwicycles. Continuing this process, we finally end up with a vertex p which is not connected to any other vertex. Thus there exists a p such that the outdegree of p is zero. Let $S_1(\gamma)$ be the set of all vertices of γ with outdegree zero.

Delete all vertices belonging to $S_1(\gamma)$ from the graph γ and also all the edges leading from or to such vertices. The resulting graph may be called the first truncation γ_1 . Clearly it is a transgraph without dwicycles. Compute $S_1(\gamma_1)$ and denote it by $S_2(\gamma)$. Delete from γ_1 , the points of $S_2(\gamma)$ and all edges leading from or to them, thus obtaining the second truncation γ_2 . Continue this process until all vertices of γ are exhausted. The last set $S_k(\gamma)$ will be such that all its points have outdegrees zero in the $(k-1)$th truncation of γ . Write

$$(S)_\gamma = \left(S_1(\gamma), S_2(\gamma), \ldots, S_k(\gamma)\right) .$$

Thus, corresponding to γ we have an ordered partition of non-empty subsets of X . We write $\eta(\gamma) = (S)_\gamma$. Note that η of the discrete graph is (X) .

PROPOSITION 3. *(i)* η *is onto the set of all ordered partitions of non-empty subsets of* X .

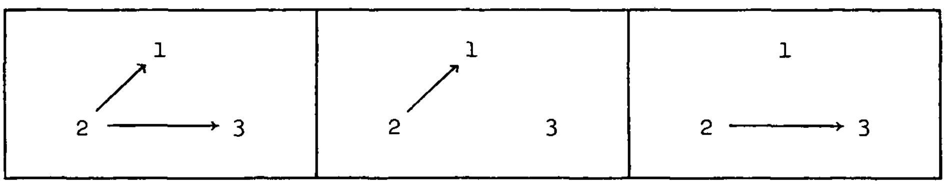
(ii) η *is many-one.*

Proof. *(i)* Given an ordered partition $(S) = \left(S_1, S_2, \ldots, S_k\right)$ of non-empty subsets of X , we produce a transgraph γ as follows. A directed edge goes from every point of S_p to one or more points of S_q , $q < p$, in such a way that, whenever $p > 2$, $x \in S_p$, $y \in S_{p-1}$, and $(x, y) \in \gamma$, it is true that, for every $i \leq p-2$,

$$z \in S_i \text{ and } (y, z) \in \gamma \Rightarrow (x, z) \in \gamma .$$

The resulting γ is clearly transitive. It has no dwicycles because all directed edges go from points of S_i to points of S_j , $i > j$, and never in the opposite direction. The points of S_1 are all of outdegree zero. So $S_1 = S_1(\gamma)$. $\gamma_1 = \gamma \backslash S_1(\gamma)$, the first truncation of γ , has the points of S_2 as its set of vertices with zero outdegree. Hence $S_2 = S_2(\gamma)$; and so on. Thus $\eta(\gamma) = (S)$ and *(i)* is proved.

(ii) To prove *(ii)* look at $X = \{1, 2, 3\}$. Suppose $(S) = (\{1, 3\}, \{2\})$. Then all the following 3-transgraphs have (S) as their η-image.



Let $N(S)$ be the number of $\gamma \in \Gamma_n$ such that $\eta(\gamma) = (S)$. $N(S)$ can be computed for each (S) (see Section 4). Given $(S) = (S_1, S_2, \ldots, S_k)$ where $|S_i| = s_i$, the number of ways in which the n labelled vertices of γ can be distributed into $S_1, S_2, \ldots, S_k$ is $\dfrac{n!}{s_1!s_2!\ldots s_k!}$. This proves the theorem of Comtet [1] as stated below.

THEOREM 3.

$$d_n = \sum_{1 \leq k \leq n} \sum_{\substack{(S)=(S_1,S_2,\ldots,S_k) \\ |S_i|=s_i>0, S_i \subset X \\ \Sigma s_i = n}} \frac{n!}{s_1!s_2!\ldots s_k!} N(S)$$

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where $N(S)$ *is the number of transgraphs* γ *on* X *, without dwicycles such that* $\eta(\gamma) = (S)$.

