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# $C^*$-Algebras of Infinite Graphs and Cuntz-Krieger Algebras

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*Abstract.* The Cuntz-Krieger algebra $\mathcal{O}_B$ is defined for an arbitrary, possibly infinite and infinite valued, matrix $B$. A graph $C^*$-algebra $G^*(E)$ is introduced for an arbitrary directed graph $E$, and is shown to coincide with a previously defined graph algebra $C^*(E)$ if each source of $E$ emits only finitely many edges. Each graph algebra $G^*(E)$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_B$ where $B$ is the vertex matrix of $E$.

## 0 Introduction

In [3] Cuntz and Krieger introduced $C^*$-algebras $\mathcal{O}_A$ associated with square matrices $A$ with entries in the nonnegative integers $\mathbb{Z}_+$ and with each row and column nonzero. In defining these algebras they first restricted themselves to the case where $A$ had entries in $\{0,1\}$ only, and then extended the definition to the more general $A$ described above by forming a certain $\{0,1\}$ matrix $A'$ from the matrix $A$. A description of this process is perhaps better understood in light of subsequent approaches to these algebras.

Building on the results of [7], Kumjian, Pask, and Raeburn define the $C^*$-algebras $C^*(E)$ of directed graphs $E$ whose vertices each emit only a finite number of edges ([6]). This $C^*$-algebra is the universal $C^*$-algebra generated by mutually orthogonal projections indexed by the vertices of $E$, along with a set of partial isometries indexed by the edges of $E$, and satisfying certain relations governed by relationships between the edges and vertices. These graph algebras are clearly related to the Cuntz-Krieger algebras. First note that the edge matrix $\mathbf{A}_E$ of the graph $E$, is a square matrix with entries in $\{0,1\}$, and that zero rows of $\mathbf{A}_E$ occur if there is a vertex of $E$ with an incoming edge but no outgoing edges. It follows that if $E$ is a finite graph, so when the vertex and edge sets of $E$ are both finite, and if each vertex of $E$ both receives and emits edges then the graph $C^*$-algebra $C^*(E)$ is the Cuntz-Krieger algebra $\mathcal{O}_{\mathbf{A}_E}$. Thus the graph algebras $C^*(E)$ may be viewed as extending the setting of Cuntz-Krieger algebras to graphs under the sole restriction that each vertex emits at most a finite number of edges. In particular, infinite directed graphs, and graphs with vertices which either emit or receive no edges are allowed.

This viewpoint is strengthened by subsequent results. In [4] much effort and ingenuity is taken to extend—and then work with—the definition of a Cuntz-Krieger algebra $\mathcal{O}_A$ to the case of possibly infinite $\{0,1\}$-valued matrices with no zero rows.

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In [5] the definition of a graph $C^*$-algebra $C^*(E)$ is extended in a rather straightforward fashion to the case of a general directed graph $E$. It is furthermore shown in [5] that for graphs $E$ in which every vertex both receives and emits edges, the graph $C^*$-algebra $C^*(E)$ coincides with this newly defined Cuntz-Krieger algebra $\mathcal{O}_{\mathbf{A}_E}$ where $\mathbf{A}_E$ is again the edge matrix of $E$ and is a possibly infinite $\{0,1\}$-valued matrix with no zero rows or columns.

These results have led to the conviction that graph $C^*$-algebras are Cuntz-Krieger algebras, or at least the natural generalization of Cuntz-Krieger algebras to a wider context. However, even in the case of certain finite graphs, this conviction is not as straightforward as it seems. For example, consider the finite directed graph $E$ with two vertices $v_0$ and $v_1$ with one of them, $v_0$ say, only emitting edges, the other, $v_1$, only receiving edges. If $E$ has $n$ edges, $\{e_i \mid 1 \le i \le n\}$, then the $C^*$-algebra $C^*(E)$ is the universal one generated by two orthogonal projections $\rho_0$, $\rho_1$ and $n$ partial isometries $\sigma_i$, $1 \le i \le n$, with initial projections $\rho_1$ and orthogonal final projections with sum $\rho_0$. The edge matrix $A = \mathbf{A}_E$ of this graph is the $n \times n$ zero matrix. Even if $\mathcal{O}_A$ were actually defined for such $A$ it is clear that $C^*(E)$ cannot be $\mathcal{O}_A$.

In this report we adopt as a guiding principle that graph algebras for arbitrary directed graphs and Cuntz-Krieger algebras for arbitrary matrices should coincide, via a natural linking of the data, as classes of algebras. To see this as a reality requires overcoming several obstacles. Initially this involves extending the context of Cuntz-Krieger algebras $\mathcal{O}_A$ to completely arbitrary matrices, namely possibly infinite square matrices $A$ with nonnegative integer or infinite valued entries and any number of zero rows or columns. The approach taken here not only simplifies the approach of [4], it also parallels the original approach used by Cuntz and Krieger. A translation of the relations governing the algebra $\mathcal{O}_A$ into graph theoretical terms then yields an alternate approach to defining the graph $C^*$-algebra, denoted here by $G^*(E)$, associated with an arbitrary directed graph $E$. If every source vertex of $E$, namely those vertices of $E$ that receive no edges, emits only finitely many edges then $G^*(E)$ coincides with $C^*(E)$. In particular $G^*(E)$ and $C^*(E)$ coincide if there are no sources. Thus one may view $G^*(E)$ as an alternate way to extend the graph algebras $C^*(E)$ of [6] for row finite graphs $E$ to arbitrary graphs $E$. In general $G^*(E)$ is an ideal of $C^*(E)$. The algebra $G^*(E)$ is also invariant under a standard operation on graphs $E$ for certain $E$, as is the case for the classical Cuntz-Krieger algebras, while there are examples of $E$, necessarily with sources emitting an infinite number of edges, where $C^*(E)$ does not display this invariance.

We briefly discuss the procedure Cuntz and Krieger employed to extend the $C^*$-algebra from the context of $\{0,1\}$ matrices with no zero rows or columns to that of $\mathbb{Z}_+$ valued matrices $B$. The $\{0,1\}$ matrix $B'$ formed from $B$ can be viewed in various ways. For example, a viewpoint that in some form or another has been around for some while is that if $E = \mathbf{E}_B$ is the directed graph associated to $B$, *i.e.*, $E$ has a vertex matrix $B$, then $B'$ is the edge matrix $\mathbf{A}_E$ of $E$. Thus $\mathcal{O}_B$ was defined as $\mathcal{O}_{\mathbf{A}_E}$, which (for these matrices) is $C^*(E)$.

In [2] a slightly different perspective is taken, which in this project then led directly to extending the Cuntz-Krieger approach to arbitrary matrices $B$. The view in [2] is that the $\{0,1\}$ matrix $B'$ is the complete in-split matrix $B_w$ of $B$. This in-splitting process can be naturally extended to completely arbitrary matrices $B$, including those with any number of zero rows or columns, allowing one to then define $\mathcal{O}_B$ in terms of relations determined by the matrix $B_w$. The congruence between the in-split matrix $B_w$ and the edge matrix $\mathbf{A}_E$ of the graph $E = \mathbf{E}_B$ fails to be preserved once one has naturally extended the in-splitting process. What one does have is that if $E$ is a directed graph with no sinks, so each vertex of $E$ emits edges, then $B_w = \mathbf{A}_E$, where $B = \mathbf{B}_E$ is the vertex matrix of $E$ and $\mathbf{A}_E$ is the edge matrix of $E$.

A quick overview of the paper follows. In Section 1 the processes connecting square matrices, square bipartite graphs and directed graphs are described. The in-splitting process, which plays a major role in the sequel, is introduced in a more general setting than that of [2], where the basic structure of a partial order for this procedure and its inverse was explored. Here those results are briefly described in this more general context. In Section 2 the Cuntz-Krieger algebra $\mathcal{O}_B$ is defined for an arbitrary square matrix $B$. Translating these relations to a graph context defines a graph $C^*$-algebra $G^*(E)$ which is isomorphic to $\mathcal{O}_B$, $B$ the vertex matrix for $E$. We then show that $G^*(E)$ exists as a universal $C^*$-algebra and that it is an ideal of $C^*(E)$. In Section 3 we describe how the algebra $\mathcal{O}_B$ behaves under proper in-splitting of matrices, extending known results for the usual finite matrix Cuntz-Krieger algebras. We conclude with a simple example where $G^*(E)$ is not isomorphic to $C^*(E)$.

