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# Schubert Calculus on a Grassmann Algebra

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*Abstract.* The (*classical*, *small quantum*, *equivariant*) cohomology ring of the grassmannian $G(k, n)$ is generated by certain derivations operating on an exterior algebra of a free module of rank $n$ (Schubert calculus on a Grassmann algebra). Our main result gives, in a unified way, a presentation of all such cohomology rings in terms of generators and relations. Using results of Laksov and Thorup, it also provides a presentation of the universal factorization algebra of a monic polynomial of degree $n$ into the product of two monic polynomials, one of degree $k$.

## 1 Introduction

In [4] the first author showed that the cohomology ring of the complex grassmannian $G(k, n)$, parametrizing $k$-dimensional subspaces of $\mathbb{C}^n$, can be realized as a commutative ring of endomorphism of the $k$-th exterior power of a free $\mathbb{Z}$-module $M$ of rank $n$. Such a result was achieved by studying a natural Hasse–Schmidt derivation on the exterior algebra of $M$; Laksov and Thorup [9, 10] generalized it to the more interesting situation regarding the cohomology of Grassmann bundles. Their point of view is quite different, as it is based on the fact that the $k$-th exterior power of a free $A$-module of rank $n$ can be endowed with a natural module structure over the ring $S$ of symmetric polynomials (with $A$-coefficients). Indeed it is a free module of rank 1 over $S$. In particular, they show that the $k$-th exterior power of a free $A$-module of rank $n$ is a free module of rank 1 over $S$. They also supply a beautiful description of the cohomology of $G(k, E)$, the Grassmann bundle of $k$-dimensional subspaces in the fibers of a vector bundle $E$, in terms of the universal factorization algebra of a certain monic polynomial $\mathrm{p}$ (encoding the Chern classes of $E$) into the product of two monic polynomials, one of degree $k$ (cf. Remark 3.6).

The main goal of this paper is to generalize [4], via a translation of Laksov and Thorup's formalism, into the language of derivations. A *derivation* on $\bigwedge M$ (the exterior algebra of a module $M$ over a commutative ring with unit) is a sequence $D := (D_0, D_1, \dots)$ of endomorphisms such that the $h$-th order Leibniz's rule,

$$D_h(\alpha \wedge \beta) = \sum_{\substack{h_1+h_2=h \\ h_i \geq 0}} D_{h_1}\alpha \wedge D_{h_2}\beta, \tag{1.1}$$

holds for each integer $h \geq 0$ and each $\alpha, \beta \in \bigwedge M$ (see 2.2). In [12], any such a derivation is called a *Schubert calculus on a Grassmann algebra*. The terminology is

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motivated by the fact that if one takes $M$ to be a finite free module over a graded commutative $\mathbb{Z}$-algebra of characteristic 0, there is a *canonical derivation* on $\bigwedge M$ (generalizing that studied in [4]; see Section 3) describing, within a unified framework, different kinds of cohomology theories on complex grassmannian varieties, such as, *e.g.*, the classical, the small quantum, or the equivariant one. Working on the exterior algebra, instead of on a single exterior power, many formal manipulations get easier. As an example we offer Theorem 4.7, the main result of this paper, which consists of a simple formula giving, in a unified way, the presentation of the classical, small quantum and equivariant cohomology ring of the complex grassmannian $G(k, n)$. In fact, the (classical, small quantum, equivariant) cohomology ring of all the grassmannians $G(k, n)$, $1 \leq k \leq n$, is a quotient of a same commutative ring of endomorphisms of the exterior algebra of a free module of rank $n$ (see Sect. 3.4). As the latter is generated by derivations, the (classical, small quantum, equivariant) Schubert calculus on $G(k, n)$ can be reduced to the much simpler one on $G(1, n) = \mathbb{P}^{n-1}$ (as in [4]; see also [5]). Our best application of such a philosophy is an elementary description, as in [6] (see also [12]), of the equivariant Schubert calculus on a grassmannian acted on by a torus with isolated fixed locus, recovering, in particular, the case studied in [7] (see also [8]).

## 2 Derivations on Exterior Algebras

***2.1*** Let $M$ be an $A$-module, $A[[t]]$ be the ring of formal power series in an indeterminate $t$ over $A$ and $\bigwedge M[[t]] := (\bigwedge M)[[t]]$ be the $A[[t]]$-module of formal power series with coefficients in $\bigwedge M = \bigoplus_{k\geq 0} \bigwedge^k M$, the exterior algebra of $M$. The former becomes an $A[[t]]$-algebra by setting

$$\sum_{i\geq 0} \alpha_i t^i \wedge \sum_{j\geq 0} \beta_j t^j = \sum_{h\geq 0} \sum_{i+j=h} (\alpha_i \wedge \beta_j) t^h.$$

***2.2*** An $A$-module homomorphism $D_t : \bigwedge M \to \bigwedge M[[t]]$ is said to be a *derivation* on $\bigwedge M$ if it is an $A$-algebra homomorphism, *i.e.*, if for each $\alpha, \beta \in \bigwedge M$:

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta. \tag{2.1}$$

The algebra homomorphism $D_t$ can be written as a formal power series $\sum_{i\geq 0} D_i t^i$, with coefficients in the $A$-algebra $\mathrm{End}_A(\bigwedge M)$. Let $D = (D_0, D_1, \dots)$ denote the sequence of the coefficients of $D_t$. Equation (2.1) implies that for each $h \geq 0$, the $A$-endomorphism $D_h$ of $\bigwedge M$ satisfies the $h$-th order Leibniz rule (1.1), obtained by expanding both sides of (2.1) and equating the coefficients of $t^h$ occurring on both sides.

***2.3*** Let $\jmath$: $\mathrm{Hom}_A(\bigwedge M, \bigwedge M[[t]]) \to \mathrm{End}_A(\bigwedge M[[t]])$ be the natural map sending any $\Psi_t = \sum_{i\geq 0} \psi_i t^i \in \mathrm{Hom}_A(\bigwedge M, \bigwedge M[[t]])$ to the endomorphism $\jmath(\Psi)$ of $\bigwedge M[[t]]$ defined on each $\sum_{i\geq 0} \alpha_i t^i \in \bigwedge M[[t]]$ by

$$\jmath(\Psi)(\sum_{i\geq 0} \alpha_i t^i) = \sum_{i\geq 0} \Psi(\alpha_i) \cdot t^i = \sum_{h\geq 0} \Big( \sum_{i+j=h} \psi_i(\alpha_j)\Big) t^h.$$