5. Computation of d_n , $\lambda(n : r : p)$, and t_n

NOTE. In this section and in the Tables at the end, we use (xyz) for $\{x, y, z\}$.

In the computation of d_n the main problem is to calculate $N(S)$ for each possible form of $(S) = (S_1, S_2, \ldots, S_k)$ where $(|S_1|, |S_2|, \ldots, |S_k|)$ is an ordered partition of the integer n . Given (S) we proceed as follows. For each point x of S_p and every $q < p$ we have to choose a point or points of S_q to which lines from x will lead. In other words, for each $x \in S_p$, one has to choose a non-empty subset of S_q . It helps to write all the possible choices for all p and q , $p > q$, in the form of a tableau as below with $k - 1$ rows and $k - 1$ columns where the square at the row titled S_p and the column titled S_q lists all the choices for the map $S_p \to$ set of non-empty subsets of S_q . Then a case by case checking is done for transitivity.

Let us illustrate this with two examples.

EXAMPLE 1. $n = 5$, $(S) = (12)\ (3)\ (4)\ (5)$.

(3)	3 → (1) 3 → (2) 3 → (12)		
(4)	4 → (1) 4 → (2) 4 → (12)	4 → (3)	
(5)	5 → (1) 5 → (2) 5 → (12)	5 → (3)	5 → (4)
	(12)	(3)	(4)

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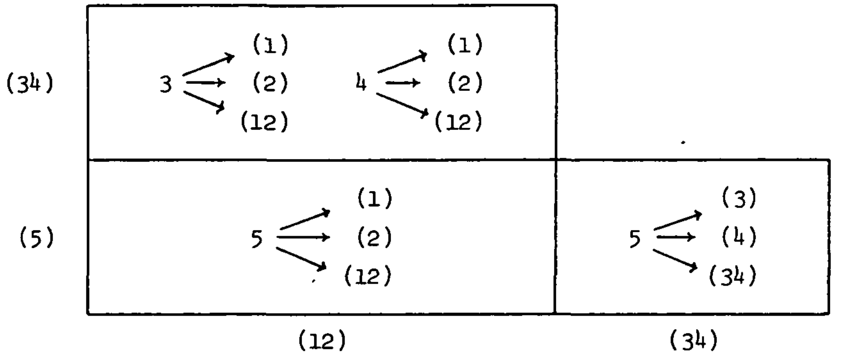
The choice $3 \to (1)$ implies $4 \to (1)$ and $5 \genfrac{}{}{0pt}{}{\nearrow (1)}{\searrow (12)}$ or $4 \to (12)$ and $5 \to (12)$. This gives 3 choices with $3 \to (1)$. Similarly there are 3 choices with $3 \to (2)$. The choice $3 \to (12)$ implies $4 \to (12)$ and $5 \to (12)$. Thus $N(S) = 3 + 3 + 1 = 7$.

EXAMPLE 2. $n = 5$, $(S) = (12)\ (34)\ (5)$.



Start with $5 \to (3)$ and $3 \to (1)$. This goes with any choice for the image of 4 and implies $5 \genfrac{}{}{0pt}{}{\nearrow (1)}{\searrow (12)}$. Hence there are 6 choices.

Similarly there are 6 choices with $5 \to (3)$ and $3 \to (2)$. There are only 3 choices that go with $5 \to (3)$ and $3 \to (12)$. Thus there are, in all, 15 choices for $5 \to (3)$. Similarly there are 15 choices for $5 \to (4)$.

Now take up $5 \to (34)$. Then

$$3 \to (1)\ ,\quad 4 \to (1) \quad \Rightarrow 5 \genfrac{}{}{0pt}{}{\nearrow (1)}{\searrow (12)}\ ,$$

$$3 \to (1),,\quad 4 \to (2) \quad \Rightarrow 5 \to (12)\ ,$$

$$3 \to (1)\ ,\quad 4 \to (12) \quad \Rightarrow 5 \to (12)\ .$$

Thus $3 \to (1)$ gives 4 choices. Similarly $3 \to (2)$ gives 4 choices. Finally $3 \to (12)$ gives 3 choices. Thus $5 \to (34)$ gives, in all, $4 + 4 + 3 = 11$ choices. Hence $N(S) = 15 + 15 + 11 = 41$.

