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Ergodic Properties of Randomly Coloured Point Sets

We dedicate this work to Hajo Leschke on the occasion of his 67th birthday.

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Abstract. We provide a framework for studying randomly coloured point sets in a locally compact second-countable space on which a metrizable unimodular group acts continuously and properly. We first construct and describe an appropriate dynamical system for uniformly discrete uncoloured point sets. For point sets of finite local complexity, we characterize ergodicity geometrically in terms of pattern frequencies. The general framework allows us to incorporate a random colouring of the point sets. We derive an ergodic theorem for randomly coloured point sets with finite-range dependencies. Special attention is paid to the exclusion of exceptional instances for uniquely ergodic systems. The setup allows for a straightforward application to randomly coloured graphs.

1 Introduction

Delone sets are subsets of Euclidean space that are uniformly discrete and relatively dense. In the natural sciences, they are used to model pieces of matter. Recently, geometric and spectral properties of Delone sets have been studied by many authors using methods from topological dynamical systems; see *e.g.,* [RaW, Ho1, Ho2, So, BelHZ, LeMS, LP, LenS1, KlLS, LenS3, LeS]. Here, the dynamical system arises from the closure of the translation orbit of the given Delone set with respect to a suitable topology. This approach is particularly useful if the dynamical system is uniquely ergodic, since the uniform ergodic theorem can then be used to infer properties of the original Delone set. For a Delone set of finite local complexity, a geometric characterization of unique ergodicity in terms of uniform pattern frequencies appears in [LeMS]. If the Delone set is not periodic, then such a characterization cannot be achieved with a discrete periodic subgroup of the Euclidean group as the group acting on the dynamical system. Therefore one has to rely upon an ergodic theorem for the action of a more general group than the multi-dimensional integers.

This approach has been generalized considerably in recent years. Euclidean space has been replaced by a σ-compact, locally compact Abelian group, which admits a suitable averaging sequence, and on which the same group acts by translations [S]. Within that setup, the important subclass of repetitive regular model sets (see *e.g.,* [Mo]), which have a pure point diffraction spectrum such as periodic point sets,

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could be characterized by certain properties of the underlying dynamical system including strict ergodicity and pure point dynamical spectrum [BLM]. More generally, dynamical systems of translation bounded Borel measures [BL, LenRi] on such spaces have been considered. Discrete subsets of a general locally compact topological space have been studied in [Yo] via group actions of a locally compact group, focussing on finite local complexity and on repetitivity.

As we are interested in discrete geometry, our setup will be formulated in terms of uniformly discrete sets. We will use a locally compact, second-countable space as our basic space of points, which we will simply refer to as the point space. Choosing a metric allows us to define a notion of uniform discreteness. Local compactness of the point space ensures sufficient structure for the space of uniformly discrete point sets. The *first main goal* of this paper is to establish geometric criteria for (unique) ergodicity of the dynamical system associated to a collection of uniformly discrete point sets in terms of pattern frequencies. To do so, we rely on properness of the group action. In addition, we will measure the "size" of a subset of the point space in terms of the Haar measure on the group, which is pushed forward by the group action and a reference point in the point space. We require that size to be the same for group-equivalent reference points. This is ensured for unimodular groups. Thus our setup comprises non-Abelian groups such as the Euclidean group. In particular, this accommodates the pinwheel tilings of the plane [Ra1, Ra2, Ra3] and their relatives [Fr]. Apart from [Yo], it seems that non-Abelian groups have not been treated in our general context so far.

In order to define an appropriate dynamical system, we require that uniformly discrete point sets, which are group-equivalent, have the same radius of discreteness. This will be guaranteed if the metric on the point space is group-invariant. Indeed, it has recently been shown [AMN, Thm. 1.1] that under our assumptions on the group action, such a metric always exists among the set of all metrics that are compatible with the topology of the point space. We will supply the space of uniformly discrete point sets (of a given radius of discreteness) with the vague topology. This ensures compactness of the relevant dynamical systems. In [Yo], a stronger "local matching topology" is favoured instead. For a proper and transitive group action, both topologies coincide if the point sets are of finite local complexity. This follows from Lemma 2.27; see also [BL] for the Abelian case.

If the point space M admits a uniform structure compatible with the given topology, but is not necessarily metrizable, it is still possible to define uniform discreteness via the uniformity. In fact, a continuous and proper action of a group T on M gives rise to a uniform structure on M, which induces the topology on M and has the desirable invariance properties with respect to T, as follows from [AMN, Thm. 1.2]. A corresponding framework could include our setup for metrizable spaces M, as well as the approach of [S, BL, Yo] for a not-necessarily-metrizable space M as special cases. We will not strive for such a generality here.

Our structural assumptions on the group and its action are minimal in a sense. By properness, the group inherits local compactness and σ-compactness from the point space. But local compactness is needed for the existence of a (well-behaved) Haar measure, and σ-compactness is required for amenability. The role of unimodularity has been discussed above, and Lindenstrauss' pointwise ergodic theorem [Lin], which we rely upon to a great extent, requires metrizability of the group. We will, however, not assume transitivity of the group action. This is motivated by our desire to describe uniformly discrete sets, coloured versions thereof, and graphs built from such sets – all within the same framework. Here, coloured point sets with possibly infinitely many colours and also graphs will appear as point sets in some suitably enlarged point space on which one cannot expect to have a transitive group action. Due to the absence of transitivity we were also prompted to free the point space from being a group by itself. We mention that coloured Delone sets of finite local complexity – and thus with at most finitely many colours – have been studied by different methods in [BelHZ, LeMS, LenS1].

As our choice of spaces is also canonical in stochastics, the connection to stochastic geometry [SKM] may be broadened. Indeed, the setup allows us to study random colourings of a point set on a rather general level. Ergodic properties of random colourings of a repetitive Delone set in the Euclidean plane have already been studied by Hof [Ho3], motivated by the problem of site percolation on the Penrose tiling. His approach has been used to infer diffraction properties of random Euclidean point sets of finite local complexity [BZ] with finite-range dependencies and beyond [BBM]; see also [Len] for an alternative approach. A recent extension to certain Delone sets in σ-compact, locally compact Abelian groups is the subject of [AI]. Another recent generalization to infinite-range dependencies, based on the theory of Gibbs measures and stochastic geometry, can be found in [Ma]. Diffraction properties of certain non-periodic stochastic point sets are also discussed in [Kü1, Kü2], where large-deviation estimates and concentration inequalities for the finite-volume scattering measure are derived.

We are not concerned with diffraction in this paper, however. In fact, our *second main goal* is to provide an optimal ergodic theorem for dynamical systems of randomly coloured point sets with finite-range stochastic dependencies. To do so, we also pursue an idea of Hof [Ho3], who used the law of large numbers for reducing the problem to that of the dynamical system for the underlying uncoloured point sets. Unfortunately, Hof's approach only works for point sets of finite local complexity, and is thus also restricted to finitely many colours. On the other hand, Lenz [Len] proved an ergodic theorem for randomly coloured translation bounded measures on Euclidean space without the need for finite local complexity. In combining the two approaches, we obtain an ergodic theorem for randomly coloured point sets without requiring finite local complexity. And, in contrast to [Len], it is optimal in the sense that exceptional instances are excluded as far as possible in the case of uniquely ergodic systems and continuous functions.

In a subsequent article, we will apply the aforementioned ergodic results to describe spectral properties of subcritical percolation graphs over such general point sets; *cf.* [KM] for the periodic case. As we are able to treat rather general colour spaces and group actions, our approach also opens the possibility to study finite-range operators on uniformly discrete point sets with quite general internal degrees of freedom and their randomized versions, such as (random) Schrödinger operators with magnetic finite-range interactions on non-periodic point sets.

This paper is organized as follows. In Section 2, we recapitulate properties of dynamical systems of uncoloured point sets, which carry over to our more general setup. As our general group actions have apparently not been studied before in this context, we provide proofs for the convenience of the reader. Based on the general pointwise ergodic theorem of Lindenstrauss [Lin], we state characterizations of ergodicity in Theorem 2.14, which are handy to use and which we have not found in the literature in sufficient generality. The same remark applies to Theorem 2.16, which is the abstract analogue for uniquely ergodic systems, extending the well-documented case of $\mathbb{Z}$-actions [KB, O, Wa, Fu]. Whereas these two theorems are only of a propaedeutic nature, our main results of Section 2 are Theorem 2.29 and Proposition 2.32. They provide geometric characterizations of ergodicity and of unique ergodicity in terms of pattern frequencies for uniformly discrete point sets of finite local complexity.

In Section 3, we construct an ergodic measure for randomly coloured point sets and present an optimal ergodic theorem as the second main result in Theorem 3.11. Finite local complexity is not required in this section.

The formalism developed in Sections 2 and 3 is applied to randomly coloured graphs in Section 4. Proofs are provided in the remaining sections.

2 Dynamical Systems for Point Sets

Here, we introduce our setup, discuss the basic ergodic theorem, and give a geometric characterization of ergodic point sets in terms of pattern frequencies.

2.1 Topology on Collections of Point Sets

For the convenience of the reader, Section 5.1 contains proofs of the material that we present here.

A *point space* M is a non-empty, locally compact, and second-countable topological space. Throughout this paper we stick to the convention that every locally compact topological space enjoys the Hausdorff property. We recall that in locally compact topological spaces, second countability is equivalent to σ-compactness and metrizability [Bou2, Chap. IX, §2.9, Cor.].

In addition to the point space M we consider a metrizable topological group T with left action $\alpha := \alpha_M \colon T \times M \to M$, $(x, m) \mapsto xm$ on M. Throughout this paper we will rely on the following assumption.

Assumption 2.1 The group T is non-compact, and its action on M is continuous and proper. Moreover, we fix a T-invariant proper metric d on M that generates the topology on M.

Remarks 2.2 (i) A metric d on M is T-invariant if and only if $d(xm, xm') = d(m, m')$ for all $x \in T$ and all $m, m' \in M$. A metric is proper if and only if every metric ball has a compact closure. The existence of a T-invariant proper metric d on the metrizable space M that generates the topology of M follows from σ-compactness of M and properness of the group action [AMN, Thm. 4.2].

(ii) We recall that the group action is continuous if and only if the map $\alpha \colon T \times M \to M$ is continuous with respect to the product topology on $T \times M$. Properness of the group action means that the (continuous) map

$$\widetilde{\alpha}\colon T \times M \to M \times M,\ (x, m) \mapsto (xm, m),$$

is proper; that is, pre-images of compact sets are compact (see [Bou1, Chap. III.4]). We refer to Lemma 2.3 for different characterizations of properness.

(iii) All results of this section (and their proofs) remain valid for compact groups, but reduce to trivial statements. For the following sections, however, non-compactness will be crucial. It will be used in the proof of the strong law of large numbers: Theorem 3.10.

(iv) Properness of the group action and local compactness of M imply that T is also locally compact; see [AMN, Proposition 1.3] and subsequent comments. Furthermore, since M is also σ-compact, so is T.

(v) We require metrizability of the group T in order to satisfy the hypotheses of Lindenstrauss' ergodic theorem. The latter is essential for large parts of this paper. Thus, by the previous remark and by what was recalled at the beginning of this section, the group T is also second countable.

(vi) Below we want to define nice T-orbits of uniformly discrete subsets in M with a given radius of uniform discreteness. It is precisely for this purpose that we work with a T-invariant metric d on M that is compatible with the topology.

For convenience we recall the following alternative characterizations of properness. We use the notation $xU := \{xm : m \in U\}$ for $x \in T$ and $U \subseteq M$ and introduce the *transporter*

$$S_{U_1,U_2} := \{x \in T : xU_1 \cap U_2 \neq \varnothing\} \subseteq T \tag{2.1}$$

of subsets $U_1, U_2 \subseteq M$. The following lemma does not rely on Assumption 2.1.

Lemma 2.3 *Assume that M and T are both locally compact and second-countable, and that T acts continuously on M from the left. Then the following are equivalent.*

(i) *T acts properly on M.*

(ii) *For every choice of compact subsets $V_1, V_2 \subseteq M$, the transporter S_{V_1,V_2} is compact in T.*

(iii) *For every choice of relatively compact subsets $U_1, U_2 \subseteq M$, the transporter S_{U_1,U_2} is relatively compact in T.*

(iv) *Given any two sequences $(x_n)_{n\in\mathbb{N}} \subseteq T$ and $(m_n)_{n\in\mathbb{N}} \subseteq M$ such that both $(m_n)_{n\in\mathbb{N}}$ and $(x_n m_n)_{n\in\mathbb{N}}$ converge in M, then $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence in T.*

Remarks 2.4 (i) We will only use the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) of the lemma in the sequel. But for the purpose of completeness, we have included the proof of all implications in Section 5.1.

(ii) Similar characterizations can be found, *e.g.*, in [Bou1, Chap. III.4.5, Thm. 1] or [AMN, Def. and Prop. 2.3].

(iii) Condition (ii) of the lemma implies that the map $\alpha(\,\cdot\,, m)\colon T \to M$ is proper for every $m \in M$, since $(\alpha(\,\cdot\,, m))^{-1}(V) = S_{\{m\},V}$ for every compact $V \subseteq M$.

Example 2.5 An often studied special case of our setup arises when M is a topological group by itself. Then one may choose $T := M$ and α as the group multiplication from the left. An important example for this special case is the Abelian group $M = \mathbb{R}^{\mathrm{d}}$ for $\mathrm{d} \in \mathbb{N}$, equipped with the Euclidean metric, which acts on itself canonically by translation. This action is transitive, free, and proper, and the Euclidean metric is T-invariant and also proper. (Recall that a group action is said to be *transitive* if and only if for every $m, m' \in M$ there exists $x \in T$ such that $xm = m'$. It is *free* if and only if for any $x \in T$ and any $m \in M$ the property $xm = m$ implies $x = e$, the neutral element in T.) Another prime example is $M = \mathbb{R}^{\mathrm{d}}$ and $T = E(\mathrm{d})$, the Euclidean group, with the Euclidean metric. Then the canonical group action from the left is transitive and proper, but not free. The Euclidean metric is also T-invariant and proper. We note, however, that in situations relevant to us, M will not be a group.

The open ball (with respect to the metric d) of radius $s > 0$ about $m \in M$ is denoted by $B_s(m)$. A subset $P \subseteq M$ is *uniformly discrete* with radius $r \in]0, \infty[$, if every open ball in M of radius r contains at most one element of P. A subset $P \subseteq M$ is called *relatively dense* with radius $R \in]0, \infty[$, if every closed ball of radius R has non-empty intersection with P. If P is both uniformly discrete and relatively dense, then it is called a *Delone set*. The collection of all subsets of M, which are uniformly discrete of radius r, is denoted by $\mathcal{P}_r(M)$. We call every element of $\mathcal{P}_r(M)$ a *point set*. Throughout this paper, the radius of uniform discreteness r will be fixed.

We define a topology on $\mathcal{P}_r(M)$ by requiring certain functions on $\mathcal{P}_r(M)$, which are of the form (2.2), to be continuous. These functions will serve as a "scanning device" on a point set. Let $C_c(M)$ denote the set of all real valued, continuous functions on M with compact support.

Definition 2.6 With $\varphi \in C_c(M)$ we associate

$$f_\varphi : \begin{array}{ccc} \mathcal{P}_r(M) & \to & \mathbb{R} \\ P & \mapsto & f_\varphi(P) := \sum_{p \in P} \varphi(p) \end{array} . \tag{2.2}$$

The *vague topology* on $\mathcal{P}_r(M)$ is the weakest topology such that f_φ in (2.2) is continuous for every $\varphi \in C_c(M)$.

Remarks 2.7 (i) Even though the set $\mathcal{P}_r(M)$ itself depends on the metric d on M, the nature of the vague topology on $\mathcal{P}_r(M)$ is solely determined by the topology on M.

(ii) Particular examples of open sets in $\mathcal{P}_r(M)$ are given by pre-images of open balls in $\mathbb{R}$. For $P \in \mathcal{P}_r(M)$, $\varphi \in C_c(M)$, and $\varepsilon > 0$ we define the open set

$$U_{\varphi,\varepsilon}(P) := \left\{ \widetilde{P} \in \mathcal{P}_r(M) : \left| f_\varphi(\widetilde{P}) - f_\varphi(P) \right| < \varepsilon \right\}.$$

It is readily checked that the family obtained from finite intersections of open sets $U_{\varphi,\varepsilon}(P)$ as above forms a neighbourhood base of the vague topology.

(iii) The above neighbourhood base arises naturally when identifying a point set P with a point measure on M that has an atom of unit mass at each point of P; see *e.g.*, [BelHZ, S, BL, Len]. It is from this perspective that the topology of Definition 2.6 appears as the vague topology on this space of measures. For the case where M is also a group, [BL] coined the term *local rubber topology* for the vague topology (and they defined it using transitivity of the canonical group action of M on itself). For the particular example $M = \mathbb{R}^{\mathrm{d}}$ the vague topology was studied under the name *natural topology* in [LenS2] and earlier on in [BelHZ, LP].

(iv) Instead of uniformly discrete subsets of M, one may consider more general *locally finite* sets. These are sets $P \subseteq M$ for which $P \cap V$ is finite for every compact set $V \subseteq M$. But the space of locally finite sets equipped with the vague topology is not closed. For example, a sequence of locally finite point sets may give rise to accumulation points in M.

(v) Local compactness and second countability of M imply metrizability of the topology on $\mathcal{P}_r(M)$, see [Bau, Thm. 31.5].

Convergence in the topological space $\mathcal{P}_r(M)$ is characterized in the following lemma.

Lemma 2.8 *Fix a sequence $(P_k)_{k\in\mathbb{N}} \subseteq \mathcal{P}_r(M)$. Then the following statements are equivalent.*

(i) *The sequence $(P_k)_{k\in\mathbb{N}}$ converges in $\mathcal{P}_r(M)$.*

(ii) *There exists $P \in \mathcal{P}_r(M)$ such that for all $\varphi \in C_c(M)$ we have*

$$\lim_{k\to\infty} f_\varphi(P_k) = f_\varphi(P).$$

(iii) *For every $m \in M$ exactly one of the following two cases occurs:*

(a) *for every $\varepsilon > 0$ we have $P_k \cap B_\varepsilon(m) \neq \varnothing$ for finally all $k \in \mathbb{N}$;*

(b) *There exists $\varepsilon > 0$ such that $P_k \cap B_\varepsilon(m) = \varnothing$ for finally all $k \in \mathbb{N}$.*

(iv) *There exists $P \in \mathcal{P}_r(M)$ such that for every compact set $V \subseteq M$ we have, for every $\varepsilon > 0$ and finally all $k \in \mathbb{N}$, the inclusions*

$$P_k \cap V \subseteq (P)_\varepsilon \qquad \textit{and} \qquad P \cap V \subseteq (P_k)_\varepsilon.$$

Here, the "thickened" point set $(P)_\varepsilon := \bigcup_{p\in P} B_\varepsilon(p)$ is the set of points in M lying within distance less than ε to P.

In either case, the limit P is the set of all points $m \in M$ satisfying (iii)(a).

Below we will be concerned with ergodic properties of $\mathcal{P}_r(M)$ as a topological dynamical system. This relies on the following proposition.

Proposition 2.9 *The space $\mathcal{P}_r(M)$ is compact with respect to the vague topology.*

Remarks 2.10 (i) In order to give a self-contained presentation, we prove (sequential) compactness of the metrizable space $\mathcal{P}_r(M)$ in Section 5.1. Thus, Proposition 2.9 yields complete metrizability of $\mathcal{P}_r(M)$, in other words, $\mathcal{P}_r(M)$ is even a Polish space. For the more general case of M being only σ-compact and locally compact, compactness of $\mathcal{P}_r(M)$ has already been shown in [BL, Thm. 3]; see also [Bau, Thm. 31.2] and [BelHZ].

(ii) In tiling dynamical systems, the topology on $\mathcal{P}_r(M)$ is often characterized in terms of a particular metric resembling a connection to symbolic dynamics, see *e.g.*, [RaW, Ho3, LeMS]. The corresponding notion of distance means that two point sets are close if they almost agree on a large ball in the point space. This can be formalized as follows. The map dist: $\mathcal{P}_r(M) \times \mathcal{P}_r(M) \to \mathbb{R}_{\geqslant 0}$, given by

$$\operatorname{dist}(P, \widetilde{P}) := \min\Big\{ \tfrac{1}{\sqrt{2}}, \inf\big\{ \varepsilon > 0 : P \cap B_{\frac{1}{\varepsilon}} \subseteq (\widetilde{P})_\varepsilon \text{ and } \widetilde{P} \cap B_{\frac{1}{\varepsilon}} \subseteq (P)_\varepsilon \big\} \Big\},$$

where $B_{\frac{1}{\varepsilon}} := B_{\frac{1}{\varepsilon}}(m_o)$ for some fixed reference point $m_o \in M$, defines a metric on $\mathcal{P}_r(M)$. The topology induced by the above metric does not depend on the choice of reference point m_o and coincides with the vague topology. (Since $\mathcal{P}_r(M)$ is compact, it is complete with respect to the above metric, and the metric is proper.) In this paper we prefer to work with the vague topology instead of the metric.

2.2 Ergodic Theorems for Group Actions

Our basic workhorse will be the general ergodic theorem of Lindenstrauss [Lin]. In order to apply it we need to recall some further notions. We fix a left Haar measure on the locally compact and second-countable group T and write $\operatorname{vol}(S) = \int_S \mathrm{d}x$ for this Haar measure of a Borel set $S \subseteq T$. Below we will also require the group T to be *unimodular*. This is equivalent to the requirement that the Haar measure is inversion invariant, *i.e.*, $\int_T f(x^{-1})\mathrm{d}x = \int_T f(x)\mathrm{d}x$ for every measurable function $f \colon T \to [0, \infty]$ (and hence also for every integrable function $f \colon T \to \mathbb{R}$). In particular, this implies $\operatorname{vol}(S^{-1}) = \operatorname{vol}(S)$ for every Borel set $S \subseteq T$, where $S^{-1} := \{x \in T : \exists s \in S \text{ such that } x = s^{-1}\}$.

Since we want to compute certain group means below, we require that T admits suitable averaging sequences. As usual, for $K \subseteq T$ we denote by $\mathring{K} = \operatorname{int}(K)$ the interior of K and by $\overline{K}$ the closure of K, and for $A, B \subseteq T$ we write $AB := \{x \in T : \exists (a, b) \in A \times B \text{ such that } x = ab\}$ for the Minkowski product of A and B.

Definition 2.11 Let $(D_n)_{n\in\mathbb{N}}$ be an increasing sequence of non-empty, compact subsets of T such that $\bigcup_{n\in\mathbb{N}} \mathring{D}_n = T$.

(i) The sequence $(D_n)_{n\in\mathbb{N}}$ is called a *Følner sequence* if for every compact $K \subseteq T$ we have

$$\lim_{n\to\infty} \frac{\operatorname{vol}(\delta^K D_n)}{\operatorname{vol}(D_n)} = 0, \tag{2.3}$$

where $\delta^K D_n$ is the symmetric difference of D_n and KD_n,

$$\delta^K D_n := (KD_n) \setminus D_n \cup (KD_n)^c \setminus D_n^c.$$

(ii) The sequence $(D_n)_{n\in\mathbb{N}}$ is called a *van Hove sequence* if for every compact $K \subseteq T$ we have

$$\lim_{n\to\infty} \frac{\operatorname{vol}(\partial^K D_n)}{\operatorname{vol}(D_n)} = 0, \tag{2.4}$$

where $\partial^K D_n := (KD_n) \setminus \mathring{D}_n \cup (K\overline{D_n^c}) \setminus D_n^c$.

(iii) The sequence $(D_n)_{n\in\mathbb{N}}$ is *tempered* (or obeys *Shulman's condition*) if there exists $C \geqslant 1$ such that for all $n \in \mathbb{N}$ we have the estimate

$$\operatorname{vol}\Big(\bigcup_{k=1}^{n-1} D_k^{-1} D_n\Big) \leqslant C \operatorname{vol}(D_n).$$

Remarks 2.12 (i) For every $n \in \mathbb{N}$, the set $\partial^K D_n$ in (2.4) is compact. If T is Abelian, then our definition of van Hove sequence is equivalent to that in [S].

(ii) We have $\delta^K D \subseteq \partial^K D$, which follows from the inclusion $(AB)^c \subseteq AB^c$ for arbitrary $A, B \subseteq T$. Consequently, every van Hove sequence is a Følner sequence.

(iii) The existence of a Følner sequence in T is equivalent to amenability of the group [P, Thm. 4.16]. Every Følner sequence has a tempered subsequence [Lin, Prop. 1.4].