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If $D_t$ is a derivation, then $\jmath(D_t)$ is itself an $A[[t]]$-algebra endomorphism of $\bigwedge M[[t]]$. In fact it is obviously an $A[[t]]$-module endomorphism. Moreover,

$$\begin{aligned}(2.2)\quad \jmath(D_t)\Big(\sum_{i\geq 0}\alpha_i t^i \wedge \sum_{j\geq 0}\beta_j t^j\Big) &= \jmath(D_t)\sum_{h\geq 0}\Big(\sum_{i+j=h}\alpha_i\wedge\beta_j\Big)t^h \\ &= \sum_{h\geq 0}\Big(\sum_{i+j=h} D_t(\alpha_i\wedge\beta_j)\Big)t^h = \sum_{h\geq 0}\Big(\sum_{i+j=h} D_t\alpha_i\wedge D_t\beta_j\Big)t^h \\ &= \sum_{i\geq 0} D_t\alpha_i\cdot t^i \wedge \sum_{j\geq 0} D_t\beta_j\cdot t^j = \jmath(D_t)\sum_{i\geq 0}\alpha_i\cdot t^i\wedge \jmath(D_t)\sum_{j\geq 0}\beta_j\cdot t^j.\end{aligned}$$

***2.4*** For each pair $D_t, D'_t \in \mathrm{Hom}_A(\bigwedge M, \bigwedge M[[t]])$, define a product $D_t * D'_t$ through the equality: $(D_t * D'_t)\alpha = \jmath(D_t)(D'_t\alpha)$. Clearly $\jmath(D_t)\alpha = D_t\alpha$ for each $\alpha \in \bigwedge M$ and

$$\begin{aligned}(2.3)\qquad (D_t * D'_t)(\alpha) &= \sum_{h\geq 0}\big(\sum_{i+j=h} D_i(D'_j\alpha)\big)t^h = \jmath(D_t)(\sum_{j\geq 0} D'_j\alpha\cdot t^j) \\ &= \jmath(D_t)(D'_t\alpha) = (\jmath(D_t)\circ \jmath(D'_t))\alpha.\end{aligned}$$

***2.5*** The product $D_t * D'_t$ of two derivations on $\bigwedge M$ is a derivation on $\bigwedge M$. Indeed, using (2.2) and (2.3):

$$\begin{aligned}(D_t * D'_t)(\alpha\wedge\beta) &= \jmath(D_t)(D'_t(\alpha\wedge\beta)) = \jmath(D_t)(D'_t\alpha\wedge D'_t\beta) = \\ &= \jmath(D_t)(D'_t\alpha)\wedge \jmath(D_t)(D'_t\beta) = (D_t * D'_t)\alpha\wedge(D_t * D'_t)\beta,\end{aligned}$$

as desired. Now let $D^{(1)} = (D_i^{(1)})_{i\geq 0}$ be any (possibly finite) sequence of endomorphisms of $M$ and, for each $m \in M$, let $D_t^{(1)}(m) = \sum_{i\geq 0} D_i^{(1)}(m)t^i$. Then $D_t^{(1)}\colon M \to M[[t]]$ is an $A$-module homomorphism.

***Proposition 2.1*** *There exists a unique derivation $D_t\colon \bigwedge M \to \bigwedge M[[t]]$ such that $D_{t|_M} = D_t^{(1)}$ (or, equivalently, $D_{i|_M} = D_i^{(1)}$).*

**Proof** For each $k \geq 1$, consider the $A$-multilinear map $M^{\otimes k} \to (\bigwedge^k M)[[t]]$ defined by $m_{i_1}\otimes\cdots\otimes m_{i_k} \mapsto D_t^{(1)}m_{i_1}\wedge\cdots\wedge D_t^{(1)}m_{i_k}$, which is clearly alternating. By the universal property of exterior powers, it factors through a unique $A$-module homomorphism $\bigwedge^k M \to (\bigwedge^k M)[[t]]$, given by $D_t^{(k)}(m_{i_1}\wedge\cdots\wedge m_{i_k}) = D_t^{(1)}m_{i_1}\wedge\cdots\wedge D_t^{(1)}m_{i_k}$ on monomials. Let $D_t\alpha = D_t^{(k)}\alpha$ for all $\alpha \in \bigwedge^k M$ and all $k \geq 0$. It follows that if $\alpha \in \bigwedge^{k_1} M$ and $\beta \in \bigwedge^{k_2} M$, equation (2.1) holds by the definition of $D_t$ and the fact that $\alpha\wedge\beta$ is a finite $A$-linear combination of elements of the form

$$\{m_{i_1}\wedge\cdots\wedge m_{i_{k_1}}\wedge m_{i_{k_1+1}}\wedge\cdots\wedge m_{i_{k_1+k_2}};\quad 1\leq i_1<\cdots<i_{k_1+k_2}\}.$$

Since any element of $\bigwedge M$ is a finite sum of homogeneous ones, equation (2.1) holds for any arbitrary pair as well. The unicity part is straightforward: were $D'_t$ another extension of $D_t^{(1)}$, one would have $D'_t(m_{i_1}\wedge\cdots\wedge m_{i_k}) = D_t^{(1)}m_{i_1}\wedge\cdots\wedge D_t^{(1)}m_{i_k} = D_t(m_{i_1}\wedge\cdots\wedge m_{i_k})$, for each $m_{i_1}\wedge\cdots\wedge m_{i_k}$ and each $k\geq 1$. Hence $D'_t = D_t$. ■

***2.6*** Let $\mathcal{S}_t(\bigwedge M)$ be the set of all derivations $D_t := \sum_{i\geq 0} D_i t^i$ such that $D_{i|_M} \in \mathrm{End}_A(M)$ (*i.e.,* the submodule $M$ of $\bigwedge M$ is $D_i$-stable) and $D_{0|_M}$ is an isomorphism. Hence $D_0 \colon \bigwedge M \to \bigwedge M$ is an isomorphism too.

***Proposition 2.2*** *The pair* $(\mathcal{S}_t(\bigwedge M), *)$ *is a group.*

**Proof** By 2.5, $\mathcal{S}_t(\bigwedge M)$ is closed under $*$. By its very definition, $*$ is associative. The map $\mathbf{1}\colon \bigwedge M \to (\bigwedge M)[[t]]$, sending any $\alpha \in \bigwedge M$ to itself, thought of as a constant formal power series, is the $*$-neutral element. Thinking of $D_t$ as a formal power series with coefficients in $\mathrm{End}_A(\bigwedge M)$, the formal inverse $D_t^{-1}$ of $D_t$ (existing because of the invertibility of $D_0$) is a derivation as well. In fact

$$\begin{aligned} D_t^{-1}(\alpha \wedge \beta) &= \jmath(D_t^{-1})\big((D_t * D_t^{-1})\alpha \wedge (D_t * D_t^{-1})\beta\big)\\ &= \jmath(D_t^{-1})(\jmath(D_t)D_t^{-1}\alpha \wedge \jmath(D_t)D_t^{-1}\beta)\\ &= (\jmath(D_t^{-1}) \circ \jmath(D_t))(D_t^{-1}\alpha \wedge D_t^{-1}\beta) = D_t^{-1}\alpha \wedge D_t^{-1}\beta, \end{aligned}$$

