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ONE-SIDED INVERSES IN RINGS

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Introduction. Following Herstein [**2**], we will call a ring R with identity *von Neumann finite* (vNf) provided that $xy = 1$ implies $yx = 1$ in R. Kaplansky [**4**] showed that group algebras over fields of characteristic zero are vNf rings, and further, that full matrix rings over such rings are also vNf. Herstein [**2**] has posed the problem for group algebras over fields of arbitrary characteristic. If group algebras over fields are always vNf, then it is easily seen that group algebras over commutative rings are always vNf. What conditions on the underlying ring of scalars would force the vNf property for all group rings over it?

The vNf condition is rather weak. We consider here stronger versions of the property, defined for an arbitrary ring R with identity:

(1) R is *fully vNf*: Every full matrix ring over R is vNf.

(2) R is *strongly vNf*: Given a finitely-generated unital left R-module, its ring of R-endomorphisms is vNf.

Sections 1, 2, 3 of this paper deal with general properties of vNf, strongly vNf, and fully vNf rings. In section 4, we consider fully vNf group rings. Let $\mathscr{C}$ be the collection of all groups G such that the group ring RG is fully vNf whenever the ring R is. Belonging to $\mathscr{C}$ is a local property and also a residual property. In particular, $\mathscr{C}$ contains every abelian group and every locally finite group.

Conjecture. $\mathscr{C}$ contains every solvable group.

Proposition 4.6 reduces this question to the case where the group is a semidirect product of a solvable group in $\mathscr{C}$ by the infinite cyclic group.

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Notation. Throughout this paper, R will denote a ring with identity. The nilpotent radical of R, denoted by $\mathscr{N}(R)$, is the ideal generated by the nilpotent ideals of R, and $J(R)$ is the Jacobson radical of R. $Z(R)$ is the center of R. Given R, we can form the opposite ring R^{op} and the ring R_n of all $n \times n$ matrices over R.

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All modules will be unital. Given a left R-module M, let $\mathrm{End}_R(M)$ be its ring of R-endomorphisms. All functions will be written on the left. Thus $R_n = \mathrm{End}_R(F)^{\mathrm{op}}$, where F is the free left R-module of rank n. If ϕ is a homomorphism, then $\mathrm{Im}(\phi)$ and $\mathrm{Ker}(\phi)$ are its image and kernel, respectively.

The notation for groups and group rings follows Passman [**6**]. Given a group G, we denote the group ring for G over R by RG. Given α in RG, we write $\alpha = \sum_{g \in G} \alpha_g g$, where $\alpha_g (g \in G)$ is a sequence in R with all but finitely many of its terms zero. The support of α is defined by $\mathrm{supp}(\alpha) = \{g \in G | \alpha_g \neq 0\}$. If H is a subgroup of G, $N_G(H)$ is its normalizer and $C_G(H)$ is its centralizer (in G). The center of G is $Z(G)$.

1. vNF Rings. Most basic properties of vNf rings follow from the following module characterization:

PROPOSITION 1.1. *Suppose M is a unital left R-module. Then* $\mathrm{End}_R(M)$ *is not a* vNf *ring* $\Leftrightarrow$ *M has R-submodules M', N such that $M = M' \oplus N$ with $M' \cong M$ and $N \neq 0$.*

Proof. ($\Rightarrow$) Choose $\alpha, \beta \in \mathrm{End}_R(M)$ with $\alpha\beta = 1 \neq \beta\alpha$. Let $M' = \mathrm{Im}(\beta)$ and let $N = \mathrm{Ker}(\alpha)$. Since β is monic and α is not monic, we have $M' \cong M$ and $N \neq 0$. Since $\mathrm{Im}(1 - \beta\alpha) \subseteq \mathrm{Ker}(\alpha)$, $M = M' + N$. Clearly, $M' \cap N = 0$.

($\Leftarrow$) Let $\phi : M' \to M$ be an R-isomorphism (onto). Define $\alpha : M \to M$ so that $\alpha(x) = \phi(x)$ for x in M' and $\alpha(x) = 0$ for x in N. Let $\beta : M \to M$, $x \mapsto \phi^{-1}(x)$. Then $\alpha, \beta \in \mathrm{End}_R(M)$, $\alpha\beta = 1$, and $\beta\alpha \neq 1$. We are done.