(iv) According to [St], every second countable, locally compact group has a left-invariant proper metric that generates the topology. Suppose the sequence $(B_n)_{n\in\mathbb{N}}$ of closed balls (with respect to this metric) about the neutral element $e \in T$ of radius $n \in \mathbb{N}$ constitutes a (tempered) Følner sequence in T satisfying $B_m B_n = B_{m+n}$ for all $m, n \in \mathbb{N}$. Assume in addition that $\operatorname{vol}(\partial B_n)/\operatorname{vol}(B_n) \to 0$ as $n \to \infty$, with ∂B_n the topological boundary of B_n. It can be shown that $(B_n)_{n\in\mathbb{N}}$ is also a (tempered) van Hove sequence under these assumptions.

(v) If T is Abelian, the existence of a tempered van Hove sequence in T is guaranteed under our hypotheses. Indeed, [S, p. 145] ensures the existence of a van Hove sequence in T, which is also a Følner sequence by (ii). As every Følner sequence has a tempered subsequence, the argument is complete.

(vi) Consider the semidirect product [HR], denoted by $T = N \rtimes H$, of a unimodular group N and a compact group H. Then T is unimodular. It can be shown that if $(D_n)_{n\in\mathbb{N}}$ is an H-invariant tempered van Hove sequence in N, then $(D_n \times H)_{n\in\mathbb{N}}$ is a tempered van Hove sequence in T. This can be used to provide examples of a non-Abelian non-compact unimodular group T with a tempered van Hove sequence. (Take H non-Abelian and N Abelian but not compact.) A prominent example is the Euclidean group $E(\mathrm{d}) = \mathbb{R}^{\mathrm{d}} \rtimes O(\mathrm{d})$, with centered closed balls of radius $n \in \mathbb{N}$ as tempered van Hove sequence in $\mathbb{R}^{\mathrm{d}}$. The existence of Følner sequences in semidirect products is discussed in [J, Wi].

The following lemma states that a Følner sequence $(D_n)_{n\in\mathbb{N}}$ and its "thickened" version $(LD_n)_{n\in\mathbb{N}}$, where $L \subseteq T$ is a compact set, have asymptotically the same volume. It also states that thickened versions of van Hove boundaries $\partial^K D_n$, with $K \subseteq T$ compact, are of small volume, asymptotically as $n \to \infty$. These properties will be used repeatedly below.

Lemma 2.13 *Let $L \subseteq T$ be a compact set. Then the following statements hold.*

(i) *If $(D_n)_{n\in\mathbb{N}}$ is a Følner sequence in T, we have the asymptotic estimate*

$$\operatorname{vol}(LD_n) = \operatorname{vol}(D_n) + o(\operatorname{vol}(D_n)) \qquad (n \to \infty).$$

(ii) *If* $(D_n)_{n\in\mathbb{N}}$ *is a van Hove sequence in* T*, we have for every compact* $K \subseteq T$ *the asymptotic estimate*

$$\mathrm{vol}(L\partial^K D_n) = o(\mathrm{vol}(\mathrm{D_n})) \qquad (\mathrm{n} \to \infty).$$

Next, we state the basic pointwise ergodic theorem that will be applied several times in the sequel. Let $\mathcal{Q}$ be a compact metrizable space (hence, with a countable base of the topology and complete with respect to every metric generating the topology), and assume that the group T acts measurably from the left on $\mathcal{Q}$, *i.e.*, there exists a measurable map $\alpha_{\mathcal{Q}}\colon T \times \mathcal{Q} \to \mathcal{Q}$, $(x, q) \mapsto \alpha_{\mathcal{Q}}(x, q) =: xq$. Here, $T \times \mathcal{Q}$ is endowed with the product topology.

A T-invariant probability measure on the Borel σ-algebra of $\mathcal{Q}$ is called (T-) *ergodic* if every T-invariant Borel set has either measure 0 or 1. The existence of an ergodic probability measure on $\mathcal{Q}$ follows from the compactness of $\mathcal{Q}$ by standard arguments (compare [Wa, §6.2] for the discrete case). In other words, $\mathcal{Q}$ is ergodic with respect to the group T. A dynamical system is called *uniquely* (T-) *ergodic*, if it carries exactly one T-invariant probability measure, which is then ergodic; see below.

We rely on the general Birkhoff ergodic theorem of Lindenstrauss [Lin, Thm. 1.2]. For related abstract ergodic theorems, see also [Ch, Kr, P, N]. The shorthand $\mu(f) := \int_{\mathcal{Q}} \mathrm{d}\mu(q)\, f(q)$ in the next theorem denotes the μ-integral of a function f on $\mathcal{Q}$. We remark that the assumptions on the group T in the next theorem are more general than those required by Assumption 2.1.

Theorem 2.14 (Pointwise Ergodic Theorem) *Let* $\mathcal{Q}$ *be a compact metrizable space on which a locally compact second-countable group* T *acts measurably from the left. Assume that* T *admits a tempered Følner sequence* $(D_n)_{n\in\mathbb{N}}$*. Fix a* T*-invariant Borel probability measure* μ *on* $\mathcal{Q}$ *and let* $f \in L^1(\mathcal{Q}, \mu)$ *arbitrary be given. Then*

$$I_n(q, f) := \frac{1}{\mathrm{vol}(D_n)} \int_{D_n} \mathrm{d}x\ f(xq) \tag{2.5}$$

is finite for μ*-a.a.* $q \in \mathcal{Q}$ *and all* $n \in \mathbb{N}$*. Furthermore, there exists a* T*-invariant function* $f^\star \in L^1(\mathcal{Q}, \mu)$ *such that* $\mu(f^\star) = \mu(f)$ *and*

$$\lim_{n\to\infty} I_n(q, f) = f^\star(q) \qquad \text{for } \mu\text{-a.a. } q \in \mathcal{Q}. \tag{2.6}$$

Moreover, the following statements are equivalent.

(i) *The measure* μ *is ergodic.*

(ii) *For every* $f \in L^1(\mathcal{Q}, \mu)$*, equation* (2.6) *holds with* $f^\star = \mu(f)$*.*

(iii) *There exists a* ($\|\cdot\|_\infty$-) *dense subset* $\mathcal{D} \subseteq C(\mathcal{Q})$ *such that for every* $f \in \mathcal{D}$*, equation* (2.6) *holds with* $f^\star = \mu(f)$*.*

Remarks 2.15 (i) For μ ergodic, the limit (2.6) is obviously independent of the tempered Følner sequence.

(ii) As the proof shows, the statement of the theorem remains true for locally compact Polish spaces (with $\mathcal{D} \subseteq C_c(\mathcal{Q})$ in (iii)). Moreover, our proof uses local compactness only for the implication (iii)⇒(i). Non-compact dynamical systems have been studied in [O]. The reason we assume even compactness of $\mathcal{Q}$ in the hypotheses of the theorem is to guarantee the existence of an ergodic probability measure on $\mathcal{Q}$. It is not obvious to us how to dispense with metrizability of $\mathcal{Q}$.

In the case of a uniquely ergodic system, one may adapt arguments from [Fu, Wa] to exclude the exceptional set in (2.6), provided that f is continuous. Note that the next theorem is stated under more general assumptions on the group T than those required by Assumption 2.1.

Theorem 2.16 (Unique ergodicity) *Let $\mathcal{Q}$ be a compact metrizable space on which a locally compact group T acts measurably from the left. Assume that T admits a Følner sequence $(D_n)_{n\in\mathbb{N}}$ and define $I_n(\cdot\,,\cdot)$ as in (2.5). Then the following statements are equivalent.*

(i) *For every $f \in C(\mathcal{Q})$ the sequence $\big(I_n(q, f)\big)_{n\in\mathbb{N}}$ converges uniformly in $q \in \mathcal{Q}$, and there is a constant $I(f) \in \mathbb{R}$ such that*

$$\lim_{n\to\infty} I_n(q, f) = I(f)$$

for all $q \in \mathcal{Q}$.

(ii) *There exists a dense subset $\mathcal{D} \subseteq C(\mathcal{Q})$ and for every $f \in \mathcal{D}$ there exists a constant $I(f) \in \mathbb{R}$ such that pointwise for every $q \in \mathcal{Q}$ we have*

$$\lim_{n\to\infty} I_n(q, f) = I(f).$$

(iii) *There exists exactly one T-invariant Borel probability measure μ on $\mathcal{Q}$.*

In either case, the measure μ is ergodic, and the above statements hold with $I(f) = \mu(f)$.

Remark 2.17 In particular, the limits in the above theorem are again independent of the choice of the Følner sequence. In contrast to Theorem 2.14, the Følner sequence here does not need to be tempered. Neither does one need second countability of the group T.

The role of the compact space $\mathcal{Q}$ in the above ergodic theorems will be played by the closure of T-orbits of point sets.

Definition 2.18 Given a collection of point sets $\mathcal{P} \subseteq \mathcal{P}_r(M)$, we introduce its closed T-orbit

$$X_{\mathcal{P}} := \overline{\{xP : x \in T, P \in \mathcal{P}\}} \subseteq \mathcal{P}_r(M), \tag{2.7}$$

where $xP := \{xp : p \in P\}$. Being closed, $X_{\mathcal{P}}$ is a compact subset of the compact space $\mathcal{P}_r(M)$. The induced group action $\alpha_{X_{\mathcal{P}}} : T \times X_{\mathcal{P}} \to X_{\mathcal{P}}, (x, P) \mapsto xP$ is continuous.

Remarks 2.19 (i) The validity of the set inclusion in (2.7) depends crucially on $\mathcal{P}_r(M)$ being defined in terms of balls with respect to a T-invariant metric on M. The compatibility between the point space M and the group T is necessary in order to have a fruitful concept of orbits.

(ii) The compact metrizable space $X_{\mathcal{P}}$ is particularly useful if the closure is not too large in comparison to the (unclosed) T-orbit. This has been analyzed mainly for $M = T = \mathbb{R}^d$ with the canonical group action and the Euclidean metric. In that case, there are two simple examples where the closure does not add anything new to the (unclosed) T-orbit of $\mathcal{P}$. This is when $\mathcal{P}$ consists of a single periodic point set, or when $\mathcal{P}$ is a suitable collection of random tilings [RiHHB, GrS]. The definition of the vague topology suggests that elements in $X_{\mathcal{P}}$ added by the closure share local properties of point sets from $\mathcal{P}$. If $\mathcal{P} = \{P\}$ consists of a single point set P, there is a geometric characterization of $X_{\mathcal{P}}$ as the so-called local isomorphism class of the point set if and only if P is repetitive. The latter property is in fact equivalent to minimality of $X_{\mathcal{P}}$ if $\mathcal{P}$ is of finite local complexity as in Definition 2.26(i), *cf.* [LP] for $M = T = \mathbb{R}^d$ and [Yo] for the general case. Another criterion for a "nice" closure is unique ergodicity of $X_{\mathcal{P}}$. We give a geometric characterization of unique ergodicity in Theorem 2.29.

The triple $(X_{\mathcal{P}}, T, \alpha_{X_{\mathcal{P}}})$ constitutes a compact topological dynamical system. Thus, we have the following corollary.

Corollary 2.20 *Let $\mathcal{P} \subseteq \mathcal{P}_r(M)$ be a collection of uniformly discrete point sets of radius r. Then the Ergodic Theorems* 2.14 *and* 2.16 *hold for $\Omega = X_{\mathcal{P}}$.*

Remark 2.21 Ergodic theorems for systems of point sets in $\mathbb{R}^d$ or in a locally compact Abelian group have been given and applied before; see *e.g.,* [S, LeMS]. In addition we mention [LenS3, Thm. 1] for Banach space-valued functions in the case of minimal ergodic systems of Delone sets of finite local complexity (see below for a definition) in $\mathbb{R}^d$.

2.3 Geometric Characterization of Ergodicity

In this section we relate ergodicity of a dynamical system of point sets to the spatial frequencies with which patterns occur therein. This will require us to count the number of equivalent patterns within a given region of the point space, where equivalence is defined by the group action.

First we introduce the relevant notation. Given a point set $P \in \mathcal{P}_r(M)$, we call a finite subset $Q \subseteq P$ a *pattern of P (in M)*. Given a collection $\mathcal{P} \subseteq \mathcal{P}_r(M)$ of point sets, we say that Q is a pattern of $\mathcal{P}$, if there exists $P \in \mathcal{P}$ such that Q is a pattern of P. We write $\mathcal{Q}_{\mathcal{P}}$ for the set of all patterns of $\mathcal{P}$; see also Definition 5.3. For a pattern Q of P, every compact set $V \subseteq M$ such that $Q = P \cap \mathring{V}$ is called a *support* of Q, and we say that Q is a V-pattern of P. Two subsets $V, V' \subseteq M$ are called (T-) *equivalent*, if $xV = V'$ for some $x \in T$.

For $P \in \mathcal{P}_r(M)$, $Q \subseteq P$ a pattern of P and $D \subseteq T$, we analyze the number of equivalent patterns of Q in P. Fixing $m \in M$, one may consider the two different sets

$$M_D(Q) := \{\widetilde{Q} \subseteq P : \exists x \in D^{-1} : xQ = \widetilde{Q}\},$$

$$M'_D(Q) := \{\widetilde{Q} \subseteq P \cap D^{-1}m : \exists x \in T : xQ = \widetilde{Q}\}$$

for this purpose. Note that the set $M'_D(Q)$ depends on the choice of $m \in M$, in contrast to the set $M_D(Q)$. For this reason we will use $M_D(Q)$ for pattern counting in Definition 2.24. The set $M_D(Q)$ is a subset of the equivalence class of Q. The next lemma describes how these two sets grow with the volume of D.

Lemma 2.22 *Assume that T is even unimodular. Fix a Følner sequence $(D_n)_{n\in\mathbb{N}}$. Let Q be a pattern of $\mathcal{P}_r(M)$ and fix $P \in \mathcal{P}_r(M)$. Then we have the asymptotic estimates*

$$\left.\begin{aligned} \operatorname{card}\left(M_{D_n}(Q)\right) &= O(\operatorname{vol}(D_n)),\\ \operatorname{card}\left(M'_{D_n}(Q)\right) &= O(\operatorname{vol}(D_n)), \end{aligned}\right. \qquad (n \to \infty).$$

If $(D_n)_{n\in\mathbb{N}}$ is even a van Hove sequence, and if $Q \subseteq Tm$, then

$$\operatorname{card}\left(M'_{D_n}(Q)\right) = \operatorname{card}\left(M_{D_n}(Q)\right) + o(\operatorname{vol}(D_n)) \qquad (n \to \infty).$$

The O-terms and the o-term may be chosen uniformly in $P \in \mathcal{P}_r(M)$.

Remarks 2.23 (i) The condition $Q \subseteq Tm$ is satisfied for a transitive group action, since $Tm = M$ in that case.

(ii) The number of equivalent copies of Q in P may also be analyzed by counting corresponding group elements of the group T. One may consider the two different sets

$$T_D(Q) := \{x \in D^{-1} : xQ \subseteq P\},$$
$$T'_D(Q) := \{x \in T : xQ \subseteq P \cap D^{-1}m\}.$$

The set $T'_D(Q)$ is commonly used for pattern counting (see [S, LeMS]), but depends on the choice of $m \in M$. In order to relate this to the above approaches of pattern counting, consider the map $f\colon T_D^{(\prime)}(Q) \to M_D^{(\prime)}(Q)$, given by $x \mapsto f(x) := xQ$. This map is onto. It is readily checked that f is one-to-one, if $Q \neq \varnothing$, the group T is free on Q and does not contain a nontrivial element of finite order. Hence, in that case, both approaches coincide.

Our central notion of pattern counting is defined as follows.

Definition 2.24 Let $(D_n)_{n\in\mathbb{N}}$ be a Følner sequence in T and let $P, Q \in \mathcal{P}_r(M)$ be point sets with $|Q| < \infty$. (In particular, Q may be a pattern of P.) If the limit

$$\nu(Q) \equiv \nu^P\left(Q; (D_n)_{n\in\mathbb{N}}\right) := \lim_{n\to\infty} \frac{\operatorname{card}\left(M_{D_n}(Q)\right)}{\operatorname{vol}(D_n)}$$

exists, we call it the *pattern frequency* of Q. In most cases we suppress its dependence on P and the Følner sequence in our notation.

Lemma 2.25 *Assume that T is even unimodular. Let $(D_n)_{n\in\mathbb{N}}$ be a Følner sequence in T and let $P, Q \in \mathcal{P}_r(M)$ be point sets with $|Q| < \infty$. Then*

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(i) *the quotient that arises in the definition of the pattern frequency is bounded,*

$$\sup_{n\in\mathbb{N}} \frac{\operatorname{card}\big(M_{D_n}(Q)\big)}{\operatorname{vol}(D_n)} < \infty;$$

in other words, since $\limsup$ *and* $\liminf$ *of* $\operatorname{card}(M_{D_n}(Q))/\operatorname{vol}(D_n)$ *are always both finite, existence of the pattern frequency* $\nu(Q)$ *is only a matter of whether they coincide;*

(ii) *if* $(D_n)_{n\in\mathbb{N}}$ *is a van Hove sequence and if the pattern frequency* $\nu^P(Q;(D_n)_{n\in\mathbb{N}})$ *exists, then* $\nu^P(xQ;(D_n)_{n\in\mathbb{N}})$, $\nu^P(Q;(xD_n)_{n\in\mathbb{N}})$ *and* $\nu^{xP}(Q;(D_n)_{n\in\mathbb{N}})$ *exist and are all equal,* i.e.,

$$\nu^P\big(xQ;(D_n)_{n\in\mathbb{N}}\big) = \nu^P\big(Q;(xD_n)_{n\in\mathbb{N}}\big) = \nu^{xP}\big(Q;(D_n)_{n\in\mathbb{N}}\big) = \nu^P\big(Q;(D_n)_{n\in\mathbb{N}}\big)$$

for every $x \in T$.

In order to relate ergodicity to pattern counting, we require a certain type of rigidity for point sets.

Definition 2.26 Let $\mathcal{P} \subseteq \mathcal{P}_r(M)$ be a collection of point sets and let $\mathcal{Q}_\mathcal{P} \subseteq \mathcal{P}_r(M)$ be the collection of its patterns.

(i) $\mathcal{P}$ is of *finite local complexity* (FLC) if for every compact set $V \subseteq M$ there is a finite collection $\mathcal{F}_\mathcal{P}(V) \subseteq \mathcal{Q}_\mathcal{P}$ of (without loss of generality mutually non-equivalent) patterns, such that every pattern of $\mathcal{P}$, which admits a support equivalent to V, is equivalent to some pattern in $\mathcal{F}_\mathcal{P}(V)$.

(ii) $\mathcal{P}$ is *locally rigid* if for every $Q \in \mathcal{Q}_\mathcal{P}$ there exists $\varepsilon > 0$ such that for all $\widetilde{Q} \in \mathcal{Q}_\mathcal{P}$ and for all $x \in T$ the properties $x\widetilde{Q} \subseteq (Q)_\varepsilon$ and $Q \subseteq (x\widetilde{Q})_\varepsilon$ imply that Q and $\widetilde{Q}$ are equivalent.

The following lemma discusses and relates the above notions. For $\mathcal{P} \subseteq \mathcal{P}_r(M)$ and $V \subseteq M$, define $\mathcal{P} \wedge V := \{P \cap V : P \in \mathcal{P}\} \subseteq \mathcal{P}_r(M)$.

Lemma 2.27 *Let* $\mathcal{P} \subseteq \mathcal{P}_r(M)$ *be a collection of point sets. Then*

(i) $\mathcal{P}$ *is FLC if and only if* $X_\mathcal{P}$ *is FLC;*

(ii) *if* $\mathcal{P}$ *is FLC, then* $\mathcal{P}$ *is locally rigid;*

(iii) *if* $\mathcal{P}$ *is locally rigid and if* $\mathcal{Q}_\mathcal{P} \wedge V$ *is closed in* $\mathcal{P}_r(M)$ *for all compact* $V \subseteq M$, *then* $\mathcal{P}$ *is FLC.*

Remarks 2.28 (i) If $\mathcal{P}$ is finite, then FLC is equivalent to local rigidity. This holds, since for finite $\mathcal{P}$ the set $\mathcal{Q}_\mathcal{P} \wedge V$ is finite for all compact $V \subseteq M$, due to uniform discreteness. In particular, it is closed in $\mathcal{P}_r(M)$.

(ii) The proof of Lemma 2.27(i) shows that in the FLC case every pattern of $X_\mathcal{P}$ is equivalent to some pattern of $\mathcal{P}$.

Restricting to collections of point sets of finite local complexity, we can now state a geometric characterization of ergodicity and of unique ergodicity.

Theorem 2.29 (Ergodicity for FLC sets) *Assume that T is even unimodular and that T has a tempered van Hove sequence $(D_n)_{n\in\mathbb{N}}$. Let $\mathcal{P} \subseteq \mathcal{P}_r(M)$ be a collection of point sets of finite local complexity. Let μ be a T-invariant Borel probability measure on $X_{\mathcal{P}}$. Then the following statements are equivalent.*

(i) *The measure μ is ergodic.*
(ii) *For every pattern Q in the set $\mathcal{Q}_{\mathcal{P}}$ of all patterns of $\mathcal{P}$, there is a subset $X \subseteq X_{\mathcal{P}}$ of full μ-measure such that the pattern frequency $\nu(Q) = \nu^P(Q; (D_n)_{n\in\mathbb{N}})$ exists for all $P \in X$ and is independent of $P \in X$.*

If any of the above statements applies, then every pattern frequency $\nu(Q)$, $Q \in \mathcal{Q}_{\mathcal{P}}$, is independent of the choice of the tempered van Hove sequence.

The system $X_{\mathcal{P}}$ is uniquely *ergodic if and only if* (ii) *holds for all patterns $Q \in \mathcal{Q}_{\mathcal{P}}$ with $X = X_{\mathcal{P}}$, that is, for* every *$P \in X_{\mathcal{P}}$. In that case, the van Hove sequence need not be tempered, and every pattern frequency $\nu(Q)$, $Q \in \mathcal{Q}_{\mathcal{P}}$, is independent of the choice of the van Hove sequence. Furthermore, the convergence to the limit underlying the definition of each $\nu(Q)$ is even uniform in $P \in X_{\mathcal{P}}$.*

In the following proposition, we give a characterization of unique ergodicity in terms of properties of $\mathcal{P}$ instead of $X_{\mathcal{P}}$. This characterization is often referred to as *uniform pattern frequencies*; compare [S, Thm. 3.2], [LeMS, Thm. 2.7], and [LP, Def. 6.1].

Definition 2.30 Fix a van Hove sequence $(D_n)_{n\in\mathbb{N}}$ and let $\mathcal{P} \subseteq \mathcal{P}_r(M)$ be given. We say that $\mathcal{P}$ has *uniform pattern frequencies* if for every pattern Q of $\mathcal{P}$ the sequence $\big(\nu_n^{y,P}(Q)\big)_{n\in\mathbb{N}}$, defined by

$$\nu_n^{y,P}(Q) := \frac{\operatorname{card}\big(\{\widetilde{Q} \subseteq P : \exists x \in D_n y : x\widetilde{Q} = Q\}\big)}{\operatorname{vol}(D_n)}, \tag{2.8}$$

converges uniformly in $(y, P) \in T \times \mathcal{P}$, and if its limit is independent of $(y, P) \in T \times \mathcal{P}$.

Remarks 2.31 (i) If $M = T = \mathbb{R}^{\mathrm{d}}$ with the canonical group action, and if $\mathcal{P} = \{P\}$ is linearly repetitive, then $\mathcal{P}$ has uniform pattern frequencies; see [LP, DL].

(ii) If $\mathcal{P}$ has uniform pattern frequencies, then the limit of (2.8) is also independent of the choice of the van Hove sequence according to Theorem 2.29 and the following proposition.

Proposition 2.32 (Unique ergodicity for FLC sets) *Assume that T is even unimodular and has a van Hove sequence $(D_n)_{n\in\mathbb{N}}$. Let $\mathcal{P} \subseteq \mathcal{P}_r(M)$ be a collection of point sets of finite local complexity. Then the following statements are equivalent:*

(i) *$X_{\mathcal{P}}$ is uniquely ergodic;*
(ii) *$\mathcal{P}$ has uniform pattern frequencies.*

At the end of this section we investigate which values an ergodic measure on $X_{\mathcal{P}}$ can assign to cylinder sets. Cylinder sets play a prominent role in the constructions of [LeMS] (compare also [Len]), and are defined as follows: it is well known [Ke,

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Lemma 4.5] that the compact metrizable topological space $\mathcal{P}_r(M)$ can be embedded into the compact product space $\prod_{\varphi\in C_c(M)} f_\varphi(\mathcal{P}_r(M))$, with injection map i given by $i(P)(f_\varphi) = f_\varphi(P)$. This motivates us to call $f_\varphi^{-1}(O) \subseteq \mathcal{P}_r(M)$ an *open cylinder* if $O \subseteq \mathbb{R}$ open and $\varphi \in C_c(M)$, and finite intersections thereof are called *cylinder sets*.