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***Notation*** The nonnegative integers are denoted $\mathbb{Z}_+$ while $\mathbb{Z}_+^\infty$ denotes $\mathbb{Z}_+ \cup \{\infty\}$. Here infinite refers to countably infinite. The transpose of a matrix $A$ is $A^t$, and square matrices are $n \times n$ matrices where $n = \infty$ is also possible. For $a$, $b$ elements of a set $S$, $\delta_a$ is the function on $S$ defined by $\delta_a(b) = \delta_{a,b} = \begin{cases} 1 & \text{if } b = a \\ 0 & \text{otherwise.} \end{cases}$

## 1 Directed and Bipartite Graphs, and Their Matrices

In the following a directed graph $E = (E^0, E^1, r, s)$ has a countable set of vertices $E^0$, a countable set of edges $E^1$, and maps $r, s \colon E^1 \to E^0$, respectively denoting the range and source maps. Associate with a directed graph its vertex matrix $B = \mathbf{B}_E$, which is a square matrix of size $|E^0|$ with entries in $\mathbb{Z}_+^\infty$, where $B_E(v, w) = |\{e \in E^1 \mid s(e) = v$ and $r(e) = w\}|$. Conversely, given such a square matrix $B$, denote by $\mathbf{E}_B$ the directed graph with vertex matrix $B$. Say that $B$ has $v$-row finite if $\sum_w B(v, w) < \infty$, and $B$ is row-finite if each row of $B$ is finite. If $E = \mathbf{E}_B$ then $E$ is said to be row finite if $B$ is, so $E$ is row finite if and only if each vertex $v \in E^0$ emits only finitely many edges.

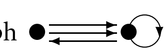
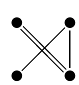
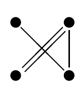
A bipartite graph $\mathcal{G} = \mathcal{G}(V, W)$ has initial state set $V$, terminal or final set $W$ and an edge set $\mathcal{E}$. If $e \in \mathcal{E}$ then $i(e) \in V$ denotes its initial vertex and $t(e) \in W$ its terminal vertex. Associate a matrix $B = \mathbf{B}_\mathcal{G}$ with entries in $\mathbb{Z}_+^\infty$ to a bipartite graph $\mathcal{G}$ by setting $B(b, a) = |\{e \in \mathcal{E} \mid i(e) = a$ and $t(e) = b\}|$. The matrix $B$ has $|W|$ rows, $|V|$ columns and is the transpose of the usual adjacency matrix of a bipartite graph. This is done for covariance reasons connected to the index of an endomorphism of a sum of type I factors [2]. It is also historically how matrices and bipartite graphs are associated with each other in the $AF$-algebra context. Write $\mathcal{G} = \mathfrak{G}_B$ for the bipartite graph corresponding to a matrix $B$.

The collection of bipartite graphs may be trivially viewed as properly contained in the collection of directed graphs, however, for our purposes it will be far more useful to view the collection of directed graphs as properly contained in the collection of bipartite graphs. The collection of directed graphs consist of those bipartite graphs $\mathcal{G} = \mathcal{G}(V, V)$ whose initial and terminal spaces are identical as sets. Call these bipartite graphs square. This is seen by viewing $E = (E^0, E^1, r, s)$ as the bipartite graph $\mathcal{G} = \boldsymbol{\mathcal{G}}_E$ with $\mathcal{G} = \mathcal{G}(E^0, E^0)$ and where an edge $e \in E^1$ with $s(e) = a$ and $r(e) = b$ corresponds to an edge $\underset{\sim}{e} \in \mathcal{E}$ of $\mathcal{G}$ with $i(\underset{\sim}{e}) = a$, $t(\underset{\sim}{e}) = b$. Intuitively speaking each vertex $v \in E^0$ is split into two halves, one part is viewed as the emitting half which is in the top row of $\mathcal{G}$, while the other is the receiving half. This process is clearly reversible: given $\mathcal{G} = \mathcal{G}(V, V)$ coalesce the corresponding vertices of $V$ to a single point $v \in E^0 = V$.

Reversing the direction of every edge in a directed graph $E$ yields a directed graph denoted $E^t$. The bipartite graph obtained by exchanging the initial and terminal state spaces and exchanging the initial and terminal vertices of each edge is denoted by $\mathcal{G}^t$. It is clear for example that $\mathbf{E}_{B^t} = (\mathbf{E}_B)^t$ for a square matrix $B$ with entries in $\mathbb{Z}_+^\infty$, and $\boldsymbol{\mathcal{G}}_{E^t} = (\boldsymbol{\mathcal{G}}_E)^t$ for a directed graph $E$. It is straightforward to check that travelling along two sides of the triangle below yields the same result as travelling along the third side and taking the 'transpose'.

$$
\begin{tikzcd}
\{B \mid B \text{ a square matrix}\} \arrow[rr, shift right, "\mathbf{E}"'] \arrow[dr, shift left, "\mathcal{G}"] & & \{E \mid E \text{ a directed graph}\} \arrow[ll, shift right, "\mathbf{B}"'] \arrow[dl, shift left, "\mathcal{G}"] \\
& \{\mathcal{G} \mid \mathcal{G} \text{ a square bipartite graph}\} \arrow[ul, shift left, "\mathbf{B}"] \arrow[ur, shift left, "\mathbf{E}"] &
\end{tikzcd}
$$

Thus, for example, $\mathbf{E}_{B^t} = \mathbf{E}_{\boldsymbol{\mathcal{G}}_B}$, $\boldsymbol{\mathcal{G}}_{E^t} = \boldsymbol{\mathcal{G}}_{B_E}$ and so on.

***Example 1.1*** Let $E$ be the directed graph  . Then its corresponding bipartite graph $\boldsymbol{\mathcal{G}}_E$ is  . The vertex matrix $\mathbf{B}_E = B$ is $\left[\begin{smallmatrix} 0 & 2 \\ 1 & 1 \end{smallmatrix}\right]$ while the bipartite graph $\boldsymbol{\mathcal{G}}_B$ is  , which is $\mathcal{G}_E^t$.

A vertex $v \in E^0$ of a directed graph $E$ is a sink if $s^{-1}(v) = \phi$, and a source if $r^{-1}(v) = \phi$. A vertex of $E$ is both a source and a sink if it emits and receives no edges. A vertex $v \in E^0$ is a sink if and only if the $v$-row of $B$ is zero, where $B = \mathbf{B}_E$ is the vertex matrix of $E$. If $\mathcal{G} = \boldsymbol{\mathcal{G}}_B$ is the bipartite graph corresponding to $B$ then $v$ is a sink if and only if there are no edges of $\mathcal{G}$ with terminal state $v$. Given the important role that sinks play in the following development we denote the subset of $E^0$ consisting of sinks by $F$.

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Given a square bipartite graph $\mathcal{G} = \mathcal{G}(V, V)$ we adopt verbatim the process of forming an in-split bipartite graph $\tilde{\mathcal{G}} = \mathcal{G}(\tilde{V}, \tilde{V})$ of $\mathcal{G}$ as described in [2] and [8], with the natural additional but already implicit condition that a vertex $l \in V$ is left untouched if $\mathcal{E}_l = \{e \in \mathcal{E} \mid t(e) = l\} = \phi$, so if, in other words, there are no edges into $l$. We quickly review this procedure. For each $v \in V$ partition $\mathcal{E}_v$, the set of edges with terminal state $v$, into sets $\mathcal{E}_v^1, \dots, \mathcal{E}_v^{m(v)}$. Replace the vertex $v$ with $m(v)$ vertices $v_1, \dots, v_{m(v)}$ of $\tilde{\mathcal{G}}$. If $e \in \mathcal{E}$ is an edge of $\mathcal{G}$ with terminal state $v$ and initial state $w$, then $e$ is an element of one of the partition sets $\mathcal{E}_v^p$ where $p \in \{1, \dots, m(v)\}$. Now replace each $e \in \mathcal{E}_v^p$ with $m(w)$ edges of $\tilde{\mathcal{G}}$, each having the same terminal state $v_p$, with the $m(w)$ initial states $w_1, \dots, w_{m(w)}$. If for each $v \in V$ the partition of the edges in $\mathcal{E}_v$ is the maximal one consisting of one element sets then the resulting bipartite graph is the complete in-split of $\mathcal{G}$ and is denoted by $\mathcal{G}_\omega$. The complete in-split of $\mathcal{G}_\omega$ is denoted by $\mathcal{G}_{\omega,\omega}$ or by $\mathcal{G}_{\omega_2}$.

