Forum of Mathematics, Sigma (2021), **Vol. 9**:e75 1–25
doi:10.1017/fms.2021.65

CrossMark

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

On the Tits–Weiss conjecture and the Kneser–Tits conjecture for $E_{7,1}^{78}$ and $E_{8,2}^{78}$ (With an Appendix by R. M. Weiss)

Seidon Alsaody[1], Vladimir Chernousov[2] and Arturo Pianzola[3,4]

[1]Department of Mathematics, Uppsala University, P.O. Box 480, 751 06 Uppsala, Sweden; E-mail: seidon.alsaody@math.uu.se.
[2]Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1, Canada; E-mail: vladimir@ualberta.ca.
[3]Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1, Canada.
[4]Centro de Altos Estudios en Ciencias Exactas, Avenida de Mayo 866, Buenos Aires, 1084 Argentina; E-mail: a.pianzola@gmail.com.

Received: 23 August 2020; **Revised:** 27 April 2021; **Accepted:** 18 September 2021

2020 Mathematics Subject Classification: *Primary* – 20G41; *Secondary* – 17C40, 20G15

Abstract
We prove that the structure group of any Albert algebra over an arbitrary field is R-trivial. This implies the Tits–Weiss conjecture for Albert algebras and the Kneser–Tits conjecture for isotropic groups of type $E_{7,1}^{78}, E_{8,2}^{78}$. As a further corollary, we show that some standard conjectures on the groups of R-equivalence classes in algebraic groups and the norm principle are true for strongly inner forms of type ${}^1E_6$.

Contents

1. Introduction

The primary aim of this article is to prove the long standing Tits–Weiss conjecture on U-operators in Albert algebras and the Kneser–Tits conjecture for algebraic groups of type $E_{7,1}^{78}$ and $E_{8,2}^{78}$.

The Tits–Weiss conjecture asserts that the structure group $\mathrm{Str}(A)$ of an arbitrary Albert algebra A is generated by the inner structure group, formed by the so-called U-operators, and the central homotheties. This problem was raised by Tits and Weiss in their 2002 book [26], where they studied spherical buildings and the corresponding generalised polygons attached to isotropic groups of relative rank 2. Despite many efforts, this problem has remained unsolved.

If $\mathbf{G}$ is an isotropic simple simply connected group over a field K of relative rank ≥ 2, then by [18] the group $\mathbf{G}(K)$ is generated by the K-points of isotropic subgroups of $\mathbf{G}$ of relative rank 1. This result allows one to reduce many problems for $\mathbf{G}(K)$ to groups of relative rank 1. For instance, this is the case for the Kneser–Tits problem (see the paragraph below). Note also that isotropic groups of relative rank 1 give rise to important examples of more general groups of rank 1. The latter were introduced by Tits in the early 1990s, who called them Moufang sets. They have proved to be important in the classification of simple groups, incidence geometry, the theory of buildings and other areas. Further still, rank 1 groups are useful in studying isotropic groups of exceptional types, where algebraic groups and their associated root subgroups are typically parametrised by a nonassociative structure and, as emphasised in [6], a rich interplay emerges between rank 1 groups, nonassociative algebras and linear algebraic groups.

The Kneser–Tits conjecture for a simple simply connected isotropic group $\mathbf{G}$ over a field K asserts that the abstract group $\mathbf{G}(K)$ of K-points of $\mathbf{G}$ coincides with its normal subgroup $\mathbf{G}(K)^+$ generated by the unipotent radicals of the minimal parabolic K-subgroups of $\mathbf{G}$. We refer to [9] for a survey of the history and recent results on this conjecture. Its importance comes from the fact that the group $\mathbf{G}(K)^+$ has a natural BN-pair structure and hence is projectively simple (i.e., simple modulo its centre) by a celebrated theorem of Tits. So if $\mathbf{G}(K) = \mathbf{G}(K)^+$, we would have many more new examples of projectively simple abstract groups. In this way, we would obtain analogues of finite simple groups of Lie type in the case of infinite fields. It is also worth mentioning that the information about the normal subgroup structure of $\mathbf{G}(K)$ is crucial in the arithmetic of algebraic groups for studying, among other things, congruence subgroups, discrete subgroups, lattices and locally symmetric spaces. In general, the Kneser–Tits conjecture does not hold and the first counterexample was constructed by V. Platonov in 1975 [17]. However, specialists believe that the conjecture does hold for many isotropic groups of exceptional type, including those of type $E_{7,1}^{78}$ and $E_{8,2}^{78}$.

The bridge connecting the Tits–Weiss conjecture and the Kneser–Tits conjecture for the abovementioned forms of type E_7 and E_8 is provided by a theorem of Tits and Weiss (see the Appendix), which states that the two conjectures are equivalent.[1] It is interesting to mention that the proof in the Appendix is characteristic free. Furthermore, using P. Gille's results on Whitehead groups [9], one can easily see that

1. the Kneser–Tits conjecture for the abovementioned groups reduces to the R-triviality of structure groups of Albert algebras and
2. the conjecture holds in arbitrary characteristic once it is established in characteristic zero.

Our main result is the following.

Theorem. *Let A be an Albert algebra over a field K. Then the structure group* $\mathbf{Str}(A)$ *of A is R-trivial; that is, for any field extension F/K the group of R-equivalence classes* $\mathbf{Str}(A)(F)/R$ *is trivial.*

[1]The proof of the equivalence of the two conjectures is not straightforward and no direct reference is available. We are grateful to Richard Weiss for writing a detailed proof of this result. It is included as an appendix to this article.

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

As explained above, this implies that the Tits–Weiss conjecture on U-operators holds for Albert algebras over any field and that the same is true for the Kneser–Tits conjecture for groups of type $E_{7,1}^{78}$ and $E_{8,2}^{78}$. Our proof is geometric in nature. We carefully analyse the properties of the natural action of the structure group $\mathbf{Str}(A)$ on the corresponding Albert algebra A. The information that we need is encoded in the Galois cohomology of the stabilisers of subalgebras of A. We compute the Galois cohomology of all these stabilisers and, using this information, we explicitly construct a system of generators of $\mathbf{Str}(A)(K)$, which we prove are R-trivial.

An important step of the proof of our main result is the weak Skolem–Noether theorem for isomorphic embeddings due to S. Garibaldi and H. Petersson (for the terminology, definitions and the precise statement we refer to [7]). In Section 5 we prove Theorem 5.3, which can be viewed as a weaker version of S. Garibaldi and H. Petersson's result.

We would also like to mention that it follows directly from our main theorem that two standard conjectures hold for simple simply connected strongly inner forms of type E_6: the abelian nature of the group of R-equivalence classes and the existence of transfers for the functor of R-equivalence classes. For these groups, the norm principle holds as well. For details we refer to the last section of the article.

Lastly, we mention that our results were proved independently by M. Thakur [24]. His proof differs from ours and is based on some explicit formulas for automorphisms of subalgebras of Albert algebras and extensions of these automorphisms.

Conventions and Notation

Throughout, K will denote a fixed field. A *K-ring* is a morphism $K \to R$ in the category of unital commutative associative rings. As is customary, by an abuse of language, the target R of such a morphism is also referred to as a K-ring.

By an *algebraic K-group*, or simply a K-group for convenience, we will understand a group scheme $\mathbf{G}$ of finite type over $\operatorname{Spec}(K)$.[2] The connected component of the identity of an algebraic K-group $\mathbf{G}$ will be denoted by $\mathbf{G}^{\circ}$.