We give an example of open cylinders in $\mathcal{P}_r(M)$ with a simple geometric interpretation. Let $U \subseteq M$ be an open ball such that $\operatorname{diam}(U) < r$. Take $\varphi \in C_c(M)$ such that $\varphi^{-1}(\{0\}) = M \setminus U$. (A possible choice is $\varphi = d(\,\cdot\,, U^c)$, where $d(\,\cdot\,,\,\cdot\,)$ denotes the metric on M.) Consider the open cylinder

$$C_U := \big\{ P \in \mathcal{P}_r(M) : f_\varphi(P) \neq 0 \big\} = f_\varphi^{-1}\big(\mathbb{R} \setminus \{0\}\big).$$

It consists of all those point sets of $\mathcal{P}_r(M)$ that have exactly one point in U. Note that C_U is independent of the particular choice of $\varphi \in C_c(M)$ with $\operatorname{supp}(\varphi) = \overline{U}$. For open cylinders $C_{U_1}, \dots, C_{U_k}$ as above, we denote the cylinder set of their intersection by

$$C_{\mathbf{U}} := \bigcap_{i=1}^{k} C_{U_i}. \tag{2.9}$$

In the case of finite local complexity, the pattern frequencies determine the values that an ergodic measure assigns to such cylinder sets. The following proposition extends [LeMS, Cor. 2.8, Lemma 4.3].

Proposition 2.33 *Assume that T is even unimodular and has a van Hove sequence. Let $\mathcal{P} \subseteq \mathcal{P}_r(M)$ be a collection of point sets of finite local complexity. Let μ be an ergodic Borel probability measure on $X_{\mathcal{P}}$. Furthermore, let $Q = \{q_1, \dots, q_k\}$, $k \in \mathbb{N}$, be a nonempty pattern of $X_{\mathcal{P}}$ of cardinality k. Choose $\varepsilon \in]0, r/2[$ such that all patterns of $X_{\mathcal{P}}$ in $(Q)_\varepsilon$ of cardinality k are equivalent to Q. For $i \in \{1, \dots, k\}$, consider the pairwise disjoint sets $U_i := B_\varepsilon(q_i)$ and define $U := \bigcup_{i=1}^k U_i$. Then the corresponding cylinder set* (2.9) *has μ-measure*

$$\mu(C_{\mathbf{U}}) = \nu(Q)\ \operatorname{vol}(D_\varepsilon),$$

with $D_\varepsilon := \{x \in T : xQ \subseteq U\} \subseteq T$ being open and relatively compact.

If T is even Abelian and acts transitively on M, then we have the equality

$$\operatorname{vol}(D_\varepsilon) = \operatorname{card}(\mathcal{S}_k(Q))\, \zeta_\varepsilon,$$

where $\zeta_\varepsilon := \operatorname{vol}(\{x \in T : xm \in B_\varepsilon(m)\})$ does not depend on $m \in M$ and where $\mathcal{S}_k(Q)$ is the group of "T-realizable" permutations of Q, i.e.,

$$\mathcal{S}_k(Q) := \big\{ \pi \in \mathcal{S}_k : \exists x \in T \ \textit{ such that } \ xq_{\pi(i)} = q_i \ \textit{ for all } \ i \in \{1, \dots, k\} \big\},$$

with $\mathcal{S}_k$ denoting the permutation group on $\{1, \dots, k\}$.

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3 Dynamical Systems for Randomly Coloured Point Sets

In this section, we supply the point sets of the previous section with a random colouring. The results obtained here will be applied to randomly coloured graphs in the next section. All proofs are deferred to Section 6.

In addition to a point space M and a metrizable group T satisfying Assumption 2.1 in the previous section, we consider a non-empty, locally compact, second-countable topological space $\mathbb{A}$, which we call a *colour space*.

Lemma 3.1 *The product space $\widehat{M} := M \times \mathbb{A}$, equipped with the product topology, constitutes a point space in the sense of Section 2. The continuous and proper action α of T on M induces a continuous and proper action $\widehat{\alpha}\colon T \times \widehat{M} \to \widehat{M}$ of T on $\widehat{M}$ by setting $\widehat{\alpha}\big(x,(m,a)\big) := (xm, a)$. Thus, $\widehat{M}$ and T satisfy Assumption 2.1. We fix a T-invariant proper metric $\widehat{d}$ on $\widehat{M}$ that is compatible with the topology on $\widehat{M}$.*

Remarks 3.2 (i) When acting on $\widehat{M}$, the group T simply transports the colour a of m along with m.

(ii) The maximum metric of the T-invariant proper metric d on M and some metric generating the topology on $\mathbb{A}$ is an admissible choice for the metric $\widehat{d}$ in Lemma 3.1, because it is T-invariant and because every metric on $\widehat{M}$ which generates the product topology is equivalent to the proper metric $\widehat{d}$ of Lemma 3.1, and hence proper itself.

Lemma 3.1 and the arguments in the previous section imply that the space $\mathcal{P}_r(\widehat{M})$ of uniformly discrete point sets with radius r in $\widehat{M}$ is a compact metrizable space with respect to the vague topology. The vague topology is defined as in Definition 2.6, but with M replaced by $\widehat{M}$. The continuous action $\widehat{\alpha}$ induces a continuous group action on $\mathcal{P}_r(\widehat{M})$ by setting

$$x\widehat{P} := \Big\{ (xm, a) \in \widehat{M} : (m, a) \in \widehat{P} \Big\} \in \mathcal{P}_r(\widehat{M}) \tag{3.1}$$

for $\widehat{P} \in \mathcal{P}_r(\widehat{M})$. Again, the group action does not lead out of $\mathcal{P}_r(\widehat{M})$ because of T-invariance of the metric $\widehat{d}$. To summarize, all results established for $\mathcal{P}_r(M)$ in Section 2 remain true for $\mathcal{P}_r(\widehat{M})$.

Rather than working with general subsets of $\widehat{M}$, we are interested in those subsets for which each point of M comes with exactly one colour.

Definition 3.3 (i) For a given point set $P \subseteq M$ we set $\Omega_P := \bigtimes_{p\in P} \mathbb{A}$ and call $P^{(\omega)} = \{(p, \omega(p)) : p \in P\} \subseteq \widehat{M}$ a *coloured point set* with colour realization $\omega \in \Omega_P$.

(ii) Given a collection $\mathcal{P} \subseteq \mathcal{P}_r(M)$ of point sets, we introduce the collection of all associated coloured point sets $\mathcal{C}_{\mathcal{P}} := \{P^{(\omega)} : P \in \mathcal{P}, \omega \in \Omega_P\}$. In particular, we write $\mathcal{C}_r(M) := \mathcal{C}_{\mathcal{P}_r(M)}$ for the space of coloured, uniformly discrete points sets of radius r and $\widehat{X}_{\mathcal{P}} := \mathcal{C}_{X_{\mathcal{P}}}$ for the space of coloured closed T-orbits.

Remarks 3.4 (i) Let $\pi\colon \widehat{M} \to M$, $(m, a) \mapsto m$, be the canonical projection onto the space M. Then $\widehat{P} \subseteq \widehat{M}$ is a coloured point set if and only if the restriction $\pi\big|_{\widehat{P}}$ is injective.

(ii) If $P \in \mathcal{P}_r(M)$, then $P^{(\omega)} \in \mathcal{P}_r(\widehat{M})$ for every $\omega \in \Omega_P$. Thus, we have $\mathcal{C}_{\mathcal{P}} \subseteq \mathcal{P}_r(\widehat{M})$ in the above definition, and $\mathcal{C}_{\mathcal{P}}$ inherits the vague topology from $\mathcal{P}_r(\widehat{M})$.

(iii) We equip Ω_P with the product topology (which is metrizable, since $\mathbb{A}$ is a metric space and the product is countable). The product topology on Ω_P and the vague topology on $\mathcal{C}_P$ coincide, when the two spaces are canonically identified by $\omega \leftrightarrow P^{(\omega)}$. This is seen by noting that for both topologies convergence means pointwise convergence.

Compactness of spaces of coloured point sets is established in the following proposition.

Proposition 3.5 *Let $\mathcal{P} \subseteq \mathcal{P}_r(M)$ be given and assume that $\mathcal{P}$ is closed in $\mathcal{P}_r(M)$. Then the metrizable space $\mathcal{C}_{\mathcal{P}}$ is closed in $\mathcal{P}_r(\widehat{M})$ and hence compact. In particular, $\mathcal{C}_r(M)$ and $\widehat{X}_{\mathcal{P}}$ are compact.*

It follows from (3.1) that the action of $x \in T$ on a coloured point set $P^{(\omega)} \in \mathcal{C}_r(M)$ can be described as

$$xP^{(\omega)} = (xP)^{(\tau_x\omega)}. \tag{3.2}$$

Here, we introduce the measurable *shift* $\tau_x \colon \Omega_P \to \Omega_{xP}, \omega \mapsto \tau_x\omega$, between probability spaces, given by $(\tau_x\omega)(xp) := \omega(p)$ for all $p \in P$.

We are particularly interested in T-invariant compact spaces of coloured point sets. Therefore the following lemma is useful.

Lemma 3.6 *For $\mathcal{P} \subseteq \mathcal{P}_r(M)$ we have*

$$\widehat{X}_{\mathcal{P}} = \overline{\left\{ xP^{(\omega)} : x \in T,\ P^{(\omega)} \in \mathcal{C}_{\mathcal{P}} \right\}},$$

where the closure is taken with respect to the vague topology. The group action $\alpha_{\widehat{X}_{\mathcal{P}}} : T \times \widehat{X}_{\mathcal{P}} \to \widehat{X}_{\mathcal{P}}$, $(x, \widehat{P}) \mapsto x\widehat{P}$ is continuous.

The preceding proposition and lemma imply the following corollary.

Corollary 3.7 *Let $\mathcal{P} \subseteq \mathcal{P}_r(M)$ be a collection of point sets. Then $(\widehat{X}_{\mathcal{P}}, T, \alpha_{\widehat{X}_{\mathcal{P}}})$ is a compact topological dynamical system and the Ergodic Theorems* 2.14 *and* 2.16 *hold for $\mathcal{Q} = \widehat{X}_{\mathcal{P}}$.* ∎

Since we want to describe randomly coloured point sets, we will now introduce suitable probability measures on the Borel spaces $(\Omega_P, \mathcal{A}_P)$ for different P. Here, $\mathcal{A}_P := \bigotimes_{p \in P} \mathcal{A}$ is the product over all points in P of the Borel σ-algebra $\mathcal{A}$ on $\mathbb{A}$. It coincides with the Borel σ-algebra on Ω_P [Ka, Lemma 1.2]. For $V \subseteq M$ and $P \in \mathcal{P}_r(M)$ we define the local σ-algebra $\mathcal{A}_P^{(V)}$ as the smallest σ-algebra on Ω_P such that the canonical projection $(\Omega_P, \mathcal{A}_P^{(V)}) \to (\Omega_{P\cap V}, \mathcal{A}_{P\cap V})$ is measurable. It is the σ-algebra of events concerning only colours attached to points in $P \cap V$.

For $\mathcal{P} \subseteq \mathcal{P}_r(M)$ consider a family of Borel probability measures $\mathbb{P}_P$ on $(\Omega_P, \mathcal{A}_P)$, which is indexed by $P \in X_{\mathcal{P}}$.

Assumption 3.8 This is a list of properties that the family $(\mathbb{P}_P)_{P\in X_{\mathcal{P}}}$ may or may not satisfy.

(i) *T-covariance.* $\mathbb{P}_{xP} = \mathbb{P}_P \circ \tau_x^{-1}$ for all $x \in T$ and all $P \in X_{\mathcal{P}}$.

(ii) *Independence at a distance.* There exists a length $\varrho > 0$ such that for every $P \in X_{\mathcal{P}}$ and every $V_1, V_2 \subset M$ with $d(V_1, V_2) > \varrho$ the local σ-algebras $\mathcal{A}_P^{(V_1)}, \mathcal{A}_P^{(V_2)}$ are $\mathbb{P}_P$-independent.

(iii) *M-compatibility.* For every $f \in C(\widehat{X}_{\mathcal{P}})$ the colour average $E_f \colon X_{\mathcal{P}} \to \mathbb{R}$, defined by

$$E_f(P) := \int_{\Omega_P} \mathrm{d}\mathbb{P}_P(\omega)\, f(P^{(\omega)}), \qquad P \in X_{\mathcal{P}},$$

is a measurable function.

(iv) *C-compatibility.* For every $f \in C(\widehat{X}_{\mathcal{P}})$ we have $E_f \in C(X_{\mathcal{P}})$.

Independence at a distance can be interpreted as interactions of finite range between points in the point set. Such types of interaction are relevant in statistical mechanics; see, for example, [Sl].

In the next lemma we give two examples for a family $(\mathbb{P}_P)_{P\in X_{\mathcal{P}}}$ of measures that satisfy all of the above assumptions. The second example involves a random field $\xi \colon \Sigma \times M \to \mathbb{A}$, $(\sigma, m) \mapsto \xi^{(\sigma)}(m)$ over M with values in $\mathbb{A}$, where $(\Sigma, \mathcal{A}', \mathbb{P}')$ is some given underlying probability space [A]. We will also need the *strong mixing coefficient* [Do] of ξ, defined by

$$\mathbb{R}_{\geqslant 0} \ni L \mapsto \kappa(L) := \sup\big\{\, \kappa(V_1, V_2) : V_1, V_2 \subseteq M,\ d(V_1, V_2) > L \big\},$$

where

$$\begin{aligned}\kappa(V_1, V_2) &:= \sup\big\{\, |\mathbb{P}'(A_1' \cap A_2') - \mathbb{P}'(A_1')\mathbb{P}'(A_2')| : A_j' \in \mathcal{A}'(V_j) \text{ for } j \in \{1, 2\} \big\}\\ &\leqslant 1/4\end{aligned}$$

measures the correlation of the local sub-σ-algebras $\mathcal{A}'(V_j)$ of events generated by the family of random variables $\{\xi^{(\,\cdot\,)}(m) : m \in V_j\}$.

Lemma 3.9 (i) *Let $\mathbb{P}$ be a Borel probability measure on $(\mathbb{A}, \mathcal{A})$, and define $\mathbb{P}_P := \bigotimes_{p\in P} \mathbb{P}$ for every $P \in X_{\mathcal{P}}$. Then the family of measures $(\mathbb{P}_P)_{P\in X_{\mathcal{P}}}$ satisfies the Assumptions* 3.8(i)–(iv).

(ii) *Let $(\Sigma, \mathcal{A}', \mathbb{P}')$ be a probability space and let $\xi \colon \Sigma \times M \to \mathbb{A}$, $(\sigma, m) \mapsto \xi^{(\sigma)}(m)$ be an $\mathbb{A}$-valued random field over M that is jointly measurable, T-stationary, has a compactly supported strong mixing coefficient, and has continuous realizations $\xi^{(\sigma)} \colon M \to \mathbb{A}$ for $\mathbb{P}'$-a.a. $\sigma \in \Sigma$. For a given point set $P \in X_{\mathcal{P}}$ we define the map $\Xi_P \colon \Sigma \to \Omega_P$, $\sigma \mapsto \Xi_P(\sigma) := \xi^{(\sigma)}\big|_P$. Then $\mathbb{P}_P := \mathbb{P}' \circ \Xi_P^{-1}$ is a Borel probability measure on $(\Omega_P, \mathcal{A}_P)$, and the family of measures $(\mathbb{P}_P)_{P\in X_{\mathcal{P}}}$ satisfies the Assumptions* 3.8(i)–(iv).

The main goal of this section is to characterize an ergodic Borel probability measure $\widehat{\mu}$ on $\widehat{X}_{\mathcal{P}}$ in terms of an ergodic Borel probability measure μ on uncoloured point sets $X_{\mathcal{P}}$ and the colour probability measures $(\mathbb{P}_P)_{P\in X_{\mathcal{P}}}$. This will be achieved in Theorem 3.11. A crucial ingredient of the proof is the following statement, which is inspired by and generalizes [Ho3, Lemma 3.1].

Theorem 3.10 (Strong law of large numbers) *Assume that T is even unimodular and admits a Følner sequence $(D_n)_{n\in\mathbb{N}}$. Fix $f \in C(\widehat{X}_{\mathcal{P}})$, $P \in X_{\mathcal{P}}$ and suppose that $(\mathbb{P}_P)_{P\in X_{\mathcal{P}}}$ has the property "independence at a distance" as in Assumption* 3.8(ii). *For $n \in \mathbb{N}$ define the random variable $Y_n\colon \Omega_P \to \mathbb{R}$ by*

$$Y_n(\omega) := \frac{1}{\operatorname{vol}(D_n)} \int_{D_n} \mathrm{d}x\, f(xP^{(\omega)}), \qquad \omega \in \Omega_P. \tag{3.3}$$

Then we have for $\mathbb{P}_P$-almost all $\omega \in \Omega_P$ the relation

$$\lim_{n\to\infty} \Big(Y_n(\omega) - \int_{\Omega_P} \mathrm{d}\mathbb{P}_P(\eta)\, Y_n(\eta) \Big) = 0. \tag{3.4}$$

We are now ready for the main result of this section.

Theorem 3.11 *Assume that T is even unimodular. Let $\mathcal{P} \subseteq \mathcal{P}_r(M)$ be a collection of point sets. Fix an ergodic Borel probability measure μ on $X_{\mathcal{P}}$ and a family of Borel probability measures $(\mathbb{P}_P)_{P\in X_{\mathcal{P}}}$ satisfying Assumptions* 3.8(i)–(iii). *Then there exists a unique ergodic probability measure $\widehat{\mu}$ on $\widehat{X}_{\mathcal{P}}$ such that the following statements hold.*

(i) *For every $f \in L^1(\widehat{X}_{\mathcal{P}}, \widehat{\mu})$ we have*

$$\int_{\widehat{X}_{\mathcal{P}}} \mathrm{d}\widehat{\mu}(P^{(\omega)})\ f(P^{(\omega)}) = \int_{X_{\mathcal{P}}} \mathrm{d}\mu(P) \int_{\Omega_P} \mathrm{d}\mathbb{P}_P(\omega)\ f(P^{(\omega)}). \tag{3.5}$$

(ii) *For every $f \in L^1(\widehat{X}_{\mathcal{P}}, \widehat{\mu})$ and every tempered Følner sequence $(D_n)_{n\in\mathbb{N}}$ in T the limit*

$$\lim_{n\to\infty} \frac{1}{\operatorname{vol}(D_n)} \int_{D_n} \mathrm{d}x\ f(xP^{(\omega)}) = \int_{\widehat{X}_{\mathcal{P}}} \mathrm{d}\widehat{\mu}(Q^{(\sigma)})\ f(Q^{(\sigma)}) \tag{3.6}$$

exists for $\widehat{\mu}$-a.a. $P^{(\omega)} \in \widehat{X}_{\mathcal{P}}$. In fact, the limit exists for μ-a.a. $P \in X_{\mathcal{P}}$ and for $\mathbb{P}_P$-a.a. $\omega \in \Omega_P$.

If $X_{\mathcal{P}}$ is even uniquely ergodic, if $(\mathbb{P}_P)_{P\in X_{\mathcal{P}}}$ satisfies also Assumption 3.8(iv) *and if f is continuous, then the limit* (3.6) *exists* for all *$P \in X_{\mathcal{P}}$ and for $\mathbb{P}_P$-a.a. $\omega \in \Omega_P$. In this case the Følner sequence does not need to be tempered.*

Remarks 3.12 (i) The asserted uniqueness of the ergodic measure $\widehat{\mu}$ in the theorem does not mean that the dynamical system $\widehat{X}_{\mathcal{P}}$ is uniquely ergodic. It only means that $\widehat{\mu}$ is uniquely determined by the given ergodic measure μ on $X_{\mathcal{P}}$ and the measures $(\mathbb{P}_P)_{P\in X_{\mathcal{P}}}$ on Ω_P.

(ii) The corresponding [Ho3, Theorem 3.1] is a statement about Bernoulli site percolation on the Penrose tiling. Our result is an extension, which covers both the aperiodic and the periodic situation, under weaker assumptions on the underlying point set, and for more general types of randomness.

(iii) In contrast to a corresponding result [Len, Lemma 10], our Theorem 3.11 does not require a group structure of the point space M. Theorem 3.11 also makes a stronger conclusion in that exceptional instances are characterized beyond being $\widehat{\mu}$-null sets. This is particularly useful in the uniquely ergodic case.

At the end of this section we discuss which values the measure $\widehat{\mu}$ assigns to cylinder sets of coloured point sets. In view of the decomposition (3.5) we are interested in the relation to the μ-measure of the corresponding uncoloured cylinder set; see Section 2.3. Consider open sets $U_1, \dots, U_k \subseteq M$ with $\operatorname{diam}(U_i) < r$ for $i \in \{1, \dots, k\}$. Choose $\varphi_i \in C_c(M)$ such that $\varphi_i^{-1}(\{0\}) = M \setminus U_i$ for $i \in \{1, \dots, k\}$. Similarly, consider open relatively compact sets $A_1, \dots, A_k \subseteq \mathbb{A}$ and choose $\psi_1, \dots, \psi_k \in C_c(\mathbb{A})$ such that $\psi_i^{-1}(\{0\}) = \mathbb{A} \setminus A_i$. With $f_{\varphi,\psi}$ as in (6.1), we define the *coloured cylinder set*

$$C_{\mathbf{U}}^{\mathbf{A}} := \left\{ P^{(\omega)} \in \mathcal{C}_r(M) : f_{\varphi_1,\psi_1}(P^{(\omega)}) \cdots f_{\varphi_k,\psi_k}(P^{(\omega)}) \neq 0 \right\}, \tag{3.7}$$

where $\mathbf{U} := U_1 \times \cdots \times U_k$ and $\mathbf{A} := A_1 \times \cdots \times A_k$. The set $C_{\mathbf{U}}^{\mathbf{A}}$ is independent of the particular choice of the functions φ_i and ψ_i with $\operatorname{supp}(\varphi_i) = \overline{U_i}$ and $\operatorname{supp}(\psi_i) = \overline{A_i}$, and it consists of all coloured point sets such that U_i contains exactly one point of the underlying point set, with corresponding colour value in A_i, for $i \in \{1, \dots, k\}$. In the case of independent and identically distributed (i.i.d.) colours, we have a nice product formula for the measure of such a cylinder set, which is stated in the following proposition.

Proposition 3.13 *Assume that T is even unimodular and admits a Følner sequence. Fix $f \in C(\widehat{X}_{\mathcal{P}})$, $P \in X_{\mathcal{P}}$ and an ergodic Borel probability measure μ on $X_{\mathcal{P}}$. Let $\mathbb{P}$ be a Borel probability measure on $(\mathbb{A}, \mathcal{A})$, and for every $P \in X_{\mathcal{P}}$ consider the product measure $\mathbb{P}_P := \bigotimes_{p \in P} \mathbb{P}$ on Ω_P, see Lemma* 3.9(i). *Assume in addition that the sets $U_1, \dots, U_k$ of the coloured cylinder set $C_{\mathbf{U}}^{\mathbf{A}}$ in* (3.7) *are pairwise disjoint. Then*

$$\widehat{\mu}(C_{\mathbf{U}}^{\mathbf{A}}) = \mu(C_{\mathbf{U}})\ \mathbb{P}(A_1) \cdots \mathbb{P}(A_k),$$

where $C_{\mathbf{U}} \subseteq X_{\mathcal{P}}$ is the corresponding uncoloured cylinder set (2.9).

4 Application to Graphs

One of our reasons for dealing with point spaces without a group structure in the previous sections is that this allows for a description of simple graphs [Di]. Most statements in this section do not require extra proofs, because they follow from the application of the general results of Sections 2 and 3. More generally, one could treat simple directed graphs or even hypergraphs [Di] by the same methods.

4.1 Graphs as Point Sets

Let $\mathbb{V}$ be a point space, *i.e.,* a non-empty, locally compact, and second-countable Hausdorff space and let T be a metrizable group. As in Section 2.1, we require the following assumption.

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Assumption 4.1 The group T is non-compact, and its left action $\alpha_{\mathbb{V}}\colon T \times \mathbb{V} \to \mathbb{V}$, $(x, v) \mapsto xv$, on $\mathbb{V}$ is continuous and proper. Moreover, we fix a T-invariant proper metric $d_{\mathbb{V}}$ on $\mathbb{V}$ that generates the topology on $\mathbb{V}$.

Hence, T is also locally compact and second countable; compare Remarks 2.2(iii) and (iv).

Next we consider the space $M := (\mathbb{V} \times \mathbb{V})/\sim$ with the quotient topology, arising from $\mathbb{V}\times\mathbb{V}$ with the product topology of $\mathbb{V}$. Here, the equivalence relation $\sim$ identifies $(v, w) \in \mathbb{V} \times \mathbb{V}$ with $(w, v) \in \mathbb{V} \times \mathbb{V}$, and we write $m = m_{v,w} = m_{w,v} \in M$ for the corresponding equivalence class. Clearly, the definition

$$d(m_{v_1,w_1}, m_{v_2,w_2}) := \min\Big\{ \max\big\{ d_{\mathbb{V}}(v_1, v_2), d_{\mathbb{V}}(w_1, w_2)\big\}, \\ \max\big\{ d_{\mathbb{V}}(v_1, w_2), d_{\mathbb{V}}(v_2, w_1)\big\}\Big\}$$

for all $v_1, v_2, w_1, w_2 \in \mathbb{V}$ provides a T-invariant proper metric d on M that is compatible with the topology on M.