The process of forming in-amalgamations described in [2] must be slightly adapted to this more general context, where vertices $v$ of $\mathcal{G}$ are allowed that are not in the range of $t$, in order to form an inverse process to in-splitting. For $\mathcal{G} = \mathcal{G}(V, V)$ we form $\mathcal{G}(\underset{\sim}{V}, \underset{\sim}{V})$ an in-amalgamation of $\mathcal{G}$ as follows: for $m \in \mathbb{Z}_+^\infty$, $v \in V$ let $S_v^m = \{w \in V \mid w \text{ is connected to } v \text{ with multiplicity } m\}$ and let $\mathcal{T} = \{T_k \mid k \in I\}$ be the 'base' partition of $V$ defined by the collection of subsets $S_v^m$ ($v \in V$, $m \in \mathbb{Z}_+^\infty$) along with the singleton subsets $\{l\}$ where $\mathcal{E}_l = \phi$. If $\mathcal{T}' = \{T_k^i \mid k \in I, i \in I_k\}$, where $\bigcup\{T_k^i \mid i \in I_k\} = T_k$, is a partition of $V$ finer than $\mathcal{T}$, collapse each set $T_k^i$ of $\mathcal{T}'$ to a single vertex $T_k^i$ of $\underset{\sim}{V}$. Each set of edges of $\mathcal{G}$ with multiplicity $m$ from the points of $\mathcal{G}$ in $T_k^i$ to a vertex $v$ of $\mathcal{G}$, where $v \in T_n^j$ some $j$ and $n$, is replaced by an edge of multiplicity $m$ from $T_k^i$ to the vertex $T_n^j$ of $\underset{\sim}{V}$. The complete in-amalgamation $\mathcal{G}_c$ of $\mathcal{G}$ denotes the bipartite graph obtained when this process is effected using the base partition $\mathcal{T}$, so when $\mathcal{T}'$ is $\mathcal{T}$.

Many of the results of Section 3 in [2] now carry over to this more general setting; for $\mathcal{G}_0, \mathcal{G}$ square bipartite graphs define $\mathcal{G}_0 \prec \mathcal{G}$ if there is a finite sequence of bipartite graphs $\mathcal{G}_k$, $k = 0, \dots, l$ with $\mathcal{G}_k$ an in-amalgamation of $\mathcal{G}_{k+1}$ and $\mathcal{G}_l = \mathcal{G}$. This defines a partial order on the collection of bipartite graphs $\mathcal{G}(V, V)$ with finite vertex and edge sets. This however fails to yield a partial order in general, since it is possible for $\mathcal{G}_0 \prec \mathcal{G}_1 \prec \mathcal{G}_0$ with $\mathcal{G}_0 \neq \mathcal{G}_1$. To see an example of this let $\mathcal{G}_0 = \mathcal{G}(\mathbb{N}, \mathbb{N})$ with $\mathcal{E} = \{e_n \mid n \in \mathbb{N}\}$ such that $i(e_n) = n$ and $t(e_{2n-1}) = t(e_{2n}) = n$. The complete in-amalgamation of $\mathcal{G}_0$ is again $\mathcal{G}_0$, so if we set $\mathcal{G}_1$ to be a partial in-amalgamation of $\mathcal{G}_0$ not equal to $\mathcal{G}_0$ we are finished. Let $\mathcal{G}_1$ be the in-amalgamation of $\mathcal{G}_0$ given by amalgamating the vertices 1 and 2 only. Then $\mathcal{G}_1$ has vertex set $\mathbb{N}$, and edges $e_n$ with $i(e_n) = n$, $n \geq 1$, $t(e_{2n}) = t(e_{2n+1}) = n$ for $n \geq 2$ while $t(e_1) = t(e_2) = t(e_3) = 1$.

This relation does however satisfy the property that any two bipartite graphs with an upper bound also have a common lower bound, and vice versa. We restate this in the following proposition.

***Proposition 1.2*** *Let $\mathcal{G}_0$ be a square bipartite graph.*

(a) *The collection $\{\mathcal{G} \mid \mathcal{G} \prec \mathcal{G}_0\}$ is lower directed.*
(b) *The collection $\{\mathcal{G} \mid \mathcal{G}_0 \prec \mathcal{G}\}$ is upper directed.*

**Proof** Part (a) follows from repeated application of Corollary 3.5 of [2], while part (b) uses Corollary 3.10 of [2]. ■

We may also define an equivalence relation on the collection of square bipartite graphs by setting $\mathcal{G}_0 \sim \mathcal{G}$ if there is a finite sequence $\mathcal{G}_k$, $k = 0, \dots, l$ with $\mathcal{G}_l = \mathcal{G}$ and $\mathcal{G}_k$ is either an in-split or an in-amalgamation of $\mathcal{G}_{k+1}$, $k = 0, \dots, l-1$. Repeated applications of Corollary 3.10 of [2] show that two bipartite graphs $\mathcal{G}_0, \mathcal{G}_1$ are equivalent if and only if there is a $\mathcal{G}$ with $\mathcal{G}_0 \prec \mathcal{G}$ and $\mathcal{G}_1 \prec \mathcal{G}$, which by Proposition 1.2 is equivalent to saying that they have a common lower bound.

Given $\mathcal{G} = \mathcal{G}(V, V)$ a square bipartite graph with a finite vertex set it follows as in Theorem 3.7 of [2] that there is a unique minimal element $\underset{\sim}{\mathcal{G}}$ with $\underset{\sim}{\mathcal{G}} \prec \mathcal{G}$. Here, however, we make no restriction on the number of edges $|\mathcal{E}|$ of $\mathcal{G}$. Given two finite square bipartite graphs with no restriction on the number of edges it is then straightforward to compute if they are equivalent or not, since there is an *a priori* bound on the maximum number of complete in-amalgamations possible before a minimal element is obtained. For square bipartite graphs that are not finite this is however no longer the case. Not only is there no *a priori* bound on the number of in-amalgamations used to attain a common lower bound, but there is also no guaranteed path to a common lower bound by using complete in-amalgamations for example.

We can transfer the relation $\prec$ to square matrices which are possibly infinite, with entries in $\mathbb{Z}_+^\infty$, using the correspondence between such matrices $B$ and bipartite graphs $\mathbf{\mathcal{G}}_B$.

***Definition 1.3*** Let $A, B$ be two square matrices with entries in $\mathbb{Z}_+^\infty$. Set $A \prec B$ if and only if $\mathbf{\mathcal{G}}_A \prec \mathbf{\mathcal{G}}_B$. We say $A$ is an in-split (in-amalgamation) of $B$ if $\mathbf{\mathcal{G}}_A$ is an in-split (in-amalgamation) of $\mathbf{\mathcal{G}}_B$.

This yields a partial order on finite square matrices with entries in $\mathbb{Z}_+$ extending the partial order introduced in [2], which was defined for finite square matrices with entries in $\mathbb{Z}_+$ with no zero rows or columns.

***Remark 1.4*** The partial order on square matrices may be transferred to such an order on directed graphs $E$ using the correspondence $B \to \mathbf{E}_B$, but because of the transpose that occurs between this route and the route connecting directed graphs with bipartite graphs via $\mathcal{G} \to \mathbf{E}_\mathcal{G}$ the in-amalgamation, or in-split of matrices is actually an out-amalgamation, or out-split process for the associated directed graphs.

## 2 Cuntz-Krieger Algebras and Graph Algebras

Beginning with a square matrix $B$ we examine more closely the complete in-split matrix $B_\omega$, along with the associated directed graphs $E = \mathbf{E}_B$ and $E_\omega$, the directed graph $\mathbf{E}_{B_\omega}$ for $B_\omega$. Note that if $E$ has no sinks then $B_\omega$ is the edge matrix of $E$. Using very minor modifications of the original relations defining Cuntz-Krieger algebras [3] we introduce the Cuntz-Krieger algebra $\mathcal{O}_B$ for a completely arbitrary square, possibly infinite, matrix with entries in $\mathbb{Z}_+^\infty$. We stress that no restrictions are placed on the possible appearance of zero rows or columns. This approach also allows for the introduction of a natural graph $C^*$-algebra $G^*(E)$ associated with an arbitrary directed graph $E$. Using this approach it is clear that $G^*(E)$ is $\mathcal{O}_B$.

A $C^*$-algebra $C^*(E)$ associated with a row finite directed graph $E$ was introduced earlier in [6] and later extended without modification to arbitrary directed graphs $E$ in [5]. We show that for graphs $E$ with finite sources $G^*(E)$ and $C^*(E)$ coincide. We show this is not the case for general graphs $E$ however, so $G^*(E)$ can be viewed as another way to extend graph $C^*$-algebras to arbitrary directed graphs $E$. Recall that the original Cuntz-Krieger algebras for finite matrices with finite entries and no zero rows or columns are naturally invariant under the in-spliting process [2]. We show this to also be the case for our graph algebras $G^*(E)$ for a family of graphs $E$ which have sources emitting an infinite number of edges. For these $E$ however $C^*(E)$ is not invariant under the in-splitting process. This suggests that $G^*(E)$ is perhaps a more natural candidate for the $C^*$-algebra of a graph $E$ which may have sources emitting an infinite number of edges.