Let A be a finite-dimensional algebra over K (not assumed associative or commutative). For any K-ring R the R-module $A \otimes_K R$ has a natural R-algebra structure. We denote by $\mathbf{Aut}(A)$ the algebraic K-group whose functor of points is given by

$$\mathbf{Aut}(A) : R \to \operatorname{Aut}_{\text{R-alg}}(A \otimes_K R).$$

Let V be a K-subspace of A. We let $\mathbf{Aut}(A, V)$ and $\mathbf{Aut}(A/V)$ be the closed algebraic subgroups of $\mathbf{Aut}(A)$ whose functor of points are given by

$$\mathbf{Aut}(A, V) : R \to \{g \in \operatorname{Aut}_{\text{R-alg}}(A \otimes_K R) : g(V \otimes_K R) = V \otimes_K R\}$$

and

$$\mathbf{Aut}(A/V) : R \to \{g \in \operatorname{Aut}_{\text{R-alg}}(A \otimes_K R) : g_{|(V \otimes_K R)} = \mathrm{id}\},$$

respectively.

If $\mathbf{G}$ is an algebraic K-group, we will denote $H^1_{\text{fppf}}(K, \mathbf{G})$ simply by $H^1(K, \mathbf{G})$. Whenever $\mathbf{G}$ is smooth, one knows that $H^1_{\text{fppf}}(K, \mathbf{G}) = H^1_{\text{ét}}(K, \mathbf{G})$ and that this last is nothing but the usual (nonabelian) Galois cohomology.

In view of point (2) above, in order to prove our main results we may assume that K is of characteristic zero. That said, many of the preliminary results are of independent interest and hold with less restrictions on the nature of the base field. For this reason, we will henceforth assume, unless specifically stated otherwise, that K of characteristic different from 2 and 3.

[2]All of the K- groups considered in our article are algebraic and affine group schemes over $\operatorname{Spec}(K)$.

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

2. Preliminaries

For later use we record some facts about Albert algebras and algebraic groups.

2.1. Albert Algebras

A *Jordan algebra* over K is a unital, commutative, not necessarily associative K-algebra A in which the Jordan identity

$$(xy)(xx) = x(y(xx))$$

is satisfied. In particular, A is power associative. Given an associative algebra B with multiplication $\cdot$, the anticommutator $\frac{1}{2}(x \cdot y + y \cdot x)$ defines on B the structure of a Jordan algebra, denoted by B^+. A Jordan algebra A is said to be *special* if it is isomorphic to a Jordan subalgebra of B^+ for an associative algebra B and *exceptional* otherwise. An *Albert algebra* is by definition a simple, exceptional Jordan algebra. It is known that any Albert algebra has dimension 27 and that, if the field K is separably closed, then all Albert algebras over K are isomorphic, as follows from [13, Proposition 37.11]. Thus, all Albert algebras over K are twisted forms of each other.

If A is an Albert algebra over K, then $\mathbf{H} = \mathbf{Aut}(A)$ is a simple K-group of type F_4. It is known that any such group arises in this fashion and that two Albert algebras are isomorphic if and only if their automorphism groups are. Moreover, A is equipped with a cubic form $N : A \to K$, called the *norm* of A. Our main object of study is the *structure group* $\mathbf{Str}(A)$ of A. This is the algebraic K-group whose functor of points is as follows: if R is a K-ring,

$$\mathbf{Str}(A)(R) = \{x \in \mathbf{GL}(A)(R) \mid N_R(x(a)) = \nu(x)N_R(a) \;\; \forall\, a \in A \otimes_K R\},$$

where N_R is the base change of N to $A \otimes_K R$ and the *multiplier* $\nu(x) \in R^\times$.

The derived subgroup

$$\mathbf{G} = [\mathbf{Str}(A), \mathbf{Str}(A)]$$

of $\mathbf{Str}(A)$ is known to be a strongly inner form of a split simple simply connected K-group of type E_6. Moreover, $\mathbf{Str}(A)$ is an almost direct product of $\mathbf{G}_m$ and $\mathbf{G}$ (the intersection being the centre of $\mathbf{G}$). Hence,

$$\overline{\mathbf{G}} = \mathbf{Str}(A)/\mathbf{G}_m$$

is an adjoint K-group of type E_6. Since in the split and even isotropic case, $\mathbf{G}$ and $\mathbf{Str}(A)$ are R-trivial (see [5]), we may, in the proof of the main theorem, assume that $\mathbf{G}$ is K-anisotropic. This amounts to the Albert algebra A being a division algebra, which is equivalent to the norm map N being anisotropic; that is, the equation $N(a) = 0$ having no nonzero solutions over K.

We denote the group of K-points of $\mathbf{Str}(A)$ (respectively $\mathbf{G}$, $\overline{\mathbf{G}}$, $\mathbf{H}$) by $\mathrm{Str}(A)$ (respectively G, $\overline{G}$, H). The group $\mathbf{H}$ coincides with the stabiliser in $\mathbf{Str}(A)$ of $1 \in A$; see e.g. [20, 5.9.4].

2.2. R-equivalence in Algebraic Groups

Let $\mathbf{G}$ be an affine algebraic K-group. Two K-points $x, y \in \mathbf{G}(K)$ are called *R-equivalent* if there is a path from x to y; that is, if there exists a rational map $f : \mathbb{A}^1_K \dashrightarrow \mathbf{G}$ defined at 0, 1 and mapping 0 to x and 1 to y. One can easily verify that this is indeed an equivalence relation on $\mathbf{G}(K)$ and that, moreover, $\mathbf{G}$ induces a group structure on the set $\mathbf{G}(K)/R$ of all R-equivalence classes. We will denote the set of elements in $\mathbf{G}(K)$ equivalent to 1 by $R\mathbf{G}(K)$.

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

2.3. *R-triviality of Cohomology Classes and the Norm Principle*

Let $\mathbf{G}$ be a semisimple K-group, $\mathbf{Z} \subset \mathbf{G}$ a central subgroup and let

$$[\xi] \in \operatorname{Ker}[H^1(K, \mathbf{Z}) \longrightarrow H^1(K, \mathbf{G})]. \tag{1}$$

Definition. We say that $[\xi]$ is *R-trivial* if there exists

$$c(t) = [\xi(t)] \in \operatorname{Ker}[H^1(K(t), \mathbf{Z}_{K(t)}) \longrightarrow H^1(K(t), \mathbf{G}_{K(t)})],$$

with $K(t)$ a purely transcendental extension of K, such that $c(t)$ is defined at $t = 0, 1$ and $c(0) = 1$ and $c(1) = [\xi]$.

Remark 2.1. The above definition requires some clarification. Here and below, if $\mathbf{G}$ is an algebraic K-group, then an element in $\mathbf{G}(K(t))$ (respectively a class in $H^1(K(t), \mathbf{G}_{K(t)})$), where t is a variable over K, is said to be *defined at 0 and 1* if it is in the image of $\mathbf{G}(\mathcal{O})$ (respectively $H^1(\mathcal{O}, \mathbf{G}_{\mathcal{O}})$), where $\mathcal{O}$ is the intersection in $K(t)$ of the localisations $K[t]_{(t)}$ and $K[t]_{(t-1)}$. In particular, via the evaluation maps $\varepsilon_0, \varepsilon_1 : \mathcal{O} \to K$, we can evaluate such an element or a class at 0 and at 1.