Lemma 4.2 *The space M is a point space and, together with the induced action $\alpha\colon T\times M \to M$, $\alpha(x, m_{v,w}) := xm_{v,w} := m_{xv,xw}$ of T on M, satisfies Assumption* 2.1.

As to the proof of the lemma, we note that M is clearly non-empty, locally compact and second-countable, and thus a point space equipped with the T-invariant proper metric above. Also, continuity of the induced action α is evident. The proof is then completed by the first part of the following lemma.

Lemma 4.3 (i) *The induced action α on M is proper.*
(ii) *If T does not contain an element of order two and $\alpha_{\mathbb{V}}$ acts freely, then so does α.*

Remarks 4.4 (i) A proof of the lemma can be found in Section 7.
(ii) Transitivity of $\alpha_{\mathbb{V}}$ does not imply transitivity of α.

Definition 4.5 A point set $G \subseteq M$ is called a (simple) *graph* (in $\mathbb{V}$) if $m_{v,w} \in G$ for $v, w \in \mathbb{V}$ implies $m_{v,v} \in G$ and $m_{w,w} \in G$. A graph $G \subseteq M$ has the vertex set $\mathcal{V}_G := \{v \in \mathbb{V} : m_{v,v} \in G\}$, which is a point set in $\mathbb{V}$, and its edge set is given by $\mathcal{E}_G := \big\{ \{v, w\} : v, w \in \mathbb{V}, v \neq w, m_{v,w} \in G\big\}$.

Remark 4.6 Every graph $G \subseteq M$ is a simple graph [Di], that is, without self-loops or multiple edges between the same pair of vertices. It is easy to see that G is a uniformly discrete subset of M with radius r if and only if $\mathcal{V}_G$ is uniformly discrete in $\mathbb{V}$ with the same radius. Also, relative denseness of G with radius R implies relative denseness of $\mathcal{V}_G$ with radius R. The converse statement does not hold. This is seen from a graph with relatively dense vertex set, but without edges. Relative denseness of the point set G implies the existence of vertices with infinitely many incident edges.

4.2 Ergodicity for Dynamical Systems of Graphs

In this section we will apply the ergodic results from Section 2 to graphs. The space $\mathcal{P}_r(M)$ of uniformly discrete point sets in $M = (\mathbb{V} \times \mathbb{V})/\sim$ with radius r is a compact metrizable space with respect to the vague topology of Definition 2.6. The action α on M induces in turn a continuous group action on $\mathcal{P}_r(M)$ by pointwise shifts as in Section 2. Consequently, all results established for $\mathcal{P}_r(M)$ in Section 2 are available in the present context.

Definition 4.7 For $r \in]0,\infty[$ we introduce the space

$$\mathcal{G}_r(M) := \left\{ G \in \mathcal{P}_r(M) : G \text{ is a graph} \right\}$$

of graphs in $\mathbb{V}$ with uniformly discrete vertex sets of radius r. It inherits the vague topology from $\mathcal{P}_r(M)$.

We omit the obvious proof of the following proposition.

Proposition 4.8 *$\mathcal{G}_r(M)$ is closed, hence compact in $\mathcal{P}_r(M)$. Moreover, $\mathcal{G}_r(M)$ is T-invariant.*

Again, we are interested in closed (hence compact) and T-invariant subsets of $\mathcal{G}_r(M)$. An example is given by

$$X_{\mathcal{G}} := \overline{\{xG : x \in T, G \in \mathcal{G}\}} \subseteq \mathcal{G}_r(M)$$

for some given subset $\mathcal{G} \subseteq \mathcal{G}_r(M)$. We denote the continuous group action of T on $X_{\mathcal{G}}$ by $\alpha_{X_{\mathcal{G}}}$ and are now ready to apply the general ergodic results of Section 2.

Corollary 4.9 *Let $\mathcal{G} \subseteq \mathcal{G}_r(M)$ be given. Then $(X_{\mathcal{G}}, T, \alpha_{X_{\mathcal{G}}})$ is a compact topological dynamical system, and the Ergodic Theorems* 2.14 *and* 2.16 *hold for $\Omega = X_{\mathcal{G}}$.*

G' is called a *subgraph* of G if $G' \subseteq G$ and if G' is a graph. A subgraph G' of G with a finite vertex set is called a *patch* of G. Every patch of G is a pattern of G. A pattern Q of G is a patch of G if and only if Q is a subgraph of G. For a collection $\mathcal{G} \subseteq \mathcal{G}_r(M)$ of graphs, we call G' a patch of $\mathcal{G}$, if there exists $G \in \mathcal{G}$ such that G' is a patch of G. Every compact set $V \subseteq \mathbb{V}$ such that $\mathcal{V}_{G'} = \mathcal{V}_G \cap \mathring{V}$ is called a *support* of the patch.

It is easy to see that a collection $\mathcal{G} \subseteq \mathcal{G}_r(M)$ of graphs has finite local complexity in the sense of Definition 2.26(i) if and only if for every compact set $V \subseteq \mathbb{V}$ there is a finite collection $\mathcal{F}_{\mathcal{G}}(V)$ of patches such that every patch of $\mathcal{G}$, which admits a support equivalent to V, is equivalent to some patch in $\mathcal{F}_{\mathcal{G}}(V)$.

In analogy to the case of point sets, we have the following characterization of ergodicity and of unique ergodicity.

Theorem 4.10 (Ergodicity for FLC graphs) *Assume that T is even unimodular and that $(D_n)_{n\in\mathbb{N}}$ is a tempered van Hove sequence in T. Let $\mathcal{G} \subseteq \mathcal{G}_r(M)$ be a collection of graphs of finite local complexity. Let μ be a T-invariant Borel probability measure on $X_{\mathcal{G}}$. Then the following statements are equivalent.*

(i) *The measure μ is ergodic.*

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(ii) *For every patch H of $\mathcal{G}$, there is a set $X \subseteq X_{\mathcal{G}}$ of full μ-measure, such that the limit*

$$\nu(H) := \lim_{n\to\infty} \frac{\operatorname{card}\left(\{\widetilde{H} \subseteq G : \exists x \in D_n : x\widetilde{H} = H\}\right)}{\operatorname{vol}(D_n)} \tag{4.1}$$

exists for all $G \in X$ and is independent of $G \in X$.

If any of the above statements applies, then the limit (4.1) *is independent of the choice of the tempered van Hove sequence.*

Furthermore, the dynamical system $X_{\mathcal{G}}$ is uniquely *ergodic if and only if* (ii) *holds for all patches H of $\mathcal{G}$ with $X = X_{\mathcal{G}}$, that is, for* all *$G \in X_{\mathcal{G}}$. In that case, the van Hove sequence need not be tempered, and the limit* (4.1) *is independent of the choice of the van Hove sequence. Moreover, the convergence to the limit in* (4.1) *is even uniform in $G \in X_{\mathcal{G}}$.*

Remarks 4.11 (i) A proof of the theorem can be found in Section 7.

(ii) Assume that condition (ii) in the above theorem is satisfied, and let H_1 and H_2 be equivalent patches of $\mathcal{G}$. We then have $\nu(H_1) = \nu(H_2)$; compare Lemma 2.25(ii).

(iii) Analogously to Proposition 2.32, there is also a characterization of unique ergodicity in terms on uniform patch frequencies.

4.3 Randomly Coloured Graphs

Dynamical systems for coloured graphs are constructed as in Section 3. The only difference is that $\mathcal{P}_r(M)$ will be replaced by $\mathcal{G}_r(M)$. A *coloured graph* $G^{(\omega)}$, where $G \subseteq M$ is a graph and $\omega \in \Omega_G$, is given as in Definition 3.3(i). Copying the proofs of Proposition 3.5 and Lemma 3.6, we get the following proposition.

Proposition 4.12 *If $\mathcal{G} \subseteq \mathcal{G}_r(M)$ is closed in $\mathcal{G}_r(M)$, then the metrizable space*

$$\mathcal{C}_{\mathcal{G}} := \{G^{(\omega)} : G \in \mathcal{G}, \omega \in \Omega_G\}$$

is closed in $\mathcal{G}_r(\widehat{M})$ and hence compact. In particular, $\mathcal{C}_{\mathcal{G}_r(M)}$ and

$$\widehat{X}_{\mathcal{G}} := \mathcal{C}_{X_{\mathcal{G}}} = \overline{\left\{xG^{(\omega)} : x \in T,\ G^{(\omega)} \in \mathcal{C}_{\mathcal{G}}\right\}}$$

are compact. Moreover, the group action $\alpha_{\widehat{X}_{\mathcal{G}}} : T \times \widehat{X}_{\mathcal{G}} \to \widehat{X}_{\mathcal{G}}$, $(x, G^{(\omega)}) \mapsto xG^{(\omega)}$, which obeys (3.2)*, is continuous.*

Corollary 4.13 *Let $\mathcal{G} \subseteq \mathcal{G}_r(M)$ be given. Then $(\widehat{X}_{\mathcal{G}}, T, \alpha_{\widehat{X}_{\mathcal{G}}})$ is a compact topological dynamical system and the Ergodic Theorems* 2.14 *and* 2.16 *hold for $\Omega = \widehat{X}_{\mathcal{G}}$.*

Finally, we turn to randomly coloured graphs and, for given $\mathcal{G} \subseteq \mathcal{G}_r(M)$, consider a family of Borel probability measures $\mathbb{P}_G$ on $(\Omega_G, \mathcal{A}_G)$, which is indexed by $G \in X_{\mathcal{G}}$. Assumptions 3.8 read exactly the same when formulated for the family $(\mathbb{P}_G)_{G\in X_{\mathcal{G}}}$. In fact, we refer to this when we cite Assumptions 3.8 below. Noting Lemma 4.3(i), the Ergodic Theorem 3.11 takes the following form for randomly coloured graphs.

Corollary 4.14 *Let $\mathcal{G} \subseteq \mathcal{G}_r(M)$ be given. Fix an ergodic Borel probability measure μ on $X_{\mathcal{G}}$ and a family of Borel probability measures $(\mathbb{P}_G)_{G\in X_{\mathcal{G}}}$ satisfying Assumptions* 3.8(i)–(iii). *Then there exists a unique ergodic probability measure $\widehat{\mu}$ on $\widehat{X}_{\mathcal{G}}$ such that the following statements hold.*

(i) *For every $f \in L^1(\widehat{X}_{\mathcal{G}}, \widehat{\mu})$ we have*

$$\int_{\widehat{X}_{\mathcal{G}}} \mathrm{d}\widehat{\mu}(G^{(\omega)})\ f(G^{(\omega)}) = \int_{X_{\mathcal{G}}} \mathrm{d}\mu(G) \int_{\Omega_G} \mathrm{d}\mathbb{P}_G(\omega)\ f(G^{(\omega)}).$$

(ii) *For every $f \in L^1(\widehat{X}_{\mathcal{G}}, \widehat{\mu})$ and every tempered Følner sequence $(D_n)_{n\in\mathbb{N}}$ in T, the limit*

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(D_n)} \int_{D_n} \mathrm{d}x\ f(xG^{(\omega)}) = \int_{\widehat{X}_{\mathcal{G}}} \mathrm{d}\widehat{\mu}(H^{(\sigma)})\ f(H^{(\sigma)}) \tag{4.2}$$

exists for $\widehat{\mu}$-a.a. $G^{(\omega)} \in \widehat{X}_{\mathcal{G}}$. In fact, the limit exists for μ-a.a. $G \in X_{\mathcal{G}}$ and for $\mathbb{P}_G$-a.a. $\omega \in \Omega_G$.

If $X_{\mathcal{G}}$ is even uniquely ergodic, if $(\mathbb{P}_G)_{G\in X_{\mathcal{G}}}$ satisfies also Assumption 3.8(iv) *and if f is continuous, then the limit* (4.2) *exists* for all *$G \in X_{\mathcal{G}}$ and for $\mathbb{P}_G$-a.a. $\omega \in \Omega_G$. In this case the Følner sequence does not need to be tempered.*

When colouring graphs randomly, one may wish to distribute colours on vertices differently from colours on edges. This is possible within our framework, as is shown by the following example.

Example 4.15 Let $\mathbb{P}_v$ and $\mathbb{P}_e$ be two Borel probability measures on $(\mathbb{A}, \mathcal{A})$. Given a graph $G \in \mathcal{G} \subseteq \mathcal{G}_r(M)$, we define

$$\mathbb{P}_G := \bigotimes_{m_{v,v}\in G:\ v\in\mathcal{V}_G} \mathbb{P}_v \bigotimes_{m_{v,w}\in G:\ e=\{v,w\}\in\mathcal{E}_G} \mathbb{P}_e$$

on $(\Omega_G, \mathcal{A}_G)$, which corresponds to an i.i.d. distribution of colours on vertices and an independent i.i.d. distribution of colours on edges. Then, the family of measures $(\mathbb{P}_G)_{G\in X_{\mathcal{G}}}$ satisfies Assumptions 3.8(i)–(iv). Indeed, T-covariance and independence at a distance are obvious, and C-compatibility can be verified as in the proof of Lemma 3.9(i). In doing so, we use that identity (6.4) in the proof of that lemma has an analogue in the present context of graphs because, if $G, G' \in \mathcal{G}_r(M)$, $m \in G$, $m' \in G'$ and $d(m, m') < r$, then m and m' are either both vertices or both edges due to uniform discreteness.

5 Proofs of Results in Section 2

For the convenience of the reader we have also included proofs of the more elementary results in Section 2.1.

5.1 Proofs of Results in Section 2.1

Proof of Lemma 2.3 (i) $\Rightarrow$ (ii). Let $V_1, V_2 \subseteq M$ be compact and observe the identity $S_{V_1,V_2} = \pi_T\big(\widetilde{\alpha}^{-1}(V_2 \times V_1)\big)$, where π_T stands for the canonical projection $T \times M \to T$. Hence, properness of the map $\widetilde{\alpha}$ and continuity of π_T yield the claim.

(ii) $\Rightarrow$ (iii). Let $U_1, U_2 \subseteq M$ be relatively compact. Then, S_{U_1,U_2} is relatively compact because it is contained in $S_{\overline{U_1},\overline{U_2}}$, which is compact by hypothesis.

(iii) $\Rightarrow$ (iv). The sets $U_1 := \{m_n \in M : n \in \mathbb{N}\}$ and $U_2 := \{x_n m_n \in M : n \in \mathbb{N}\}$ are relatively compact in M because the sequences converge. Since $\{x_n \in T : n \in \mathbb{N}\} \subseteq S_{U_1,U_2}$, and the latter is relatively compact by hypothesis, we infer the claim from the Bolzano–Weierstrass theorem.

(iv) $\Rightarrow$ (i). Let $V^{(2)} \subseteq M \times M$ be compact. Then $Z := \widetilde{\alpha}^{-1}(V^{(2)})$ is closed in $T \times M$ by continuity of $\widetilde{\alpha}$. Let $((x_n, m_n))_{n\in\mathbb{N}}$ be any sequence in Z. Then $((x_n m_n, m_n))_{n\in\mathbb{N}} \subseteq V^{(2)}$, and compactness allows us to choose a convergent subsequence $(x_{n_l} m_{n_l}, m_{n_l}) \to (m', m) \in V^{(2)}$ as $l \to \infty$. In particular, every component converges and (iv) yields the existence of a further subsequence $(x_{n_{l_k}})_{k\in\mathbb{N}} \subseteq T$, which converges in T. Let us denote its limit by x. Since Z is closed, we have $(x_{n_{l_k}}, m_{n_{l_k}}) \to (x, m) \in Z$ as $k \to \infty$. This proves compactness of Z. ■

Proof of Lemma 2.8 (i) $\Rightarrow$ (ii). This holds by continuity.

(ii) $\Rightarrow$ (iii). Fix $m \in M$.

Case 1: assume $m \in P$. Let $\varepsilon > 0$ and define $\varphi \in C_c(M)$ by $\varphi(m') = 1 - d(m,m')/\varepsilon$ for $m' \in B_\varepsilon(m)$ and $\varphi(m') = 0$ otherwise. Then we have $f_\varphi(P) \geqslant 1$ and hence $f_\varphi(P_k) > 0$ for finally all $k \in \mathbb{N}$ by (ii). But this means that $P_k \cap B_\varepsilon(m) \neq \varnothing$ for finally all $k \in \mathbb{N}$.

Case 2: assume $m \notin P$. Choose $\varepsilon > 0$ such that $P \cap B_{2\varepsilon}(m) = \varnothing$. Define $\varphi \in C_c(M)$ by $\varphi(m') = 1 - d(m,m')/2\varepsilon$ for $m' \in B_{2\varepsilon}(m)$ and $\varphi(m') = 0$ otherwise. Then we have $f_\varphi(P) = 0$ and hence $f_\varphi(P_k) < 1/2$ for finally all $k \in \mathbb{N}$ by (ii). But this means that $P_k \cap B_\varepsilon(m) = \varnothing$ for finally all $k \in \mathbb{N}$.

(iii) $\Rightarrow$ (iv). Let P be the set of points satisfying condition (iii)(a) and define $N := M \setminus P$. We first show that $P \in \mathcal{P}_r(M)$. To see this, take arbitrary $m \in M$ and assume that $p, q \in P \cap B_r(m)$. Then there exist for finally all $k \in \mathbb{N}$ points $p_k, q_k \in P_k$ such that $p_k \to p$ and $q_k \to q$ as $k \to \infty$. But then $p_k, q_k \in P_k \cap B_r(m)$ for finally all $k \in \mathbb{N}$, which implies $p_k = q_k$ for finally all $k \in \mathbb{N}$ due to uniform discreteness of P_k. Hence $p = q$ and $\mathrm{card}(P \cap B_r(m)) = 1$.

Since V is compact, $P_f := P \cap V$ is finite. For $m \in N$ denote by $\varepsilon(m) > 0$ a number satisfying $P_k \cap B_{\varepsilon(m)}(m) = \varnothing$ for finally all $k \in \mathbb{N}$. Now fix $\varepsilon > 0$. Compactness of V yields the existence of a finite set $N_f \subseteq N$ such that $V \subseteq (P_f)_\varepsilon \cup \bigcup_{m\in N_f} B_{\varepsilon(m)}(m)$. We therefore have for finally all $k \in \mathbb{N}$ the inclusions

$$P_k \cap V \subseteq P_k \cap \Big((P)_\varepsilon \cup \bigcup_{m\in N_f} B_{\varepsilon(m)}(m) \Big) = P_k \cap (P)_\varepsilon \subseteq (P)_\varepsilon,$$

where we used the assumption (iii)(b) for the equality sign. The remaining inclusion follows from $P \cap V = P_f \subseteq (P_k)_\varepsilon$ for finally all $k \in \mathbb{N}$ due to assumption (iii)(a).

(iv) $\Rightarrow$ (i). Let $\varphi_1, \dots, \varphi_n \in C_c(M)$ and $\varepsilon_1, \dots, \varepsilon_n > 0$ be given. For an arbitrary $i \in \{1, \dots, n\}$, define the compact set $V_i = \mathrm{supp}(\varphi_i)$ and denote by $n_i \in \mathbb{N}$ an upper bound for the number of points that a uniformly discrete point set of radius r may have in V_i. By continuity of φ_i, we may choose $\delta_i \in]0, r[$ such that we have $|\varphi_i(m) - \varphi_i(m')| < \varepsilon_i/n_i$ for all $m, m' \in V_i$ satisfying $d(m, m') < \delta_i$. This means in particular that $|\varphi_i(m)| < \varepsilon_i/n_i$ for all $m \in V_i$ such that $d(m, V_i^c) < \delta_i$. By assumption (iv), with $\varepsilon = \delta_i$ and $V = V_i$, this implies for finally all $k \in \mathbb{N}$ the estimate

$$|f_{\varphi_i}(P_k) - f_{\varphi_i}(P)| < \varepsilon_i.$$

Since $i \in \{1, \dots, n\}$ was arbitrary, this means that for finally all $k \in \mathbb{N}$ we have $P_k \in U_{\varphi_1,\varepsilon_1}(P) \cap \cdots \cap U_{\varphi_n,\varepsilon_n}(P)$. ■

Proof of Proposition 2.9 Let $(P_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}_r(M)$ be given. It suffices to show that $(P_n)_{n\in\mathbb{N}}$ contains a convergent subsequence, since $\mathcal{P}_r(M)$ is metrizable.

Since M is σ-compact, we can find a countable open cover $(B_r(m_j))_{j\in\mathbb{N}}$ of M with $m_j \in M$ for $j \in \mathbb{N}$. For $j \in \mathbb{N}$ fixed, consider the sequence $(P_n \cap B_r(m_j))_{n\in\mathbb{N}}$. Exactly one of the following two cases occurs. Either there is $N_j \in \mathbb{N}$ such that $P_n \cap B_r(m_j) = \varnothing$ for all $n > N_j$, or there is a subsequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\varnothing \neq P_{n_k} \cap B_r(m_j) =: \{p^{(j)}_{n_k}\}$. Due to relative compactness of $B_r(m_j)$ we assume without loss of generosity that the induced point sequence $(p^{(j)}_{n_k})_{k\in\mathbb{N}}$ converges in M.

Now, consider the sequence $(P_n \cap B_r(m_1))_{n\in\mathbb{N}}$. In the second case of the above scenario, choose a subsequence $(P_{n^{(1)}_k})_{k\in\mathbb{N}}$ of $(P_n)_{n\in\mathbb{N}}$ such that the induced point sequence $(p^{(1)}_{n^{(1)}_k})_{k\in\mathbb{N}}$ converges to some $p^{(1)} \in M$. Otherwise, set $n^{(1)}_k := N_1 + k$ for all $k \in \mathbb{N}$. We repeat this procedure with the sequence $(P_{n^{(1)}_k} \cap B_r(m_2))_{k\in\mathbb{N}}$, yielding a subsequence $(n^{(2)}_k)_k$ of $(n^{(1)}_k)_k$, and then successively for all $j \geqslant 3$. In this way we obtain nested subsequences $(n^{(j+1)}_k)_k \subseteq (n^{(j)}_k)_k$ for all $j \in \mathbb{N}$. We claim that by Cantor's diagonal sequence trick, $(P_{n^{(k)}_k})_{k\in\mathbb{N}}$ fulfills the convergence criterion of Lemma 2.8(iii) and thus converges to $P := \{p^{(j)} \in M : j \in \mathbb{N}\}$ in the vague topology. Indeed, for every j the set $B_r(m_j) \cap (\bigcup_{k\in\mathbb{N}} P_{n^{(k)}_k})$ is either empty or consists of the points of the convergent sequence $(p^{(j)}_{n^{(k)}_k})_{k\in\mathbb{N}}$ with limit $p^{(j)} \in M$. Thus, alternative (b) must hold for every $m \in B_r(m_j)$, $m \neq p^{(j)}$, otherwise (a) applies. ■

5.2 Proofs of Results in Section 2.2

Proof of Lemma 2.13 (i) This follows readily from the Følner property, as $(LD_n) \setminus D_n \subseteq \delta^L D_n$.

(ii) For $D, E \subseteq T$, it is straightforward to verify $L((KD) \setminus E) \subseteq LKD \cap LE^c$. This results in the relations

$$L\big((KD) \setminus E\big) \subseteq \begin{cases} (LKD) \setminus D \cup (LE^c) \setminus D^c, \\ (LKD) \setminus E \cup (LE^c) \setminus E^c. \end{cases} \tag{5.1}$$

Consider the first relation in (5.1) for $D = D_n$ and for $E = \mathring{D}_n$. This yields

$$L\big((KD_n) \setminus \mathring{D}_n\big) \subseteq (LKD_n) \setminus \mathring{D}_n \cup (L\overline{D^c_n}) \setminus D^c_n,$$

where we used $(\mathring{D})^c = \overline{D^c}$. Consider the second relation in (5.1) for $D = \overline{D_n^c}$ and for $E = D_n^c$. This yields

$$L\Big((K\overline{D_n^c}) \setminus D_n^c\Big) \subseteq (LK\overline{D_n^c}) \setminus D_n^c \cup (LD_n) \setminus \mathring{D}_n.$$

When combining these two implications, we obtain

$$L(\partial^K D_n) \subseteq \partial^{LK} D_n \cup \partial^L D_n.$$

Now the van Hove property yields the claim of the lemma. ■

Proof of Theorem 2.14 Until further notice in this proof we only assume that Ω is a Polish space (*i.e.*, completely metrizable with a countable base of the topology); see Remark 2.15(ii). The μ-almost-sure existence of the integral (2.5) follows from Fubini's theorem. To prove (2.6) we apply the general pointwise ergodic theorem of Lindenstrauss [Lin, Thm. 1.2] for tempered Følner sequences. Since this theorem requires to work with a standard probability space (also called Lebesgue space, see [I, Thm. 2.4.1]), we consider the completed probability space $(\Omega, \bar{\mu})$ with the completed measure $\bar{\mu}$ living on the completion of the Borel σ-algebra. Then [Lin] yields the existence of a T-invariant function $f^\star \in L^1(\Omega, \bar{\mu})$ that obeys $\bar{\mu}(f^\star) = \bar{\mu}(f)$ and

$$\lim_{n\to\infty} I_n(q, f) = f^\star(q) \qquad \text{for } \bar{\mu}\text{-a.e. } q \in \Omega. \tag{5.2}$$

But the limit on the left-hand side of (5.2) is clearly Borel measurable in q, since f is. Thus, we conclude $f^\star \in L^1(\Omega, \mu)$, $\mu(f^\star) = \mu(f)$ and that the exceptional set in (5.2) can be chosen to be a μ-null set.