Let M be a left R-module such that $\mathrm{End}_R(M)$ is not vNf. Choose R-submodules M_1, N_1 of M such that $M = M_1 \oplus N_1$ with $M_1 \cong M$ and $N_1 \neq 0$. Since M_1 and M are isomorphic, we may write $M_1 = M_2 \oplus N_2$ with $M_2 \cong M$ and $N_2 \cong N_1$. Proceeding inductively, we can find sequences M_k and N_k (k a positive integer) of R-submodules of M such that $M_k = M_{k+1} \oplus N_{k+1}$, $M_k \cong M$ (as R-modules), and $N_k \cong N_1$. Let $W_k = N_1 + \ldots + N_k$. Then the sequence $M_k (k = 1, \ldots)$ is a proper descending chain of R-submodules of M, the sequence $W_k (k = 1, \ldots)$ is a proper ascending chain of R-submodules of M, and we have $M = M_k \oplus N_k$ for each k. This proves

COROLLARY 1.2. *If M satisfies the ascending chain condition or descending chain condition on its direct summands over R, then* $\mathrm{End}_R(M)$ *is* vNf.

COROLLARY 1.3. *If R is left Noetherina, then R is strongly* vNf.

2. Strongly vNf rings. We begin with a characterization of endomorphism rings of finitely-generated modules. For each left ideal L of R, let $\mathscr{I}_R(L) = \{x \in R | Lx \subseteq L\}$, the *idealizer* of L in R. The following facts are easily verified:

(1) $\mathscr{I}_R(L)$ is a subring of R containing $Z(R)$.

(2) L is an ideal of $\mathscr{I}_R(L)$.

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(3) If T is an ideal of R, then $\mathscr{I}_R(L) \subseteq \mathscr{I}_R(L+T)$ and $\mathscr{I}_{R/T}(L+T/T) = \mathscr{I}_R(L+T)/T$.

Definition. Suppose A is a ring with identity. Then A is a *left section* of R provided that $A \cong \mathscr{I}_R(L)/L$ for some left ideal L of R.

PROPOSITION 2.1. *If M is a finitely-generated left R-module, then* $\mathrm{End}_R(M)^{\mathrm{op}}$ *is a left section of R_n for some positive integer n.*

Proof. Choose an ordered generating set $S = \{m_1, \ldots, m_n\}$ for M over R. For each $n \times n$ matrix (a_{ij}) over R and $\alpha \in \mathrm{End}_R(M)$, write

$$(a_{ij}) \text{ represents } \alpha \pmod S \Leftrightarrow \alpha(m_i) = \sum_{j=1}^{n} a_{ij}m_j,\ i = 1, \ldots, n.$$

Let $L = \{x \in R_n | x \text{ represents } 0 \pmod S\}$, and let

$$B = \{x \in R_n | x \text{ represents } \alpha \pmod S \text{ for some } \alpha \in \mathrm{End}_R(M)\}.$$

It is easily verified that L is a left ideal of R_n and $B = \mathscr{I}_{R_n}(L)$. Let $\phi : B \to \mathrm{End}_R(M)$ be the map which sends each element of B to the element it represents. Then ϕ is an anti-epimorphism of rings and $\mathrm{Ker}(\phi) = L$. We are done.

Note that if L is a left ideal of R, then $\mathrm{End}_R(R/L)^{\mathrm{op}} \cong \mathscr{I}_R(L)/L$. Since the vNf property is opposite-invariant, we get

COROLLARY 2.2. *The ring R is strongly* vNf $\Leftrightarrow$ *the left sections of R_n are* vNf *rings for every positive integer n.*

PROPOSITION 2.3. *Suppose R is integral over $Z(R)$. Then R is a* vNf *ring and every left section of R is a* vNf *ring.*

Proof. Suppose $a, b \in R$ and $ab = 1$. Write

$$a^{n+1} = \sum_{i=0}^{n} z_i a^i \text{ with } z_0, \ldots, z_n \in Z(R).$$

Then

$$a = a^{n+1}b^n = \sum_{i=0}^{n} z_i b^{n-i},$$

so that $ba = ab = 1$. The same argument clearly applies to the left sections of R.