Let $E = (E^0, E^1, r, s)$ be the directed graph associated with the matrix $B$, so $B(v, w) = |\{e \in E^1 \mid s(e) = v, r(e) = w\}|$. Label each edge $e \in E^1$ by a triple $(v, k, w)$ where $s(e) = v$, $r(e) = w$ and $k$ a nonzero element of $\mathbb{Z}_+^\infty$ with $1 \leq k \leq B(v, w)$. If $\mathcal{G} = \mathcal{G}_B$ is the bipartite graph associated with $B$ then $\mathcal{G}_\omega$, the complete in-split of $\mathcal{G}$, has associated matrix $B_\omega$. Denote by $E_\omega$ the directed graph for $B_\omega$, namely $E_\omega = \mathbf{E}_{B_\omega}$. By previous comments $E_\omega$ is $\mathbf{E}_{(\mathcal{G}_\omega)^t}$.

We describe the vertex set $E^0_\omega$ of $E_\omega$ which is also the index set for the matrix entries of $B_\omega$. A vertex $v \in E^0$ that is not a sink splits into the new set of vertices $\bigcup\{(v, k, \omega) \mid 1 \leq k \leq B(v, w), \omega \in E^0 \text{ with } B(v, w) \neq 0\}$ in $E^0_\omega$, while a vertex $v \in E^0$ that is a sink remains untouched by the in-splitting process. Thus the index set $E^0_\omega$ for the matrix entries of $B_\omega$ is identified with $E^1 \cup F$, and the number of zero rows of $B$ remains unchanged under this, or any, in-splitting process. It is useful to think of a sink $v \in F$ as the triple $(v, 0, v)$, namely an edge of multiplicity zero from $v$ to $v$. Defining $r(v) = v$ and $s(v) = v$ for $v \in F$ extends the range and source maps to maps from $E^0_\omega$ to $E^0$. We have $s^{-1}(l) = l$, $(l \in F)$ is disjoint from all other $s^{-1}(v)$, $v \in E^0_\omega \setminus \{l\}$. Also $r^{-1}(l) = \{l\} \cup (r|_{E^1})^{-1}(l)$ for $l \in F$.

For $e \in E^0_\omega$, the $e$-row of $B_\omega$ is zero if and only if $e$ is a sink of $E$. Also, for $e, f \in E^0_\omega$, $B_\omega(e, f) = 1$ if and only if $e \in E^1$ and $r(e) = s(f)$, where $s(f) = f$ if $f$ is a sink. In this later case, when $f$ is a sink, this is the only nonzero entry of the $e$-row of $B_\omega$. For $e, f \in E^0_\omega$ it follows that if the $e$-row of $B_\omega$ and the $f$-row of $B_\omega$ share a nonzero entry in the same column, say the $l$-column, then $e, f \in E^1$ and $r(e) = s(l) = r(f)$, so that $e$ and $f$ must have the same range. However, $r(e) = r(f)$ if and only if the $e$-row of $B_\omega = f$-row of $B_\omega$, $e, f \in E^1$.

We summarize some of these findings.

***Proposition 2.1*** *Let $B$ be a square matrix with entries in $\mathbb{Z}_+^\infty$ and $B_\omega$ the complete in-split matrix of $B$. Then*

(a) *$B_\omega$ is a square matrix with entries in $\{0, 1\}$,*
(b) *any two rows of $B_\omega$ are either equal or orthogonal, so that the two sets $\{h \mid B_\omega(e, h) \neq 0\}$ and $\{h \mid B_\omega(f, h) \neq 0\}$ are either equal or disjoint,*

(c) *if* $B_\omega(e,l) = 1$ *with the* $l$*-row of* $B_\omega$ *zero, then* $B_\omega(e,f) = 0$ *for all other* $f$.

It is of use to recognize from the graph $E$ when the matrix $B_\omega$ is row-finite, *i.e.*, when each row of $B_\omega$ has a finite number of nonzero entries. For $e \in E^0_\omega$ it follows from Proposition 2.1 that the $e$-row of $B_\omega$ is certainly finite whenever $e$ is a sink of $E$ or when $e \in E^1$ with $r(e)$ a sink. For $e \in E^1$, the $e$-row of $B_\omega$ is finite if and only if $|\{f \in E^1 \mid s(f) = r(e)\}| < \infty$, so exactly when the number of paths in $E^1$ with source $r(e)$ is finite. Thus $B_\omega$ is row finite if and only if each $v$-row of $B$ is finite for $v \in E^0$, $v \in r(E^1)$; in other words for $v$ not a source. It follows that if $B_\omega$ is row-finite, then so is $(B_\omega)_\omega$ and the process of forming complete in-splits preserves row finiteness.

***Proposition 2.2*** *Let* $E = (E^0, E^1, r, s)$ *be the directed graph associated with a matrix* $B$. *The complete in-split matrix* $B_\omega$ *is row-finite if and only if each* $v$*-row of* $B$ *is finite for* $v \in E^0$, $v$ *not a source.*

Note that the condition for the row-finiteness of $B_\omega$ is weaker than requiring that $B$ is row-finite. The matrix $B_\omega$ is row finite even though there may be any number of sources $v \in E^0$ of the directed graph $E = \mathbf{E}_B$ which emit infinitely many edges. For example the complete in-split matrix $B_\omega$ of $B = \left[\begin{smallmatrix} 0 & \infty \\ 0 & 0 \end{smallmatrix}\right]$ is row-finite.

***Definition 2.3*** For a matrix $B \in \mathcal{M}_m(\mathbb{Z}_+^\infty)$ with $m \in \mathbb{Z}_+^\infty$ let $B_\omega$ denote the complete in-split of $B$, a square $n \times n$ matrix with entries in $\{0,1\}$. Then $\mathcal{O}_B$ is the universal $C^*$-algebra generated by $n$ partial isometries $S_e$ with orthogonal range projections such that $S_e^* S_e = \sum B_\omega(e,f) S_f S_f^*$ if the $e$-row of $B_\omega$ is nonzero and finite, and $S_e^* S_e = S_e$ if the $e$-row of $B_\omega$ is zero. In addition we assume that the initial projections commute, that

$$S_e^* S_e S_f S_f^* = B_\omega(e,f) S_f S_f^*$$

if the $e$-row of $B_\omega$ is nonzero, and that

$$S_e^* S_e S_f^* S_f = \begin{cases} S_f^* S_f & \text{if } f\text{-row of } B_\omega = e\text{-row of } B_\omega \\ 0 & \text{otherwise} \end{cases}$$

whenever the $e$-row and $f$-row of $B_\omega$ are both nonzero.

***Proposition 2.4*** *For a matrix* $B \in \mathcal{M}_m(\mathbb{Z}_+^\infty)$, $m \in \mathbb{Z}_+^\infty$, *with* $B_\omega$ *row finite the additional assumptions defining the algebra* $\mathcal{O}_B$ *are superfluous.*

**Proof** If $B_\omega$ is row finite then $S_e^* S_e = \sum B_\omega(e,f) S_f S_f^*$ whenever the $e$-row of $B_\omega$ is nonzero, and $S_e^* S_e = S_e S_e^*$ if the $e$-row of $B_\omega$ is zero. The additional assumptions now follow easily from parts (b) and (c) of Proposition 2.1 after noting that the collection of final projections is orthogonal. ■

If $B$ is a finite square matrix with entries in $\mathbb{Z}_+$ with no zero rows or columns then this reduces to the usual definition of the Cuntz-Krieger algebra $\mathcal{O}_B$. This follows from [2] where it is shown that $\mathcal{O}_B$ is invariant under an arbitrary in-splitting of $B$, in particular for the complete in-split $B_\omega$ (see also [9]). Even in the finite case however this definition now includes matrices with zero rows or columns. For example, if $B$ is the square 0 matrix of size $n$, then $B_\omega = B$ and $\mathcal{O}_B$ is the universal $C^*$-algebra generated by $n$ orthogonal projections, namely the algebra of functions on a discrete space of $n$ points. Since such a matrix $B$ has zero rows it has not previously been included in approaches to defining Cuntz-Krieger algebras [3], [4].

Before showing that the universal $C^*$-algebra $\mathcal{O}_B$ exists we translate the defining relations for $\mathcal{O}_B$ in terms only involving the vertex graph $E = \mathbf{E}_B$ of $B$. Recall that for $e, f \in E^0_w$, $B_w(e, f) = 1$ if and only if $e \in E^1$ and $r(e) = s(f)$. Also, for $e, f \in E^1$, the $e$-row of $B_w = f$-row of $B_w$ if and only if $r(e) = r(f)$.