It remains to establish the chain of equivalences.

(i) $\Rightarrow$ (ii). Since μ is ergodic, $f^\star$ is μ-a.e. constant [BeM, Thm. 1.3]. Hence $f^\star(q) = \mu(f^\star) = \mu(f)$ for μ-a.a. $q \in \Omega$.

(ii) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (i). We are inspired by ideas in [Ho3, Thm. 3.1] and establish first an auxiliary lemma.

Auxiliary Lemma *For $k \in \mathbb{N}$ let $f_k, f \in L^1(\Omega, \mu)$ be given. Assume that $\|f - f_k\|_1 \to 0$ as $k \to \infty$ and that $f_k^\star = \mu(f_k)$ holds μ-a.e. for all $k \in \mathbb{N}$. Then $f^\star = \mu(f)$ holds μ-a.e.*

Proof of the Auxiliary Lemma Equation (2.6), the triangle inequality, Fatou's Lemma, Fubini's theorem, and T-invariance of μ provide the inequality $\|(f - f_k)^\star\|_1 \leqslant \|f - f_k\|_1$. Thus $\|(f - f_k)^\star\|_1 \to 0$ as $k \to \infty$, which in turn implies the existence of a subsequence $(k_l)_{l\in\mathbb{N}}$ such that $(f - f_{k_l})^\star \to 0$ pointwise μ-a.e. as $l \to \infty$. Now, the assertion of the Auxiliary Lemma can be seen from

$$0 = \lim_{l\to\infty} (f - f_{k_l})^\star = f^\star - \lim_{l\to\infty} \mu(f_{k_l}) = f^\star - \mu(f),$$

which holds μ-a.e. and where the rightmost equality follows from L^1-convergence. ■

From now on we assume in addition that $\mathcal{Q}$ is locally compact. We use the Auxiliary Lemma to establish μ-a.e.

$$(1_K)^\star = \mu(1_K) \tag{5.3}$$

for indicator functions 1_K of compact sets $K \subseteq \mathcal{Q}$. To see this, fix a metric on the metrizable space $\mathcal{Q}$ that is compatible with the topology, and note that by local compactness of $\mathcal{Q}$, there exists $\varepsilon > 0$ such that $(K)_\varepsilon$ is relatively compact. For $n \in \mathbb{N}$ such that $n \geqslant 1/\varepsilon$ consider the relatively compact thickened sets $K_n := (K)_{1/n}$. Using the metric, we define continuous functions $g_n \colon \mathcal{Q} \to [0,1]$ such that $g_n = 1$ on K, and $g_n = 0$ on K_n^c and $\operatorname{supp}(g_n) \subseteq \overline{K_n}$. We thus have $g_n \in C_c(\mathcal{Q})$ for all $n \geqslant N$. We also have L^1-convergence of g_n to 1_K, since

$$\|g_n - 1_K\|_1 = \int_{K_n \setminus K} \mathrm{d}\mu(q)\, g_n(q) \leqslant \mu(K_n \setminus K).$$

The latter expression vanishes as $n \to \infty$ by dominated convergence, since $\mu(\mathcal{Q}) = 1$, and since closedness of K implies $\lim_{n\to\infty} 1_{K_n \setminus K} \equiv 0$. On the other hand, denseness of $\mathcal{D}$ in $C_c(\mathcal{Q})$ with respect to $\|\cdot\|_\infty$ implies denseness with respect to $\|\cdot\|_1$ so that we infer the existence of a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ with $\|f_n - 1_K\|_1 \to 0$ as $n \to \infty$. By the hypothesis of (iii) we also have μ-a.e. the equality $f_n^\star = \mu(f_n)$ for all $n \in \mathbb{N}$. The Auxiliary Lemma then yields (5.3).

Now local compactness and second countability of $\mathcal{Q}$ ensure ([Bau, Thm. 29.12]) inner regularity of the Borel measure μ, and another application of the Auxiliary Lemma yields $(1_B)^\star = \mu(1_B)$ almost surely for arbitrary Borel sets $B \subseteq \mathcal{Q}$. In particular, for every T-invariant Borel set $B \subseteq \mathcal{Q}$ we conclude from this $\mu(B) = 1_B(q)$ for μ-a.a. $q \in \mathcal{Q}$. Hence, either $\mu(B) = 0$ or $\mu(B) = 1$, proving (i). ■

Proof of Theorem 2.16 We adapt the line of reasoning in [Wa, Thm. 6.19]. An alternative proof can be given using [Fu, Thm. 3.5]; compare also [S, Thm. 3.2].

The implication (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Let μ_j, $j \in \{1,2\}$, be two T-invariant Borel probability measures on $\mathcal{Q}$. The estimate $|I_n(q,f)| \leqslant \|f\|_\infty$ holds for all $n \in \mathbb{N}$, $q \in \mathcal{Q}$ and $f \in \mathcal{D}$. This and dominated convergence imply $\lim_{n\to\infty} \mu_j(I_n(\,\cdot\,,f)) = \mu_j(I(f)) = I(f)$. On the other hand, Fubini's theorem yields $\mu_j(I_n(\,\cdot\,,f)) = \mu_j(f)$ for all $n \in \mathbb{N}$ and $j \in \{1,2\}$. Hence, we get $\mu_1(f) = \mu_2(f)$ for all $f \in \mathcal{D}$. Now, denseness of $\mathcal{D}$ and boundedness of μ_j give $\mu_1(f) = \mu_2(f)$ for all $f \in C(\mathcal{Q})$. Thus, $\mu_1 = \mu_2$, as both belong to the dual space of $C(\mathcal{Q})$.

(iii) $\Rightarrow$ (i). We prove that (i) holds with $I(f) = \mu(f)$. Suppose this were false. Then there exists $g \in C(\mathcal{Q})$, $\varepsilon > 0$, a subsequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and a sequence $(q_k)_{k\in\mathbb{N}} \subseteq \mathcal{Q}$ such that for all $k \in \mathbb{N}$ we have

$$\left| I_{n_k}(q_k, g) - \mu(g) \right| \geqslant \varepsilon. \tag{5.4}$$

On the other hand, for every $k \in \mathbb{N}$ the linear functional $I_{n_k}(q_k,\,\cdot\,)$ belongs to the closed unit ball in the dual of the Banach space $C(\mathcal{Q})$, which is separable, since $\mathcal{Q}$ is metrizable [Bou2, Sec. X.3.3, Thm. 1]. The sequential Banach-Alaoglu theorem [Ru,

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Thm. 3.17] asserts that this closed unit ball is weak*-sequentially compact so that for every $f \in C(\Omega)$ the sequence $(I_{n_k}(q_k, f))_{k\in\mathbb{N}}$ contains a convergent subsequence. Pick a countable dense subset $\mathcal{C} \subseteq C(\Omega)$ with $g, 1 \in \mathcal{C}$. Cantor's diagonal trick gives the existence of a *common* subsequence $(n_{k_l})_{l\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\lim_{l\to\infty} I_{n_{k_l}}(q_{k_l}, f) =: J(f)$ exists for all $f \in \mathcal{C}$. Furthermore we have $|J(f)| \leqslant \|f\|_\infty$ for all $f \in \mathcal{C}$. Thus, $J\colon \mathcal{C} \to \mathbb{R}$, $f \mapsto J(f)$, is a bounded linear functional. It is T-invariant, due to the Følner property of $(D_n)_n$. It admits a unique bounded linear extension to $C(\Omega)$, which we denote again by J. This extension is positivity preserving, *i.e.*, $J(f) \geqslant 0$, if $f \geqslant 0$. Therefore the Riesz–Markov representation theorem [ReSi, Thm. IV.14] yields the existence of a positive Borel measure ν on $C(\Omega)$ such that $J(f) = \nu(f)$ for all $f \in C(\Omega)$. But $J(1) = 1$, which can be seen from the definition of J. Moreover, ν inherits T-invariance from J. Hence, ν is a T-invariant probability measure and thus $\nu = \mu$ by uniqueness. But $\nu(g) \neq \mu(g)$ on account of (5.4), which is a contradiction.

So far we have obtained the equivalences (i) to (iii) and that, in either case, the limit $I(f)$ equals $\mu(f)$. In particular, this limit does not depend on the chosen Følner sequence. If μ is not ergodic, there exists a T-invariant Borel set E such that $0 < \mu(E) < 1$. Then a T-invariant probability measure $\nu \neq \mu$ is given by $\nu(B) := \mu(B \cap E)/\mu(E)$ for all Borel sets B. Since μ is the only T-invariant probability measure, we conclude that μ is ergodic. ∎

5.3 Proofs of Results in Section 2.3

The Haar measure on T allows us to estimate how many points of a given point set $P \in \mathcal{P}_r(M)$ fall into some compact region in M. The following statement is a preparation for the proof of Lemma 2.22.

Proposition 5.1 *Assume that T is even unimodular. Let $(D_n)_{n\in\mathbb{N}}$ be a Følner sequence in T.*

(i) *For every relatively compact set $U \subseteq M$ we have the asymptotic estimate*

$$\operatorname{card}(P \cap D_n^{-1}U) = O(\operatorname{vol}(D_n)) \qquad \textit{as} \quad n \to \infty,$$

uniformly in $P \in \mathcal{P}_r(M)$.

(ii) *If $(D_n)_{n\in\mathbb{N}}$ is even a van Hove sequence, we have for every compact set $K \subseteq T$ and for every relatively compact set $U \subseteq M$ the asymptotic estimate*

$$\operatorname{card}\left(P \cap (\partial^K D_n)^{-1}U\right) = o(\operatorname{vol}(D_n)) \qquad \textit{as} \quad n \to \infty,$$

uniformly in $P \in \mathcal{P}_r(M)$.

Before we give the proof of the proposition we turn to a useful transformation property of transporters.

Remark 5.2 In slight abuse of the notation for transporters introduced in (2.1), we write $S_{m,U} := S_{\{m\},U} = \{x \in T : xm \in U\}$ for $m \in M$ and a subset $U \subseteq M$. Then, given any group element $x \in T$, we observe the identity $S_{xm,U} = S_{m,U}\, x^{-1}$.

Proof of Proposition 5.1 We fix $\varepsilon > 0$ and a relatively compact subset $U \subseteq M$. Without loss of generality we assume that $U \neq \varnothing$. The open thickened subset $U_\varepsilon := (U)_\varepsilon$ is still relatively compact due to the properness of the metric $d(\cdot\,,\,\cdot)$ on M. We define $\varphi_\varepsilon \in C_c(M)$ for $p \in M$ by $\varphi_\varepsilon(p) := d(p,(U_\varepsilon)^c)$. For given $p \in M$, the function $x \mapsto \varphi_\varepsilon(xp)$ lies in $C_c(T)$, since $\varphi_\varepsilon \in C_c(M)$ and the group action is continuous and proper. In particular, $x \mapsto \varphi_\varepsilon(xp)$ is integrable. For $D \subseteq T$ compact we evaluate

$$(5.5) \qquad \int_D \mathrm{d}x\; f_{\varphi_\varepsilon}(xP) = \int_D \mathrm{d}x \sum_{p\in P} \varphi_\varepsilon(xp) = \sum_{p\in P\cap D^{-1}U_\varepsilon} \int_D \mathrm{d}x\; \varphi_\varepsilon(xp).$$

Note that $P\cap D^{-1}U_\varepsilon$ is finite, because P is uniformly discrete and $D^{-1}\overline{U_\varepsilon}$ is compact. (This argument uses continuity of the group action.) Below we wish to approximate the last integral of (5.5) by

$$(5.6) \qquad I(p) := \int_T \mathrm{d}x\; \varphi_\varepsilon(xp) = \int_{S_{p,U_\varepsilon}} \mathrm{d}x\; \varphi_\varepsilon(xp).$$

Here, we make use of the transporter $S_{p,U_\varepsilon} \subseteq T$, which was introduced in Remark 5.2 and is relatively compact by Lemma 2.3(iii). It serves to restrict the integration in $I(p)$ to all those arguments where the integrand is strictly positive. But first, we will rewrite $I(p)$ for $p \in P\cap D^{-1}U_\varepsilon$. For such p there exists $y \in D$ and $m \in U_\varepsilon$ such that $p = y^{-1}m$. Hence, Remark 5.2 implies

$$S_{p,U_\varepsilon} = S_{m,U_\varepsilon}\, y \subseteq S_{m,U_\varepsilon} D \subseteq L_{U_\varepsilon} D,$$

where $L_{U_\varepsilon} := S_{\overline{U_\varepsilon},\,\overline{U_\varepsilon}}$ is compact in T by Lemma 2.3(ii). Therefore we have the identity

$$(5.7) \qquad I(p) = \int_{L_{U_\varepsilon} D} \mathrm{d}x\; \varphi_\varepsilon(xp)$$

for every $p \in P\cap D^{-1}U_\varepsilon$.

Next we derive a positive lower bound on the integral $I(p)$, which is uniform in $p \in P\cap TU$ (not U_ε!). For such p there is $y \in T$ and $q \in U$ such that $yp = q$. This implies

$$I(p) = \int_T \mathrm{d}x\, \varphi_\varepsilon(xy^{-1}q) = \int_T \mathrm{d}x\, \varphi_\varepsilon\big((yx)^{-1}q\big) = \int_T \mathrm{d}x\, \varphi_\varepsilon(xq) = I(q),$$

where we use unimodularity of the group in the second and third equality and left invariance of the Haar measure in the third equality. We conclude that for every $p \in P\cap TU$ we have

$$(5.8) \qquad I(p) \geqslant \inf\big\{ I(q) : q \in \overline{U} \big\} =: I_U > 0.$$

The strict positivity follows from a compactness argument, using continuity of the map $q \mapsto I(q)$, and from $I(q) > 0$ for all $q \in \overline{U}$. To see the latter we observe

$S_{q,U_\varepsilon} \supseteq S_{q,B_{\varepsilon/2}(q)}$ for every $q \in \overline{U}$. The transporter $S_{q,B_{\varepsilon/2}(q)}$ contains the identity $e \in T$ and is open by continuity of the group action and openness of the ball $B_{\varepsilon/2}(q) \subseteq M$. Therefore there exists an open ball $B^T \subseteq T$ about e such that $S_{q,B_{\varepsilon/2}(q)} \supseteq B^T$ and $\varphi_\varepsilon|_{B^T} > 0$. Since $\mathrm{vol}(B^T) > 0$ (one can cover the σ-compact group T by countably many copies of B^T, all of which have the same Haar measure), it follows that $I(q) > 0$.

Next, we establish an auxiliary estimate, which is a consequence of uniform discreteness (of a given radius r): for every relatively compact subset $U \subseteq M$ there exists a constant $N(U) < \infty$ such that for every $P \in \mathcal{P}_r(M)$ and every $x \in T$ the bound

$$\mathrm{card}(P \cap xU) \leqslant N(U) < \infty \tag{5.9}$$

holds. To prove this, we first set $x = e$, the identity in T, and note that a covering argument then implies (5.9) uniformly in $P \in \mathcal{P}_r(M)$. This and the equality $\mathrm{card}(P \cap xU) = \mathrm{card}(x^{-1}P \cap U)$ yield the desired uniformity of $N(U)$ in $x \in T$.

Now, consider the difference of the right-hand side of (5.5) and the corresponding expression where the integral is replaced by $I(p)$. This difference can be estimated as

$$\begin{aligned} \Big| \sum_{p \in P \cap D^{-1}U_\varepsilon} \int_{(L_{U_\varepsilon}D)\setminus D} \mathrm{d}x\, \varphi_\varepsilon(xp) \Big| &\leqslant \int_{(L_{U_\varepsilon}D)\setminus D} \mathrm{d}x \sum_{p \in P \cap D^{-1}U_\varepsilon} |\varphi_\varepsilon(xp)| \\ &\leqslant \int_{L_{U_\varepsilon}D} \mathrm{d}x \sum_{p \in P \cap x^{-1}U_\varepsilon} |\varphi_\varepsilon(xp)| \\ &\leqslant \mathrm{vol}(L_{U_\varepsilon}D)\, F_{U_\varepsilon}, \end{aligned} \tag{5.10}$$

where, using (5.9), the constant $F_{U_\varepsilon} := N(U_\varepsilon)\, \|\varphi_\varepsilon\|_\infty < \infty$ does not depend on P, nor on D. Therefore (5.8) and (5.10) imply

$$\begin{aligned} \mathrm{card}(P \cap D^{-1}U)\, I_U \leqslant \sum_{p \in P \cap D^{-1}U} I(p) &\leqslant \sum_{p \in P \cap D^{-1}U_\varepsilon} I(p) \\ &\leqslant \int_D \mathrm{d}x\, f_{\varphi_\varepsilon}(xP) + \mathrm{vol}(L_{U_\varepsilon}D)\, F_{U_\varepsilon} \\ &\leqslant F_{U_\varepsilon} \big[\, \mathrm{vol}(D) + \mathrm{vol}(L_{U_\varepsilon}D) \big]. \end{aligned}$$

Thus, the first claim of the proposition follows with $D = D_n$ and Lemma 2.13(i), while the second claim follows with $D = \partial^K D_n$, the van Hove property, and Lemma 2.13(ii). ∎

Proof of Lemma 2.22 Fix any $\varnothing \neq D \subseteq T$ compact, assume without loss of generality that $Q \neq \varnothing$ and fix $q \in Q$. Then the number of points $\widetilde{q} \in P \cap D^{-1}m$ such that $\widetilde{q} = xq$ for some $x \in T$ is at most $\mathrm{card}(P \cap D^{-1}m)$. For $\widetilde{q} \in P \cap D^{-1}m$ we introduce the set

$$A_{q,\widetilde{q}} := \Big\{ \widetilde{Q} \subseteq P : \exists x \in T : xQ = \widetilde{Q} \ \text{ and } \ xq = \widetilde{q} \Big\}. \tag{5.11}$$

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Clearly, the estimate

$$\mathrm{card}\left(M'_D(Q)\right) \leqslant \mathrm{card}(P \cap D^{-1}m) \cdot \max\left\{ \mathrm{card}(A_{q,\widetilde{q}}) : \widetilde{q} \in P \cap D^{-1}m \right\}$$

holds with the fixed $q \in Q$. In order to estimate the cardinality of $A_{q,\widetilde{q}}$ for a given $\widetilde{q}$ (assuming $A_{q,\widetilde{q}} \neq \varnothing$), we fix a reference pattern $\widetilde{Q}_r \in A_{q,\widetilde{q}}$ and consider an arbitrary $\widetilde{Q} \in A_{q,\widetilde{q}}$. Thus there exist $x_r, x \in T$ such that $x_r Q = \widetilde{Q}_r$, $xQ = \widetilde{Q}$, and $x_r q = \widetilde{q}$, $xq = \widetilde{q}$. The latter imply $x_r^{-1}x \in S_{q,\{q\}} =: S_q$, the compact stabilizer group of q by Lemma 2.3(ii). In addition, $\widetilde{Q} = x_r x_r^{-1} x Q \subseteq x_r S_q Q$. By definition, we have $\widetilde{Q} \subseteq P$, hence $\widetilde{Q} \subseteq P \cap x_r S_q Q$ for every $\widetilde{Q} \in A_{q,\widetilde{q}}$.

We conclude from (5.9) that $\mathrm{card}(P \cap x_r S_q Q) \leqslant N(\bigcup_{q \in Q} S_q Q)$, uniformly in $P \in \mathcal{P}_r(M)$ and uniformly in $q \in Q$ and in $\widetilde{q} \in P \cap D^{-1}m$ (which enters through x_r). Therefore there are at most

$$F(Q) := \binom{N\left(\bigcup_{q \in Q} S_q Q\right)}{\mathrm{card}(Q)}$$

possibilities to choose a subset $\widetilde{Q}$ with $\mathrm{card}(\widetilde{Q}) = \mathrm{card}(Q)$ points out of the pattern $P \cap x_r S_q Q$. We conclude

$$\mathrm{card}(A_{q,\widetilde{q}}) \leqslant F(Q) \tag{5.12}$$

uniformly in $q \in Q$ and $\widetilde{q} \in P \cap D^{-1}m$ and $P \in \mathcal{P}_r(M)$, and thus

$$\mathrm{card}\left(M'_D(Q)\right) \leqslant F(Q)\ \mathrm{card}(P \cap D^{-1}m).$$

Hence the second estimate follows with $D = D_n$ from Proposition 5.1(i), since T is unimodular and the group action is proper. For the first estimate, we note

$$M_D(Q) \subseteq \left\{ \widetilde{Q} \subseteq P \cap D^{-1}Q : \exists x \in T : xQ = \widetilde{Q} \right\}. \tag{5.13}$$

Therefore we can argue as above and obtain

$$\mathrm{card}\left(M_D(Q)\right) \leqslant F(Q)\ \mathrm{card}(P \cap D^{-1}Q). \tag{5.14}$$

Since Q is compact, we may now set $D = D_n$ and apply Proposition 5.1(i), which uses unimodularity and properness.

So it remains to prove that $\mathrm{card}(M_D(Q))$ and $\mathrm{card}(M'_D(Q))$ differ by a $o(\mathrm{vol}(D))$-term under the specified stronger hypotheses. Assume without loss of generality that $Q \neq \varnothing$. By assumption, we may write $Q = Km$ for some non-empty finite set $K \subseteq T$. Let $S := S_m := S_{m,\{m\}}$ denote the stabilizer group of $m \in M$, which is compact by Lemma 2.3(ii). We have $S = S^{-1}$ and $Am \cap Bm \subseteq (A \cap BS)m$ for $A, B \subseteq T$ arbitrary.

Note that $\widetilde{Q} \in M'_{D_n}(Q) \setminus M_{D_n}(Q)$ implies that there exists $x \in (D_n^{-1})^c$ such that $xQ = \widetilde{Q} \subseteq P \cap D_n^{-1}m$. Hence we have

$$\widetilde{Q} \subseteq D_n^{-1}m \cap (D_n^{-1})^c Km \subseteq \left(D_n^{-1} \cap (D_n^c)^{-1} KS\right) m \subseteq (\partial^{SK^{-1}} D_n)^{-1} m.$$

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Now the same argument as in (i) yields

$$\operatorname{card}\left(M'_{D_n}(Q) \setminus M_{D_n}(Q)\right) \leqslant F(Q) \operatorname{card}\left(P \cap (\partial^{SK^{-1}} D_n)^{-1} m\right),$$

and the latter term is recognized as $o(\operatorname{vol}(D_n))$ by Proposition 5.1(ii), since T is unimodular and the group action is proper.

Similarly, $\widetilde{Q} \in M_{D_n}(Q) \setminus M'_{D_n}(Q)$ implies $\widetilde{Q} \subseteq D_n^{-1}Q$ and $\widetilde{Q} \nsubseteq D_n^{-1}m$. Thus, there exist $x \in T$ and $q \in Q$ such that $xq \in \widetilde{Q}$ and $xq \in (D_n^{-1}m)^c$, which implies

$$xq \in D_n^{-1}Km \cap (D_n^c)^{-1}m \subseteq \left(D_n^{-1}KS \cap (D_n^c)^{-1}\right) m \subseteq (\partial^{SK^{-1}} D_n)^{-1}m.$$

We have

$$M_{D_n}(Q) \setminus M'_{D_n}(Q)$$
$$\subseteq \left\{\widetilde{Q} \subseteq P : \exists (x,q) \in T \times Q : xQ = \widetilde{Q} \wedge xq \in (\partial^{SK^{-1}} D_n)^{-1}m\right\} =: A.$$

This set can be represented as

$$A = \bigcup_{q \in Q} \bigcup_{\widetilde{q} \in P \cap (\partial^{SK^{-1}} D_n)^{-1}m} A_{q,\widetilde{q}}$$

with $A_{q,\widetilde{q}}$ given by (5.11). Therefore we use (5.12) to conclude

$$\operatorname{card}\left(M_{D_n}(Q) \setminus M'_{D_n}(Q)\right) \leqslant F(Q) \operatorname{card}(Q) \operatorname{card}\left(P \cap (\partial^{SK^{-1}} D_n)^{-1} m\right),$$

and the latter term is recognized as $o(\operatorname{vol}(D_n))$ by Proposition 5.1(ii), since T is unimodular and the group action is proper. ■

Proof of Lemma 2.25 (i). The finiteness of the supremum follows from inequality (5.14) and Proposition 5.1(i).