Example 2.2. Let D be a central simple algebra of degree n over K and set $\mathbf{G} = \mathbf{SL}(1, D)$. The centre $\mathbf{Z}$ of $\mathbf{G}$ is isomorphic to $\boldsymbol{\mu}_n$ and thus $H^1(K, \mathbf{Z}) \simeq K^\times/K^{\times n}$. Moreover, $H^1(K, \mathbf{G}) \simeq K^\times/\operatorname{Nrd}(D^\times)$. Hence,

$$\operatorname{Ker}[H^1(K, \mathbf{Z}) \longrightarrow H^1(K, \mathbf{G})] \simeq \operatorname{Nrd}(D^\times)/K^{\times n},$$

and as D is the affine space $\mathbb{A}_K^{n^2}$, any element of the above kernel is R-trivial.

Example 2.3. Let f be a Pfister form over K, set $\mathbf{G} = \mathbf{Spin}(f)$ and let again $\mathbf{Z} \subset \mathbf{G}$ be its centre. Then all cohomology classes in (1) are R-trivial; indeed, by [14, Proposition 7] the group $\mathbf{G}/\mathbf{Z} = \mathbf{PGO}^+(f)$ is stably rational, and hence R-trivial, and since the canonical map

$$(\mathbf{G}/\mathbf{Z})(K) \to \operatorname{Ker}[H^1(K, \mathbf{Z}) \to H^1(K, \mathbf{G})]$$

is surjective, any element in the kernel is R-trivial.

Example 2.4. Let f be an n-fold Pfister form over K, and let g be a nondegenerate subform of $f \oplus \mathbb{H}$ of codimension 2, where $\mathbb{H}$ is the hyperbolic plane. If d is the determinant of g, then we have a decomposition

$$g \oplus a\langle 1, -d\rangle \simeq f \oplus \mathbb{H} \tag{2}$$

for some scalar $a \in K^\times$. We claim that the group $\mathbf{PGO}^+(g)$ is R-trivial or, equivalently (see [14]), that the multiplier of any similitude with respect to g is R-trivial. In particular, if $\mathbf{G} = \mathbf{Spin}(g)$ and $\mathbf{Z}$ is its centre, then arguing as in the previous example, one finds that every element in the kernel (1) is R-trivial.

To compute the group of multipliers of g we first recall that every multiplier of g is contained in the set $N_{K(\sqrt{d})/K}(K(\sqrt{d})^\times)$ (see [13, Theorem 13.38]). Therefore, it follows from (2) that a multiplier m with respect to g is a multiplier with respect to f as well and hence is contained in the value group $D(f)$ of f. Conversely, (2) implies that every element

$$m \in N_{K(\sqrt{d})/K}(K(\sqrt{d})^\times) \cap D(f)$$

is a multiplier of g. Let now $U \subset K(\sqrt{d})$ be the open subvariety consisting of all elements with nonzero norm and let $X \subset U \times \mathbb{A}_K^{2^n}$ be the K-variety consisting of the elements (x, y) satisfying $N_{K(\sqrt{d})/K}(x) = f(y)$. Consider the map $X \to \mathbf{G}_m$ given by $(x, y) \to N_{K(\sqrt{d})/K}(x)$. Then the group of multipliers of g is the image of the K-points of X. Since X is K-rational, the group of multipliers of g is R-trivial.

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

Let $\mathbf{G}$ be a semisimple K-group and let $\mathbf{Z} \subset \mathbf{G}$ be a central subgroup. For any finite extension L/K we have the restriction map

$$res_K^L : H^1(K, \mathbf{Z}) \to H^1(L, \mathbf{Z})$$

and the corestriction map

$$cor_K^L : H^1(L, \mathbf{Z}) \to H^1(K, \mathbf{Z}).$$

Definition. Let L/K be a finite field extension. We say that the *norm principle* holds for a cohomology class

$$[\eta] \in \operatorname{Ker}[H^1(L, \mathbf{Z}) \longrightarrow H^1(L, \mathbf{G})]$$

if

$$cor_K^L([\eta]) \in \operatorname{Ker}[H^1(K, \mathbf{Z}) \longrightarrow H^1(K, \mathbf{G})].$$

We also say that the norm principle holds for the pair $(\mathbf{Z}, \mathbf{G})$ if it holds for every $[\eta] \in \operatorname{Ker}[H^1(L, \mathbf{Z}) \longrightarrow H^1(L, \mathbf{G})]$ whenever L/K is a finite extension.

Theorem 2.5 (P. Gille [8].). *Let K be a field of arbitrary characteristic and let L/K be a finite field extension. The norm principle holds for all R-trivial elements $[\eta] \in Ker[H^1(L, \mathbf{Z}) \longrightarrow H^1(L, \mathbf{G})]$. Moreover, $cor_K^L([\eta])$ is R-trivial.*

2.4. *Groups of Type* D_4

First we recall a statement about groups of type D_4 inside split groups of type F_4, proved in [1].

Proposition 2.6. *Let $\mathbf{F}$ be a split K-group of type F_4. Then any K-subgroup $\mathbf{M} \subset \mathbf{F}$ of type D_4 is quasi-split.*

For later use we need one more fact about groups of type ${}^{3,6}\mathrm{D}_4$ inside F_4. Let thus $\mathbf{M} \subset \mathbf{F}$ be a split subgroup of type D_4 and consider its normaliser $\mathbf{N} = N_{\mathbf{F}}(\mathbf{M})$. The quotient group $\mathbf{N}/\mathbf{M}$ is isomorphic to the group of outer automorphisms $\mathbf{Out}(\mathbf{M})$ of $\mathbf{M}$. This is the symmetric group S_3, which we view as a constant finite group scheme over K.

Let $[\xi] \in H^1(K, \mathbf{N})$ be an arbitrary cohomology class and consider its image $[\overline{\xi}] \in H^1(K, \mathbf{Out}(\mathbf{M}))$. Since $\mathbf{Out}(\mathbf{M})$ is a constant group scheme, any cocycle $\overline{\xi}$ representing it corresponds to a homomorphism $\phi_\xi : \mathrm{Gal}(K^{sep}/K) \to S_3$. The image $\operatorname{Im}\phi_\xi$ is then isomorphic to the Galois group of the minimal Galois extension F/K over which the twisted group ${}^\xi\mathbf{M}$ becomes a group of inner type. It follows that $\operatorname{Im}\phi_\xi$ is generated by the cycle (123) if ${}^\xi\mathbf{M}$ has type ${}^3\mathrm{D}_4$ and is equal to S_3 if ${}^\xi\mathbf{M}$ has type ${}^6\mathrm{D}_4$.

Lemma 2.7. *Assume that ${}^\xi\mathbf{M}$ has type ${}^{3,6}\mathrm{D}_4$. Then the natural map $\phi : {}^\xi\mathbf{N}(K) \to {}^{\overline{\xi}}(S_3)(K)$ is surjective.*

Proof. If $\operatorname{Im}\phi_\xi = S_3$, we have ${}^{\overline{\xi}}(S_3)(K) = 1$ and there is nothing to prove. Assume next that $\operatorname{Im}\phi_\xi = \langle(123)\rangle$. In this case ${}^{\overline{\xi}}(S_3)(K)$ consists of three elements. Note that ϕ is the composition of the two natural maps ${}^\xi\mathbf{N}(K) \to \mathbf{Aut}({}^\xi\mathbf{M})(K)$ and $\mathbf{Aut}({}^\xi\mathbf{M})(K) \to {}^{\overline{\xi}}(S_3)(K)$. By [7], the group $\mathbf{Aut}({}^\xi\mathbf{M})(K)$ contains an outer automorphism, say a, and it suffices to lift it modulo inner automorphisms to ${}^\xi\mathbf{N}(K)$.