***Definition 2.5*** Let $E = (E^0, E^1, r, s)$ be a directed graph with $E^0$, $E^1$ countable and denote the set of sinks by $F$. Define the graph $C^*$-algebra $G^*(E)$ to be the universal $C^*$-algebra generated by partial isometries $S_e$ ($e \in E^1 \cup F$) with orthogonal range projections such that $S_e^* S_e = S_e$ if $e \in F$ and, for $e \in E^1 \cup F$, $S_e^* S_e = \sum_{f \in s^{-1}(r(e))} S_f S_f^*$ if the set $s^{-1}\big(r(e)\big)$ is finite. Assume in addition that for $e, f \in E^1$

$$S_e^* S_e S_f^* S_f = \begin{cases} S_f^* S_f & \text{if } r(e) = r(f) \\ 0 & \text{otherwise} \end{cases}$$

and that for $e \in E^1$ and $f \in E^1 \cup F$

$$S_e^* S_e S_f S_f^* = \begin{cases} S_f S_f^* & \text{if } r(e) = s(f) \\ 0 & \text{otherwise.} \end{cases}$$

***Remark 2.6*** If $l$ is a sink and there is an edge $k$ with $r(k) = l$ then the sum condition implies $S_k^* S_k$ is the projection $S_l$. We have $S_e^* S_e S_l S_l^* = S_e^* S_e S_l = S_e^* S_e S_k^* S_k$ which is zero unless $r(e) = r(k)$, so unless $r(e) = l = r(l)$. Thus if $I$ denotes the subset of sinks $F$ consisting of the isolated vertices of $E$, namely those sinks with no incoming edges, we see that the set $F$ in Definition 2.5 may be replaced by the smaller set $I$. If $I$ is empty, so if $E$ has no isolated points, the graph $C^*$-algebra $G^*(E)$ is then the universal $C^*$-algebra generated by partial isometries $S_e$ ($e \in E^1$) with orthogonal range projections such that for $e \in E^1$, $S_e^* S_e = \sum_{f \in s^{-1}(r(e))} S_f S_f^*$ if $s^{-1}\big(r(e)\big)$ is finite, and the above additional conditions for $e, f \in E^1$ only. If $E_{\mathrm{ess}} = E \setminus I$ then clearly $G^*(E) \cong G^*(E_{\mathrm{ess}}) \oplus C_0(I)$.

As before, the additional relations in Definition 2.5 are unnecessary if the graph $E$ has $s^{-1}(r(e))$ finite for all $e \in E^1$, since if $l \in F$, $s^{-1}(r(l)) = l$ which is finite. In particular these relations are unnecessary if the only vertices emitting infinitely many edges are sources. They are necessary however for the alternate definition of Remark 2.6.

***Theorem 2.7*** *If $E$ is a directed graph and $B$ is the vertex matrix of $E$ then $G^*(E) \cong \mathcal{O}_B$.*

**Proof** This basically follows from how we arrived at the definition of $G^*(E)$. ■

For example, if $B$ is the one by one matrix with $\infty$ as its entry, the directed graph $E = \mathbf{E}_B$ corresponding to $B$, so the graph with vertex matrix $B$, is a loop of infinite multiplicity on a single vertex. We see that the in-split matrix $B_w$ is the infinite square matrix with each entry equal to one and that $\mathcal{O}_B = G^*(E)$ is the Cuntz algebra $\mathcal{O}_\infty$, where the unit is the initial projection $S_e^* S_e$ of any of the partial isometries $S_e$, $e \in E^1$.

Given a directed graph $E = (E^0, E^1, r, s)$ a universal $C^*$-algebra $C^*(E)$ was previously defined in [5], extending without change the definition for the algebra $C^*(E)$ of [6] for $E$ row finite. The $C^*$-algebra $C^*(E)$ is the universal $C^*$-algebra generated by mutually orthogonal projections $\rho_v$, $(v \in E^0)$ and partial isometries $\{\sigma_e \mid e \in E^1\}$ with orthogonal ranges such that $\sigma_e^* \sigma_e = \rho_{r(e)}$, $(e \in E^1)$, $\sigma_e \sigma_e^* \leq \rho_{s(e)}$ and $\rho_v = \sum_{e \in s^{-1}(v)} \sigma_e \sigma_e^*$ $(v \in s(E^1))$ whenever this sum is finite.

We show the existence of $\mathcal{O}_B$, or equivalently of $G^*(E)$ by showing that a certain $*$-subalgebra of $C^*(E)$ satisfies the universal property of Definition 2.5. We also show that if $B$ is the vertex matrix for $E$ and if every source of $E$ emits only finitely many edges then $\mathcal{O}_B \cong C^*(E)$, so that the two graph $C^*$-algebras $G^*(E)$ and $C^*(E)$ coincide for these directed graphs. In general however, $G^*(E)$ is an ideal of $C^*(E)$.

***Theorem 2.8*** *Let $E = (E^0, E^1, r, s)$ be a directed graph and $B = \mathbf{B}_E$ the vertex matrix of $E$. The $C^*$-algebra $\mathcal{O}_B$, or equivalently $G^*(E)$, may be identified with an ideal of $C^*(E)$. If every source of $E$ emits only finitely many edges then $G^*(E) = C^*(E)$.*

**Proof** If $\{\rho_v, \sigma_e \mid v \in E^0, e \in E^1\}$ are generators of the $C^*$-algebra $C^*(E)$, set $S_e = \sigma_e$ $(e \in E^1)$ and $S_e = \rho_e$ $(e \in F)$ and define $G$ to be the $*$-subalgebra of $C^*(E)$ generated by these partial isometries $S_e$ $(e \in E^1 \cup F)$.

We first show that these generators of $G$ satisfy the relations of Definition 2.5. The range projections of $S_e$ are mutually orthogonal. If $s^{-1}\big(r(e)\big)$ is finite for $e \in E^1$ then $S_e^* S_e = \sigma_e^* \sigma_e = \rho_{r(e)} = \sum_{f \in s^{-1}(r(e))} S_f S_f^*$ where $r(e) \in s^{-1}\big(r(e)\big)$ if $r(e)$ is a sink. Also $S_e^* S_e = S_e = \rho_e = S_e S_e^*$ if $e \in F$. The additional relations in Definition 2.5 follow similarly.

We next show that $G$ satisfies the universal property. Assume that $A$ is a $C^*$-algebra generated by partial isometries $t_e$ $(e \in E^1 \cup F)$ satisfying the relations in Definition 2.5. We shall show that there is a $*$-homomorphism $\varphi\colon G \to A$ mapping $S_e$ to $t_e$.

Let $\pi\colon A \to \mathcal{B}(\mathcal{H})$ be a faithful $*$-representation of $A$ as bounded operators on a Hilbert space $\mathcal{H}$. Define

$$\psi(\sigma_e) = \pi(t_e) \quad \text{for } e \in E^1$$

$$\psi(\rho_l) = \pi(t_l) \quad \text{for } l \in F$$

$$\psi(\rho_{r(e)}) = \pi(t_e^* t_e) \quad \text{for } e \in E^1 \text{ and } r(e) \notin F$$

$$\psi(\rho_v) = \sum_{e \in s^{-1}(v)} \pi(t_e t_e^*) \quad \text{for } v \in E^0 \text{ a source, } v \notin F.$$

The sum converges in the weak topology of $\mathcal{B}(\mathcal{H})$ to a projection in the weak closure $\overline{\pi(A)}^w$ since the projections occuring in the sum are orthogonal. The operator $\psi(\rho_{r(e)})$ is well defined since $t_e^* t_e = t_l^* t_l$ if $r(e) = r(l)$.

We check that the elements $\{\psi(\sigma_e), \psi(\rho_v) \mid e \in E^1, v \in E^0\}$ satisfy the relations defining $C^*(E)$, thus defining a $*$-homomorphism $\psi \colon C^*(E) \to \overline{\pi(A)}^w$.

Since $\pi$ is a representation of $A$ the elements $\psi(\rho_v)$ $(v \in E^0)$ are projections and $\psi(\sigma_e)$ $(e \in E^1)$ are partial isometries in $\mathcal{B}(\mathcal{H})$. If $e \in E^1$ with $r(e)$ not a sink $\psi(\sigma_e)$ has initial space $\pi(t_e^* t_e) = \psi(\rho_{r(e)})$ while if $r(e) = v$ is a sink then $t_e^* t_e = t_v^* t_v = t_v$ by Definition 2.5, so $\psi(\sigma_e)$ has initial space $\psi(\rho_{r(e)})$ in this case also.