(ii). Fix $y \in T$. It suffices to show $\nu^P(yQ; (D_n)_{n\in\mathbb{N}}) = \nu^P(Q; (D_n)_{n\in\mathbb{N}})$. Since $M_{D_n}(yQ) = M_{yD_n}(Q)$ and

$$\left| \operatorname{card}\left(M_{yD_n}(Q)\right) - \operatorname{card}\left(M_{D_n}(Q)\right) \right| \leqslant \operatorname{card}\left(M_{A_n}(Q)\right),$$

where $A_n := \delta^{\{y\}} D_n \subseteq \partial^{\{y\}} D_n$, the claim follows from inequality (5.14) and Proposition 5.1(ii). ■

Proof of Lemma 2.27 (i). If $X_{\mathcal{P}}$ is FLC, then $\mathcal{P} \subseteq X_{\mathcal{P}}$ is FLC by definition. Conversely, assume that $\mathcal{P}$ is FLC. Take $V \subseteq M$ compact and a corresponding finite set $\mathcal{F}_{\mathcal{P}}(V) \subseteq \mathcal{Q}_{\mathcal{P}}$ of patterns of $\mathcal{P}$. Now let Q be any xV-pattern of $X_{\mathcal{P}}$. Then there is $P \in X_{\mathcal{P}}$ such that $Q = P \cap x\mathring{V}$. Since $P \in X_{\mathcal{P}}$, there is a sequence $((x_n, P_n))_{n\in\mathbb{N}} \subseteq T \times \mathcal{P}$ such that $x_n P_n \to P$ as $n \to \infty$. Hence, for every $n \in \mathbb{N}$, the pattern $\widetilde{Q}_n := P_n \cap x_n^{-1} x\mathring{V}$ is equivalent to some pattern in $\mathcal{F}_{\mathcal{P}}(V)$, and $x_n \widetilde{Q}_n \to Q$ as $n \to \infty$. Since $\mathcal{F}_{\mathcal{P}}(V)$ is finite, there is $\widetilde{Q} \in \mathcal{F}_{\mathcal{P}}(V)$, a sequence $(y_k)_{k\in\mathbb{N}}$ in T

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and a subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $\widetilde{Q}_{n_k} = y_k\widetilde{Q}$ for all $k \in \mathbb{N}$, implying that $x_{n_k}y_k\widetilde{Q} \to Q$ as $k \to \infty$. Local compactness of M and properness of the group action imply that a subsequence of $(x_{n_k}y_k)_{k\in\mathbb{N}}$ converges to some $z \in T$. Continuity of the group action then yields $z\widetilde{Q} = Q$. Thus $z^{-1}Q \in \mathcal{F}_{\mathcal{P}}(V)$.

(ii). For patterns $Q, \widetilde{Q} \in \mathcal{P}_r(M)$ define $\varepsilon(Q,\widetilde{Q})$ by

$$\varepsilon(Q,\widetilde{Q}) := \inf\left\{ \delta > 0 : \exists x \in T : Q \subseteq (x\widetilde{Q})_\delta \text{ and } x\widetilde{Q} \subseteq (Q)_\delta \right\}.$$

If $\widetilde{Q}$ is not equivalent to Q, we have $\varepsilon(Q,\widetilde{Q}) > 0$. Indeed, write $Q = \{q_1, \dots, q_k\}$ and $\widetilde{Q} = \{\widetilde{q}_1, \dots, \widetilde{q}_k\}$ and assume that $\varepsilon(Q,\widetilde{Q}) = 0$. Invoking the local compactness of M, we find a sequence $(x_n)_{n\in\mathbb{N}} \subseteq T$ such that we have $x_n\widetilde{q}_i \to q_i$ for $i \in \{1, \dots, k\}$ as $n \to \infty$ (possibly after some permutation of indices). Local compactness of M and properness of the group action imply that a subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in T$. By continuity of the group action, we thus get $x\widetilde{Q} = Q$, which is a contradiction.

Now take an arbitrary $Q \in \mathcal{Q}_{\mathcal{P}}$ and define the compact support $V := \overline{(Q)_r}$ of Q in M. Let $\mathcal{F}_{\mathcal{P}}(V)$ be a finite set of patterns corresponding to V in the FLC condition and define $\varepsilon > 0$ by

$$2\varepsilon := \min\left\{ \varepsilon(Q,\widetilde{Q}) : \widetilde{Q} \in \mathcal{F}_{\mathcal{P}}(V) \text{ and } \forall y \in T : Q \neq y\widetilde{Q} \right\}.$$

Now assume that there exist $x \in T$ and $\widetilde{Q} \in \mathcal{Q}_{\mathcal{P}}$ such that $x\widetilde{Q} \subseteq (Q)_\varepsilon$ and $Q \subseteq (x\widetilde{Q})_\varepsilon$. Then Q is equivalent to $\widetilde{Q}$, by definition of ε.

(iii). Assume that $\mathcal{P}$ is not FLC. Then there exist a compact set $V_0 \subseteq M$ and an infinite collection $(Q_n)_{n\in\mathbb{N}}$ of mutually non-equivalent patterns of $\mathcal{P}$ supported on T-shifted copies of V_0. Due to the compactness of $\mathcal{P}_r(M)$, a subsequence $\widetilde{Q}_k := Q_{n_k}$, $k \in \mathbb{N}$, of $(Q_n)_{n\in\mathbb{N}} \subseteq \mathcal{Q}_{\mathcal{P}}$ converges to some $Q \in \mathcal{P}_r(M)$. Let $V \subseteq M$ be a compact set satisfying $Q \subseteq \mathring{V}$. Since $\mathcal{Q}_{\mathcal{P}} \wedge V$ is closed in $\mathcal{P}_r(M)$ by assumption, we have $Q \in \mathcal{Q}_{\mathcal{P}} \wedge V$, which implies $Q \in \mathcal{Q}_{\mathcal{P}}$. By construction, we have $\varepsilon(Q,\widetilde{Q}_k) \to 0$ as $k \to \infty$. If Q is not equivalent to any $\widetilde{Q}_k$, this contradicts the local rigidity of $\mathcal{P}$. Otherwise, $Q = x\widetilde{Q}_\ell$ for exactly one ℓ and some $x \in T$ and $\varepsilon(\widetilde{Q}_\ell,\widetilde{Q}_k) = \varepsilon(Q,\widetilde{Q}_k) \to 0$ as $k \to \infty$. This is also a contradiction to the local rigidity of $\mathcal{P}$, since $\widetilde{Q}_\ell$ is not equivalent to $\widetilde{Q}_k$ for $k \neq \ell$. ∎

Before we can approach auxiliary results for the proof of Theorem 2.29 we introduce some more systematic notation for pattern collections of point sets.

Definition 5.3 Let $U \subseteq M$ and $D \subseteq T$. For $P \in \mathcal{P}_r(M)$ we define

$$\mathcal{Q}_P(U;D) := \left\{ Q \subseteq P : \exists x \in D \text{ such that } xQ \subseteq U \right\}$$

and, in slight abuse of notation for $\mathcal{P} \subseteq \mathcal{P}_r(M)$,

$$\mathcal{Q}_{\mathcal{P}}(U;D) := \bigcup_{P\in\mathcal{P}} \mathcal{Q}_P(U;D).$$

It is also convenient to fix in addition the number of points $k \in \mathbb{N}$ of the patterns, in symbols,

$$\mathcal{Q}_P^k(U;D) := \left\{ Q \in \mathcal{Q}_P(U;D) : \operatorname{card}(Q) = k \right\},$$

and similarly $\mathcal{Q}_{\mathcal{P}}^k(U;D)$. In particular, we have $M_D(Q) = \mathcal{Q}_P^{\operatorname{card}(Q)}(Q;D)$ for every given point set P. We write $\mathcal{Q}_{\mathcal{P}}^k(U) := \mathcal{Q}_{\mathcal{P}}^k(U;T)$ and $\mathcal{Q}_{\mathcal{P}}^k := \mathcal{Q}_{\mathcal{P}}^k(M)$, which has already been used.

The next lemma and the subsequent proposition will be needed in the course of the proof of Theorem 2.29. But they also enter into the proof of Theorem 3.10, which is the main ingredient for the ergodic theorem of randomly coloured point sets.

Lemma 5.4 *Given $P \in \mathcal{P}_r(M)$ and a relatively compact subset $U \subseteq M$, there exists a constant $\Gamma_U > 0$ that depends only on U and the radius of relative discreteness r, but not on $P \in \mathcal{P}_r(M)$, such that for every $k \in \mathbb{N}$ and every compact subset $D \subseteq T$ the following estimate holds:*

$$\operatorname{card}\left(\mathcal{Q}_P^k(U;D)\right) \leqslant \Gamma_U^{k-1} \operatorname{card}(P \cap D^{-1}U).$$

Proof We start by observing that $\mathcal{Q}_P^k(U;D) \subseteq \bigcup_{q \in P \cap D^{-1}U} A_q$, where

$$\begin{aligned} A_q &:= \Big\{ Q \subseteq P : \ \operatorname{card}(Q) = k,\ q \in Q \ \text{ and } \ \exists\, x \in T \ \text{ such that } \ xQ \subseteq U \Big\} \\ &= \Big\{ \{q, q_2, \dots, q_k\} \subseteq P : \ \exists\, x \in S_{q,U} \ \text{ with } \ xq_i \in U \ \forall i \in \{2, \dots, k\} \Big\} \\ &\subseteq \Big\{ \{q, q_2, \dots, q_k\} \subseteq P : \ q_i \in P \cap S_{q,U}^{-1} U \ \forall i \in \{2, \dots, k\} \Big\}. \end{aligned}$$

This implies

$$\begin{aligned} \operatorname{card}\left(\mathcal{Q}_P^k(U;D)\right) &\leqslant \sum_{q \in P \cap D^{-1}U} \Big[\operatorname{card}\left(P \cap S_{q,U}^{-1}U\right) \Big]^{k-1} \\ &\leqslant \operatorname{card}(P \cap D^{-1}U) \Big[\sup_{q \in P \cap TU} \operatorname{card}\left(P \cap S_{q,U}^{-1}U\right) \Big]^{k-1}. \end{aligned}$$

For every $q \in P \cap TU$ there exists $x_q \in T$ and $m_q \in U$ such that $q = x_q m_q$. Thus, we conclude from the transformation property of transporters in Remark 5.2 that

$$\begin{aligned} \operatorname{card}\left(P \cap S_{q,U}^{-1}U\right) &= \operatorname{card}\left(P \cap x_q S_{m_q,U}^{-1}U\right) \leqslant \operatorname{card}\left(x_q^{-1}P \cap S_{U,U}^{-1}U\right) \\ &\leqslant N\big(S_{U,U}^{-1}U\big), \end{aligned}$$

where the last inequality rests on (5.9) and holds uniformly in $P \in \mathcal{P}_r(M)$ and $x_q \in T$, and therefore uniformly in $q \in P \cap TU$. Here, the application of (5.9) is justified, because $S_{U,U}^{-1}U$ is relatively compact in M. This follows from Lemma 2.3(iii) and continuity of the group action. So the claim holds with $\Gamma_U = N(S_{U,U}^{-1}U)$. ■

We write $L^0_{b,c}(M)$ to denote the set of all real-valued, Borel-measurable, and bounded functions φ on M, whose set-theoretic support $\{m \in M : \varphi(m) \neq 0\}$ is relatively compact. For $\varphi \in L^0_{b,c}(M)$, we consider $f_\varphi \colon \mathcal{P}_r(M) \to \mathbb{R}$ as in Definition 2.6.

Proposition 5.5 *Let $(D_n)_{n\in\mathbb{N}}$ be a Følner sequence in T. Fix $k \in \mathbb{N}$ and consider functions $\varphi_i \in L^0_{b,c}(M)$, $i \in \{1,\dots,k\}$, whose set-theoretic supports U_i, $i \in \{1,\dots,k\}$, are relatively compact and pairwise disjoint. Let $U := \bigcup_{i=1}^k U_i$. Then we have the equality*

$$\int_{D_n} \mathrm{d}x \, \Big(\prod_{i=1}^k f_{\varphi_i} \Big)(xP) = \sum_{Q \in \mathcal{Q}^k_P(U;D_n)} I(Q) + o(\mathrm{vol}(D_n)),$$

asymptotically as $n \to \infty$. Here the $o(\mathrm{vol}(D_n))$-term can be chosen uniformly in $P \in \mathcal{P}_r(M)$, and the leading term

$$I(Q) := \sum_{\pi \in \mathcal{S}_k} \int_T \mathrm{d}x \, \prod_{i=1}^k \varphi_i(xq_{\pi(i)}) \tag{5.15}$$

involves a sum over all permutations from the symmetric group $\mathcal{S}_k$ so that the fixed choice for enumerating the points of the pattern $Q = \{q_1,\dots,q_k\}$ is irrelevant.

Proof We fix $P \in \mathcal{P}_r(M)$ arbitrary. Note first that, for $p \in M$ and $i \in \{1,\dots,k\}$ fixed, the function $x \mapsto \varphi_i(xp)$ is integrable, since φ_i is measurable, bounded, has a relatively compact support, and since the group action is continuous and proper. Hence, $x \mapsto \prod_{i=1}^k \varphi_i(xq_{\pi(i)})$ is integrable, too. Moreover, since the supports U_i, $i \in \{1,\dots,k\}$, are pairwise disjoint, we have

$$\Big(\prod_{i=1}^k f_{\varphi_i} \Big)(xP) = \prod_{i=1}^k \Big(\sum_{p \in P} \varphi_i(xp) \Big) = \sum_{Q \in \mathcal{Q}^k_P(U;D_n)} \sum_{\pi \in \mathcal{S}_k} \prod_{i=1}^k \varphi_i(xq_{\pi(i)}) \tag{5.16}$$

for every $x \in D_n$. By Lemma 5.4 the set $\mathcal{Q}^k_P(U;D_n)$ is finite, and integrating (5.16) gives

$$\int_{D_n} \mathrm{d}x \, \Big(\prod_{i=1}^k f_{\varphi_i} \Big)(xP) = \sum_{Q \in \mathcal{Q}^k_P(U;D_n)} \sum_{\pi \in \mathcal{S}_k} \int_{D_n} \mathrm{d}x \, \prod_{i=1}^k \varphi_i(xq_{\pi(i)}). \tag{5.17}$$

Now we wish to replace the sum over permutations on the right-hand side of (5.17) by $I(Q)$ asymptotically as $n \to \infty$. This is achieved in analogy to the argument leading from (5.6) to (5.7). We start with the observation

$$\int_T \mathrm{d}x \, \prod_{i=1}^k \varphi_i(xq_{\pi(i)}) = \int_{S(Q)} \mathrm{d}x \, \prod_{i=1}^k \varphi_i(xq_{\pi(i)}), \tag{5.18}$$

where we introduced

$$S(\widehat{Q}) := \{x \in T : x\widehat{Q} \subseteq U\} = \bigcap_{\widehat{q} \in \widehat{Q}} S_{\widehat{q},U} \subseteq T$$

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for general $\widehat{Q} \subseteq M$. In this way we excluded parts of the domain of integration in (5.18) where the integrand vanishes anyway. In the special case $\widehat{Q} \subseteq U$ we have $S(\widehat{Q}) \subseteq S_{\overline{U},\overline{U}} =: L_U$, which is compact by Lemma 2.3(ii). Next suppose that $Q \subseteq D_n^{-1}U$ (as is the case for $Q \in \mathcal{Q}^k_{D_n}(P)$). Then there exists $y \equiv y(Q) \in D_n$ and $\widehat{Q} \subseteq U$ such that $Q = y^{-1}\widehat{Q}$. Hence we conclude from Remark 5.2 that

$$S(Q) = S(y^{-1}\widehat{Q}) = S(\widehat{Q})y \subseteq L_U y \tag{5.19}$$

$$\subseteq L_U D_n. \tag{5.20}$$

Therefore (5.18) and (5.20) yield the identity

$$I(Q) = \sum_{\pi \in \mathcal{S}_k} \int_{L_U D_n} \mathrm{d}x \prod_{i=1}^{k} \varphi_i(xq_{\pi(i)}) \tag{5.21}$$

for every $Q \in \mathcal{Q}^k_{D_n}(P)$. From (5.21) and (5.17) we deduce the estimate

$$\begin{aligned} &\Bigg| \sum_{Q \in \mathcal{Q}^k_P(U;D_n)} I(Q) - \int_{D_n} \mathrm{d}x \Big(\prod_{i=1}^{k} f_{\varphi_i} \Big)(xP) \Bigg| \\ &\qquad \leqslant \sum_{\pi \in \mathcal{S}_k} \int_{(L_U D_n)\setminus D_n} \mathrm{d}x \sum_{Q \in \mathcal{Q}^k_P(U;\{x\})} \prod_{i=1}^{k} |\varphi_i(xq_{\pi(i)})| \\ &\qquad \leqslant k!\ \mathrm{vol}\big((L_U D_n) \setminus D_n\big) \sup_{x \in T} \Big[\mathrm{card}\big(\mathcal{Q}^k_P(U;\{x\})\big) \Big] \prod_{i=1}^{k} \|\varphi_i\|_\infty. \end{aligned}$$

In order to get the first inequality above we have used the identity

$$\sum_{Q \in \mathcal{Q}^k_P(U;D_n)} \prod_{i=1}^{k} \varphi_i(xq_{\pi(i)}) = \sum_{Q \in \mathcal{Q}^k_P(U;\{x\})} \prod_{i=1}^{k} \varphi_i(xq_{\pi(i)}),$$

which holds for every fixed $x \in T$. The assertion of the proposition now follows from the Følner property (2.3), the fact that L_U is independent of P, and the estimate

$$\mathrm{card}\big(\mathcal{Q}^k_P(U;\{x\})\big) \leqslant \Gamma_U^{k-1}\,\mathrm{card}(P \cap x^{-1}U) \leqslant \Gamma_U^{k-1} N(U) < \infty,$$

which is based on Lemma 5.4 and holds uniformly in $P \in \mathcal{P}_r(M)$ and $x \in T$ by (5.9). ■

The following proposition refines the asymptotic evaluation of Proposition 5.5 in terms of pattern frequencies. For that reason, T is assumed to be unimodular, and the FLC assumption is imposed.

Proposition 5.6 *Let $(D_n)_{n\in\mathbb{N}}$ be a van Hove sequence in the unimodular group T. Fix $k \in \mathbb{N}$ and consider functions $\varphi_i \in L^0_{b,c}(M)$, $i \in \{1,\dots,k\}$, whose set-theoretic*

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supports U_i, $i \in \{1,\dots,k\}$, are relatively compact and pairwise disjoint. Set $U := \bigcup_{i=1}^k U_i$. Let $\mathcal{P} \subseteq \mathcal{P}_r(M)$ be of finite local complexity and let $\mathcal{F}^k_{X_{\mathcal{P}}}(U)$ be a maximal subset of mutually non-equivalent patterns in $\mathcal{Q}^k_{X_{\mathcal{P}}}(U)$. Then we have for every $P \in X_{\mathcal{P}}$ the asymptotic estimate

$$\int_{D_n} \mathrm{d}x \,\Big(\prod_{i=1}^k f_{\varphi_i}\Big)(xP) = \sum_{Q \in \mathcal{F}^k_{X_{\mathcal{P}}}(U)} I(Q)\ \mathrm{card}\,\big(M_{D_n}(Q)\big) + o(\mathrm{vol}(D_n))$$

as $n \to \infty$. Here, the finite set $\mathcal{F}^k_{X_{\mathcal{P}}}(U)$ and the integral $I(Q)$ are independent of the particular choice of $P \in X_{\mathcal{P}}$, and the error term can be chosen uniformly in $P \in X_{\mathcal{P}}$.

Proof Fix $P \in X_{\mathcal{P}}$. By Proposition 5.5, we have

$$\int_{D_n} \mathrm{d}x \,\Big(\prod_{i=1}^k f_{\varphi_i}\Big)(xP) = \sum_{\widetilde{Q} \in \mathcal{Q}^k_P(U;D_n)} I(\widetilde{Q}) + o(\mathrm{vol}(D_n)), \tag{5.22}$$

asymptotically as $n \to \infty$, where the error term can be chosen uniformly in $P \in X_{\mathcal{P}}$. In order to establish a connection to pattern frequencies, we partition the set $\mathcal{Q}^k_P(U;D_n)$ into subsets of equivalent patterns. Due to FLC of $X_{\mathcal{P}}$ (*cf.* Lemma 2.27), the set $\mathcal{F}^k_{X_{\mathcal{P}}}(U)$ is finite. Given an arbitrary pattern $Q \in \mathcal{F}^k_{X_{\mathcal{P}}}(U)$ we consider the collection $\mathcal{Q}^k_{P\cap D_n^{-1}U}(Q) \subseteq \mathcal{Q}^k_{P\cap D_n^{-1}U} = \mathcal{Q}^k_P(U;D_n)$ of all its translates in $P \cap D_n^{-1}U$. Then the sum in (5.22) decomposes

$$\int_{D_n} \mathrm{d}x \,\Big(\prod_{i=1}^k f_{\varphi_i}\Big)(xP) = \sum_{Q \in \mathcal{F}^k_{X_{\mathcal{P}}}(U)} \ \sum_{\widetilde{Q} \in \mathcal{Q}^k_{P\cap D_n^{-1}U}(Q)} I(\widetilde{Q}) + o(\mathrm{vol}(D_n)). \tag{5.23}$$

But the integral $I(\widetilde{Q})$ is independent of the particular choice of $\widetilde{Q} \in \mathcal{Q}^k_{P\cap D_n^{-1}U}(Q)$, as we now show. By definition there exists $y = y(\widetilde{Q}) \in T$ and enumerations of the points in the two patterns $Q = \{q_1,\dots,q_k\}$ and $\widetilde{Q} = \{\widetilde{q}_1,\dots,\widetilde{q}_k\}$ such that $y\widetilde{q}_i = q_i$ for all $i \in \{1,\dots,k\}$. Then we get

$$\begin{aligned} I(\widetilde{Q}) &= \sum_{\pi \in \mathcal{S}_k} \int_T \mathrm{d}x \prod_{i=1}^k \varphi_i(x\widetilde{q}_{\pi(i)}) = \sum_{\pi \in \mathcal{S}_k} \int_T \mathrm{d}x \prod_{i=1}^k \varphi_i\big((yx)^{-1} q_{\pi(i)}\big) \\ &= \sum_{\pi \in \mathcal{S}_k} \int_T \mathrm{d}x \prod_{i=1}^k \varphi_i(xq_{\pi(i)}) = I(Q), \end{aligned} \tag{5.24}$$

where we used unimodularity of the group for the second and third equality and left invariance of the Haar measure for the third equality.

In order to analyze the cardinality of $\mathcal{Q}^k_{P\cap D_n^{-1}U}(Q)$ for $\mathrm{card}(Q) = k$, consider the set

$$S := \{x \in T : xQ \subseteq U\} = \bigcap_{i=1}^k S_{q_i,U},$$

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which is relatively compact in T by Lemma 2.3(iii). Then we claim

$$\mathcal{Q}^k_{P\cap D_n^{-1}U}(Q) = \left\{ \widetilde{Q} \subseteq P : \exists y \in S^{-1}D_n \text{ with } y\widetilde{Q} = Q \right\} = M_{S^{-1}D_n}(Q).$$

Indeed, to verify the inclusion $\mathcal{Q}^k_{P\cap D_n^{-1}U}(Q) \subseteq M_{S^{-1}D_n}(Q)$, take $\widetilde{Q} \in \mathcal{Q}^k_{P\cap D_n^{-1}U}(Q)$ and choose $x \in D_n$ and $y \in T$ such that $x\widetilde{Q} \subseteq U$ and $y\widetilde{Q} = Q$. Then we have $xy^{-1} \in S$. But this means that $y \in S^{-1}D_n$, whence $\widetilde{Q} \in M_{S^{-1}D_n}(Q)$. For the reverse inclusion, take $\widetilde{Q} \in M_{S^{-1}D_n}(Q)$ and choose $y \in S^{-1}D_n$ with $y\widetilde{Q} = Q$. Then $y = s^{-1}x$ for some $s \in S$ and some $x \in D_n$. Hence $x\widetilde{Q} = sQ \subseteq U$. This means that $Q \in \mathcal{Q}^k_{P\cap D_n^{-1}U}(Q)$.