The exact sequence

$$1 \longrightarrow \mathbf{Z} \longrightarrow {}^\xi\mathbf{N} \longrightarrow \mathbf{Aut}({}^\xi\mathbf{M}) \longrightarrow 1,$$

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

where $\mathbf{Z}$ is the centre of ${}^{\xi}\mathbf{N}$, induces the connecting map

$$\psi : \mathbf{Aut}({}^{\xi}\mathbf{M})(K) \longrightarrow H^1(K, \mathbf{Z}).$$

Let $\eta := \psi(a)$. Let L/K be the cubic field extension over which ${}^{\xi}\mathbf{M}$ becomes an inner form. By construction, we can say even more: it is a strongly inner form of type D_4; hence, ${}^{\xi}\mathbf{M}_L \simeq \mathrm{Spin}(f_L)$ where f_L is a 3-fold Pfister form. Therefore, a viewed over L can be lifted modulo inner automorphisms to an element of ${}^{\xi}\mathbf{N}(L)$. In other words, there is an element $a' \in (\mathbf{Aut}({}^{\xi}\mathbf{M}))^{\circ}(L)$ such that its image under the connecting map ψ_L is η_L.

According to Example 2.3, η_L is R-trivial. This means that if t is a variable over K, then there exists a class

$$[\eta(t)] \in \mathrm{Ker}[H^1(L(t), \mathbf{Z}) \to H^1(L(t), {}^{\xi}\mathbf{M})]$$

such that $\eta(t)$ is defined at $t = 0, 1$ and $\eta(0) = 1, \eta(1) = \eta_L$. We now pass to $[\widetilde{\eta(t)}] := \mathrm{cor}_K^L([\eta(t)]$. By a result of P. Gille (see Theorem 2.5), the class $[\widetilde{\eta(t)}]$ belongs to

$$\mathrm{Ker}[H^1(K(t), \mathbf{Z}) \to H^1(K(t), (\mathbf{Aut}({}^{\xi}\mathbf{M}))^{\circ}]$$

and can be specialised at $t = 0, 1$. Since L/K has degree 3 and since $H^1(K, \mathbf{Z})$ is a 2-group, we have $\widetilde{\eta(t)}(1) = \eta$.

Choose an element $b(t) \in (\mathbf{Aut}({}^{\xi}\mathbf{M}))^{\circ}(K(t))$ which maps to $\widetilde{\eta(t)}$ under $\psi_{K(t)}$ and which can be specialised at $t = 0, 1$. By construction, $\psi(b(1)) = [\eta]$. This implies that $\psi(ab(1)^{-1}) = 1$. In other words, $ab(1)^{-1}$ has a lifting to ${}^{\xi}\mathbf{N}(K)$, as required. □

2.5. *Conjugacy of Maximal Tori*

Let $\mathbf{G}$ be an absolutely simple semisimple K-group. Let $\mathbf{T}$ and $\mathbf{T}'$ be maximal tori in $\mathbf{G}$. Since all maximal tori become conjugate upon extension to K^{sep}, there exists $g \in \mathbf{G}(K^{sep})$ such that $\mathbf{T}' = g\mathbf{T}g^{-1}$. Since $\mathbf{T}$ and $\mathbf{T}'$ are K-subgroups, we have $(g^{-1})^{\tau}g \in N_{\mathbf{G}}(\mathbf{T})(K^{sep})$ for all $\tau \in \mathrm{Gal}(K^{sep}/K)$. Thus, the class of the cocycle $(\xi_{\tau}) = ((g^{-1})^{\tau}g)$ with coefficients in $N_{\mathbf{G}}(\mathbf{T})$ is a cohomological obstruction to the conjugacy of $\mathbf{T}$ and $\mathbf{T}'$. Note that since $\mathbf{T}' = g\mathbf{T}g^{-1}$, the twisted tori ${}^{(\xi_{\tau})}\mathbf{T}$ and $\mathbf{T}'$ are isomorphic K-groups.

Next we will show that under some additional assumptions, one can choose g such that the cocycle (ξ_{τ}) takes values in $\mathbf{T}(K^{sep}) \subset N_{\mathbf{G}}(\mathbf{T})(K^{sep})$. Note that for such a choice of g we have ${}^{(\xi_{\tau})}\mathbf{T} \simeq \mathbf{T}$. Therefore, a necessary condition for this is that $\mathbf{T}$ and $\mathbf{T}'$ be isomorphic over K, since ${}^{(\xi_{\tau})}\mathbf{T} \simeq \mathbf{T}'$. Furthermore, our claim will hold true in $\mathbf{G}$ if it does in $\mathbf{G}/\mathbf{Z}$ for some central subgroup $\mathbf{Z}$ (because $\mathbf{Z}$ is contained in $\mathbf{T}$). This reduces the problem to the adjoint case.

Assume thus that $\mathbf{T} \simeq \mathbf{T}'$ and that $\mathbf{G}$ is adjoint. Let F/K be the minimal splitting field of $\mathbf{T}$ (and hence of $\mathbf{T}'$) and let $\Gamma = \mathrm{Gal}(F/K)$. The group Γ acts naturally on the character lattices $X(\mathbf{T})_*$ and $X(\mathbf{T}')_*$ and these actions preserve the root systems $\Sigma = \Sigma(G, \mathbf{T})$ and $\Sigma' = \Sigma(G, \mathbf{T}')$. Thus, we have two canonical embeddings $\rho_1 : \Gamma \hookrightarrow \mathrm{Aut}(\Sigma)$ and $\rho_2 : \Gamma \hookrightarrow \mathrm{Aut}(\Sigma')$. Since $\mathbf{G}$ is adjoint, Σ and Σ' generate $X(\mathbf{T})_*$ and $X(\mathbf{T}')_*$, respectively. Since Σ and Σ' are root systems of the same type we may identify them, which in turn gives rise to an identification $X(\mathbf{T})_* = X(\mathbf{T}')_*$. After all of these identifications we obtain two actions of Γ on each of Σ and $X(\mathbf{T})_*$ through ρ_1 and ρ_2.

Lemma 2.8. *Assume that there is an inner automorphism* $\rho : \mathrm{Aut}(\Sigma) \to \mathrm{Aut}(\Sigma)$ *such that* $\rho|_{Im\,\rho_1} = \rho_2 \circ \rho_1^{-1}$. *Then there is a* Γ*-equivariant automorphism* $X(\mathbf{T})_* \to X(\mathbf{T})_*$ *preserving the root system* Σ*, where* Γ *acts on the domain through* ρ_1 *and on the codomain through* ρ_2.

Proof. Let $\rho = \mathrm{Int}(a)$ where $a \in \mathrm{Aut}(\Sigma)$. The map $a : \Sigma \to \Sigma$ can be extended uniquely to an automorphism $a_{X(\mathbf{T})_*} : X(\mathbf{T})_* \to X(\mathbf{T})_*$ preserving roots. It is straightforward to check that it is Γ-equivariant. □

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

We are now ready to conclude this section with the following theorem. Since we will mainly be concerned with outer forms of type A_2, it is stated for groups of outer type.