We check that the projections $\psi(\rho_v)$, $v \in E^0$ are orthogonal. For example, if $l \in F$ and $e \in E^1$ with $r(e) \notin F$ then $l$ and $r(e)$ are distinct so the projections $t_l^* t_l$ and $t_e^* t_e$ are orthogonal in $A$ and $\psi(\rho_l)$, $\psi(\rho_{r(e)})$ are orthogonal. Also if $l \in F$ and $v \in E^0$ is a source and not a sink then $\psi(\rho_l)\psi(\rho_v) = \sum_{e \in s^{-1}(v)} \pi(t_l)\pi(t_e t_e^*)$ is zero since the range projections of the generators of $A$ are orthogonal. Similarly with $v$ as before and $e \in E^1$ with $r(e) \notin F$ then $r(e) \neq s(k)$ for any edge $k$ with $s(k) = v$. Thus by Definition 2.5 the projections $t_e^* t_e$ and $t_k t_k^*$ are orthogonal, implying that $\psi(\rho_{r(e)})$ and $\psi(\rho_v)$ are orthogonal.

We next show that $\psi(\sigma_e)\psi(\sigma_e)^* \leq \psi(\rho_{s(e)})$ for $e \in E^1$. If $s(e)$ is a source then $\psi(\sigma_e)\psi(\sigma_e)^*$ occurs in the sum defining $\psi(\rho_{s(e)})$. If $s(e)$ is not a source then $s(e) = r(l)$ for some edge $l$ and $\psi(\rho_{s(e)}) = \psi(\rho_{r(l)}) = \pi(t_l^* t_l)$ so by Definition 2.5, $\psi(\sigma_e)\psi(\sigma_e)^* \leq \psi(\rho_{s(e)})$ in this case also.

We lastly show that if $0 < |s^{-1}(v)| < \infty$, so if $v$ is not a sink and emits only finitely many edges, then $\psi(\rho_v) = \sum_{l \in s^{-1}(v)} \psi(\sigma_l)\psi(\sigma_l)^*$. If $v$ is not a source then $v = r(e)$ for an edge $e$ and $\psi(\rho_v) = \psi(\rho_{r(e)}) = \pi(t_e^* t_e)$ which is the sum by Definition 2.5. If $v$ is a source then the equality holds by definition.

This completes the argument showing the existence of the homomorphism $\psi$. The restriction of $\psi$ to $G$ has range in $\pi(A)$ and $\varphi = \pi^{-1} \circ \psi|_G$ is a $*$-homomorphism of $G$ to $A$ mapping generators to generators. Thus $G^*(E) = G$ exists and is a $*$-subalgebra of $C^*(E)$.

Since $\rho_v = \sigma_e^* \sigma_e \in G$ if $v = r(e) \notin F$ or if $v$ is a source with $0 < |s^{-1}(v)| < \infty$ it is clear that $G^*(E) = C^*(E)$ if every source emits a finite number of edges. To see that $G^*(E)$ is an ideal of $C^*(E)$ it is only necessary to check that both $\rho_v \sigma_l$ and $\sigma_l \rho_v \in G^*(E)$ for $v$ a source and $l$ an edge. We have $\sigma_l \sigma_l^* \leq \rho_{s(l)}$ and so $\rho_v \sigma_l = \rho_v \sigma_l \sigma_l^* \sigma_l$ which is $\sigma_l \sigma_l^* \sigma_l = \sigma_l$ if $s(l) = v$, and zero otherwise. Also $\sigma_l \rho_v = \sigma_l \sigma_l^* \sigma_l \rho_v = \sigma_l \rho_{r(l)} \rho_v$ which is $\sigma_l$ if $r(l) = v$ and zero otherwise. ■

Even in the finite case this theorem asserts something new, namely that one has not obtained new $C^*$-algebras by considering $C^*$-algebras $C^*(E)$ of graphs $E$, since if $B = \mathbf{B}_E$ is the vertex matrix of $E$ then $C^*(E) \cong \mathcal{O}_B$. Returning for a moment to the graph $E$ with $n$ edges, $n \in \mathbb{Z}_+$ and $n \neq 0$, say $e_1, \dots, e_n$, and two vertices $v_0$, $v_1$ with $r(e_i) = v_1$, $s(e_i) = v_0$ we see that $B = \mathbf{B}_E$ is the matrix $\left[\begin{smallmatrix} 0 & n \\ 0 & 0 \end{smallmatrix}\right]$. The complete in-split matrix $B_\omega$ is the $n+1$ by $n+1$ square matrix $\begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ & & & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$. The algebra $C^*(E)$ is the universal $C^*$-algebra generated by two orthogonal projections $\rho_0$, $\rho_1$ and $n$

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partial isometries $\sigma_1, \dots, \sigma_n$ with initial space $\rho_1$ and $\rho_0 = \sum^n \sigma_i \sigma_i^*$. The universal $C^*$-algebra $\mathcal{O}_B$ is generated by $n+1$ partial isometries $S_1, \dots, S_{n+1}$ with orthogonal ranges, $S_{n+1} = S_{n+1}^* S_{n+1}$ is a projection $p$ and $S_i^* S_i = \sum B_\omega(i,j) S_j S_j^* = S_{n+1} S_{n+1}^* = p$ for all $i$. It is clear these algebras are isomorphic.

Note that the complete in-split matrix $B_\omega$ of the vertex matrix $B$ is just the edge matrix $A_E$ of the directed graph $E$ if $E$ has no sinks. Thus, if $E$ has no sinks or sources it follows immediately from Theorem 2.8 that $C^*(E) \cong \mathcal{O}_B$, while from [5] we have $C^*(E) \cong \mathcal{O}_{A_E}$ for such $E$, where $\mathcal{O}_{A_E}$ is the Cuntz-Krieger algebra defined in [4]. Thus $\mathcal{O}_B$, which is defined in terms of the matrix $B_\omega = A_E$, is just the same as the algebra $\mathcal{O}_{A_E}$ of [4] in this case. Theorem 2.8 is valid however even when sinks and sources are present.

We make a few remarks concerning the Cuntz-Krieger algebra defined in [4]. This algebra is associated with a possibly infinite $\{0,1\}$-valued matrix $B$ with no zero rows, so the directed graph $E = \mathbf{E}_B$ has no sinks. For now we denote it by $\mathcal{O}(B)$. Using that $B$ is a $\{0,1\}$ matrix one can show that there is a $*$-homomorphism $\varphi\colon \mathcal{O}_B \to \mathcal{O}(B)$ mapping $S_e$ to $s_{s(e)}(s_{r(e)} s_{r(e)}^*)$ where $S_e$ $(e \in E^1)$ are generators of $\mathcal{O}_B$ satisfying the relations of Definition 2.3 and where $s_w$ $(w \in E^0)$ are partial isometries generating $\mathcal{O}(B)$ satisfying the relations in [4]. The image $\varphi(\mathcal{O}_B)$ is a left ideal in $\mathcal{O}(B)$. In addition, if $B_w$ is a row finite matrix, so if the vertices of $E$ which are not sources emit a finite number of edges, then $\psi(\mathcal{O}_B)$ is an ideal in $\mathcal{O}(B)$. If the matrix $B$ is row finite then $\varphi$ is a surjection since $s_w = s_w(s_w^* s_w) = s_w\left(\sum B(w,z) s_z s_z^*\right) = \sum s_{s(l)} s_{r(l)} s_{r(l)}^* = \sum \varphi(S_l)$, where the last two sums are over $l \in E^1$ with $s(l) = w$. It also follows easily from Theorem 10 of [5] and Theorem 3.4 below that if $E$ has no sources in addition to $B_w$ being row finite, then $\mathcal{O}_{B_w} \cong \mathcal{O}(B_w)$.

## 3 Invariance under Matrix In-Splits

The Cuntz-Krieger algebras $\mathcal{O}_A$ for $A$ a finite matrix with entries in $\mathbb{Z}_+$ are invariant under in-splits and in-amalgamations of the matrix $A$ [2]. In the following we examine how the general Cuntz-Krieger algebras $\mathcal{O}_B$ behave under in-splits, where $B$ is an arbitrary square matrix with entries in $\mathbb{Z}_+^\infty$.

For the remainder of this section we fix the following notation. Let $E = (E^0, E^1, r, s,)$ be the directed graph with vertex matrix $B$, and let $\underset{\sim}{E} = (\underset{\sim}{E}^0, \underset{\sim}{E}^1, \underset{\sim}{r}, \underset{\sim}{s})$ be the directed graph with vertex matrix $C$ where $C$ is an in-split of the matrix $B$.

We first restate the process of forming an in-split $C$ of the matrix $B$ in terms of the associated directed graphs $\underset{\sim}{E}$ and $E$. Recall from Remark 1.4 that this will describe what is more accurately called an out-split procedure for directed graphs.