But the sets $M_{S^{-1}D_n}(Q)$ and $M_{D_n}(Q)$ are asymptotically of the same cardinality. This can be seen from

$$\begin{aligned} M_{S^{-1}D_n}(Q) \vartriangle M_{D_n}(Q) &= \left\{ \widetilde{Q} \subseteq P : \exists x \in \left(\delta^{S^{-1}} D_n\right)^{-1} : xQ = \widetilde{Q} \right\} \\ &\subseteq \left\{ \widetilde{Q} \subseteq P \cap \left(\partial^{S^{-1}} D_n\right)^{-1} Q : \exists x \in T : xQ = \widetilde{Q} \right\}, \end{aligned}$$

where $\vartriangle$ denotes the symmetric difference. We argue as in the proof of Lemma 2.22; compare equations (5.13)–(5.14), to show

$$\operatorname{card}\left(M_{S^{-1}D_n}(Q) \vartriangle M_{D_n}(Q)\right) \leqslant F(Q) \operatorname{card}\left(P \cap (\partial^{S^{-1}} D_n)^{-1} Q\right).$$

A final appeal to Proposition 5.1(ii) yields

$$\operatorname{card}\left(\mathcal{Q}^k_{P\cap D_n^{-1}U}(Q)\right) = \operatorname{card}\left(M_{S^{-1}D_n}(Q)\right) = \operatorname{card}\left(M_{D_n}(Q)\right) + o(\operatorname{vol}(D_n))$$

as $n \to \infty$, where the error term can be chosen uniformly in $P \in X_{\mathcal{P}}$. This holds by the van Hove property of $(D_n)_{n\in\mathbb{N}}$, where we used unimodularity and properness of the group action. Thus, the claim follows together with (5.23) and (5.24). ■

Proof of Theorem 2.29 Let μ be a T-invariant Borel probability measure on $X_{\mathcal{P}}$. We first prove the asserted characterization of ergodicity of μ. Our arguments rely on Theorem 2.14, which requires a tempered Følner sequence. In addition, the van Hove property enters through Proposition 5.6.

(i) ⇒ (ii) Without loss of generality fix a non-empty pattern $Q = \{q_1, \dots, q_k\}$, $k \in \mathbb{N}$, of $\mathcal{P}$. By FLC of $X_{\mathcal{P}}$ (*cf* Remarks 2.28), we may choose $\varepsilon \in]0, r/2[$ such that all patterns of $X_{\mathcal{P}}$ of cardinality k, which admit a support equivalent to the compact set $\overline{(Q)_\varepsilon}$, are equivalent to Q. For $i \in \{1, \dots, k\}$ define the mutually disjoint sets $U_i := B_\varepsilon(q_i)$. Choose $\varphi_i \in C_c(M)$ of compact support $\overline{U_i}$ for $i \in \{1, \dots, k\}$ and consider the function $f := f_{\varphi_1} \cdots f_{\varphi_k} \in C(X_{\mathcal{P}})$. Setting $U := \bigcup_{i=1}^k U_i$, we can now apply Proposition 5.6 with $\mathcal{F}^k_{X_{\mathcal{P}}}(U) = \{Q\}$. This yields

$$\int_{D_n} \mathrm{d}x\, f(xP) = I(Q) \operatorname{card}\left(M_{D_n}(Q)\right) + o(\operatorname{vol}(D_n)), \tag{5.25}$$

where P enters only through $M_{D_n}(Q)$ on the right-hand side. Since μ is ergodic and $f \in L^1(X_{\mathcal{P}}, \mu)$, Theorem 2.14 (ii) guarantees the existence of a set $X \subseteq X_{\mathcal{P}}$ of full μ-measure such that for all $P \in X$ we have

$$\lim_{n\to\infty} \frac{1}{\operatorname{vol}(D_n)} \int_{D_n} \mathrm{d}x\, f(xP) = \mu(f),$$

and this limit is independent of $P \in X$. Hence condition (ii) of the theorem is satisfied.

(ii) $\Rightarrow$ (i) We will apply the characterization of ergodicity in Theorem 2.14(iii). First, we define a suitable $\|\cdot\|_\infty$-dense subset $\mathcal{D}$ of $C(X_{\mathcal{P}})$. It will be constructed from the set

$$\mathcal{D}_0 := \left\{ f_\varphi : \varphi \in C_c(M), \operatorname{diam}\big(\operatorname{supp}(\varphi)\big) < r/2 \right\} \cup \{1\},$$

where $1 \in C(X_{\mathcal{P}})$ denotes the constant function equal to one, and with f_φ as in Definition 2.6. The set $\mathcal{D}_0$ separates points in $X_{\mathcal{P}}$. Hence the Stone–Weierstrass theorem [Ke, Prob. 7R] assures that the algebra $\mathcal{D} := \operatorname{alg}(\mathcal{D}_0)$ generated by $\mathcal{D}_0$ is dense in $C(X_{\mathcal{P}})$ with respect to the supremum norm.

Without loss of generality consider $f \in \mathcal{D}$ of the form $f = f_{\varphi_1} \cdots f_{\varphi_k}$, where $k \in \mathbb{N}$ and $f_{\varphi_j} \in \mathcal{D}_0 \setminus \{1\}$ for $j \in \{1, \dots, k\}$. Write $V_i := \operatorname{supp}(\varphi_i)$ for the compact supports of the functions φ_i for $i \in \{1, \dots, k\}$. Note that $V_i \cap V_j \neq \varnothing$ implies that $f_{\varphi_i} \cdot f_{\varphi_j} = f_{\varphi_i \cdot \varphi_j}$. We thus assume without loss of generality that $V_i \cap V_j = \varnothing$ for $i \neq j$. Write $U_i \subseteq V_i$ for the set-theoretical support of the function φ_i, for $i \in \{1, \dots, k\}$, and define $U := \bigcup_{i=1}^k U_i$.

Theorem 2.14 guarantees the existence of $X' \subseteq X_{\mathcal{P}}$ of full μ-measure and of a T-invariant function $f^\star \in L^1(X_{\mathcal{P}}, \mu)$ such that $\mu(f^\star) = \mu(f)$, and such that for all $P \in X'$ we have

$$\lim_{n\to\infty} \frac{1}{\operatorname{vol}(D_n)} \int_{D_n} \mathrm{d}x\, f(xP) = f^\star(P). \tag{5.26}$$

In order to show that the right-hand side of (5.26) is constant in P, we will evaluate the left-hand side of (5.26) using Proposition 5.6. To do so, we note that $\mathcal{F}^k_{X_{\mathcal{P}}}(U)$ is a finite set. Thus, there exists a set $X \subseteq X_{\mathcal{P}}$ of full μ-measure such that hypothesis (ii) is satisfied for all $Q \in \mathcal{F}^k_{X_{\mathcal{P}}}(U)$ and for all $P \in X$. Then the set $X'' := X' \cap X$ has full μ-measure (and is in particular non-empty), and equation (5.26) holds for all $P \in X''$. Now, Proposition 5.6. and hypothesis (ii) imply that the value $f^\star(P)$ is indeed independent of $P \in X''$, which in turn yields $f^\star = \mu(f^\star) = \mu(f)$ on X''. Therefore μ is ergodic. The asserted independence of the pattern frequency of the choice of the tempered van Hove sequence follows from the corresponding independence in the Ergodic Theorem 2.14.

In the simpler case of unique ergodicity one can argue as above, now with an arbitrary van Hove sequence. To prove (i) $\Rightarrow$ (ii), one uses Theorem 2.16(i). To prove (ii) $\Rightarrow$ (i), one can apply the characterization of unique ergodicity in Theorem 2.16(ii). Independence of the choice of the van Hove sequence follows from Theorem 2.16.

If $X_{\mathcal{P}}$ is uniquely ergodic, then the convergence to $\nu(Q)$ in Definition 2.24 is even uniform in $P \in X_{\mathcal{P}}$, since, after dividing by $\operatorname{vol}(D_n)$ in (5.25), the convergence on the left-hand side is uniform in $P \in X_{\mathcal{P}}$ by Theorem 2.16(i), and since the error term can be chosen uniformly in $P \in X_{\mathcal{P}}$. ■

Proof of Proposition 2.32 Let $(D_n)_{n\in\mathbb{N}}$ be a van Hove sequence in T.

(i) ⇒ (ii). Note first that uniform convergence of $\nu(y, P)$ in $(y, P) \subseteq T \times \mathcal{P}$, with a limit independent of (y, P), is equivalent to the existence of the limit

$$\lim_{n\to\infty} \nu_n^{y_n,P_n}(Q) = \lim_{n\to\infty} \frac{\operatorname{card}\left(\left\{\widetilde{Q} \subseteq P_n : \exists x \in D_n y_n : x\widetilde{Q} = Q\right\}\right)}{\operatorname{vol}(D_n)} \tag{5.27}$$

for every sequence $((y_n, P_n))_{n\in\mathbb{N}} \subseteq T \times \mathcal{P}$, with independence of the limit of $((y_n, P_n))_{n\in\mathbb{N}}$. Assume now that $X_{\mathcal{P}}$ is uniquely ergodic and fix a pattern $Q \in \mathcal{Q}_{\mathcal{P}}$. Then condition (5.27) is satisfied, because for every sequence $((y_n, P_n))_{n\in\mathbb{N}} \subseteq T \times \mathcal{P}$ we have

$$\nu_n^{y_n,P_n}(Q) = \frac{\operatorname{card}\left(\left\{\widetilde{Q} \subseteq y_nP_n : \exists x \in D_n : x\widetilde{Q} = Q\right\}\right)}{\operatorname{vol}(D_n)} = \nu_n^{e,y_nP_n}(Q),$$

and because the convergence in the limit underlying the definition of $\nu(Q)$ is uniform in $P \in X_{\mathcal{P}}$ by unique ergodicity of $X_{\mathcal{P}}$; see the second part of Theorem 2.29. Hence $\mathcal{P}$ has uniform pattern frequencies.

(ii) ⇒ (i). We use the characterization in Theorem 2.16(ii) with the dense algebra of functions $\mathcal{D}$ from the proof of Theorem 2.29, (ii) ⇒ (i). As explained there, it suffices to consider products $\prod_{i=1}^{k} f_{\varphi_i}$, $k \in \mathbb{N}$, with $\varphi_i \in L^0_{b,c}(M)$ for $i \in \{1, \dots, k\}$ having pairwise disjoint set-theoretical supports U_i. Given $P \in X_{\mathcal{P}}$, we abbreviate

$$I_n(P) := \frac{1}{\operatorname{vol}(D_n)} \int_{D_n} \mathrm{d}x \left(\prod_{i=1}^{k} f_{\varphi_i}\right)(xP)$$

and take a sequence $\big((y_m, P_m)\big)_{m\in\mathbb{N}} \subseteq T \times \mathcal{P}$ such that $(y_mP_m)_{m\in\mathbb{N}}$ converges to P. Then, for every $n \in \mathbb{N}$, the sequence $(I_n(y_mP_m))_{m\in\mathbb{N}}$ converges to $I_n(P)$ by dominated convergence. On the other hand, uniform pattern frequencies, Remark 2.28(ii) and Proposition 5.6 imply that

$$\lim_{n\to\infty} I_n(\widetilde{P}) = \sum_{Q\in\mathcal{F}^k_{X_{\mathcal{P}}}(U)} I(Q)\,\nu(Q) =: \mathcal{J},$$

the convergence being uniform in $\widetilde{P} \in X_{\mathcal{P}}$ and the limit $\mathcal{J}$ independent of $\widetilde{P} \in X_{\mathcal{P}}$. In particular, uniformity allows the interchange of limits in

$$\lim_{n\to\infty} I_n(P) = \lim_{n\to\infty}\lim_{m\to\infty} I_n(y_mP_m) = \lim_{m\to\infty}\lim_{n\to\infty} I_n(y_mP_m) = \mathcal{J},$$

showing that $\lim_{n\to\infty} I_n(P)$ exists for every $P \in X_{\mathcal{P}}$ and is independent of P. ■

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Proof of Proposition 2.33 With 1_A denoting the indicator function of a set A, the pointwise ergodic theorem Theorem 2.14(ii) (together with a tempered subsequence $(D_n)_{n\in\mathbb{N}}$ of the given van Hove sequence in T) yields for μ-a.a. $P \in X_{\mathcal{P}}$

$$\mu(C_{\mathbf{U}}) = \mu(1_{C_{\mathbf{U}}}) = \lim_{n\to\infty} \frac{1}{\operatorname{vol}(D_n)} \int_{D_n} \mathrm{d}x\, 1_{C_{\mathbf{U}}}(xP).$$

On the other hand, the indicator function of the cylinder set $C_{\mathbf{U}}$ can be expressed as

$$1_{C_{\mathbf{U}}} = f_{1_{U_1}} \cdots f_{1_{U_k}},$$

since $\operatorname{diam}(U_i) < r$. Apart from Q itself, the region $(Q)_\varepsilon$ contains, up to equivalence, by hypothesis no other pattern of $\mathcal{Q}_{X_{\mathcal{P}}}$ of the same cardinality as Q. Therefore we can apply Proposition 5.6 with $\mathcal{F}^k_{X_{\mathcal{P}}}(U) = \{Q\}$, which yields for all $P \in X_{\mathcal{P}}$

$$\int_{D_n} \mathrm{d}x\, 1_{C_{\mathbf{U}}}(xP) = I(Q)\ \operatorname{card}\big(M_{D_n}(Q)\big) + o(\operatorname{vol}(D_n))$$

as $n \to \infty$, where the integral $I(Q)$ is given by

$$I(Q) = \sum_{\pi\in\mathcal{S}_k} \int_T \mathrm{d}x\, \prod_{i=1}^{k} 1_{U_i}(xq_{\pi(i)}).$$

Therefore we conclude from Theorems 2.29 and 2.14 that the pattern frequency $\nu(Q)$ exists for μ-a.a. $P \in X_{\mathcal{P}}$, and for any such P we have

$$\mu(C_{\mathbf{U}}) = I(Q)\,\nu(Q).$$

Next we show that $I(Q) = \operatorname{vol}(D_\varepsilon)$. To do so, we use the notation of Remark 5.2 and introduce $T^{\pi}_{Q,U} := \bigcap_{i=1}^{k} S_{q_{\pi(i)},U_i}$ for $\pi \in \mathcal{S}_k$. Since each U_i can accommodate at most one point of a pattern, we obtain

$$\begin{aligned} D_\varepsilon &= \{x \in T : xQ \subseteq U\} \\ &= \bigcup_{\pi\in\mathcal{S}_k} \big\{x \in T : xq_{\pi(i)} \in U_i \ \text{ for all } \ i \in \{1,\dots,k\}\big\} \\ &= \bigcup_{\pi\in\mathcal{S}_k} T^{\pi}_{Q,U} = \bigcup_{\pi\in\mathcal{S}_k(Q)} T^{\pi}_{Q,U}. \end{aligned} \tag{5.28}$$

The restriction to $\mathcal{S}_k(Q) \subseteq \mathcal{S}_k$ in the last equality of (5.28) is justified, because if for some $\pi \in \mathcal{S}_k$ there is $x \in T$ such that $xq_{\pi(i)} \in U_i$ for all i, then there must exist $x_\pi \in T$ such that $x_\pi q_{\pi(i)} = q_i$, due to our hypothesis on the smallness of ε and the uniqueness of Q. Hence $\pi \in \mathcal{S}_k(Q)$. The representation (5.28) also implies that D_ε is open and relatively compact in T; compare Lemma 2.3(iii).

On the other hand, since $\pi \in \mathcal{S}_k \setminus \mathcal{S}_k(Q)$ does not contribute to $I(Q)$ either (by the same argument as above), we conclude

$$I(Q) = \sum_{\pi\in\mathcal{S}_k(Q)} \int_T \mathrm{d}x\, \prod_{i=1}^{k} 1_{U_i}(xq_{\pi(i)}) = \sum_{\pi\in\mathcal{S}_k(Q)} \operatorname{vol}\big(T^{\pi}_{Q,U}\big).$$

Thus, the desired equality $I(Q) = \mathrm{vol}(D_\varepsilon)$ follows if the rightmost union in (5.28) is disjoint. To see this we take $\pi, \widetilde{\pi} \in \mathcal{S}_k(Q)$. By definition, there exist $x_\pi, x_{\widetilde{\pi}} \in T$ such that

$$x_\pi q_{\pi(i)} = q_i \qquad \text{and} \qquad x_{\widetilde{\pi}} q_{\widetilde{\pi}(i)} = q_i$$

for all $i \in \{1, \dots, k\}$. On account of Remark 5.2, this implies

$$T^{\pi}_{Q,U} = T^{\mathrm{id}}_{Q,U} x_\pi \qquad \text{and} \qquad T^{\widetilde{\pi}}_{Q,U} = T^{\mathrm{id}}_{Q,U} x_{\widetilde{\pi}}. \tag{5.29}$$

Assuming $T^{\pi}_{Q,U} \cap T^{\widetilde{\pi}}_{Q,U} \neq \varnothing$, we see that (5.29) then ensures the existence of $y, \widetilde{y} \in T^{\mathrm{id}}_{Q,U}$, which obey $y x_\pi = \widetilde{y} x_{\widetilde{\pi}}$. This implies in turn

$$y q_i = y x_\pi q_{\pi(i)} = \widetilde{y} x_{\widetilde{\pi}} q_{\pi(i)} = \widetilde{y} x_{\widetilde{\pi}} q_{\widetilde{\pi}\left((\widetilde{\pi}^{-1} \circ \pi)(i)\right)} = \widetilde{y} q_{(\widetilde{\pi}^{-1} \circ \pi)(i)}$$

for all $i \in \{1, \dots, k\}$. Hence, $y q_i \in U_i \cap U_{(\widetilde{\pi}^{-1} \circ \pi)(i)}$ for all $i \in \{1, \dots, k\}$. Since $U_i \cap U_j = \varnothing$ for $i \neq j$, we infer that $\pi = \widetilde{\pi}$. Hence the rightmost union in (5.28) is disjoint and $I(Q) = \mathrm{vol}(D_\varepsilon)$ holds. This completes the proof of the first statement of the proposition.

To show the remaining statement of the proposition we assume that T is Abelian and note that

$$\begin{aligned} S_{ym, B_\varepsilon(ym)} &= \{x \in T : d(xym, ym) < \varepsilon\} = \{x \in T : d(xm, m) < \varepsilon\} \\ &= S_{m, B_\varepsilon(m)} \end{aligned}$$

for all $y \in T$ and all $m \in M$ due to T-invariance of the metric. Hence, if the group also acts transitively on M, we infer $T^{\mathrm{id}}_{Q,U} = S_{m, B_\varepsilon(m)}$ for every $m \in M$. Together with (5.29) and (5.28) this implies

$$D_\varepsilon = \dot{\bigcup_{\pi \in \mathcal{S}_k(Q)}} S_{m, B_\varepsilon(m)} x_\pi,$$

and the statement follows from the unimodularity of T (which yields right invariance of the Haar measure). ■

6 Proofs of Results in Section 3

Proof of Lemma 3.1 We only comment on properness of the induced action $\widehat{\alpha}$, since all other claims are evident. Properness of α follows from [Bou1, Prop. 5(ii), Chap. III.4.2], where we choose $G = G' = T$, $\varphi = \mathrm{id}$, $X = \widehat{M}$, $X' = M$, and $\psi \colon \widehat{M} \to M$, $(m, a) \mapsto m$. ■

Proof of Proposition 3.5 Compactness follows from closedness of $\mathcal{C}_{\mathcal{P}}$ in the compact metrizable space $\mathcal{P}_r(\widehat{M})$. Let $(P_n^{(\omega_n)})_{n \in \mathbb{N}} \subseteq \mathcal{C}_{\mathcal{P}}$ be a sequence with

$$\lim_{n \to \infty} P_n^{(\omega_n)} = \widehat{P} \in \mathcal{P}_r(\widehat{M}).$$

Let $P := \pi(\widehat{P}) \subseteq M$. We show that $P \in \mathcal{P}$ and that $\widehat{P}$ is a coloured point set, which implies $\widehat{P} \in \mathcal{C}_{\mathcal{P}}$.

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Continuity of the projection π yields $\lim_{n\to\infty} P_n = P$. Therefore we have $P \in \mathcal{P}$ by closedness of $\mathcal{P}$. Now assume that $\widehat{p}_1 := (p, a_1) \in \widehat{P}$ and $\widehat{p}_2 := (p, a_2) \in \widehat{P}$, where $p \in P$ and $a_1, a_2 \in \mathbb{A}$. Thus, there exist two sequences $(\widehat{p}_j^{\,n})_{j\in\mathbb{N}}$, $j = 1, 2$, such that $\widehat{p}_j^{\,n} \in P_n^{(\omega_n)}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \widehat{d}(\widehat{p}_j^{\,n}, \widehat{p}_j) = 0$ for both $j = 1, 2$. Continuity of π yields $\lim_{n\to\infty} d(p_1^n, p) = 0 = \lim_{n\to\infty} d(p_2^n, p)$. This implies $d(p_1^n, p_2^n) < r$ for finally all $n \in \mathbb{N}$. Uniform discreteness of P_n then yields $p_1^n = p_2^n =: p^n$, and we must have $a_1^n = \omega_n(p^n) = a_2^n$ for finally all $n \in \mathbb{N}$. This shows $a_1 = a_2$. ∎

Proof of Lemma 3.6 Let $Y := \left\{ xP^{(\omega)} : x \in T,\ P^{(\omega)} \in \mathcal{C}_{\mathcal{P}} \right\}$. Since $\mathcal{C}_{\mathcal{P}} \subseteq \widehat{X}_{\mathcal{P}}$ and $\widehat{X}_{\mathcal{P}}$ is T-invariant and closed, we deduce $\overline{Y} \subseteq \widehat{X}_{\mathcal{P}}$.

To prove the converse inclusion, let $P^{(\omega)} \in \widehat{X}_{\mathcal{P}}$ be arbitrary. This means $P \in X_{\mathcal{P}}$, so there exists a sequence $(P_n)_{n\in\mathbb{N}}$ in $\mathcal{P}$ converging to P. By choosing appropriate $\omega_n \in \Omega_{P_n}$, we obtain a sequence $(P_n^{(\omega_n)})_{n\in\mathbb{N}}$ in $\mathcal{C}_{\mathcal{P}} \subseteq Y$ that converges to P^{ω}. Hence, $P^{(\omega)} \in \overline{Y}$.

Continuity of the group action $\alpha_{\widehat{X}_{\mathcal{P}}}$ follows from continuity of action $\widehat{\alpha}$ on $\widehat{M}$. ∎

Proof of Lemma 3.9 We give a detailed proof for the first example only. The proof for the second example follows along the same lines, where T-stationarity ensures T-covariance, the compactly supported strong mixing coefficient ensures independence at a distance and where the continuous realizations $\xi^{(\sigma)}(\,\cdot\,)$ ensure C-compatibility and hence M-compatibility.

For the first example, T-covariance and independence at a distance are clear. It remains to verify C-compatibility, from which M-compatibility follows. First, we construct a $\|\cdot\|_\infty$-dense subset $\mathcal{D}$ of $C(\widehat{X}_{\mathcal{P}})$. For $\varphi \in C_c(M)$ and $\psi \in C_c(\mathbb{A})$ define $f_{\varphi,\psi} \colon \widehat{X}_{\mathcal{P}} \to \mathbb{R}$ by

$$f_{\varphi,\psi}(P^{(\omega)}) := \sum_{p\in P} \varphi(p)\cdot\psi\big(\omega(p)\big), \qquad P^{(\omega)} \in \widehat{X}_{\mathcal{P}}. \tag{6.1}$$

Continuity of $f_{\varphi,\psi}$ is obvious from the definition of the vague topology. We also introduce the constant function $1 \in C(\widehat{X}_{\mathcal{P}})$ equal to one and the set

$$\mathcal{D}_0 := \Big\{ f_{\varphi,\psi} : \varphi \in C_c(M) \text{ with } \operatorname{diam}(\operatorname{supp}(\varphi)) < r/2,\ \psi \in C_c(\mathbb{A}) \Big\} \cup \{1\}, \tag{6.2}$$

which separates points in $\widehat{X}_{\mathcal{P}}$. The Stone–Weierstrass theorem [Ke, Prob. 7R] then assures that the algebra $\mathcal{D} := \operatorname{alg}(\mathcal{D}_0)$ generated by $\mathcal{D}_0$ is dense in $C(\widehat{X}_{\mathcal{P}})$ with respect to the supremum norm.

Since

$$E_f(P) - E_f(P') = \int_{\Omega_P} \mathrm{d}\mathbb{P}_P(\omega) \int_{\Omega_{P'}} \mathrm{d}\mathbb{P}_{P'}(\sigma)\,[\,f(P^{(\omega)}) - f(P'^{(\sigma)})]$$

for all $P, P' \in X_{\mathcal{P}}$ and since the algebra $\mathcal{D}$ is uniformly dense in $C(\widehat{X}_{\mathcal{P}})$, it suffices to prove continuity of E_f for functions f of the form $g_k := \prod_{i=1}^k f_{\varphi_i,\psi_i}$, where $k \in \mathbb{N}$ and $f_{\varphi_i,\psi_i} \in \mathcal{D}_0$ for all $i \in \{1, \dots, k\}$. Furthermore, since $X_{\mathcal{P}}$ is metrizable, it suffices to show sequential continuity of E_{g_k}.

