On the Summability of Series by a Method of Valiron

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§ 1. *Introduction.*

The method of summability with which I shall be concerned here is denoted by (V, α) and is defined[1] as follows:—The series Σa_n is said to be summable (V, α) to the sum s if

$$\lim_{\mu \to \infty} \frac{\mu^{-\alpha}}{\sqrt{(2\pi)}} \sum_{-\mu}^{\infty} \exp(-\tfrac{1}{2}n^2 \mu^{-2\alpha}) s_{n+\mu} = s.$$

This is a particular case of a method due to Valiron[2] in which $\mu^{-2\alpha}$ is replaced by a function of μ.

In § 2 the (V, α) consistency theorem is proved. This theorem exhibits the curious property that any convergent series whose sum is s is summable (V, α) to the sum $s\,\phi(\alpha)$, where $\phi(\alpha)$ is a certain step function.

It has been shown by Hardy[3] that, if the first Cesaro mean of the series Σa_n is of the form $s + o(n^{-\frac{1}{2}})$, the series is summable by Borel's method to the sum s. Under a more general hypothesis a similar theorem is found to be true for Valiron summability. This is given in § 3.

The Tauberian condition for the (V, α) method is known[4] to be $a_n = O(n^{-\alpha})$. In § 4 I give a more general condition which is analogous to that obtained by Vijayaraghavan[5] for Borel summability.

My thanks are due to Professor Hardy for his helpful suggestions and criticisms during the course of this work.

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[1] Summability (V, α) is usually defined by means of the limit

$$\lim_{\mu \to \infty} \frac{\mu^{\frac{1}{2}\alpha - 1}}{\sqrt{(2\pi)}} \sum_{-\mu}^{\infty} \exp(-\tfrac{1}{2}n^2 \mu^{\alpha - 2}) s_{n+\mu}.$$

The definition which I have given makes for greater compactness throughout the paper.

[2] G. Valiron, *Rendiconti di Palermo*, **42** (1917), 267-284.

[3] G. H. Hardy, *Quarterly Journal*, **35** (1904), 22-66.

[4] G. Valiron, *loc. cit.*

[5] T. Vijayaraghavan, *Proc. London Math. Soc.* (2), **27** (1927-28), 316-326.

§ 2. *The Consistency Theorem.*

We define the function $\phi(\alpha)$ as follows:

$$\phi(\alpha) = \tfrac{1}{2}, \qquad (\alpha > 1)$$
$$\phi(1) = \tfrac{1}{2} + \frac{1}{2\sqrt{\pi}} \int_0^{1/2} x^{-1/2} e^{-x} dx,$$
$$\phi(\alpha) = 1, \qquad (0 < \alpha < 1)$$
$$\phi(0) = 1 + 2 \sum_1^\infty \exp(-2n^2 \pi^2),$$
$$\phi(\alpha) = \infty, \qquad (\alpha < 0).$$

LEMMA A. *For each fixed value of* α,

$$E(\mu, \alpha) = \frac{\mu^{-\alpha}}{\sqrt{(2\pi)}} \sum_{-\mu}^{\infty} \exp(-\tfrac{1}{2} n^2 \mu^{-2\alpha}) \to \phi(\alpha)$$

as μ *tends to infinity.*

When $\alpha < 0$, $E(\mu, \alpha) > \dfrac{\mu^{-\alpha}}{\sqrt{(2\pi)}} \to \infty$.

When $\alpha = 0$,

$$E(\mu, 0) = \frac{1}{\sqrt{(2\pi)}} \sum_{-\mu}^{\infty} \exp(-\tfrac{1}{2} n^2) \to \frac{1}{\sqrt{(2\pi)}} \sum_{-\infty}^{\infty} \exp(-\tfrac{1}{2} n^2) = \phi(0),$$

by the transformation formula for the Theta function[1].

When $\alpha > 0$, we have

$$\begin{aligned} E(\mu, \alpha) &= \frac{\mu^{-\alpha}}{\sqrt{(2\pi)}} \sum_0^\infty \exp(-\tfrac{1}{2} n^2 \mu^{-2\alpha}) + \frac{\mu^{-\alpha}}{\sqrt{(2\pi)}} \sum_1^\mu \exp(-\tfrac{1}{2} n^2 \mu^{-2\alpha}) \\ &= \frac{\mu^{-\alpha}}{\sqrt{(2\pi)}} \left\{ \int_0^\infty \exp(-\tfrac{1}{2} y^2 \mu^{-2\alpha}) \, dy + O(1) \right\} \\ &\qquad + \frac{\mu^{-\alpha}}{\sqrt{(2\pi)}} \left\{ \int_1^\mu \exp(-\tfrac{1}{2} y^2 \mu^{-2\alpha}) \, dy + O(1) \right\} \\ &= \tfrac{1}{2} + \frac{1}{2\sqrt{\pi}} \int_{\frac{1}{2}\mu^{-2\alpha}}^{\frac{1}{2}\mu^{2-2\alpha}} x^{-1/2} e^{-x} dx + O(\mu^{-\alpha}) \\ &\to \phi(\alpha). \end{aligned}$$

The lemma is therefore proved.