The Cayley-Hamilton theorem gives

COROLLARY 2.4. *If R is commutative, then R is strongly* vNf.

The Jacobson radical is superfluous with respect to the vNf and fully vNf properties. For the strongly vNf property, we at least get the superfluousness of $\mathscr{N}(R)$.

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PROPOSITION 2.5. *Suppose I is an ideal of R and $I \subseteq \mathcal{N}(R)$. Then R is strongly* vNf $\Leftrightarrow$ *R/I is strongly* vNf.

Proof. ($\Rightarrow$) Every factor ring of R is strongly vNf.

($\Leftarrow$) Suppose L is a left ideal of R. Let $A = R/I$ and $T = \mathscr{I}_R(L) \cap I$. Then $\mathscr{I}_R(L)/(L + T) \subseteq \mathscr{I}_R(L + I)/(L + I) \cong \mathscr{I}_A(L')/L'$, where $L' = L + I/I$. Thus, $\mathscr{I}_R(L)/(L + T)$ is a vNf ring. But $T \subseteq J(\mathscr{I}_R(L))$, so that $\mathscr{I}_R(L)/L$ is also vNf. So every left section of R is vNf. Since $R_n/I_n \cong (R/I)_n$ is also strongly vNf and since $I_n \cong \mathcal{N}(R_n)$, the same argument shows that the left sections of R_n are vNf.

COROLLARY 2.6. *If X is a finitely-generated abelian group, then* $\operatorname{End}(X)$ *is a strongly* vNf *ring.*

Proof. Let $R = \operatorname{End}(X)$. Write $X = T \times F$, where T is the torsion part of X and F is torsion free. Then T is an R-invariant subgroup of X. Let

$$\phi : R \to \operatorname{End}(T) + \operatorname{End}(F), \quad \alpha \mapsto \alpha' \oplus (p \circ \alpha \circ j),$$

where α' is the restriction of α to T, $p : X \to T$ is the canonical projection, and $j : F \to X$ is the canonical insertion. Then ϕ is a ring epimorphism and $\operatorname{Ker}(\phi)^2 = 0$. Since $R/\operatorname{Ker}(\phi)$ is left Noetherian and $\operatorname{Ker}(\phi) \subseteq \mathcal{N}(R)$, it follows that R is strongly vNf.

PROPOSITION 2.7. *If R is strongly* vNf *and M is a finitely-generated left R-module, then* $\operatorname{End}_R(M)$ *is fully* vNf.

Proof. The fully vNf property is opposite-invariant. Hence, we need only show that the left sections of R are fully vNf. But if A is a left section of R, then it is easy to show that A_n is a left section of R_n, and R_n is a strongly vNf ring. Hence, every left section of R is fully vNf.

3. Fully vNf rings. Most properties of vNf rings carry over to fully vNf rings. In addition, we have weak consequences of the strongly vNf property:

PROPOSITION 3.1. (i) *Suppose A is a subring of R and $1 \in A$. If R is free and finitely-generated as a left A-module, then R is fully* vNf.$\Leftrightarrow$ *A is fully* vNf.

(ii) *If R is fully* vNf *and P is a finitely-generated projective left R-module, then* $\operatorname{End}_R(P)$ *is a* vNf *ring.*

Proof. (i) ($\Rightarrow$) Every subring of R is fully vNf. ($\Leftarrow$) R is a subring of $\operatorname{End}_A(R)^{\mathrm{op}} = A_n$ for some n.

(ii) Let F be a free finitely-generated left R-module such that P is a direct summand of F over R. Since $\operatorname{End}_R(F)$ is a vNf ring, F cannot be essentially a proper direct summand of itself over R, by 1.1. Hence, P is also not essentially a proper direct summand of itself over R, and so $\operatorname{End}_R(P)$ must be vNf.

Shepherdson [**7**] showed that a full matrix ring over a vNf ring need not be a vNf ring. We construct an example of a ring which is fully vNf but not strongly vNf.