since $D_t^{-1} * D_t = D_t * D_t^{-1} = \mathbf{1}$. ■

***2.7*** We fix another piece of notation. Let $A[\mathbf{T}]$ be the polynomial ring in infinitely many indeterminates $\mathbf{T} = (T_1, T_2, \dots)$. For each $k$-tuple $I := (i_1, \dots, i_k)$ of positive integers, we denote by $\Delta_I(\mathbf{T}) := \Delta_{(i_1,\dots,i_k)}(\mathbf{T})$ the *Schur polynomial* $\det[(T_{i_j-i})_{1\leq i,j\leq k}] \in A[\mathbf{T}]$ (setting $T_0 = 1$ and $T_j = 0$, if $j < 0$). By expanding $\Delta_I(\mathbf{T})$ along the last column, one sees that $\Delta_I(\mathbf{T})$ belongs to the ideal $(T_{i_k-1}, \dots, T_{i_k-k})$ of $A[\mathbf{T}]$. In particular $\Delta_{(2,3,\dots,h+1)}(\mathbf{T}) \in (T_1, \dots, T_h)$. If $D := (D_0, D_1, \dots, )$ is the sequence of coefficients of some $D_t \in \mathcal{S}_t(\bigwedge M)$ such that $D_0 = id_{\bigwedge M}$, one defines $\Delta_I(D)$ to be the evaluation of $\Delta_I(\mathbf{T})$ at $D$ (via the substitution $T_i \mapsto D_i$).

***2.8*** For each $i \geq 0$, define $\overline{D}_i \in \mathrm{End}_A(\bigwedge M)$ via the equality

$$D_t^{-1} = \sum_{i\geq 0}(-1)^i \overline{D}_i t^i.$$

By equating the coefficients of the same power of $t$ on both sides of the equation $D_t * D_t^{-1} = 1$, one gets $\overline{D}_0 = D_0^{-1}$, while, for each $h \geq 1$,

$$\overline{D}_h - \overline{D}_{h-1}D_1 + \cdots + (-1)^h D_h = 0,$$

so that, *e.g.,* $\overline{D}_1 = D_1, \overline{D}_2 = D_1^2 - D_2$. In general, one has (see [2, Appendix A])

$$\overline{D}_h = \Delta_{(2,3,\dots,h+1)}(D).$$

***Proposition 2.3*** **(Integration by parts)** *Let* $D_t \in \mathcal{S}_t(\bigwedge M)$. *Then*

$$\begin{aligned} D_h\alpha \wedge \beta &= \sum_{i\geq 0}(-1)^i D_{h-i}(\alpha \wedge \overline{D}_i\beta) \qquad (2.4)\\ &= D_h\alpha \wedge \beta - D_{h-1}\alpha \wedge \overline{D}_1\beta + \cdots + (-1)^i D_0\alpha \wedge \overline{D}_h\beta. \end{aligned}$$

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**Proof** One expands both sides of the equality $\jmath(D_t)(\alpha\wedge D_t^{-1}\beta) = D_t\alpha\wedge\beta$, and then compares the coefficients of $t^h$ occurring on each side. ■

***Example 2.4*** One has $D_1\alpha\wedge D_0\beta = D_1(\alpha\wedge\beta) - D_0\alpha\wedge\overline{D}_1\beta$ and

$$D_2\alpha\wedge D_0\beta = D_2(\alpha\wedge\beta) - D_1(\alpha\wedge\overline{D}_1\beta) + D_0\alpha\wedge\overline{D}_2\beta.$$

## 3 Schubert Calculus on a Grassmann Algebra

***3.1*** From now on, $A$ will be assumed to be any graded ring $\bigoplus_{i\geq 0} A_i$ such that $A_0 = \mathbb{Z}$. Let $X$ be an indeterminate over $A$, $M := XA[X]$ and $M(\mathrm{p}) := M/\mathrm{p}M$, where $\mathrm{p}$ is either the 0 polynomial or a monic polynomial $X^n - e_1X^{n-1} + \cdots + (-1)^n e_n \in A[X]$ such that $e_i \in A_i$. Then $M(\mathrm{p})$ is a free $A$-module generated by $\epsilon = (\epsilon^i)_{1\leq i\leq n}$, where $n$ is either $\deg(\mathrm{p})$ if $\mathrm{p}\neq 0$, or $\infty$ if $\mathrm{p} = 0$.

***3.2*** Let $\mathcal{I}^k = \{I = (i_1,\dots,i_k)\in\mathbb{N}^k \mid 1\leq i_1 < \cdots < i_k\}$ (as in [3, §, Section 1] and [4]). The *weight* of $I\in\mathcal{I}^k$ is $\mathrm{wt}(I) = \sum_{j=1}^k (i_j - j)$. It coincides with the weight of the associated partition $(i_k - k, i_{k-1}-(k-1),\dots,i_1 - 1)$. If $I := (i_1,\dots,i_k)\in\mathcal{I}^k$, let $\wedge^I\epsilon$ denote $\epsilon^{i_1}\wedge\cdots\wedge\epsilon^{i_k}$. Each exterior power $\bigwedge^k M(\mathrm{p})$ is a free $A$-module with basis $\bigwedge^k\epsilon := \{\wedge^I\epsilon : I\in\mathcal{I}^k_n\}$. If $a\in A_h$, the *weight* of $a\cdot\wedge^I\epsilon$ is, by definition, $h+\mathrm{wt}(I)$. Set $(\bigwedge^k M(\mathrm{p}))_w = \bigoplus_{0\leq h\leq w}\big(\bigoplus_{\mathrm{wt}(I)=h} A_{w-h}\cdot\wedge^I\epsilon\big)$. Then $\bigwedge^k M(\mathrm{p}) = \bigoplus_{w\geq 0}(\bigwedge^k M(\mathrm{p}))_w$, which is a graded $A$-module via *weight*.

***3.3*** By Proposition 2.1 there is a unique sequence $D := (D_0, D_1,\dots)$ of $A$-endomorphisms of $\bigwedge M(\mathrm{p})$ such that (i) (the $h$-th order) Leibniz's rule (1.1) holds for each $h\geq 0$ and each $\alpha,\beta\in\bigwedge^k M(\mathrm{p})$ and (ii) the initial conditions $D_h\epsilon^i = \epsilon^{i+h}$ are satisfied, for each $h\geq 0$ and each $i\geq 1$. Notice that $D_i\circ D_j = D_j\circ D_i$ in $\mathrm{End}_A(\bigwedge M(\mathrm{p}))$, as a simple induction shows.

***Proposition 3.1*** *The following formula holds:*

$$D_h(\epsilon^{i_1}\wedge\cdots\wedge\epsilon^{i_k}) = \sum \epsilon^{i_1+h_1}\wedge\cdots\wedge\epsilon^{i_k+h_k}, \tag{3.1}$$

*summing over all $h$-tuples $(h_i)_{1\leq i\leq k}$ of non negative integers such that $h_1+\cdots+h_k = h$.*

**Proof** See [9] or, since equality (3.1) is defined over the integers, use the same inductive proof as in [4, Proposition 2.6]. ■

***Example 3.2*** When expanding $D_h(\epsilon^{i_1}\wedge\cdots\wedge\epsilon^{i_k})$, cancellations may occur on the right hand side of (3.1), due to the $\mathbb{Z}_2$-symmetry of the $\wedge$-product. For instance:

$$D_2(\epsilon^1\wedge\epsilon^2) = \epsilon^3\wedge\epsilon^2 + \epsilon^2\wedge\epsilon^3 + \epsilon^1\wedge\epsilon^4 = \epsilon^1\wedge\epsilon^4.$$

The surviving summands are predicted by *Pieri's formula* for $D_h$, a rule to speed up computations of "derivatives" of $k$-vectors.