THEOREM 1. *If the series* Σa_n *converges to* s, *then it is summable* (V, α) *to the sum* $s\,\phi(\alpha)$.

We may suppose, without loss of generality, that s is positive.

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[1] E. T. Whittaker and G. N. Watson, *Modern Analysis* (1927), 475-476. For a proof of the particular case used above see T. M. MacRobert, *Functions of a Complex Variable* (1925), 116.

Case (*i*), $\alpha < 0$.

Since Σa_n converges, we can find ν such that $s_{n+\mu} > \frac{1}{2}s$ whenever $n + \mu \geqq \nu$. Thus

$$\frac{\mu^{-\alpha}}{\sqrt{(2\pi)}} \sum_{-\mu}^{\infty} \exp\left(-\tfrac{1}{2}n^2 \mu^{-2\alpha}\right) s_{n+\mu}$$
$$= \frac{\mu^{-\alpha}}{\sqrt{(2\pi)}} \sum_{-\mu+\nu}^{\infty} \exp\left(-\tfrac{1}{2}n^2 \mu^{-2\alpha}\right) s_{n+\mu} + \frac{\mu^{-\alpha}}{\sqrt{(2\pi)}} \sum_{-\mu}^{-\mu+\nu-1} \exp\left(-\tfrac{1}{2}n^2 \mu^{-2\alpha}\right) s_{n+\mu}$$
$$> \frac{s\mu^{-\alpha}}{2\sqrt{(2\pi)}} \sum_{-\mu+\nu}^{\infty} \exp\left(-\tfrac{1}{2}n^2 \mu^{-2\alpha}\right) + o(1),$$

which tends to infinity with μ.

Case (*ii*), $\alpha \geqq 0$.

Given ϵ, there exists M such that $|s_m - s| < \epsilon$ for all values of $m > M$. Choose $\mu > 2M$. Then

$$\left|\frac{\mu^{-\alpha}}{\sqrt{(2\pi)}} \left\{\sum_{-\mu}^{\infty} (s_{n+\mu} - s) \exp\left(-\tfrac{1}{2}n^2 \mu^{-2\alpha}\right)\right\}\right| \leqq \frac{\mu^{-\alpha}}{\sqrt{(2\pi)}} \left\{\sum_{-\mu}^{-\mu+M} + \sum_{-\mu+M+1}^{\infty} |s_{n+\mu} - s| \exp\left(-\tfrac{1}{2}n^2 \mu^{-2\alpha}\right)\right\}$$
$$< \frac{A\mu^{-\alpha}}{\sqrt{(2\pi)}} \sum_{\mu-M}^{\mu} \exp\left(-\tfrac{1}{2}n^2 \mu^{-2\alpha}\right) + \frac{\epsilon\mu^{-\alpha}}{\sqrt{(2\pi)}} \sum_{-\mu}^{\infty} \exp\left(-\tfrac{1}{2}n^2 \mu^{-2\alpha}\right)$$
$$\to \epsilon\phi(\alpha)$$

as μ tends to infinity. Since ϵ is arbitrary, the result follows from Lemma A.

It will be observed from this theorem that the (V, α) method is consistent, in the ordinary sense of the term, only when $0 < \alpha < 1$.

§ 3. *A connection with the Cesaro method.*

THEOREM 2. *If p is a positive integer, $0 < \rho < p$, and if the pth Cesaro mean of the series Σa_n is such that*

$$c_n^{(p)} = s + o(n^{-\rho}),$$

then the series is summable (V, α) to the sum s, for any value of α in the range $\beta = (p - \rho)/p \leqq \alpha < 1$.

We shall prove first that the series is summable (V, β) to the sum s.