PROPOSITION 3.2. *Assume R is not the trivial ring, and let G be the free group on three generators. Then some factor ring of RG is not a* vNf *ring. In particular, RG is not strongly* vNf.

Proof. In order to avoid totally superfluous complications, assume R is commutative. Let M be the free left R-module with basis $\{m_i | i = 1, \ldots\}$, and let $A = \mathrm{End}_R(M)$. As in [**3**], let α, β be those elements of A such that $\alpha(m_i) = 0$ if $i = 1$, $\alpha(m_i) = m_{i-1}$ if $i > 1$, and $\beta(m_i) = m_{i+1}$ $(i = 1, \ldots)$. We have $\alpha\beta = 1 \neq \beta\alpha$. Let γ, γ', γ'' be those elements of A such that

$$\gamma(m_i) = \begin{cases} m_1, & \text{if } i = 1 \\ m_i + m_{i-1}, & \text{if } i > 1 \end{cases} \quad \text{and} \quad \gamma'(m_i) = \begin{cases} m_i + m_{i-1}, & \text{if } i \text{ is even} \\ m_{i+1}, & \text{if } i \text{ is odd} \end{cases}$$

and $\gamma'' = \beta + \gamma - \gamma'$. Then $\gamma, \gamma', \gamma'' \in U$, where U is the group of units of A. We have $\alpha = \gamma - 1$ and $\beta = -\gamma + \gamma' + \gamma''$. Let H be the subgroup of U generated by γ, γ', γ''. We have the obvious "evaluation" map $\phi : RH \to A$. Since α, β are in $\mathrm{Im}(\phi)$, $RH/\mathrm{Ker}(\phi)$ is not a vNf ring.

COROLLARY 3.3. *Let F be a field of characteristic zero, and let G be the free group on three generators. Then FG is fully* vNf *but not strongly* vNf.

4. Fully vNf group rings. Given a fully vNf ring R, for what groups G is RG fully vNf? Let $\mathscr{C}$ be the collection of all such groups. We note first that belonging to $\mathscr{C}$ is a local property.

PROPOSITION 4.1. (i) *RG is fully* vNf $\Leftrightarrow$ *RH is a fully* vNf *for every finitely-generated subgroup H of G.*

(ii) *If H is a subgroup of G of finite index, then RG is fully* vNf $\Leftrightarrow$ *RH is fully* vNf.

(iii) *If A is an ascending chain of subgroups whose union is G, then RG is fully* vNf $\Leftrightarrow$ *RH is fully* vNf *for all $H \in A$.*

(iv) *If $G = G_1 x \ldots x G_n$, then $G \in \mathscr{C} \Leftrightarrow G_i \in \mathscr{C}$, $i = 1, \ldots, n$.*

Proof. (i) ($\Rightarrow$) RH is a subring of RG. ($\Leftarrow$) Suppose $a, b \in RG$ and $ab = 1$. Let H be the subgroup of G generated by $\mathrm{supp}(a) \cup \mathrm{supp}(b)$. Then $a, b \in RH$, forcing $ba = 1$. Since $(RG)_n \cong R_nG$, the same argument shows that RG is fully vNf.

(ii) This is immediate from 3.1.

(iii) A finitely-generated subgroup of G belongs to some element of A.

(iv) If $G = H \times K$, then $RG \cong AK$, where $A = RH$. The result follows by vacuous induction.

COROLLARY 4.2. *If G is a locally finite group, then RG is fully* vNf $\Leftrightarrow$ *R is fully* vNf.

THEOREM 4.3. *Belonging to $\mathscr{C}$ is a residual property, i.e. if $\mathscr{S}$ is a collection of normal subgroups of G such that $G/N \in \mathscr{C}$ for all $N \in \mathscr{S}$ and $\bigcap \mathscr{S} = 1$, then $G \in \mathscr{C}$.*

Proof. Suppose X is a finite subset of G, $X = \{g_1, \ldots, g_m\}$. For $1 \leqq i < j \leqq m$, there exists $N_{ij} \in \mathscr{S}$ such that $g_i g_j^{-1} \notin N_{ij}$. Let

$$N_X = \bigcap_{i<j} N_{ij}.$$

By 4.1 (iv), we have $G/N_X \in \mathscr{C}$, and also the elements of X lie in distinct cosets of G (mod N_X).