***Theorem 3.3*** **(Pieri's formula)** *Pieri's formula holds:*

$$D_h(\epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}) = \sum_{(h_i)\in P(I,h)} \epsilon^{i_1+h_1} \wedge \cdots \wedge \epsilon^{i_k+h_k}, \tag{3.2}$$

*where, if* $I = (i_1, \ldots, i_k) \in \mathcal{I}^k$*, we denote by* $\mathcal{P}(I, h)$ *the set of all k-tuples of non negative integers* $(h_1, \ldots, h_k)$ *such that* $i_1 + h_1 < i_2 \leq i_2 + h_2 < \cdots < i_{k-1} \leq i_k$ *and* $h_1 + \cdots + h_k = h$.

**Proof** See [9] or, since formula (3.2) is defined over the integers, use the same proof as in [4, Theorem 2.4]. ■

***3.4*** Let $A$ be as in 3.1 and $A[\mathbf{T}]$ be as in 2.8. If $a \in A_l$, the *degree* of the monomial $aT_{i_1}^{m_1} \cdots T_{i_j}^{m_j}$ is defined to be $l + m_1 i_1 + \cdots + m_j i_j$. Then $A[\mathbf{T}]$ is itself a graded ring $\bigoplus_{h\geq 0} A[\mathbf{T}]_h$, where $A[\mathbf{T}]_h$ is the submodule of all elements of $A[\mathbf{T}]$ of degree $h$. There is a natural evaluation map, $\mathrm{ev}_D\colon A[\mathbf{T}] \to \mathrm{End}_A(\bigwedge M(\mathrm{p}))$, sending $P \in A[\mathbf{T}]$ to $P(D)$ (obtained by "substituting" $T_i \mapsto D_i$ into $P$). We denote by $\mathcal{A}^*(\bigwedge M(\mathrm{p}))$ the image of $\mathrm{ev}_D$ in $\mathrm{End}_A(\bigwedge M(\mathrm{p}))$ and by $\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$ the image of the natural restriction map

$$\rho_k\colon \mathcal{A}^*\big(\bigwedge M(\mathrm{p})\big) \to \mathrm{End}_A\big(\bigwedge^k M(\mathrm{p})\big),$$

given by $P(D) \mapsto P(D)_{|_{\bigwedge^k M(\mathrm{p})}}$. Pieri's formula implies *Giambelli's formula*, a special case of the general determinantal formula stated in [9, Main Theorem], which reads, in this case, as:

$$\epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k} = \Delta_{(i_1,\ldots,i_k)}(D) \cdot \epsilon^1 \wedge \cdots \wedge \epsilon^k, \tag{3.3}$$

where, as in 2.8, $\Delta_{(i_1,\ldots,i_k)}(D) = \mathrm{ev}_D(\Delta_{(i_1,\ldots,i_k)}(\mathbf{T}))$.

Hence, we have shown the following.

***Theorem 3.4*** *The natural evaluation map*

$$\mathrm{ev}_{\epsilon^1\wedge\cdots\wedge\epsilon^k}\colon \mathcal{A}^*(\bigwedge M(\mathrm{p})) \to \bigwedge^k M(\mathrm{p}),\ \textit{mapping}\ P(D) \mapsto P(D)\epsilon^1 \wedge \cdots \wedge \epsilon^k$$

*is surjective.*

***3.5*** It follows that $\ker(\rho_k) = \ker(\mathrm{ev}_{\epsilon^1\wedge\cdots\wedge\epsilon^k})$, and then

$$\mathcal{A}^*(\bigwedge^k M(\mathrm{p})) = \frac{\mathcal{A}^*(\bigwedge M(\mathrm{p}))}{\ker(\mathrm{ev}_{\epsilon^1\wedge\cdots\wedge\epsilon^k})}.$$

We call the induced map $\Pi_k : \mathcal{A}^*(\bigwedge^k M(\mathrm{p})) \to \bigwedge^k M(\mathrm{p})$, defined by

$$P(D) + \ker \mathrm{ev}_{\epsilon^1\wedge\cdots\wedge\epsilon^k} \mapsto P(D)\epsilon^1 \wedge \cdots \wedge \epsilon^k,$$

the *Poincaré isomorphism*.

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*Remark* 3.5. Let $\mathcal{I}^k_n = \{I \in \mathcal{I}^k \mid i_k \leq n\}$. A routine check shows that if $I = (i_1, \ldots, i_k) \in \mathcal{I}^k_n$ and $H \in \mathcal{P}(I, h)$ then $I + H := (i_1 + h_1, \ldots, i_k + h_k) \in \mathcal{I}^k$. Denote by $\mathcal{I}^{k,w}$ the set of all $I \in \mathcal{I}^k$ such that $\operatorname{wt}(I) = w$. Combining Pieri's formula (3.2) with Giambelli's formula (3.3), one has, for each $I \in \mathcal{I}^k$ and each $h \geq 0$:

$$D_h \Delta_I(D)\epsilon^1 \wedge \cdots \wedge \epsilon^k = D_h \cdot \wedge^I \epsilon = \sum_{H \in \mathcal{P}(I,h)} \wedge^{I+H} \epsilon = \sum_{H \in \mathcal{P}(I,h)} \Delta_{I+H}(D)\epsilon^1 \wedge \cdots \wedge \epsilon^k,$$

proving the equality $D_h \Delta_I(D) = \sum_{H \in \mathcal{P}(I,h)} \Delta_{I+H}(D)$ in the ring $\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$.