If m is some integer greater than p we have, by summing partially p times,

$$\frac{\mu^{-\beta}}{\sqrt{(2\pi)}} \sum_{0}^{m} (s_n - s) \exp\left\{-\tfrac{1}{2}(n-\mu)^2 \mu^{-2\beta}\right\}$$
$$= \frac{\mu^{-\beta}}{\sqrt{(2\pi)}} \sum_{0}^{m-p} \left\{s_n^{(p)} - \binom{n+p}{n} s\right\} \Delta^p \exp\left\{-\tfrac{1}{2}(n-\mu)^2 \mu^{-2\beta}\right\} + \Sigma'.$$

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Σ' is a finite sum of terms of the form

$$\frac{\mu^{-\beta}}{\sqrt{(2\pi)}}\left\{s_r^{(p)}-\binom{r+p}{r}s\right\}\Delta^q \exp\{-\tfrac{1}{2}(m-s-\mu)^2\mu^{-2\beta}\},$$

where q, r, s satisfy the inequalities

$$m-p<r\leqq m,\quad 0\leqq s<p,\quad 0\leqq q<p.$$

Each of these terms tends to zero as m tends to infinity so that

$$F(\mu)=\frac{\mu^{-\beta}}{\sqrt{(2\pi)}}\sum_0^\infty (s_n-s)\exp\{-\tfrac{1}{2}(n-\mu)^2\mu^{-2\beta}\}$$
$$=\frac{\mu^{-\beta}}{\sqrt{(2\pi)}}\sum_0^\infty\left\{s_n^{(p)}-\binom{n+p}{n}s\right\}\Delta^p\exp\{-\tfrac{1}{2}(n-\mu)^2\mu^{-2\beta}\}. \qquad (1)$$

By hypothesis we have

$$s_n^{(p)}-\binom{n+p}{n}s=o(n^{p\beta})$$

as n tends to infinity. It easily follows that, as μ tends to infinity,

$$F(\mu)=o\{G(\mu)\}+o(1), \qquad (2)$$

where

$$G(\mu)=\frac{\mu^{-\beta}}{\sqrt{(2\pi)}}\sum_0^\infty n^{p\beta}\,|\,\Delta^p\exp\{-\tfrac{1}{2}(n-\mu)^2\mu^{-2\beta}\}\,|\,.$$

We proceed to show that $G(\mu)$ is bounded for all large values of μ.

If

$$f(x)=\exp\{-\tfrac{1}{2}(x-\mu)^2\mu^{-2\beta}\},$$

it is easy to verify that $f^{(p)}(x)$ is of the form

$$\sum_{r=0}^{t} b_{p-2r}(\mu-x)^{p-2r}\mu^{-2\beta(p-r)}\exp\{-\tfrac{1}{2}(x-\mu)^2\mu^{-2\beta}\},$$

where b_p, b_{p-2}, ..., are constants, and t is $\tfrac{1}{2}(p-1)$ or $\tfrac{1}{2}p$ according as p is odd or even.

If $n+\theta$ is that value of x which gives the upper bound of $|f^{(p)}(x)|$ in the range $n\leqq x\leqq n+p$, we have, by repeated application of the Mean Value Theorem,

$$|\,\Delta^p f(n)\,|\leqq A\,|\,f^{(p)}(n+\theta)\,|,$$

where A is a positive constant. Accordingly

$$G(\mu)\leqq\frac{A\mu^{-\beta}}{\sqrt{(2\pi)}}\sum_{r=0}^{t}\sum_{n=0}^{\infty}n^{p\beta}\,|\,b_{p-2r}\,|\,|\,\mu-n-\theta\,|^{p-2r}\mu^{-2\beta(p-r)}\exp\{-\tfrac{1}{2}(n+\theta-\mu)^2\mu^{-2\beta}\},$$

and our assertion will be proved if we show that

$$H(\mu)=\mu^{-\beta}\sum_{n=0}^{\infty}n^{p\beta}\,|\,\mu-n-\theta\,|^{p-2r}\mu^{-2\beta(p-r)}\exp\{-\tfrac{1}{2}(n+\theta-\mu)^2\mu^{-2\beta}\}$$

is bounded for all large values of μ and $0\leqq r\leqq\tfrac{1}{2}p$.

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Write

$$\mu^{\beta(2p-2r+1)} H(\mu) = \sum_{0}^{\mu-p-1} + \sum_{\mu+1}^{\infty} + \sum_{\mu-p}^{\mu} [n^{p\beta} \,|\, \mu - n - \theta \,|^{p-2r} \exp\{-\tfrac{1}{2}(n+\theta-\mu)^2 \mu^{-2\beta}\}]$$
$$= S_1 + S_2 + S_3.$$

Clearly $S_3 = O(\mu^{p\beta})$.