Fix $P \in X_{\mathcal{P}}$ and take a sequence $(P_n)_{n\in\mathbb{N}} \subseteq X_{\mathcal{P}}$ which converges to P. Define the compact set $V := \bigcup_{i=1}^{k} \operatorname{supp}(\varphi_i)$ and the (finite) pattern $Q := P \cap \mathring{V}$. Then the pattern Q and $(\mathring{V})^c$ have a positive distance $\delta_0 := d(Q, (\mathring{V})^c) > 0$. For arbitrary fixed $\delta \in]0, \min(\delta_0, r)[$, we find by Lemma 2.8(iv) an $N = N(\delta)$ such that we have for all $n \geqslant N$ the inclusions

$$P_n \cap V \subseteq (P)_\delta, \qquad P \cap V \subseteq (P_n)_\delta. \tag{6.3}$$

For $n \geqslant N$ we consider the finite patterns

$$Q_n := \Big\{ p \in P_n : \ \exists\, q \in Q \text{ with } d(p,q) < \delta \Big\} \subseteq \mathring{V}.$$

By (6.3), there exists a bijection $h_n \colon Q \to Q_n$ with $d\big(q, h_n(q)\big) < \delta$ for all $q \in Q$ and for all $n \geqslant N$. Thus, we get

$$\begin{aligned} E_{g_k}(P_n) &= \sum_{(q_1,\dots,q_k)\in Q^k} \Big(\prod_{i=1}^{k} \varphi_i\big(h_n(q_i)\big)\Big) \int_{\Omega_{P_n}} \mathrm{d}\mathbb{P}(\sigma) \prod_{i=1}^{k} \psi_i\big(\sigma\big(h_n(q_i)\big)\big) \\ &= \sum_{(q_1,\dots,q_k)\in Q^k} \Big(\prod_{i=1}^{k} \varphi_i\big(h_n(q_i)\big)\Big) \int_{\Omega_{P}} \mathrm{d}\mathbb{P}(\omega) \prod_{i=1}^{k} \psi_i\big(\omega(q_i)\big), \end{aligned} \tag{6.4}$$

where the last equality follows from the fact that all random variables are independently and identically distributed. This implies, for all $n \geqslant N$, the estimate

$$\begin{aligned} &|E_{g_k}(P) - E_{g_k}(P_n)| \\ &\quad\leqslant \sum_{(q_1,\dots,q_k)\in Q^k} \Big| \prod_{i=1}^{k} \varphi_i(q_i) - \prod_{i=1}^{k} \varphi_i\big(h_n(q_i)\big)\Big| \int_{\Omega_P} \mathrm{d}\mathbb{P}(\omega) \prod_{i=1}^{k} \big|\psi_i\big(\omega(q_i)\big)\big| \\ &\quad\leqslant \Big(\prod_{i=1}^{k} \|\psi_i\|_\infty \Big) \sum_{(q_1,\dots,q_k)\in Q^k} \Big| \prod_{i=1}^{k} \varphi_i(q_i) - \prod_{i=1}^{k} \varphi_i\big(h_n(q_i)\big)\Big|. \end{aligned}$$

Since the functions φ_i are continuous with compact support, we can make this difference as small as we want uniformly in $n \geqslant N$, by choosing δ sufficiently close to zero. ∎

Proof of Theorem 3.10 The map $Y_n \colon \Omega_P \to \mathbb{R}$ is continuous (hence measurable) for every $n \in \mathbb{N}$, as can be seen by applying Lebesgue's dominated convergence theorem.

Below we prove (3.4) for random variables Y_n corresponding to functions f in the $\|\cdot\|_\infty$-dense subalgebra $\mathcal{D} \subseteq C(\widehat{X}_{\mathcal{P}})$, which was introduced after equation (6.2). This and an $\varepsilon/3$-argument establish the lemma for all $f \in C(\widehat{X}_{\mathcal{P}})$ because, given an approximating sequence $(f_k)_{k\in\mathbb{N}} \subseteq \mathcal{D}$, we have $\big|Y_n^{(k)}(\omega) - Y_n(\omega)\big| \leqslant \|f_k - f\|_\infty$ uniformly in n and in ω. Here, $Y_n^{(k)}$ denotes the random variable (3.3) corresponding to f_k.

Thus, it suffices to prove (3.4) for random variables Y_n corresponding to functions f of the form $f = f_{\varphi_1,\psi_1} \cdots f_{\varphi_k,\psi_k}$, where $k \in \mathbb{N}$ and $f_{\varphi_j,\psi_j} \in \mathcal{D}_0$ for $j \in \{1,\dots,k\}$.

To do so we fix $P \in X_{\mathcal{P}}$ and $\omega \in \Omega_P$ arbitrary and set $U_j := \operatorname{int}(\operatorname{supp}(\varphi_j))$, which is relatively compact for $j \in \{1, \dots, k\}$. Then we can apply Proposition 5.5 (with $\widehat{M}$ playing the role of M there) and obtain

$$(6.5)\qquad \int_{D_n} \mathrm{d}x \Big(\prod_{i=1}^{k} f_{\varphi_i,\psi_i} \Big) (xP^{(\omega)}) = \sum_{\pi \in \mathcal{S}_k} \sum_{Q \in \mathcal{Q}_P^k(U;D_n)} I^{\pi}(Q)\, Z_Q^{\pi}(\omega) + o(\operatorname{vol}(D_n)),$$

asymptotically as $n \to \infty$, where the $o(\operatorname{vol}(D_n))$-term can be chosen uniformly in $P \in X_{\mathcal{P}}$ and $\omega \in \Omega_P$. In (6.5) we have used the notation of Proposition 5.5 and Definition 5.3, except that we have singled out the sum over permutations π from the integral (5.15) as well as the part involving the random variables

$$\Omega_P \ni \omega \mapsto Z_Q^{\pi}(\omega) := \prod_{i=1}^{k} \psi_i\big(\omega(q_{\pi(i)})\big),$$

which amounts to setting

$$I^{\pi}(Q) := \int_T \mathrm{d}x \prod_{i=1}^{k} \varphi_i(xq_{\pi(i)}).$$

The lemma will now follow from (6.5) and the relation

$$(6.6)\qquad \lim_{n\to\infty} \frac{1}{\operatorname{vol}(D_n)} \sum_{Q \in \mathcal{Q}_P^k(U;D_n)} I^{\pi}(Q) \left[Z_Q^{\pi}(\omega) - \int_{\Omega_P} \mathrm{d}\mathbb{P}_P(\eta)\, Z_Q^{\pi}(\eta) \right] = 0$$

for $\mathbb{P}_P$-a.a. $\omega \in \Omega_P$, every $P \in$ and every permutation $\pi \in \mathcal{S}_k$.

If the set $\mathcal{Q}_P^k(U) := \mathcal{Q}_P^k(U;T)$ is finite, then (6.6) follows from $\operatorname{vol}(D_n) \to \infty$ as $n \to \infty$. Hence we assume in the remainder that the set $\mathcal{Q}_P^k(U)$ is infinite. The relation (6.6) will then follow from the strong law of large numbers, as we now show. We note, first, that the variances $\operatorname{Var}(Z_Q^{\pi}) \leqslant \prod_{i=1}^{k} \|\psi_i\|_\infty^2$ are bounded uniformly in Q (and π). Second, the cardinality of the finite set $\mathcal{Q}_P^k(U;D_n)$ grows at most with $\operatorname{vol}(D_n)$. This can be seen from the relative compactness of $U = \bigcup_{i=1}^{k} U_i$, Lemma 5.4, and Proposition 5.1(i), which both require unimodularity. Third, we show that the coefficients $I^{\pi}(Q)$ are uniformly bounded in $Q \in \mathcal{Q}_P^k(U)$ (and $\pi \in \mathcal{S}_k$). This will follow from (5.18), which yields the estimate

$$|I^{\pi}(Q)| \leqslant \operatorname{vol}\big(S(Q)\big) \prod_{i=1}^{k} \|\varphi_i\|_\infty,$$

the inclusion (5.19), compactness of $L_U := S_{\overline{U},\overline{U}}$ by Lemma 2.3(ii), and right invariance $\operatorname{vol}(L_U y) = \operatorname{vol}(L_U) < \infty$ of the Haar measure on the unimodular group T.

Having these three properties in mind, the desired relation (6.6) follows from the strong law of large numbers and Kolmogorov's criterion [Bau, Thm. 14.5], provided we know that the family $(I^{\pi}(Q)Z_Q^{\pi})_{Q\in\mathcal{Q}_P^k(U)}$ consists of pairwise independent random variables.

If pairwise independence happens not to be the case, then we argue below that the index set $\mathcal{Q}_P^k(U)$ can be partitioned into a *finite* number J of mutually disjoint subsets

$$\mathcal{Q}_P^k(U) = \dot{\bigcup_{j=1}^{J}} F_j \tag{6.7}$$

such that for each $j \in \{1, \dots, J\}$ the subfamily $\big(I^\pi(Q) Z_Q^\pi\big)_{Q \in F_j}$ consists of pairwise independent random variables. Assuming this decomposition for the time being, we rewrite (6.6) as

$$\lim_{n\to\infty} \sum_{j=1}^{J} \frac{\operatorname{card}(F_j^{(n)})}{\operatorname{vol}(D_n)}\, \mathcal{Z}_j^{(n)}(\omega) = 0 \qquad \text{for } \mathbb{P}_P\text{-almost all } \omega \in \Omega_P, \tag{6.8}$$

where $F_j^{(n)} := F_j \cap \mathcal{Q}_P^k(U; D_n)$ and

$$\mathcal{Z}_j^{(n)}(\omega) := \frac{1}{\operatorname{card}(F_j^{(n)})} \sum_{Q \in F_j^{(n)}} I^\pi(Q) \left[Z_Q^\pi(\omega) - \int_{\Omega_P} \mathrm{d}\mathbb{P}_P(\eta)\, Z_Q^\pi(\eta) \right]$$

for $j \in \{1, \dots, J\}$. But (6.8) is indeed true. This follows from $\operatorname{vol}(D_n) \to \infty$ as $n \to \infty$ for those $j \in \{1, \dots, J\}$ such that F_j is finite, and from the $\mathbb{P}_P$-almost sure relation $\lim_{n\to\infty} \mathcal{Z}_j^{(n)} = 0$ for those $j \in \{1, \dots, J\}$ such that F_j is infinite, thanks to pairwise independence by the strong law of large numbers and Kolmogorov's criterion.

It remains to verify the existence of the partition (6.7). This may be seen by a graph-colouring argument: construct a graph $\mathcal{T}$ with infinite vertex set $\mathcal{Q}_P^k(U)$. Two vertices Q and Q' of $\mathcal{T}$ are joined by an edge if and only if Z_Q^π and $Z_{Q'}^\pi$ are $\mathbb{P}_P$-dependent. Clearly, a vertex colouring of $\mathcal{T}$ (with finitely many colours and with adjacent vertices having different colours) provides an example for the partition that we are seeking. Due to uniform discreteness of P, independence at a distance of $\mathbb{P}_P$, and because U is contained in some compact set in M, we infer that the degree of any vertex in $\mathcal{T}$ is bounded by some number $d_{\mathcal{T},\max} < \infty$. Thus, the vertex-colouring theorem [Di] ensures the existence of such a colouring with $J \leqslant 1 + d_{\mathcal{T},\max}$ different colours. ∎

Proof of Theorem 3.11 First, we prove the existence of a unique T-invariant Borel probability measure $\widehat{\mu}$ on $\widehat{X}_{\mathcal{P}}$ which obeys (i).

Thanks to M-compatibility, Assumption 3.8(iii), the integral

$$I(f) := \int_{X_{\mathcal{P}}} \mathrm{d}\mu(P)\, E_f(P)$$

is well defined and finite for every $f \in C(\widehat{X}_{\mathcal{P}})$. Moreover, the map $I\colon C(\widehat{X}_{\mathcal{P}}) \to \mathbb{R}$, $f \mapsto I(f)$ is a positive, bounded linear functional that is also normalized, $I(1) = 1$, and T-invariant because of (3.2), T-covariance of $\mathbb{P}_P$, and T-invariance of μ. By the Riesz–Markov theorem there exists a unique Borel probability measure $\widehat{\mu}$ on $\widehat{X}_{\mathcal{P}}$ such that

$$\widehat{\mu}(f) = \int_{X_{\mathcal{P}}} \mathrm{d}\mu(P) \int_{\Omega_P} \mathrm{d}\mathbb{P}_P(\omega)\; f(P^{(\omega)}) \tag{6.9}$$

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for all $f \in C(\widehat{X}_{\mathcal{P}})$. Since $\widehat{X}_{\mathcal{P}}$ is a compact metric space and $\widehat{\mu}$ is a Borel measure, the continuous functions $C(\widehat{X}_{\mathcal{P}})$ lie dense in $L^1(\widehat{X}_{\mathcal{P}}, \widehat{\mu})$ with respect to $\|\cdot\|_1$. Thus, given $f \in L^1(\widehat{X}_{\mathcal{P}}, \widehat{\mu})$ there exists a sequence $(f_k)_{k\in\mathbb{N}} \subseteq C(\widehat{X}_{\mathcal{P}})$ that converges pointwise and in $\|\cdot\|_1$-sense towards f. This and dominated convergence yield for all $f \in L^\infty(\widehat{X}_{\mathcal{P}}, \widehat{\mu})$ measurability of the map $E_f \colon X_{\mathcal{P}} \to \mathbb{R} \cup \{\pm\infty\}$ and that (6.9) holds. Finally, these conclusions hold also for $f \in L^1(\widehat{X}_{\mathcal{P}}, \widehat{\mu})$ by decomposing f into its positive and negative part and using monotone convergence for a sequence of L^∞-approximants.

In what remains we prove ergodicity of the T-invariant probability measure $\widehat{\mu}$. The additional statements about exceptional sets will be obtained along the way. Fix $f \in C(\widehat{X}_{\mathcal{P}})$ arbitrary. On the one hand, the Ergodic Theorem 2.14 for $\Omega = \widehat{X}_{\mathcal{P}}$ provides the existence of $f^\star \in L^1(\widehat{X}_{\mathcal{P}}, \widehat{\mu})$ and of a $\widehat{\mu}$-null set $\widehat{N} \subseteq \widehat{X}_{\mathcal{P}}$ such that

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(D_n)} \int_{D_n} \mathrm{d}x \, f(xP^{(\omega)}) = f^\star(P^{(\omega)}) \tag{6.10}$$

for all $P^{(\omega)} \in \widehat{X}_{\mathcal{P}} \setminus \widehat{N}$. On the other hand, we apply the Ergodic Theorem 2.14 for $\Omega = X_{\mathcal{P}}$ to the function $E_f \in L^\infty(X_{\mathcal{P}}, \mu)$ and combine it with Theorem 3.10 (which requires unimodularity of T). This yields the existence of a set $\widetilde{X} \subseteq X_{\mathcal{P}}$ of full μ-measure and, for every $P \in \widetilde{X}$, of a set $\widetilde{\Omega}_P \subseteq \Omega_P$ of full $\mathbb{P}_P$-measure such that the equality

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{\mathrm{vol}(D_n)} \int_{D_n} \mathrm{d}x \, f(xP^{(\omega)}) &= \int_{X_{\mathcal{P}}} \mathrm{d}\mu(Q) \int_{\Omega_Q} \mathrm{d}\mathbb{P}_Q(\sigma) \, f(Q^{(\sigma)}) \\ &= \widehat{\mu}(f) \end{aligned} \tag{6.11}$$

holds for all $P \in \widetilde{X}$ and for all $\omega \in \widetilde{\Omega}_P$. In the uniquely ergodic case we rely on C-compatibility $E_f \in C(X_{\mathcal{P}})$ and apply the Ergodic Theorem 2.16 instead. This gives (6.11) with $\widetilde{X} = X_{\mathcal{P}}$ (and without requiring temperedness for the Følner sequence). But

$$\begin{aligned} &\int_{\widehat{X}_{\mathcal{P}}} \mathrm{d}\widehat{\mu}(P^{(\omega)}) \, |f^\star(P^{(\omega)}) - \widehat{\mu}(f)| \\ &\qquad = \int_{\widehat{X}_{\mathcal{P}}} \mathrm{d}\widehat{\mu}(P^{(\omega)}) \, 1_{\widehat{X}_{\mathcal{P}} \setminus \widehat{N}}(P^{(\omega)}) \, |f^\star(P^{(\omega)}) - \widehat{\mu}(f)| \\ &\qquad = \int_{X_{\mathcal{P}}} \mathrm{d}\mu(P) \int_{\Omega_P} \mathrm{d}\mathbb{P}_P(\omega) \, 1_{\widehat{X}_{\mathcal{P}} \setminus \widehat{N}}(P^{(\omega)}) \, |f^\star(P^{(\omega)}) - \widehat{\mu}(f)| \\ &\qquad = 0 \end{aligned}$$

on account of (3.5), (6.10), and (6.11), showing $\widehat{\mu}$-a.e. $f^\star = \widehat{\mu}(f)$ for all $f \in C(\widehat{X}_{\mathcal{P}})$. The implication (iii) $\Rightarrow$ (i) in the Ergodic Theorem 2.14 for $\Omega = \widehat{X}_{\mathcal{P}}$ now completes the proof. ■

Proof of Proposition 3.13 Let $(D_n)_{n\in\mathbb{N}}$ be a tempered subsequence of a Følner sequence in T. By Theorem 3.11 we have for μ-a.a. $P \in X_{\mathcal{P}}$ and for $\mathbb{P}_P$-a.a. $\omega \in \Omega_P$ that

$$\widehat{\mu}(C_{\mathbf{U}}^{\mathbf{A}}) = \int_{\widehat{X}_{\mathcal{P}}} \mathrm{d}\widehat{\mu}(Q^{(\sigma)})\, 1_{C_{\mathbf{U}}^{\mathbf{A}}}(Q^{(\sigma)}) = \int_{X_{\mathcal{P}}} \mathrm{d}\mu(Q) \int_{\Omega_Q} \mathrm{d}\mathbb{P}_Q(\sigma)\, 1_{C_{\mathbf{U}}^{\mathbf{A}}}(Q^{(\sigma)})$$
$$= \lim_{n\to\infty} \frac{1}{\mathrm{vol}(D_n)} \int_{D_n} \mathrm{d}x\, 1_{C_{\mathbf{U}}^{\mathbf{A}}}(xP^{(\omega)}).$$

Since the coloured cylinder set $C_{\mathbf{U}}^{\mathbf{A}}$ contains precisely all those coloured point sets that possess exactly one point in each of the U_i (thanks to $\mathrm{diam}(U_i) < r$), and with corresponding colour value in A_i for $i \in \{1,\dots,k\}$, we can express its indicator function as

$$1_{C_{\mathbf{U}}^{\mathbf{A}}} = f_{1_{U_1},1_{A_1}} \cdots f_{1_{U_k},1_{A_k}}.$$

Thus, we conclude from (6.5), which, according to the hypotheses of Proposition 5.5, is also valid for indicator functions $\varphi_i = 1_{U_i}$, $\psi_i = 1_{A_i}$, $i \in \{1,\dots,k\}$, of open, relatively compact sets, that for every $P^{(\omega)} \in \widehat{X}_{\mathcal{P}}$ the equality

$$\int_{D_n} \mathrm{d}x\, 1_{C_{\mathbf{U}}^{\mathbf{A}}}(xP^{(\omega)}) = \sum_{Q\in\mathcal{Q}_P^k(U;D_n)} \sum_{\pi\in\mathcal{S}_k} I^{\pi}(Q)\, Z_Q^{\pi}(\omega) + o(\mathrm{vol}(D_n)),$$

holds asymptotically as $n \to \infty$. Here we used the notation introduced in (6.5). Likewise, the law of large numbers (6.6) continues to hold for $\varphi_i = 1_{U_i}$, $\psi_i = 1_{A_i}$, $i \in \{1,\dots,k\}$. This amounts to

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(D_n)} \sum_{Q\in\mathcal{Q}_P^k(U;D_n)} I^{\pi}(Q)\, \big[Z_Q^{\pi}(\omega) - \mathbb{P}(A_1)\cdots\mathbb{P}(A_k)\big] = 0$$

for $\mathbb{P}_P$-a.a. $\omega \in \Omega_P$, for every $P \in X_{\mathcal{P}}$ and every permutation $\pi \in \mathcal{S}_k$, because the expectation of Z_Q^{π} factorizes due to the product structure of $\mathbb{P}_P$ and disjointness of the U_i. Now we benefit from $\mathbb{P}_P$ being a product of identical factors and summarize the arguments so far as

$$\widehat{\mu}(C_{\mathbf{U}}^{\mathbf{A}}) = \mathbb{P}(A_1)\cdots\mathbb{P}(A_k) \lim_{n\to\infty} \frac{1}{\mathrm{vol}(D_n)} \sum_{Q\in\mathcal{Q}_P^k(U;D_n)} I(Q), \tag{6.12}$$

where $I(Q) := \sum_{\pi\in\mathcal{S}_k} I^{\pi}(Q) = \sum_{\pi\in\mathcal{S}_k} \int_T \mathrm{d}x \prod_{i=1}^k 1_{U_i}(xq_{\pi(i)})$. Equation (6.12) holds for μ-a.a. $P \in X_{\mathcal{P}}$. Since $f_{1_{U_1}} \cdots f_{1_{U_k}} = 1_{C_{\mathbf{U}}}$, Proposition 5.5 yields

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(D_n)} \sum_{Q\in\mathcal{Q}_P^k(U;D_n)} I(Q) = \lim_{n\to\infty} \frac{1}{\mathrm{vol}(D_n)} \int_{D_n} \mathrm{d}x\, 1_{C_{\mathbf{U}}}(xP) = \mu(C_{\mathbf{U}}),$$

where the last equality holds for μ-a.a. $P \in X_{\mathcal{P}}$ as a consequence of the Pointwise Ergodic Theorem 2.14 applied to μ (Cor. 2.20). The claim then follows together with (6.12). ∎

7 Proofs of Results in Section 4

Proof of Lemma 4.3 (i) First, properness of the action $\alpha_{\mathbb{V}}$ of T on $\mathbb{V}$ implies that the action $\alpha_{\mathbb{V}\times\mathbb{V}}$ of T on $\mathbb{V}\times\mathbb{V}$, defined by $\alpha_{\mathbb{V}\times\mathbb{V}}(x,(v,w)) := x(v,w) := (xv,xw)$, is also proper. This follows from [Bou1, Prop. 5(ii), Chap. III.4.2], where we chose $G = G' = T$, $\varphi = \mathrm{id}$, $X = \mathbb{V}\times\mathbb{V}$, $X' = \mathbb{V}$, and $\psi\colon \mathbb{V}\times\mathbb{V} \to \mathbb{V}$, $(v,w) \mapsto v$. Secondly, properness of the action $\alpha_{\mathbb{V}\times\mathbb{V}}$ of T on $\mathbb{V}\times\mathbb{V}$ implies that the action α of T on M is proper. This follows from [Bou1, Prop. 5(i) Chap. III.4.2], where we chose $G = G' = T$, $\varphi = \mathrm{id}$, $X = \mathbb{V}\times\mathbb{V}$, $X' = M$, and $\psi\colon \mathbb{V}\times\mathbb{V} \to$, $(v,w) \mapsto \{v,w\}$. Here, the map ψ is required to be continuous, onto and proper. While the first two properties are instantly clear, the third one follows from [Bou1, Prop. 2(c), Chap. III.4.1]: setting there $X = \mathbb{V}\times\mathbb{V}$ and $K = \{e,\pi\}$, the permutation group of 2 objects that acts on $(v,w) \in \mathbb{V}\times\mathbb{V}$ according to $e(v,w) := (v,w)$ and $\pi(v,w) := (w,v)$, we recognize ψ as the canonical map $X \to X/K$.

(ii) Let $m_{v,w} \in M$ and $x \in T$ be given such that $xm_{v,w} = m_{v,w}$. This means that $\{(xv,xw),(xw,xv)\} = \{(v,w),(w,v)\}$. If $(xv,xw) = (v,w)$, we have $xv = v$, and freeness on $\mathbb{V}$ implies $x = e$. Otherwise, we have $(xv,xw) = (w,v)$, implying $w = x(xw) = x^2 w$. Freeness on $\mathbb{V}$ yields $x^2 = e$, from which $x = e$ follows by assumption. ■

Proof of Theorem 4.10 Theorem 2.29 gives a characterization of (unique) ergodicity in terms of uniform *pattern* frequencies. In order to prove Theorem 4.10, it suffices to show that the frequency of every pattern of $X_{\mathcal{G}}$ can be expressed in terms of frequencies of certain patches from $X_{\mathcal{G}}$.

Indeed, for every pattern Q of $\mathcal{G}$ that is not a patch there exists a uniquely determined minimal patch H of $\mathcal{G}$ by "adding the missing vertices on the diagonal". Then, the pattern Q occurs if and only if the patch H occurs. ■

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