Thus, for each nonempty finite subset X of G, there is a normal subgroup N_X of G such that $G/N_X \in \mathscr{C}$ and the elements of X lie in distinct cosets of G (mod N_X). Consider the direct product ring

$$\prod_X RG/N_X,$$

where X ranges over the nonempty finite subsets of G. Since each component is fully vNf (assuming that R is), it follows that the direct product is fully vNf. But the obvious map $RG \rightarrow \prod_X RG/N_X$ is a monomorphism. Thus RG is fully vNf whenever R is.

COROLLARY 4.4. *If G is abelian, then $G \in \mathscr{C}$.*

Definition. For each group G, let $\Delta(G)$ be the *FC-subgroup* of G (the collection of all elements of G possessing only finitely many conjugates in G).

COROLLARY 4.5. *Suppose $\Delta(G)$ has finite index in G. Then $G \in \mathscr{C}$.*

Proof. Assume R is fully vNf. If H is a finitely-generated subgroup of $\Delta(G)$, then $Z(H)$ has finite index in H, so that H is in $\mathscr{C}$, by 4.1 and 4.4. The result follows.

PROPOSITION 4.6. *Suppose R is fully* vNf *and H is a subgroup of G such that RH is fully* vNf. *Then G has a subgroup K such that*

(i) *$H \subseteq K$ and RK is fully* vHf; *and*

(ii) *if L is a subgroup of G and $K \subset L \subseteq G$, then RL is not fully* vNf.

Further, if K is a subgroup of G satisfying (i) *and* (ii), *then $C_G(K) \subseteq K$ and $N_G(K)/K$ is torsion free.*

Proof. The set $\{L | H \subseteq L$, L is a subgroup of G, and RL is fully vNf$\}$ is inductively ordered. Thus G has a subgroup K satisfying (i) and (ii). Let K be any such group. Suppose $g \in N_G(K)$ and $g \notin K$. Let $L = K\langle g \rangle$. Since RL is not fully vNf, we must have $K \cap \langle g \rangle = 1$ with $\langle g \rangle$ infinite. If $g \in C_G(K)$, then $L = Kx\langle g \rangle$ and $RL = A\langle g \rangle$, where $A = RK$. But then RL is fully vNf, a contradiction. Hence, $N_G(K)/K$ is torsion free and $C_G(K) \subseteq K$.

Remark. If K is a subgroup of G satisfying conditions (i) and (ii) in the above proposition, then it is easy to show that $\Delta(G) \subseteq K$. In this sense, the *FC*-subgroup is superfluous with respect to the fully vNf property.

COROLLARY 4.7. *Let G' be the commutator subgroup of G. If G/G' is a torsion group, then RG is fully* vNf $\Leftrightarrow$ *RG' is fully vNf.*

Added in proof. Professor Joe Fischer has kindly pointed out that much of section 4 is contained in a paper by G. Losey, *Are one-sided inverses two-sided inverses in matrix rings over group rings*, Can. Math. Bull. *13* (1970), 475–479.

References

1. R. Baer, *Inverses and zero divisors*, Bull. Amer. Math. Soc. *48* (1942), 630–638.

2. I. N. Herstein, *Notes from a ring theory conference*, Regional Conference Series in Mathematics No. *9* (Amer. Math. Soc., Providence, 1971).

3. N. Jacobson, *Some remarks on one-sided inverses*, Proc. Amer. Math. Soc. *1* (1950), 352–355.

4. I. Kaplansky, *Fields and rings*, Chicago Lectures in Mathematics (Univ. Chicago Press, Chicago, 1969).

5. M. S. Montgomery, *Left and right inverses in group algebras*, Bull. Amer. Math. Soc. *75* (1969), 539–540.

6. D. S. Passman, *Infinite group rings* (Marcel Dekker, New York, 1971).

7. J. C. Shepherdson, *Inverses and zero divisors in matrix rings*, Proc. London Math. Soc. *1* (1951), 71–85.

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