To arrive at the graph $\underset{\sim}{E}$ we replace each vertex $v \in E^0$ with $m(v)$ vertices $v_1, \dots, v_{m(v)}$ where the set of edges $\{f \mid s(f) = v\} = s^{-1}(v)$ are partitioned into $m(v)$ sets. Here we mean that $v$ is replaced by $\{v_n \mid n \in \mathbb{N}\}$ if $m(v) = \infty$. Also, if an edge $e \in E^1$ lies in some partition set, say the $p$-th, of $s^{-1}(v)$ where $s(e) = v$ and $r(e) = w$ then $e$ is replaced with $m(w)$ edges $e_1, \dots, e_{m(w)}$, each with the same source $v_p$, but with ranges $w_1, \dots, w_{m(w)}$ respectively. Notice that if $v$ is a sink then $s^{-1}(v) = v$ using the extended range and source maps $r, s\colon E^1 \cup F \to E^0$, and so $v$ is left untouched and becomes a sink in $\underset{\sim}{E}$ also.

We will find it useful to restate this process in terms of equivalence relations on $E^1 \cup F$ rather than partitions of the disjoint sets $s^{-1}(v)$, $v \in E^0$ . Begin by choosing any equivalence relation $\approx$ on $E^1 \cup F$ so that the source map yields a well defined map of $E^1 \cup F/\approx$ to $E^0$; in other words $e \approx f$ implies $s(e) = s(f)$. Then each vertex $v \in E^0$ is replaced by the set $\{[e] \mid e \in s^{-1}(v)\}$ and each edge $e \in E^1$ is replaced with the edges $\{(e, [f]) \mid f \in E^1 \cup F, r(e) = s(f)\}$. Here note that if $l \in F$ and $r(e) = s(l)$ then $r(e) = l$ and $[l]$ consists of the single point $l$, so $e$ is replaced by a single edge $(e, l)$. The source $\underset{\sim}{s}$ of the edge $(e, [f])$ is the vertex $[e]$ and the range $\underset{\sim}{r}$ of $(e, [f])$ is the vertex $[f]$.

***Lemma 3.1*** *Let $E$ be a directed graph with vertex matrix $B$ and $\underset{\sim}{E}$ the directed graph with vertex matrix $C$ with $C$ an in-split of $B$. Then $\underset{\sim}{E} = (\underset{\sim}{E}^0, \underset{\sim}{E}^1, \underset{\sim}{r}, \underset{\sim}{s})$ with $\underset{\sim}{E}^0 = E^1 \cup F/\approx$, $\underset{\sim}{E}^1 = \{(e, [f]) \mid e \in E^1,\ f \in E^1 \cup F,\ r(e) = s(f)\}$, $\underset{\sim}{r}(e, [f]) = [f]$, and $\underset{\sim}{s}(e, [f]) = [e]$ where $\approx$ is an equivalence relation satisfying $e \approx f$ implies $s(e) = s(f)$.*

Note that the sinks of $\underset{\sim}{E}$ are $\{[l] \mid l \in F\}$ where $[l]$ consists of the single element $l$. Thus $F$ consists of the sinks of $\underset{\sim}{E}$. If $l \in F$ then a pair $(l, [f])$ satisfying $f \in E^1 \cup F$ and $r(l) = s(f)$ can only be $(l, [l])$, so $F$ may be identified with $\{(l, [l]) \mid l \in F\}$ and the set $\underset{\sim}{E}^1 \cup F = \{(e, [f]) \mid e, f \in E^1 \cup F, r(e) = s(f)\}$. This set is the pullback of the range map $r\colon E^1 \cup F \to E^0$ and the well defined source map $s\colon E^1 \cup F/{\approx} \to E^0$.

We wish to single out those in-splits that satisfy the requirement that the partition of the edges in $s^{-1}(v)$ where $v \in E^0$ is not a source consists of sets containing only a finite number of edges. Note though that the number of sets of the partition may still be infinite. Also, the complete in-split is an example of this type of in-split. If $B_\omega$ is row finite then $s^{-1}(v)$ is finite for $v \in E^0$, $v$ not a source, so in this case any in-split of $B$ is also of this type. In terms of the equivalence relation $\approx$ on $E^1 \cup F$ this type of in-split is characterized by stipulating that only finitely many edges occur in each equivalence class $[f]$ with $s(f) = r(e)$ for some $e \in E^1$, so for $s(f)$ not a source. We define an equivalent condition for the matrix $C$.

***Definition 3.2*** The in-split matrix $C$ of $B$ is a proper in-split if the $[f]$ row of $C$ has only finite entries whenever the $[f]$ column is nonzero.

The above comments show that if $B_\omega$ is row finite then $C$ is always a proper in-split of $B$. Also if $C$ is row finite then $B_\omega$ is also row finite, but the converse is false in general, as one sees by letting $C = B$ for example. The following theorem shows there is a $*$-homomorphism $\varphi\colon G^*(\underset{\sim}{E}) \to G^*(E)$ whenever $C$ is a proper in-split of $B$.

***Theorem 3.3*** *Let $E$ be a directed graph with vertex matrix $B$ and $\underset{\sim}{E}$ a directed graph with vertex matrix $C$ where $C$ is a proper in-split of $B$. Then there is a $*$-homomorphism $\varphi\colon \mathcal{O}_C \to \mathcal{O}_B$. In particular there is a $*$-homomorphism $\mathcal{O}_C \to \mathcal{O}_B$ if $C = B_w$, the complete in-split of $B$.*

**Proof** Let $\{S_e \mid e \in E^1 \cup F\}$ be generators for $G^*(E)$ satisfying the relations of Definition 2.5, and $\{T_{(e,[f])} \mid (e, [f]) \in \underset{\sim}{E}^1 \cup F\}$ generators for $G^*(\underset{\sim}{E})$ satisfying the relations of Definition 2.5. Define $\varphi(T_{(e,[f])}) = S_e(\sum_{h\in[f]} S_h S_h^*)$. It is sufficient to

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check that $\{\varphi(T_{(e,[f])}) \mid (e,[f]) \in \underset{\sim}{E}^1 \cup F\}$ satisfies the relations of Definition 2.5 to conclude that this defines a $*$-homomorphism $\varphi$. The sum occurring in the definition of $\varphi$ is finite since the in-split is proper. We compute $\varphi(T_{(e,[f])})^*\varphi(T_{(e',[f'])}) = (\sum_{h\in[f]} S_hS_h^*)S_e^*S_{e'}(\sum_{h'\in[f']} S_{h'}S_{h'}^*)$ which is zero unless $e = e'$. If $e = e'$ then $S_e^*S_eS_{h'} = S_e^*S_eS_{h'}S_{h'}^*S_{h'} = S_{h'}$ by Definition 2.5, so the entire expression $= (\sum_{h\in[f]} S_hS_h^*)(\sum_{h'\in[f']} S_{h'}S_{h'}^*)$ which is the projection $\sum S_hS_h^*$ if $[f] = [f']$, and is zero if $[f] \neq [f']$ since equivalence classes are then disjoint. Thus $\varphi(T_{(e,[f])})$ are partial isometries with orthogonal final ranges. For $(l,[l]) \in F$ with $l \in F$, $\varphi(T_{(l,[l])}) = S_lS_lS_l^* = S_l$ which is a projection, so the next property of Definition 2.5 is true. The calculation above shows that the initial projection of $\varphi(T_{(e,[f])})$ is $\sum_{h\in[f]} S_hS_h^*$, so

$$\begin{aligned}\varphi(T_{(e,[f])})^*&\varphi(T_{(e,[f])})\varphi(T_{(e',[f'])})^*\varphi(T_{(e',[f'])})\\ &= \begin{cases}\varphi(T_{(e',[f'])})^*\varphi(T_{(e',[f'])}) & \text{if } \underset{\sim}{r}(e,[f]) = \underset{\sim}{r}(e',[f'])\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Since

$$\Big(\sum_{h\in[f]} S_hS_h^*\Big)S_{e'} = \begin{cases}S_{e'} & \text{if } e' \in [f]\\ 0 & \text{otherwise}\end{cases}$$

and $e' \in [f]$ if and only if $[e'] = [f]$ it follows that

$$\begin{aligned}\varphi(T_{(e,[f])})^*&\varphi(T_{(e,[f])})\varphi(T_{(e',[f'])})\varphi(T_{(e',[f'])})^*\\ &= \begin{cases}\varphi(T_{(e',[f'])})\varphi(T_{(e',[f'])})^* & \text{if } \underset{\sim}{r}(e,[f]) = \underset{\sim}{s}(e',[f'])\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