Theorem 2.9. *Let* $\mathbf{G}$ *be an absolutely simple semisimple* K*-group of outer type with* $|Out(\mathbf{G})| = 2$ *and let* $\mathbf{T}$ *and* $\mathbf{T}'$ *be two isomorphic maximal tori in* $\mathbf{G}$*, with corresponding root systems* Σ *and* Σ'*, respectively. Assume that there is an inner automorphism* $\rho : \mathrm{Aut}(\Sigma) \to \mathrm{Aut}(\Sigma)$ *such that* $\rho|_{Im\,\rho_1} = \rho_2 \circ \rho_1^{-1}$*, where* ρ_1 *and* ρ_2 *are the above embeddings of* Γ *into* $\mathrm{Aut}(\Sigma)$*. If there is* $f \in \mathrm{Aut}(\mathbf{G})(K) \setminus Int(\mathbf{G})(K)$ *such that* $f(\mathbf{T}) = \mathbf{T}$*, then there is* $g \in \mathbf{G}(K^{sep})$ *such that* $g\mathbf{T}g^{-1} = \mathbf{T}'$ *and* $(g^{-1})^{\tau}g \in \mathbf{T}(K^{sep})$ *for all* $\tau \in Gal(K^{sep}/K)$*.*

Proof. Without loss of generality, we may assume that $\mathbf{G}$ is adjoint. The assumptions of Lemma 2.8 are satisfied. Let thus

$$a_{X(\mathbf{T})_*} : X(\mathbf{T})_* \to X(\mathbf{T})_*$$

be the Γ-equivariant map constructed in that lemma. Using the identification of $X(\mathbf{T})_*$ and $X(\mathbf{T}')_*$, we obtain a Γ-equivariant map $X(\mathbf{T})_* \to X(\mathbf{T}')_*$, which can be extended to a K-group isomorphism $a : \mathbf{T} \to \mathbf{T}'$ that induces an isomorphism between Σ and Σ'. By [11, Theorem 32.1] the map $a_{\mathbf{T}}$ can be further extended to an automorphism $a_{\mathbf{G}} : \mathbf{G} \to \mathbf{G}$. Replacing $a_{\mathbf{G}}$ with $a_{\mathbf{G}} \circ f$, if necessary, we may assume that $a_{\mathbf{G}}$ is inner, say $a_{\mathbf{G}} = \mathrm{Int}(g)$, where $g \in \mathbf{G}(K^{sep})$. Since $\mathrm{Int}(g)|_{\mathbf{T}} : \mathbf{T} \to \mathbf{T}'$ is a K-group isomorphism and $\mathrm{Int}((g^{-1})^{\tau}g)$ fixes Σ, it follows that $(g^{-1})^{\tau}g \in \mathbf{T}(K^{sep})$ for all $\tau \in \mathrm{Gal}(K^{sep}/K)$. □

Example 2.10. We keep the above notation. Let E/K be a quadratic étale extension and let B be a central simple algebra of degree 3 over E equipped with an involution σ of the second kind. Consider two isomorphic cubic subfields $L, L' \subset B_{\sigma}$ where $B_{\sigma} \subset B$ is the subset consisting of all σ-invariant elements. Since the maximal subfields $L \cdot E$ and $L' \cdot E$ of B are σ-stable, they give rise to two maximal K-tori $\mathbf{T}$ and $\mathbf{T}'$ in $\mathbf{G} = \mathbf{SU}(B, \sigma)$, given by

$$\mathbf{T} = \{x \in L \cdot E \mid \sigma(x)x = 1,\ \ \mathrm{Nrd}(x) = 1\};$$

$$\mathbf{T}' = \{x \in L' \cdot E \mid \sigma(x)x = 1,\ \ \mathrm{Nrd}(x) = 1\}.$$

Clearly, $\mathbf{T} \simeq \mathbf{T}'$ and the Galois group Γ of the minimal splitting field of $\mathbf{T}$ (and hence of $\mathbf{T}'$) are of order divisible by 6. Now $\mathbf{SU}(B, \sigma)$ is of type A_2, with $W(A_2) \simeq S_3$ and the automorphism group of its root system Σ,

$$\mathrm{Aut}(\Sigma) \simeq W(A_2) \times \mathbb{Z}/2 \simeq S_3 \times \mathbb{Z}/2,$$

is of order 12. Thus, Γ has order 6 or 12.

Case 1: $|\Gamma| = 12$. Then $\mathrm{Im}\,\rho_1$ and $\mathrm{Im}\,\rho_2$ coincide with $\mathrm{Aut}(\Sigma)$. Note that

$$\rho_2 \circ \rho_1^{-1} : \mathrm{Aut}(\Sigma) \to \mathrm{Aut}(\Sigma)$$

preserves the Weyl group $W(A_2)$ since $\rho_1^{-1}(W(A_2))$ (respectively $\rho_2^{-1}(W(A_2))$) coincides with $\mathrm{Gal}(F/E) < \mathrm{Gal}(F/K)$, where F/K is the Galois closure of $L \cdot E/K$. Hence, $\rho_2 \circ \rho_1^{-1}$ obviously satisfies all of the assumptions in Theorem 2.9 and the map $f(x) = \sigma(x)^{-1}$ is an outer automorphism of $\mathbf{G}$ preserving $\mathbf{T}$.

Case 2: $|\Gamma| = 6$. The automorphism group $\mathrm{Aut}(\Sigma)$ has 3 subgroups of order 6, namely, $\Gamma_1 = W(A_2) = S_3 \times 0$, the subgroup $\Gamma_2 \subset S_3 \times \mathbb{Z}/2$ generated by the two elements $((123), 0)$ and $((12), 1)$, where (123) and (12) are standard cycles in S_3 and the cyclic subgroup $\Gamma_3 \subset S_3 \times \mathbb{Z}/2$ of order 6 generated by the two elements $((123), 0)$ and $(\mathrm{Id}, 1)$.

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

Since $\mathbf{G}$ has outer type, we know from [19, Lemma 4.1] that Γ does not embed into $\Gamma_1 = W(\mathrm{A}_2) \simeq S_3$. If $\rho_1(\Gamma) = \Gamma_2 \simeq S_3$, then $\operatorname{Im}\rho_1 = \operatorname{Im}\rho_2$ and since every automorphism of Γ_2 is obviously inner, the automorphism $\rho_2 \circ \rho_1^{-1}$ of Γ_2 can be extended to an inner automorphism of $\operatorname{Aut}(\Sigma)$. If instead $\rho_1(\Gamma) = \Gamma_3 \simeq \mathbb{Z}/3\times\mathbb{Z}/2$, then again $\operatorname{Im}\rho_1 = \operatorname{Im}\rho_2$. The group Γ_3 has a unique nontrivial automorphism given by $x \mapsto x^{-1}$ and one easily checks that it is the restriction of an inner automorphism of $\operatorname{Aut}(\Sigma)$.

Thus, in all cases, the hypothesis of Theorem 2.9 is satisfied.

3. Subgroups of the Automorphism Group of an Albert Algebra

In this section, we will study automorphisms of Albert algebras related to 9-dimensional subalgebras. Recall that for our purposes it suffices to consider Albert algebras that are division algebras. Therefore, throughout this section, A is an arbitrary division Albert algebra over K. The main result of this section is the rationality, hence R-triviality, of the group of all automorphisms stabilising a 9-dimensional subalgebra.

