MATHEMATICAL NOTES

A NOTE ON DUAL EQUATIONS WITH TRIGONOMETRICAL KERNELS

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1. Sneddon (**1**) has recently pointed out that the dual integral equations

$$\int_0^\infty \xi^{-1}\psi(\xi)\cos(x\xi)d\xi = f(x) \quad (0 \leqq x \leqq 1), \qquad \text{.................(1)}$$

$$\int_0^\infty \psi(\xi)\cos(x\xi)d\xi = 0 \qquad (x > 1), \qquad \text{.......................(2)}$$

are not covered by Busbridge's solution (**2**) of the more general equations with Bessel function kernels and he has given a solution to equations (1) and (2) in the form of a Neumann series. The purpose of this note is to show that a very simple formal solution can be obtained by using the well-known Bessel function integral representations ((**3**), pp. 48 and 170)

$$\frac{\pi}{2}J_0(r\xi) = \int_0^r \frac{\cos(\xi x)dx}{(r^2-x^2)^{\frac{1}{2}}} = \int_r^\infty \frac{\sin(\xi x)dx}{(x^2-r^2)^{\frac{1}{2}}}. \qquad \text{.................(3)}$$

The similar problem in which $\sin(x\xi)$ replaces $\cos(x\xi)$ in equations (1) and (2) is also solved in the same way.

2. We start by integrating equation (2) with respect to x to give

$$\int_0^\infty \xi^{-1}\psi(\xi)\sin(x\xi)d\xi = C\ (\text{constant}) \quad (x > 1). \qquad \text{..............(4)}$$

As $x \to \infty$, the value of this integral tends to $\frac{1}{2}\pi\psi(+0)$ and, since $\psi(+0) = 0$ if the integral in (1) is to exist, it follows that $C = 0$ and equation (4) becomes

$$\int_0^\infty \xi^{-1}\psi(\xi)\sin(x\xi)d\xi = 0 \quad (x > 1). \qquad \text{.......................(5)}$$

Multiplication of equations (1) and (5) respectively by $(r^2-x^2)^{-\frac{1}{2}}$ and $(x^2-r^2)^{-\frac{1}{2}}$, integration with respect to x between 0, r and r, ∞, and use of the integral representations (3) leads to

$$\frac{\pi}{2}\int_0^\infty \xi^{-1}\psi(\xi)J_0(r\xi)d\xi = \begin{cases} \displaystyle\int_0^r \frac{f(x)dx}{(r^2-x^2)^{\frac{1}{2}}} & (0 < r < 1), \\ 0 & (r > 1). \end{cases} \qquad \text{..............(6)}$$

Application of the Hankel inversion theorem then gives

$$\frac{\pi}{2}\xi^{-2}\psi(\xi)=\int_0^1 rJ_0(\xi r)dr\int_0^r \frac{f(x)dx}{(r^2-x^2)^{\frac{1}{2}}}. \quad \text{.....................(7)}$$

3. In the solution of the dual equations

$$\int_0^\infty \xi^{-1}\psi(\xi)\sin(x\xi)d\xi=f(x)\quad (0\leqq x\leqq 1), \quad \text{.....................(8)}$$

$$\int_0^\infty \psi(\xi)\sin(x\xi)d\xi=0\qquad (x>1), \quad \text{..........................(9)}$$

we first differentiate equation (8) to give

$$\int_0^\infty \psi(\xi)\cos(x\xi)d\xi=f'(x)\qquad (0\leqq x\leqq 1). \quad \text{..................(10)}$$

Using the same procedure on equations (10) and (9) as was used with equations (1) and (5) we find

$$\frac{\pi}{2}\int_0^\infty \psi(\xi)J_0(r\xi)d\xi=\begin{cases}\displaystyle\int_0^r \frac{f'(x)dx}{(r^2-x^2)^{\frac{1}{2}}} & (0<r<1),\\ 0 & (r>1),\end{cases} \quad \text{...............(11)}$$

so that, in this case,

$$\frac{\pi}{2}\xi^{-1}\psi(\xi)=\int_0^1 rJ_0(\xi r)dr\int_0^r \frac{f'(x)dx}{(r^2-x^2)^{\frac{1}{2}}}. \quad \text{.....................(12)}$$

REFERENCES

(1) I. N. Sneddon, *Proc. Glasgow Math. Assoc.*, **5** (1962), 147-152.

(2) I. W. Busbridge, *Proc. London Math. Soc.*, **44** (1938), 115-129.

(3) G. N. Watson, *Theory of Bessel functions* (second edition, Cambridge University Press, 1944).

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