It remains to show $\varphi(T_{(e,[f])})^*\varphi(T_{(e,[f])}) = \sum \varphi(T_{(a,[b])})\varphi(T_{(a,[b])})^*$ if the sum is finite and where the sum is over those $(a,[b])$ with $\underset{\sim}{s}(a,[b]) = \underset{\sim}{r}(e,[f])$, so those $(a,[b])$ with $[a] = [f]$. Since this is a proper in-split, the number of $a \in [f]$ is finite, so this set of $(a,[b])$ is finite if and only if the set $\{[b] \mid r(a) = s(b)\}$ is finite for each $a \in [f]$. Again, however, each class $[b]$ is finite, so this set is finite if and only if $\{\, b \mid b \in s^{-1}\big(r(a)\big)\}$ is finite for each $a \in [f]$. Thus, if the index set for the sum on the right side is finite, so is $s^{-1}\big(r(a)\big)$, and by the corresponding property for $G^*(E)$, $S_a^*S_a = \sum_{h\in s^{-1}(r(a))} S_hS_h^*$. Expanding the right side we obtain $\sum_{(a,[b])} S_a(\sum_{h\in[b]} S_hS_h^*)S_a^* = \sum_{a\in[f]} S_a(\sum_{h\in[b],b\in s^{-1}(r(a))} S_hS_h^*)S_a^* = \sum_{a\in[f]} S_a(\sum_{h\in s^{-1}(r(a))} S_hS_h^*)S_a^* = \sum_{a\in[f]} S_a(S_a^*S_a)S_a^* = \sum_{a\in[f]} S_aS_a^*$ which is the initial projection of $\varphi(T_{(e,[f])})$. ■

The $*$-homomorphism $\varphi\colon \mathcal{O}_C \to \mathcal{O}_B$ for $C$ a proper in-split of $B$ is an isomorphism if the complete in-split matrix $B_\omega$ is row finite. This extends to the infinite matrix case the known isomorphism results for the finite matrix Cuntz-Krieger algebras [2].

***Theorem 3.4*** *Let $C$ be an in-split matrix for $B$ where the matrix $B_\omega$ is row finite. The map $\varphi\colon \mathcal{O}_C \to \mathcal{O}_B$ of Theorem 3.3 is an isomorphism.*

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**Proof** Let $E$ and $\underset{\sim}{E}$ denote the directed graphs with vertex matrices $B$ and $C$ respectively. We define $\psi\colon G^*(E) \to G^*(\underset{\sim}{E})$ by specifying $\psi$ on generators $\{S_e \mid e \in E^1 \cup F\}$ of $G^*(E)$ and showing that $\{\psi(S_e) \mid e \in E^1 \cup F\}$ satisfy the relations of Definition 2.5.

Define $\psi(S_e) = \sum T_{(e,[f])}$ where the sum is over $[f]$ with $(e,[f]) \in \underset{\sim}{E}^1 \cup F$. This sum is finite if $B_\omega$ is row finite. For $l \in F$ we have $\psi(S_l) = T_{(l,[l])}$ and so $\psi(S_l)^*\psi(S_l) = \psi(S_l)$. In $G^*(\underset{\sim}{E})$, $T_{(e,[f])}T^*_{(e,[h])} = 0$ unless $\underset{\sim}{r}(e,[f]) = \underset{\sim}{r}(e,[h])$, so unless $[f] = [h]$, and it follows that $\psi(S_e)\psi(S_e)^*$ is a finite sum of orthogonal projections. Thus it is a projection and $\psi(S_e)$ is a partial isometry. Also in $G^*(\underset{\sim}{E})$, $T^*_{(e,[f])}T_{(a,[b])} = 0$ unless $(e,[f]) = (a,[b])$, and a quick computation shows that the range projections of $\psi(S_e)$ are orthogonal.

Since $B_\omega$ is row finite, $s^{-1}\big(r(e)\big)$ is finite for $e \in E^1 \cup F$. Thus the additional relations of Definition 2.5 are superfluous and we only need to check that $\psi(S_e)^*\psi(S_e) = \sum_{a\in s^{-1}(r(e))} \psi(S_a)\psi(S_a)^*$ for $e \in E^1 \cup F$. First note that $T^*_{(e,[f])}T_{(e,[f])} = \sum T_{(a,[b])}T^*_{(a,[b])}$ where the sum is over those $(a,[b]) \in \underset{\sim}{E}^1 \cup F$ with $[a] = \underset{\sim}{s}(a,[b]) = \underset{\sim}{r}(e,[f]) = [f]$; in other words those $(a,[b])$ with $a \in [f]$. As $[f]$ varies under the restriction that $(e,[f]) \in \underset{\sim}{E}^1 \cup F$ we thus obtain all possible $a \in s^{-1}\big(r(e)\big)$. We have

$$\begin{aligned}\psi(S_e)^*\psi(S_e) &= \sum_{[f]} T^*_{(e,[f])}T_{(e,[f])}\\ &= \sum_{[f]}\Big(\sum_{\substack{a\in[f]\\(a,[b])}} T_{(a,[b])}T^*_{(a,[b])}\Big)\\ &= \sum_{a\in s^{-1}(r(e))}\Big(\sum_{(a,[b])} T_{(a,[b])}T^*_{(a,[b])}\Big)\\ &= \sum_{a\in s^{-1}(r(e))} \psi(S_a)\psi(S_a)^*.\end{aligned}$$

Since $B_\omega$ is row finite the in-split $C$ is proper and the map $\varphi\colon \mathcal{O}_C \to \mathcal{O}_B$ of Theorem 3.3 exists. The composition $\varphi\circ\psi$ maps $S_e$ to $\sum_{[f]}\varphi(T_{(e,[f])}) = \sum S_e \sum_{h\in[f]} S_hS_h^* = S_e\sum_{h\in s^{-1}(r(e))} S_hS_h^* = S_eS_e^*S_e = S_e$ while $\psi\circ\varphi$ maps $T_{(e,[f])}$ to $\psi(S_e\sum_{a\in[f]} S_aS_a^*) = (\sum_{[g]} T_{(e,[g])})\big(\sum_{a\in[f]}(\sum_{[b]} T_{(a,[b])}T^*_{(a,[b])})\big) = T_{(e,[f])}(\sum_{\substack{a\in[f]\\(a,[b])}} T_{(a,[b])}T^*_{(a,[b])})$ since $T_{(e,[g])}T_{(a,[b])} = 0$ unless $[g] = \underset{\sim}{r}(e,[g]) = \underset{\sim}{s}(a,[b]) = [a] = [f]$. This latter expression is $T_{(e,[f])}(T^*_{(e,[f])}T_{(e,[f])}) = T_{(e,[f])}$. Thus $\varphi = \psi^{-1}$ is an isomorphism. ■

This last result, or Theorem 2.8, gives easy access to examples of directed graphs $E$ with $G^*(E)$ not isomorphic to $C^*(E)$. For example let $E$ have two vertices $v$, $w$ with edges $\{e_n \mid n \in \mathbb{N}\}$ and $s(e_n) = v$, $r(e_n) = w$, $(n \in \mathbb{N})$. The vertex matrix of $E$ is $B = \left[\begin{smallmatrix} 0 & \infty\\ 0 & 0\end{smallmatrix}\right]$ which has complete in-split $B_\omega = \begin{bmatrix} 0 & \cdots & 0 & 1\\ \vdots & & \vdots & \vdots\\ 0 & \cdots & 0 & 1\\ 0 & \cdots & 0 & 0\end{bmatrix}$, a row finite matrix. If $\underset{\sim}{E}$ is the directed graph with vertex matrix $B_\omega$ then $G^*(E) \cong G^*(\underset{\sim}{E})$ by Theorem 3.4. Theorem 2.8 implies that $G^*(\underset{\sim}{E}) \cong C^*(\underset{\sim}{E})$. We show that $C^*(E)$ is not isomorphic to

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$C^*(\underset{\sim}{E})$, so Theorem 3.4 does not apply to $C^*(E)$ in general. This is equivalent to $C^*(E)$ and $G^*(E)$ being nonisomorphic. For this specific example, $C^*(E)$ is $G^*(E)$ with an adjoined unit. The $C^*$-algebra $C^*(E)$ is generated by two orthogonal projections $\rho_1$, $\rho_2$ and partial isometries $\sigma_n$, $(n \in \mathbb{N})$ with orthogonal ranges $\sigma_n\sigma_n^* \leq \rho_1$ and initial spaces all equal to $\rho_2$. Thus $\rho_1 + \rho_2$ is the unit for $C^*(E)$. The algebra $G^*(E)$ is the nonunital algebra generated by partial isometries $\{S_n \mid n \in \mathbb{N}\}$ along with a projection $S_\omega$ so that the projections $\{S_nS_n^* \mid n \in \mathbb{N}\} \cup \{S_\omega\}$ are orthogonal and the initial projections $S_n^*S_n$ are all equal to $S_w$.

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