*Remark* 3.6. (see [10]) Let $\mathrm{Fact}^k_A(\mathrm{p})$ be the universal factorization algebra of the monic polynomial $\mathrm{p}$ into the product of two monic polynomials, one of degree $k$. Let $\mathrm{p} = \mathrm{p}_1 \mathrm{q}$ be the universal factorization of $\mathrm{p}$ in $\mathrm{Fact}^k_A(\mathrm{p})$, where $\deg(\mathrm{p}_1) = k$, and denote by $s_i$ the complete symmetric polynomial of degree $i$ in the universal roots of $\mathrm{p}_1$. Then $\mathrm{Fact}^k_A(\mathrm{p})$ is generated, as an $A$-algebra, by $(s_i)_{i \geq 1}$ and the map $\mathcal{A}^*(\bigwedge^k M(\mathrm{p})) \to \mathrm{Fact}^k_A(\mathrm{p})$, defined by $D_i \mapsto s_i$, is an $A$-algebra isomorphism. This is because of the module structure of $\bigwedge^k A[X]$ over the ring of symmetric functions defined and studied in [9]. In fact our formula (3.2) is the same as Pieri's formula (2.1.1) in [9], after replacing $s_i$ with $D_i$. Let $\pi\colon E \to \mathcal{Y}$ be a vector bundle of rank $n$ and let $\pi_k\colon G(k, E) \to \mathcal{Y}$ be the Grassmann bundle over $\mathcal{Y}$ of $k$-planes in the fibers of $E$. In [10] the authors show that, if $A := A^*(\mathcal{Y})$ is the Chow ring of $\mathcal{Y}$ and $\mathrm{p} = X^n + c_1 X^{n-1} + \cdots + c_n \in A[X]$ is such that $c_i := c_i(E)$ are the Chern classes of $E$, there is an isomorphism $\mathrm{Fact}^k_A(\mathrm{p}) \to A^*(G(k, E))$. Let $\mathcal{Q}_k$ be the universal quotient bundle over $G(k, E)$. Then, the same proof as in [10] works using derivations. By the basis theorem ([2, p. 268]) the unique $A$-module homomorphism $\iota_k\colon A^*(G(k, E)) \to \mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$, mapping $\Delta_I(c(\mathcal{Q}_k - p_k^* E)$ to $\Delta_I(D)$, is certainly an isomorphism. To check that it is also a ring homomorphism, it is sufficient to check it on products of the form $c_h(\mathcal{Q}_k - p_k^* E) \cdot \Delta_I(c(\mathcal{Q}_k - p_k^* E)$:

$$\begin{aligned}
\iota_k(c_h(\mathcal{Q}_k - p_k^* E) \cdot \Delta_I(c(\mathcal{Q}_k - p_k^* E)) &= \iota_k\Big(\sum_{H \in \mathcal{P}(I,h)} \Delta_{I+H}(c(\mathcal{Q}_k - p_k^* E))\Big) \\
&= \sum_{H \in \mathcal{P}(I,h)} \Delta_{I+H}(D) = D_h \Delta_I(D) \\
&= \iota_k(c_h(\mathcal{Q}_k - p_k^* E) \cdot \iota_k(\Delta_I((c(\mathcal{Q}_k - p_k^* E)),
\end{aligned}$$

by [2, Proposition 14.6.1] and 3.5.

*Remark* 3.7. Theorem 3.4 can be proved by showing that for each $I \in \mathcal{I}^k$, there exists $G_I \in A[\mathbf{T}]$ such that $\wedge^I \epsilon = G_I(D) \cdot \epsilon^1 \wedge \cdots \wedge \epsilon^k$. This can be achieved via *integration by parts* (2.4), as follows. We say that $\bigwedge^k M(\mathrm{p})$ enjoys the property $\mathbf{G}_j$, for some $1 \leq j \leq k$, if, for each $i_{j+1} < \cdots < i_k$ such that $j < i_{j+1}$, there exists a polynomial $G_{j,i_{j+1},\ldots,i_k} \in A[\mathbf{T}]$ such that $\epsilon^1 \wedge \cdots \wedge \epsilon^j \wedge \epsilon^{i_{j+1}} \wedge \cdots \wedge \epsilon^{i_k} = G_{j,i_{j+1},\ldots,i_k}(D) \cdot \epsilon^1 \wedge \cdots \wedge \epsilon^k$. We shall show, by descending induction, that $\bigwedge^k M(\mathrm{p})$ enjoys $\mathbf{G}_j$ for each $1 \leq j \leq k$. In fact $\mathbf{G}_k$ is trivially true.

Let us suppose that $\mathbf{G}_j$ holds for some $2 \le j \le k-1$. Then $\mathbf{G}_{j-1}$ holds. In fact, for each $j-1 < i_j < \cdots < i_k$,

$$\epsilon^1 \wedge \cdots \wedge \epsilon^{j-1} \wedge \epsilon^{i_j} \wedge \cdots \wedge \epsilon^{i_k} = D_{i_j-j}(\epsilon^1 \wedge \cdots \wedge \epsilon^{j-1} \wedge \epsilon^j) \wedge \epsilon^{i_{j+1}} \wedge \cdots \wedge \epsilon^{i_k},$$

basically by [4, Corollary 2.5]. By applying integration by parts (2.4), one gets:

$$\epsilon^1 \wedge \cdots \wedge \epsilon^{j-1} \wedge \epsilon^{i_j} \wedge \cdots \wedge \epsilon^{i_k} = \sum_{h=0}^{i_j-j} D_{i_j-j-h}(\epsilon^1 \wedge \cdots \wedge \epsilon^j \wedge \overline{D}_h(\epsilon^{i_{j+1}} \wedge \cdots \wedge \epsilon^{i_k})).$$

But $\overline{D}_h(\epsilon^{i_{j+1}} \wedge \cdots \wedge \epsilon^{i_k})$ is a sum of elements of the form $\epsilon^{h_{j+1}} \wedge \cdots \wedge \epsilon^{h_k}$, with $j < h_{j+1} < \cdots < h_j$. Then, by the inductive hypothesis, one concludes that $\mathbf{G}_{j-1}$ holds, too. In particular $\mathbf{G}_1$ holds and the claim is proved.

## 4 Presentations for Intersection Rings

***Proposition 4.1*** *Let $D_t^{-1} := \sum_{j\ge 0}(-1)^j \overline{D}_j t^j$ be the inverse of $D_t \in \mathcal{S}_t(\bigwedge M(\mathrm{p}))$. Then $\overline{D}_{h|_{\bigwedge^k M(\mathrm{p})}} = 0$, for each $h > k$.*

**Proof** By induction on $k$. If $k = 0$ one has $\overline{D}_h(m) = 0$, for each $h \ge 2$ and each $m \in M(\mathrm{p})$. In fact, if $m \in M(\mathrm{p})$, $D_t m = \sum_{i\ge 0} D_1^i m \cdot t^i$. Therefore $D_t^{-1} m = m - D_1 m \cdot t$, *i.e.*, $\overline{D}_{h|_{M(\mathrm{p})}} = 0$ for each $h \ge 2$. Suppose now the property is true for $k-1$ and let $h > k$. Any $m_k \in \bigwedge^k M(\mathrm{p})$ is a finite $A$-linear combination of elements of the form $m \wedge m_{k-1}$ for suitable $m \in M(\mathrm{p})$ and $m_{k-1} \in \bigwedge^{k-1} M(\mathrm{p})$. It then suffices to check the property for elements of this form. One has $\overline{D}_h(m \wedge m_{k-1}) = \sum_{j=0}^{h} \overline{D}_j m \wedge \overline{D}_{h-j} m_{k-1}$. As $\overline{D}_j m = 0$, for $j \ge 2$, it follows that $\sum_{j=0}^{h} \overline{D}_j m \wedge \overline{D}_{h-j} m_{k-1} = \overline{D}_1 m \wedge \overline{D}_{h-1} m_{k-1}$. By the inductive hypothesis, this last term vanishes as well, because $h-1 > k-1$. ∎

In the sequel $M$ will be as in 3.1 (*i.e.*, $M(\mathrm{p})$ for $\mathrm{p} = 0$).