Also

$$S_1 \leqq \sum_{0}^{\mu-p-1} n^{p\beta} (\mu - n)^{p-2r} \exp\{-\tfrac{1}{2}(n+p-\mu)^2 \mu^{-2\beta}\}$$
$$= O[\sum_{\nu=p}^{\mu-1} \nu^{p\beta} (\mu-\nu)^{p-2r} \exp\{-\tfrac{1}{2}(\mu-\nu)^2 \mu^{-2\beta}\}]$$
$$= O[\mu^{p\beta} \int_{p}^{\mu-1} (\mu - x)^{p-2r} \exp\{-\tfrac{1}{2}(\mu-x)^2 \mu^{-2\beta}\} dx] + O\{\mu^{\beta(2p-2r)}\}$$
$$= O\{\mu^{\beta(2p-2r+1)}\}.$$

Finally

$$S_2 \leqq \sum_{\mu+1}^{\infty} n^{p\beta} (n+p-\mu)^{p-2r} \exp\{-\tfrac{1}{2}(n-\mu)^2 \mu^{-2\beta}\}$$
$$= O[\sum_{\mu+1}^{\infty} n^{p\beta} (n-\mu)^{p-2r} \exp\{-\tfrac{1}{2}(n-\mu)^2 \mu^{-2\beta}\}]$$
$$= O[\sum_{\mu+1}^{2\mu}] + O[\sum_{2\mu+1}^{\infty}] = S_{2,1} + S_{2,2}.$$

As in the case of S_1 it is easy to show that

$$S_{2,1} = O\{\mu^{\beta(2p-2r+1)}\},$$

and

$$S_{2,2} = O\{\sum_{\mu+1}^{\infty} \nu^{p\beta+p-2r} \exp(-\tfrac{1}{2}\nu^2 \mu^{-2\beta})\}$$
$$= O\{\mu^{\beta(p\beta+p-2r+1)} \int_{\frac{1}{2}\mu^{2(1-\beta)}}^{\infty} u^{\frac{1}{2}(p\beta+p-2r+1)} e^{-u} du\} + o(1)$$
$$= o(1).$$

It follows that $H(\mu)$ is bounded. Hence, by (2), (1) and Lemma A, the series Σa_n is summable (V, β) to the sum s.

To prove that it is summable (V, α) to the sum s for $\beta < \alpha < 1$, we observe that the hypothesis implies

$$c_n^{(p)} = s + o(n^{-\rho'})$$

for $0 < \rho' < \rho$. The series is therefore summable $\{V, (p-\rho')/p\}$ to the sum s.

The proof of this theorem applies, with trivial modifications, to the case $\rho = 0$, when we have the following interesting result:

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THEOREM 3. *If Σa_n is summable (C, p) to the sum s, then it is summable $(V, 1)$ to the sum $s\,\phi(1)$.*

When $\rho = 0$, $p = 0$, the hypothesis of Theorem 2 reduces to the convergence of Σa_n, and the proof becomes simply the proof of Theorem 1, case (ii).

§ 4. *The Tauberian Theorem.*

THEOREM 4. *If $0 < a < 1$, and Σa_n is summable (V, a) to the sum s, and if*

$$\varliminf_{n \to \infty} (s_{n+p} - s_n) \geqq 0$$

whenever $p = o(n^a)$, then Σa_n converges to s.

The truth of this theorem for $0 < a \leqq \frac{1}{2}$ was conjectured by Hardy and Littlewood[1].

The proof is similar to the proof of the corresponding theorem[2] for Borel summability. Several of the necessary lemmas are obtained from the corresponding lemmas in Vijayaraghavan's paper by putting a, or in some cases $1 - a$, for $\frac{1}{2}$. Others are particular cases of more general lemmas due to Valiron[3], his function $H(\mu)$ being replaced by μ^{-2a}. Important parts of the proof are also to be found in a paper[4] by Hardy and Littlewood. The analogues for (V, a) summability of Vijayaraghavan's first four lemmas cannot be obtained however from these sources. The first two may be proved after the manner of Lemma A, while, from these and Lemma A, the third may easily be deduced. By defining the sequence $M, M_1, M_2, \ldots,$ analogous to the sequence which occurs in Lemma a of Vijayaraghavan's paper, and by dividing the range (M, ∞) into the components (M, M_1), $(M_1, M_2), \ldots,$ it is not difficult to prove the fourth.

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[1] G. H. Hardy and J. E. Littlewood, *Annali di Pisa* (2), 3 (1934), 54.

[2] T. Vijayaraghavan, *loc. cit.*

[3] G. Valiron, *loc. cit.*

[4] G. H. Hardy and J. E. Littlewood, *Rendiconti di Palermo*, 41 (1915), 1-18.