The completed results are tabulated in Table 1 for $2 \le n \le 5$.

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Table 2 gives the results for $\lambda(n : r : p)$ and the computations for obtaining t_n , $2 \leq n \leq 5$.

Table 1

Computation of d_n , $2 \leq n \leq 5$

n	ordered partition of n	Typical form of (S)	$N(S)$	Number of S's of the same form	Contribution to d_n	d_n
2	2	(12)	1	1	1	
	11	(1) (2)	1	2	2	3
3	3	(123)	1	1	1	
	12	(1) (23)	1	3	3	
	21	(12) (3)	3	3	9	
	111	(1) (2) (3)	1	6	6	19
4	4	(1234)	1	1	1	
	13	(1) (234)	1	4	4	
	31	(123) (4)	7	4	28	
	22	(12) (34)	9	6	54	
	112	(1) (2) (34)	1	12	12	
	121	(1) (23) (4)	3	12	36	
	211	(12) (3) (4)	5	12	60	
	1111	(1) (2) (3) (4)	1	24	24	219
5	5	(12345)	1	1	1	
	14	(1) (2345)	1	5	5	
	41	(1234) (5)	15	5	75	
	23	(12) (345)	27	10	270	
	32	(123) (45)	49	10	490	
	221	(12) (34) (5)	41	30	1230	
	212	(12) (3) (45)	9	30	270	
	122	(1) (23) (45)	9	30	270	
	311	(123) (4) (5)	19	20	380	
	131	(1) (234) (5)	7	20	140	
	113	(1) (2) (345)	1	20	20	
	1112	(1) (2) (3) (45)	1	60	60	
	1121	(1) (2) (34) (5)	3	60	180	
	1211	(1) (23) (4) (5)	5	60	300	
	2111	(12) (3) (4) (5)	7	60	420	
	11111	(1) (2) (3) (4) (5)	1	120	120	4231

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Table 2

Values of $\lambda(n : r : p)$ and t_n

n	Unordered partition of n	$\lambda(n : r : p)$	$S(n, p)$	d_p	Contribution to t_n	t_n
2	11	$\lambda(2 : 0 : 2) = 1$	$S(2, 2) = 1$	3	3	
	2	$\lambda(2 : 1 : 1) = 1$	$S(2, 1) = 1$	1	1	4
3	111	$\lambda(3 : 0 : 3) = 1$	$S(3, 3) = 1$	19	19	
	21	$\lambda(3 : 1 : 2) = 3$	$S(3, 2) = 3$	3	9	
	3	$\lambda(3 : 3 : 1) = 1$	$S(3, 1) = 1$	1	1	29
4	1111	$\lambda(4 : 0 : 4) = 1$	$S(4, 4) = 1$	219	219	
	211	$\lambda(4 : 1 : 3) = 6$	$S(4, 3) = 6$	19	114	
	22	$\lambda(4 : 2 : 2) = 3$	$S(4, 2) = 7$	3	9	
	31	$\lambda(4 : 3 : 2) = 4$		3	12	
	4	$\lambda(4 : 6 : 1) = 1$	$S(4, 1) = 1$	1	1	355
5	11111	$\lambda(5 : 0 : 5) = 1$	$S(5, 5) = 1$	4231	4231	
	2111	$\lambda(5 : 1 : 4) = 10$	$S(5, 4) = 10$	219	2190	
	221	$\lambda(5 : 2 : 3) = 15$	$S(5, 3) = 25$	19	285	
	311	$\lambda(5 : 3 : 3) = 10$		19	190	
	32	$\lambda(5 : 4 : 2) = 10$	$S(5, 2) = 15$	3	30	
	41	$\lambda(5 : 6 : 2) = 5$		3	15	
	5	$\lambda(5 : 10 : 1) = 1$	$S(5, 1) = 1$	1	1	6942

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