***Proposition 4.2*** *The ring $\mathcal{A}^*(\bigwedge^k M)$ is generated by $(D_1, D_2, \ldots, D_k)$ as an $A$-algebra.*

**Proof** Let $D_t^{-1} = \sum_{i\ge 0}(-1)^i \overline{D}_i t^i$ be the inverse of $D_t$. First one observes that, for each $h \ge 1$, $\overline{D}_h = \mathrm{ev}_D(\Delta_{(2,3,\ldots,h+1)}(\mathbf{T})) = \Delta_{(2,3,\ldots,h+1)}(D)$ and that $\Delta_{(2,3,\ldots,h+1)}(\mathbf{T}) \in A[\mathbf{T}]$ in fact lands in the subring $A[T_1, \ldots, T_h]$ of $A[\mathbf{T}]$, by Remark 2.7. One knows that $\overline{D}_{k+j} = 0$ in $\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$ for each $j \ge 1$ (Proposition 4.1). Working modulo $\ker(\rho_k)$ (see 3.4) we may hence write

$$\sum_{i\ge 0} D_i t^i = \frac{1}{1 - \overline{D}_1 t + \overline{D}_2 t^2 + \cdots + (-1)^k \overline{D}_k t^k}.$$

Define $\widetilde{D}_j(\mathbf{T}_k) \in A[T_1, \ldots, T_k] \subseteq A[\mathbf{T}]$ as

$$\sum_{j\ge 0} \widetilde{D}_j(\mathbf{T}_k) t^i = \frac{1}{1 - \Delta_{(2)}(\mathbf{T})t + \Delta_{(23)}(\mathbf{T})t^2 + \cdots + (-1)^k \Delta_{(23\cdots k+1)}(\mathbf{T})t^k}. \tag{4.1}$$

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One clearly has that $D_j - \widetilde{D}_j(\mathbf{D}_k) \in \ker(\rho_k)$ for each $j \geq 0$. Moreover, if $1 \leq j \leq k$, $\widetilde{D}_j(\mathbf{T}_k) = T_j$, proving the claim. ■

***Example 4.3*** In $\mathcal{A}^*(\bigwedge^2 M)$ one has, using the recipe (4.1):

$$\widetilde{D}_3(\mathbf{T}_2) := \widetilde{D}_3(T_1, T_2) = T_2T_1 - T_1(T_1^2 - T_2) = -T_1^3 + 2T_1T_2 \tag{4.2}$$

and

$$\begin{aligned} \widetilde{D}_4(\mathbf{T}_2) &= \widetilde{D}_3(\mathbf{T}_2)T_1 - T_2(T_1^2 - T_2) \\ &= (-T_1^3 + 2T_1T_2)T_1 - T_2(T_1^2 - T_2) = -T_1^4 + T_1^2T_2 + T_2^2. \end{aligned} \tag{4.3}$$

***Proposition 4.4*** *Let $P \in A[T_1, \dots, T_k]_w \subset A[\mathbf{T}]_w$ such that $P(D)\epsilon^1 \wedge \cdots \wedge \epsilon^k = 0$ ($w \geq 0$). Then $P = 0$.*

**Proof** Any polynomial $P \in A[T_1, \dots, T_k]$ of degree $w$ is a unique $A$-linear combination of $\Delta_I(\mathbf{T})$, with $I \in \mathcal{I}^{k,w}$ (since the Schur polynomials $\{\Delta_I(\mathbf{T}) \mid I \in \mathcal{I}^k\}$ are a $\mathbb{Z}$-basis of $\mathbb{Z}[\mathbf{T}]$). Hence $P = \sum_{I \in \mathcal{I}^{k,w}} a_I \Delta_I(\mathbf{T})$ for some (unique!) $a_I \in A_{w-\mathrm{wt}(I)}$ and if $P(D)\epsilon^1 \wedge \cdots \wedge \epsilon^k = 0$, then

$$0 = P(D) \cdot \epsilon^1 \wedge \cdots \wedge \epsilon^k = \sum_{I \in \mathcal{I}^{k,w}} a_I \Delta_I(D) \cdot \epsilon^1 \wedge \cdots \wedge \epsilon^k = \sum_{I \in \mathcal{I}^{k,w}} a_I \cdot \wedge^I \epsilon.$$

Since $\{\bigwedge^I \epsilon\}_{I \in I^{k,w}}$ are $A$-linearly independent, $a_I = 0$ for all $I \in \mathcal{I}^{k,w}$, *i.e.*, $P = 0$. ■

***Corollary 4.5*** *The map* $\mathrm{ev}_D \colon A[\mathbf{T}] \longrightarrow \mathcal{A}^*(\bigwedge M)$ *in subsection* 3.4 *is an isomorphism. Hence:*

$$\mathcal{A}^*(\textstyle\bigwedge M) = A[D] := A[D_1, D_2, \dots] \cong A[\mathbf{T}], \tag{4.4}$$

*the polynomial ring in infinitely many indeterminates, while*

$$\mathcal{A}^*(\textstyle\bigwedge^k M) = A[\mathbf{D}_k] := A[D_1, D_2, \dots, D_k]. \tag{4.5}$$

**Proof** Apply Proposition 4.4. One may assume that $P \in A[\mathbf{T}]$ is homogeneous of degree $w \geq 0$. Suppose that $\mathrm{ev}_D(P) = P(D) = 0 \in \mathcal{A}^*(\bigwedge M)$. There is $k \geq 1$ such that $P \in A[T_1, T_2, \dots, T_k]$. But then $P(D) \cdot \epsilon^1 \wedge \cdots \wedge \epsilon^k = 0$ implies $P = 0$, because otherwise one would have a relation (of degree $w$), whence (4.4). Since $\mathcal{A}^*(\bigwedge^k M) = \rho_k\bigl(\mathcal{A}^*(\bigwedge M)\bigr)$ and, by Proposition 4.1, $\rho_k(\overline{D}_h) = 0$ for all $h \geq k+1$, one gets the presentation (4.5). ■

***4.1*** For each $i \geq 1$, let $\nu^{qn+i} = (\mathrm{p}(X))^q X^i$. Then $\nu = (\nu^1, \nu^2, \dots)$ is an $A$-basis of $M := M(0)$, such that $\nu^i = X^i$ for each $1 \leq i \leq n$. Let $\bigwedge M \wedge \mathrm{p}M := \bigoplus_{k \geq 1} \bigwedge^{k-1} M \wedge \mathrm{p}M$ be the bilateral ideal of $\bigwedge M$ generated by p. As p$M$ is the submodule of $M$ generated by $\nu^i$ with $i > n$, the submodule $\bigwedge^{k-1} M \wedge \mathrm{p}M$ is the $A$-submodule of $\bigwedge^k M$ generated by $\nu^{i_1} \wedge \cdots \wedge \nu^{i_k}$, with $i_k > n$. The natural map $\bigwedge M \to \bigwedge M(\mathrm{p})$ (resp.