3.1. 9*-dimensional Subalgebras and Their Automorphisms*

By [13, Theorem 37.12 (2)], any proper nontrivial subalgebra of A is either a cubic field extension $K \subset L \subset A$ or a 9-dimensional subalgebra $K \subset S \subset A$. Furthermore, S is of the form $S = D^+$ where D is a central simple algebra of degree 3 over K or $S = B^+_\sigma$ where B is a central division algebra of degree 3 over a quadratic field extension E/K equipped with an involution σ of the second kind. For later use we record some facts related to automorphism groups of D^+, B^+_σ and their extensions to automorphisms of A.

First, let $S = D^+$. By [13, Theorem 39.14 (2)], the algebra A has a presentation $A = D \oplus D \oplus D$ (as a vector space) where the subalgebra S coincides with the first component. By [13, formula (37.7)] , we have an exact sequence

$$1 \longrightarrow \mathbf{Aut}(D) \longrightarrow \mathbf{Aut}(D^+) \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

This sequence is split if and only if D is split. Since D is a division algebra, any K-automorphism of D^+ thus comes from $\mathbf{Aut}(D)(K) = \operatorname{Aut}(D)$; that is, is given by conjugation $x \mapsto dxd^{-1}$ for some $d \in D^\times$. Moreover, such an automorphism can be extended to A by the formula

$$(x, y, z) \mapsto (gxg^{-1}, gyg^{-1}, gzg^{-1}).$$

Thus, the sequence implies that

$$\mathbf{Aut}(D^+)^\circ \simeq \mathbf{PGL}(1, D).$$

Note that $\mathbf{PGL}(1, D)$ is rational and hence, in particular, R-trivial.

Next, let $S = B^+_\sigma$. Here the situation is completely analogous to that of $S = D^+$. Namely, by [13, Theorem 39.18 (2)], the algebra A admits the presentation $A = B^+_\sigma \oplus B$ as a vector space, with the first component a subalgebra. By [13, Section 37.B], the algebraic K-group $\mathbf{Aut}(B^+_\sigma)$ is smooth and

$$\mathbf{Aut}(B^+_\sigma)^\circ \simeq \mathbf{PGU}(B, \sigma).$$

Passing to the quadratic field extension E/K, we conclude that

$$\mathbf{Aut}(B^+_\sigma)(K) = \mathbf{Aut}(B^+_\sigma)^\circ(K).$$

Thus, any K-automorphism of B^+_σ is given by conjugation $x \mapsto bxb^{-1}$ for some $b \in B^\times$ satisfying $b\sigma(b) \in K$. Since $\mathbf{PGU}(B, \sigma)$ has rank 2 it is rational and hence R-trivial. In Corollary 3.2 we shall see that any K-automorphism of B^+_σ can be extended to a K-automorphism of A.

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

3.2. *The Group* $\mathbf{Aut}(A/B^+_\sigma)$

Note that if $E = K \times K$ and $B = D \otimes_K E$ with the flip involution σ, then B^+_σ is equal to D^+ embedded diagonally into B. This provides a unified treatment of both kinds of 9-dimensional subalgebras. Therefore, here and in what follows in this section, we will let E/K be a quadratic étale extension, including the possibility of E being split; doing so, any 9-dimensional subalgebra of A is of the form B^+_σ.

As above, let $A = B^+_\sigma \oplus B$. Recall the algebraic K-group

$$\mathbf{Aut}(A/B^+_\sigma) \subset \mathbf{Aut}(A) = \mathbf{H}$$

(see Introduction). One knows (see [13, Section 39.B]) that $\mathbf{Aut}(A/B^+_\sigma)$ is simple simply connected of type A_2 (hence connected). Thus,

$$\mathbf{Aut}(A/B^+_\sigma) \simeq \mathbf{SU}(B, \tau)$$

where τ is some involution of the second kind on B, which in general is different from σ. Being an algebraic group of rank 2, this group is rational and hence R-trivial.

3.3. *The Group* $\mathbf{Aut}(A, B^+_\sigma)$

Recall from the Introduction the K-group

$$\mathbf{Aut}(A, B^+_\sigma) \subset \mathbf{Aut}(A) = \mathbf{H}.$$

By [13, Proposition 39.16], we have an exact sequence

$$1 \longrightarrow \mathbf{Aut}(A/B^+_\sigma) \longrightarrow \mathbf{Aut}(A, B^+_\sigma) \longrightarrow \mathbf{Aut}(B^+_\sigma) \longrightarrow 1.$$

Moreover, by [13, Corollary 39.12] ,

$$\mathbf{Aut}(A, B^+_\sigma)^\circ(K) = \mathbf{Aut}(A, B^+_\sigma)(K).$$

Furthermore, the above sequence induces the exact sequence

$$1 \longrightarrow \mathbf{Aut}(A/B^+_\sigma) \longrightarrow \mathbf{Aut}(A, B^+_\sigma)^\circ \longrightarrow \mathbf{Aut}(B^+_\sigma)^\circ \longrightarrow 1. \tag{3}$$

Thus,

$$\mathbf{Aut}(A, B^+_\sigma)^\circ/\mathbf{Aut}(A/B^+_\sigma) \simeq \mathbf{Aut}(B^+_\sigma)^\circ \simeq \mathbf{PGU}(B, \sigma) \simeq \mathbf{SU}(B, \sigma)/\mathbf{Z},$$

where $\mathbf{Z} \subset \mathbf{SU}(B, \sigma)$ is the centre.

From the point of view of algebraic groups, (3) implies that the algebraic K-group $\mathbf{G} := \mathbf{Aut}(A, B^+_\sigma)^\circ$ is semisimple and is an almost direct product of two simple simply connected groups of type A_2: the first is

$$\mathbf{G}_1 := \mathbf{Aut}(A/B^+_\sigma) = \mathbf{SU}(B, \tau)$$

and the second is isomorphic to $\mathbf{G}_2 := \mathbf{SU}(B, \sigma)$. The centres $\mathbf{Z}_1$ and $\mathbf{Z}_2$ of $\mathbf{G}_1$ and $\mathbf{G}_2$, respectively, are both isomorphic to $\mathbf{Z} = R^{(1)}_{E/K}(\mu_3)$ and $\mathbf{G}_1 \cap \mathbf{G}_2 = \mathbf{Z}$ (see [13, Corollary 39.12]). Thus, we have the exact sequence

$$1 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{G}_1 \times \mathbf{G}_2 \stackrel{\phi}{\longrightarrow} \mathbf{G} \longrightarrow 1, \tag{4}$$

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

where $\mathbf{Z}$ is embedded *codiagonally*; that is, via $z \mapsto (z, z^{-1})$. Identifying the image $\phi(\mathbf{G}_1 \times 1) \subset \mathbf{G}$ with $\mathbf{G}_1$, we recover the sequence (3) in the form

$$1 \longrightarrow \mathbf{G}_1 \longrightarrow \mathbf{G} \xrightarrow{\psi} \mathbf{G}/\mathbf{G}_1 \longrightarrow 1. \quad (5)$$

Note that $\mathbf{G}/\mathbf{G}_1 \simeq \mathbf{G}_2/\mathbf{Z}$.