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$\bigwedge^k M \to \bigwedge^k M(\mathtt{p}))$ is surjective and has kernel $\bigwedge M \wedge \mathtt{p}M$ (resp. $\bigwedge^{k-1} M \wedge \mathtt{p}M$). Hence, one has canonical isomorphisms

$$\bigwedge M(\mathtt{p}) = \frac{\bigwedge M}{\bigwedge M \wedge \mathtt{p}M} \qquad \textit{and} \qquad \bigwedge^k M(\mathtt{p}) = \frac{\bigwedge^k M}{\bigwedge^{k-1} M \wedge \mathtt{p}M}.$$

Let $\phi_k \colon \bigwedge^k M \to \bigwedge^k M(\mathtt{p})$ be the canonical projection and let

$$J_k(\mathtt{p}) := \{P(D) \in A[D_1, \dots, D_k] \mid P(D)\epsilon^1 \wedge \dots \wedge \epsilon^k \in \bigwedge^{k-1} M \wedge \mathtt{p}M\},$$

which is an ideal of $A[D_1, \dots, D_k] = \mathcal{A}^*(\bigwedge^k M)$.

***Theorem 4.6*** *For each $j \geq 1$, let*

$$\widetilde{D}_{n-k+j}(\mathbf{D}_k, \mathtt{p}) = \widetilde{D}_{n-k+j}(\mathbf{D}_k) + \sum_{i=1}^{n-k+j} c_i \widetilde{D}_{n-k+j-i}(\mathbf{D}_k).$$

*Then:*

$$J_k(\mathtt{p}) = (\widetilde{D}_{n-k+1}(\mathbf{D}_k, \mathtt{p}), \dots, \widetilde{D}_n(\mathbf{D}_k, \mathtt{p})). \tag{4.6}$$

**Proof** Let $D'_t = \sum_{i \geq 0} D'_i t^i$ be the unique derivation on $\bigwedge M$ such that $D'_t \nu^j = \sum_{i \geq 0} \nu^{i+j} t^i$. Then

(i) $D'_i \in \mathcal{A}^*(\bigwedge M(\mathtt{p}))$ for each $i \geq 0$,

(ii) $\rho_k(D'_i) = D_i$ if $1 \leq i \leq n-k$ and

(iii)

$$\rho_k(D'_{n-k+j}) = \widetilde{D}_{n-k+j}(\mathbf{D}_k, \mathtt{p}), \qquad \forall j \geq 1. \tag{4.7}$$

To check (i), is sufficient to show that each $D'_i$ is an $A$-polynomial expression in the $D_i$s. As a matter of fact, if $i \leq n-k$,

$$\begin{aligned} D'_i(X^1 \wedge \dots \wedge X^k) &= D'_i(\nu^1 \wedge \dots \wedge \nu^k) = \nu^1 \wedge \dots \wedge \nu^{k-1} \wedge \nu^{k+j} \\ &= X^1 \wedge \dots \wedge X^{k-1} \wedge X^{k+i} = D_i(X^1 \wedge \dots \wedge X^k), \end{aligned} \tag{4.8}$$

and, for each $j \geq 1$:

$$\begin{aligned} D'_{n-k+j}(X^1 \wedge \dots \wedge X^k) &= D'_{n-k+j}\nu^1 \wedge \dots \wedge \nu^k = \nu^1 \wedge \dots \wedge \nu^{k-1} \wedge \nu^{n+j} \\ &= X^1 \wedge \dots \wedge X^{k-1} \wedge (X^{n+j} + c_1 X^{n+j-1} + \dots + c_{n+j-k}X^k + \dots + c_n X^j) \\ &= X^1 \wedge \dots \wedge X^{k-1} \wedge X^{n+j} + \sum_{i=1}^{n-k+j} c_i (X^1 \wedge \dots \wedge X^{k-1} \wedge X^{n+j-i}) \\ &= (\widetilde{D}_{n-k+j}(\mathbf{D}_k) + \sum_{i=1}^{n-k+j} c_i \widetilde{D}_{n-k+j-i}(\mathbf{D}_k)) X^1 \wedge \dots \wedge X^k. \end{aligned} \tag{4.9}$$

Therefore formulas (4.8) and (4.9) show (i), (4.8) shows (ii) and (4.9) shows (iii). We can now prove equality (4.6). Clearly $(\widetilde{D}_{n-k+j}(\mathbf{D}_k, \mathtt{p}))_{j\geq 1} \subseteq J_k(\mathtt{p})$, because

$$\widetilde{D}_{n-k+j}(\mathbf{D}_k, \mathtt{p})\nu^1 \wedge \cdots \wedge \nu^k = D'_{n-k+j}\nu^1 \wedge \cdots \wedge \nu^k$$
$$= \nu^1 \wedge \cdots \wedge \nu^{k-1} \wedge \nu^{n+j} \in \bigwedge^{k-1} M \wedge \mathtt{p}M.$$

To show that $J_k(\mathtt{p}) \subseteq (\widetilde{D}_{n-k+j}(\mathbf{D}_k, \mathtt{p}))_{j\geq 1}$ as well, let $P \in A[T_1, \dots, T_k] \subseteq A[\mathbf{T}]$ such that

$$P(D')X^1 \wedge \cdots \wedge X^k \in \bigwedge^{k-1} M \wedge \mathtt{p}M.$$

Without loss of generality one may assume that $P$ is homogeneous of degree $w$. Then

$$P(D')\nu^1 \wedge \cdots \wedge \nu^k = \sum a_I \Delta_I(D')\nu^1 \wedge \cdots \wedge \nu^k = \sum \nu^{i_1} \wedge \cdots \wedge \nu^{i_k},$$

where the last sum is over all $(i_1, \dots, i_k) \in \mathcal{I}^{k,w}$ such that $i_k > n$. By subsection 2.7, $\Delta_I(D')$ belongs to the ideal $(D'_{i_k-1}, \dots, D'_{i_k-k})$ and, since $i_k > n$, one sees that if $\Delta_I(D')\nu^1 \wedge \cdots \wedge \nu^k \in \bigwedge^{k-1} M \wedge \mathtt{p}M$, then $\Delta_I(D') \in (\widetilde{D}_{n-k+j}(\mathbf{D}'_k))_{j\geq 1}$. The relation

$$D'_{n+1} - D'_n\overline{D}'_1 + \cdots + (-1)^{n-k+1}D'_{n-k+1}\overline{D}'_k = 0,$$

holding in $\mathcal{A}^*(\bigwedge^k M(\mathtt{p}))$, implies that $D'_{n+1}\big(\bigwedge^k M(\mathtt{p})\big) \subseteq (D'_{n-k+1}, \dots, D'_n)\bigwedge^k M(\mathtt{p})$ (here , as in 2.8, $(-1)^i\overline{D}'_i$ stands for the $i$-th coefficient of $(D'_t)^{-1}$).

By induction

$$D'_{n+j}\big(\bigwedge^k M(\mathtt{p})\big) \subseteq (D'_{n-k+1}, \dots, D'_n)\bigwedge^k M(\mathtt{p})$$

as well. Hence, because of (4.7), one has $J_k(\mathtt{p}) \subseteq (\rho_k(D'_{n-k+1}), \dots, \rho_k(D'_n)) = (\widetilde{D}_{n-k+1}(\mathbf{D}_k, \mathtt{p}), \dots, \widetilde{D}_n(\mathbf{D}_k, \mathtt{p}))$, *i.e.*, $J_k(\mathtt{p})$ is given precisely by (4.6). ∎

***Theorem 4.7*** *The following isomorphism holds:*

$$A^*(\bigwedge^k M(\mathtt{p})) = \frac{A[D_1, \dots, D_k]}{(\widetilde{D}_{n-k+1}(\mathbf{D}_k, \mathtt{p}), \dots, \widetilde{D}_n(\mathbf{D}_k, \mathtt{p}))}.$$

**Proof** Notation is as in 4.1. Recall that by Corollary 4.5 and formula (4.5), $D_1, \dots, D_k$, are algebraically independent elements of $\mathcal{A}^*(\bigwedge^k M)$. Clearly, $P(D) \in \ker(\phi_k)$ if and only if $P(D)\epsilon^1 \wedge \cdots \wedge \epsilon^k \in \bigwedge^{k-1} M \wedge \mathtt{p}M$, *i.e.*, if and only if $P(D) \in J_k(\mathtt{p})$. Hence $\mathcal{A}^*(\bigwedge^k M(\mathtt{p})) = \mathcal{A}^*(\bigwedge^k M)/J_k(\mathtt{p})$, and the conclusion follows by Corollary 4.5 and Theorem 4.6. ∎

*Remark* 4.8. By [10], the polynomial $\mathtt{p}$, as above, factors in the ring $\mathcal{A}^*(\bigwedge^k M(\mathtt{p}))[X]$ as the product $(X^k + D_1X^{k-1} + \cdots + D_k) \cdot \mathtt{q}$, where $\mathtt{q}$ is a monic polynomial of degree $n-k$ (with $\mathcal{A}^*(\bigwedge^k M(\mathtt{p}))[X]$-coefficients).

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***Example 4.9*** Let $M = XA[X]$ and $\mathtt{p}(X) = X^4 + c_1X^3 + c_2X^2 + c_3X + c_4$, $c_i \in A_i$. Then one has

$$\mathcal{A}^*(\bigwedge^2 M(\mathtt{p})) = \frac{A[D_1, D_2]}{(\widetilde{D}_3(\mathbf{D}_2, \mathtt{p}), \widetilde{D}_4(\mathbf{D}_2, \mathtt{p}))}, \tag{4.10}$$

where

$$\begin{aligned}\widetilde{D}_3(\mathbf{D}_2, \mathtt{p}) &= \widetilde{D}_3(\mathbf{D}_2) + c_1\widetilde{D}_2(\mathbf{D}_2) + c_2\widetilde{D}_1(\mathbf{D}_2) + c_3\\ &= 2D_1D_2 - D_1^3 + c_1D_2 + c_2D_1 + c_3\end{aligned}$$

and

$$\begin{aligned}\widetilde{D}_4(\mathbf{D}_2, \mathtt{p}) &= \widetilde{D}_4(\mathbf{D}_2) + c_1\widetilde{D}_3(\mathbf{D}_2) + c_2\widetilde{D}_2(\mathbf{D}_2) + c_3\widetilde{D}_1(\mathbf{D}_2) + c_4\\ &= D_2^2 + D_1^2D_2 - D_1^4 + c_1(2D_1D_2 - D_1^3) + c_2D_2 + c_3D_1 + c_4,\end{aligned}$$

which we obtained from (4.2) and (4.3). Let us enumerate some particular cases.

(1) If $A = \mathbb{Z}$, thought of as a graded ring concentrated in degree 0, then $c_i = 0$, $1 \le i \le 4$. Then $\mathtt{p} = X^4$ and presentation (4.10) becomes:

$$\mathcal{A}^*(\bigwedge^2 M(X^4)) = \frac{A[D_1, D_2]}{(2D_1D_2 - D_1^3, D_2^2 + D_1^2D_2 - D_1^4)},$$

which coincides (Cf. [4, 5]) with the presentation of the integral cohomology ring of the grassmannian $G(2, 4)$ of 2-planes in $\mathbb{C}^4$ (or of the grassmannian $G(1, \mathbb{P}^3)$ of lines in the complex projective 3-space).

(2) If $A = \mathbb{Z}[q]$, and $\mathtt{p}(X) = X^4 + q$, then (4.10) reads:

$$\mathcal{A}^*(\bigwedge^2 M(X^4 + q)) = \frac{\mathbb{Z}[q][D_1, D_2]}{(2D_1D_2 - D_1^3, D_2^2 + D_1^2D_2 - D_1^4 + q)},$$

which is the Witten–Siebert–Tian presentation of the *small quantum cohomology ring* $QH^*(G(2, 4))$ ([1, 13, 14]; see also [4]);

(3) If $\pi\colon E \to \mathcal{Y}$ is a holomorphic vector bundle of rank 4 on a smooth complex variety of dimension $m \ge 0$, and $\mathtt{p}(X) = X^4 + \pi^*c_1\,X^3 + \pi^*c_2\,X^2 + \pi^*c_3\,X + \pi^*c_4 \in A^*(\mathcal{Y})[X]$, where $c_i$ are the Chern classes of $E$ as in [2], p. 141, then $M(\mathtt{p}) = A_*(\mathbb{P}(E))$, $A = A^*(\mathcal{Y})$ and $D_1 = c_1(O_{\mathbb{P}(E)}(-1))$, thought of as operator on $A_*(\mathcal{Y})$; in this case (4.10) gives the presentation of $A^*(G(2, E))$ (Cf. 3.6). If $\mathcal{Y}$ is a point, then $A^*(\mathcal{Y}) = \mathbb{Z}$, $c_i = 0$ and one recovers once again the presentation of the Chow ring of the grassmannian $G(2, 4)$.

(4) Let $A = \mathbb{Z}[y_1, y_2, y_3, y_4]$ and $\mathtt{p}(X) = \prod_{i=1}^4(X - y_i + y_1) + q \in A[q]$. In this case presentation (4.10) is that the *quantum equivariant cohomology ring* $QH_T^*(G(2, 4))$ of the Grassmannian $G(2, 4)$ under the action of a 4-dimensional compact or algebraic torus via a diagonal action with only isolated fixed points, as studied by Mihalcea in [11, Theorem 4.2], setting $p = 2$ and $m = 4$. This is compatible with the main result of the paper [6, Theorem 3.7], with [4, Theorem 2.9] and is now a consequence of [8]. Notice that our generators are not the same as used in [11] (Cf. [6]).

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