3.4. *Rationality of* $\mathbf{Aut}(A, B_\sigma^+)^\circ$

We keep the notation introduced in Subsection 3.3.

Proposition 3.1. *The K-group* $\mathbf{G} = \mathbf{Aut}(A, B_\sigma^+)^\circ$ *is rational and hence R-trivial.*

Proof. Consider the exact sequences (4) and (5). The two groups $\mathbf{G}_1$ and $\mathbf{G}_2/\mathbf{Z}$ in (5), being groups of rank 2, are rational over K. Therefore, it suffices to show that ψ has a rational section. This is equivalent to proving that

$$\operatorname{Ker}\,[H^1(F, \mathbf{G}_1) \longrightarrow H^1(F, \mathbf{G})] = 1$$

for all field extensions F/K.

Fix a field extension F/K and let $[\xi] \times 1 \in H^1(F, \mathbf{G}_1 \times 1)$ be a class whose image in $H^1(F, \mathbf{G})$ is trivial. From (4) it follows that there is $[\lambda] \in H^1(F, \mathbf{Z})$ whose image in $H^1(F, \mathbf{G}_1 \times \mathbf{G}_2)$ is $[\xi] \times 1$. Since $\mathbf{Z}$ is embedded codiagonally into $\mathbf{G}_1 \times \mathbf{G}_2$, the image of $[\lambda]$ under the natural map

$$H^1(F, \mathbf{Z}) \to H^1(F, \mathbf{G}_2)$$

is trivial. We distinguish two cases.

Case 1: The quadratic étale extension E/K is split; that is, $E = K \times K$. Then up to K-isomorphism we may assume that $\mathbf{G}_1 = \mathbf{SL}(1, D_1)$ and $\mathbf{G}_2 = \mathbf{SL}(1, D_2)$, where D_1 and D_2 are central simple algebras over K of degree 3 and either $D_2 = D_1$ or $D_2 = D_1^{\mathrm{op}}$. In both cases their centres are $\boldsymbol{\mu}_3$, whence $H^1(F, \mathbf{Z}) \simeq F^\times/F^{\times 3}$, so that the class $[\lambda] \in H^1(F, \mathbf{Z})$ is represented by some $f \in F^\times$. The fact that

$$H^1(F, \mathbf{G}_2) \simeq F^\times/\mathrm{Nrd}(D_2^\times)$$

together with the image of $[\lambda]$ in $H^1(F, \mathbf{G}_2)$ being trivial then imply that f is a reduced norm in D_2, and hence also in D_1. This implies that the image $[\xi]$ of $[\lambda]$ in $H^1(F, \mathbf{G}_1)$ is trivial, which completes the proof in this case.

Case 2: E/K is a separable field extension. Let $F' = F \cdot E$ and consider the class $[\lambda]^2 \in H^1(F, \mathbf{Z})$. From Case 1 we know that $res_F^{F'}([\lambda]^2)$ is contained in $\operatorname{Ker}\,[H^1(F', \mathbf{Z}) \to H^1(F', \mathbf{G}_1)]$. It follows from Example 2.2 and the norm principle (Theorem 2.5) that

$$[\lambda] = [\lambda]^4 = cor_F^{F'}(res_F^{F'}([\lambda]^2)$$

is contained in $\operatorname{Ker}\,[H^1(F, \mathbf{Z}) \to H^1(F, \mathbf{G}_1)]$ and the proof is complete. □

As a direct consequence of our proof, we have the following.

Corollary 3.2. *For any field extension* F/K*, the canonical map* $\mathbf{G}(F) \to (\mathbf{G}_2/\mathbf{Z})(F)$ *is surjective. Hence, if* A_F *is a division Albert algebra, then*

$$\mathbf{Aut}(A, B_\sigma^+)(F) \longrightarrow \mathbf{Aut}(B_\sigma^+)(F)$$

is surjective.

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

4. Subgroups of the Structure Group of an Albert Algebra

We now turn to the structure group and consider, along the same lines as in the previous section, subgroups of it related to 9-dimensional subalgebras. Throughout this section, as in the previous, A denotes an arbitrary division Albert algebra over K. We moreover continue using the convention that the quadratic étale algebra E involved in the definition of B^+_σ may be split.

4.1. The Group $\mathbf{Str}(B^+_\sigma)$

The structure group $\mathbf{Str}(B^+_\sigma)$ of the Jordan algebra B^+_σ is a closed subgroup of the algebraic K-group $\mathbf{GL}(B^+_\sigma)$ consisting of all similitudes. More precisely, for any K-ring R,

$$\mathbf{Str}(B^+_\sigma)(R) = \{x \in \mathbf{GL}(B^+_\sigma)(R) \mid \mathrm{Nrd}_R(x(b)) = \nu(x)\mathrm{Nrd}_R(b) \ \ \forall b \in B^+_\sigma \otimes_K R\}$$

where Nrd_R is the base change of Nrd to $B \otimes_K R$ and the *multiplier* $\nu(x) \in R^\times$. By [12, Chap. V, Thm. 5.12.10], the group $\mathrm{Str}(B^+_\sigma) = \mathbf{Str}(B^+_\sigma)(K)$ consists of the linear maps of the form

$$B^+_\sigma \longrightarrow B^+_\sigma, \ \ x \mapsto \lambda b x \sigma(b),$$

where $b \in B^\times$ and $\lambda \in K^\times$. It follows that we have a surjective map of algebraic K-groups

$$\mathbf{G}_m \times R_{E/K}(\mathbf{GL}(1, B)) \longrightarrow \mathbf{Str}(B^+_\sigma)^\circ$$

and that $\mathbf{Str}(B^+_\sigma)^\circ(K) = \mathbf{Str}(B^+_\sigma)(K)$. The kernel of this map is the torus $R_{E/K}(\mathbf{G}_{m,E})$, where the embedding

$$R_{E/K}(\mathbf{G}_{m,E}) \hookrightarrow \mathbf{G}_m \times R_{E/K}(\mathbf{GL}(1, B))$$

is given by $x \mapsto (N_{E/K}(x^{-1}), x)$. Thus, we have the exact sequence

$$1 \longrightarrow R_{E/K}(\mathbf{G}_{m,E}) \longrightarrow \mathbf{G}_m \times R_{E/K}(\mathbf{GL}(1, B)) \xrightarrow{\phi} \mathbf{Str}(B^+_\sigma)^\circ \longrightarrow 1. \tag{6}$$

Lemma 4.1. *The algebraic K-group* $\mathbf{Str}(B^+_\sigma)^\circ$ *is rational and hence, in particular, R-trivial.*

Proof. We identify $\phi(\mathbf{G}_m \times 1) \subset \mathbf{Str}(B^+_\sigma)^\circ$ with $\mathbf{G}_m$. Since for any field extension F/K one has $H^1(F, \mathbf{G}_m) = 1$, the canonical map

$$\mathbf{Str}(B^+_\sigma)^\circ \to \mathbf{Str}(B^+_\sigma)^\circ/\mathbf{G}_m$$

has a rational section. Therefore, it suffices to establish the rationality of $\mathbf{Str}(B^+_\sigma)^\circ/\mathbf{G}_m$. By (6) we have

$$\mathbf{Str}(B^+_\sigma)^\circ/\mathbf{G}_m \simeq R_{E/K}(\mathbf{GL}(1, B^\times))/R_{E/K}(\mathbf{G}_{m,E}),$$

which is clearly rational. □

4.2. The Group $\mathbf{Str}(A, B^+_\sigma)$

Let $\mathbf{Str}(A, B^+_\sigma)$ (respectively $\mathbf{Str}(A/B^+_\sigma)$) be the closed subgroup of $\mathbf{Str}(A)$ consisting of those elements stabilising B^+_σ (respectively fixing B^+_σ pointwise). Note that since the elements of $\mathbf{Str}(A/B^+_\sigma)$ fix the identity element of A, it follows from [20, Proposition 5.9.4] that $\mathbf{Str}(A/B^+_\sigma) = \mathbf{Aut}(A/B^+_\sigma)$. This group is the kernel of the canonical restriction map

$$\mathbf{Str}(A, B^+_\sigma)^\circ \to \mathbf{Str}(B^+_\sigma)^\circ.$$

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

Note that the restriction map is well defined because the restriction to B_σ^+ of the norm on A is the norm on B_σ^+. Since, by [7, Proposition 7.2.4], this map is surjective on the level of K^{sep}-points, we have the exact sequence

$$1 \longrightarrow \mathbf{Aut}(A/B_\sigma^+) \longrightarrow \mathbf{Str}(A, B_\sigma^+)^\circ \xrightarrow{\phi} \mathbf{Str}(B_\sigma^+)^\circ \longrightarrow 1 \tag{7}$$

of algebraic K-groups.

Proposition 4.2. *The group* $\mathbf{Str}(A, B_\sigma^+)^\circ$ *is rational and hence R-trivial.*

Proof. By [7], for any field extension F/K the map

$$\mathbf{Str}(A, B_\sigma^+)^\circ(F) \xrightarrow{\phi_F} \mathbf{Str}(B_\sigma^+)^\circ(F)$$

is surjective, implying that ϕ has a rational section. Thus, $\mathbf{Str}(A, B_\sigma^+)^\circ$ is birationally isomorphic to $\mathbf{Aut}(A/B_\sigma^+) \times \mathbf{Str}(B_\sigma^+)^\circ$. It remains to be noted that, being a group of rank 2, the group $\mathbf{Aut}(A/B_\sigma^+)$ is rational and by Lemma 4.1 the group $\mathbf{Str}(B_\sigma^+)^\circ$ is rational. □

Corollary 4.3. *The natural map* $\mathrm{Str}(A, B_\sigma^+) \to \mathrm{Str}(B_\sigma^+)$ *is surjective.*

Proof. From the proof of Proposition 4.2 it follows that the map

$$\mathbf{Str}(A, B_\sigma^+)^\circ(K) \to \mathbf{Str}(B_\sigma^+)^\circ(K)$$

is surjective, and from 4.1 we know that $\mathbf{Str}(B_\sigma^+)(K) = \mathbf{Str}(B_\sigma^+)^\circ(K)$. The assertion follows. □

5. The Weak Skolem–Noether Property for Isomorphic Embeddings

Let A be an Albert algebra over a field K. Let $K \subset L \subset A$ and $K \subset L' \subset A$ be two isomorphic separable cubic field extensions; one says that they are *weakly equivalent* if there exists an element $g \in \mathrm{Str}(A)$ such that $g(L) = L'$. That this property holds follows from Theorem B (Skolem–Noether theorem for Albert algebras) due to S. Garibaldi and H. Petersson [7].

The goal of this section is to give a self-contained proof of the validity of weak equivalence based on the technique of conjugacy of maximal tori detailed in Section 2.

We start with the intermediate step of a 9-dimensional Jordan algebra B_σ^+.

Proposition 5.1. *Let* L *and* L' *be two isomorphic separable cubic field extensions of the base field* K *contained in the subalgebra* B_σ^+. *Then there exists an element* $s \in \mathrm{Str}(B_\sigma^+)$ *such that* $s(L) = L'$.

Proof. Let E/K be the étale quadratic extension over which B is defined. The two cubic fields L and L' give rise to the two 4-dimensional (maximal) tori

$$\mathbf{T} = \{x \in R_{L\cdot E/K}(\mathbf{G}_{m,L\cdot E}) \mid \sigma(x)x \in \mathbf{G}_{m,K}\}$$

and

$$\mathbf{T}' = \{x \in R_{L'\cdot E/K}(\mathbf{G}_{m,L'\cdot E}) \mid \sigma(x)x \in \mathbf{G}_{m,K}\}$$

of the algebraic K-group $\mathbf{Sim}(B, \sigma)$.[3] From the point of view of algebraic groups, $\mathbf{Sim}(B, \sigma)$ is a reductive group which is the almost direct product of the central 2-dimensional torus $\mathbf{P} = R_{E/K}(\mathbf{G}_{m,E})$ and the simple simply connected group $\mathbf{SU}(B, \sigma)$ of outer type A_2. Note that $\mathbf{P}$ is contained in both $\mathbf{T}$ and $\mathbf{T}'$. Let $\mathbf{T}_{ss} = \mathbf{T} \cap \mathbf{SU}(B, \sigma)$ and $\mathbf{T}'_{ss} = \mathbf{T}' \cap \mathbf{SU}(B, \sigma)$. Then $\mathbf{T} = \mathbf{P} \cdot \mathbf{T}_{ss}$ and $\mathbf{T}' = \mathbf{P} \cdot \mathbf{T}'_{ss}$.

[3]For the definition of $\mathbf{Sim}(B, \sigma)$ see [13].

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

By Theorem 2.9 and Example 2.10, there exists $b \in \mathbf{SU}(B,\sigma)(K^{sep})$ such that $b\mathbf{T}_{ss}b^{-1} = \mathbf{T}'_{ss}$ and $b^{-\tau+1} \in \mathbf{T}_{ss}(K^{sep})$ for all $\tau \in \mathrm{Gal}(K^{sep}/K)$. Since $\mathbf{P} \subset \mathbf{Sim}(B,\sigma)$ is a central torus, we also have $b\mathbf{T}b^{-1} = \mathbf{T}'$.

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R_{E/K}(\mathbf{G}_{m,E}) & \longrightarrow & \mathbf{Sim}(B,\sigma) & \xrightarrow{\phi_1} & \mathbf{Aut}(B^+_\sigma)^\circ & \longrightarrow & 1 \\ & & \downarrow \mathrm{Id} & & \downarrow \lambda_2 & & \downarrow & & \\ 1 & \longrightarrow & R_{E/K}(\mathbf{G}_{m,E}) & \longrightarrow & \mathbf{J} & \xrightarrow{\phi_2} & \mathbf{Str}(B^+_\sigma)^\circ & \longrightarrow & 1. \end{array}$$

Here $\mathbf{J} = \mathbf{G}_{m,K} \times R_{E/K}(\mathbf{GL}(1,B))$, so that the lower sequence is the exact sequence (6), ϕ_1 is given by conjugation,

$$\phi_2(x,y) : B^+_\sigma \longrightarrow B^+_\sigma, \quad a \mapsto xya\sigma(y)$$

and

$$\lambda_2(x) = (\nu(x^{-1}), x) = (x^{-1}x^{-\sigma}, x)$$

where $\nu(x)$ is the multiplier of x. The map λ_2 takes $\mathbf{T}$ and $\mathbf{T}'$ into the quasi-trivial tori

$$\widetilde{\mathbf{T}} = \mathbf{G}_{m,K} \times R_{L\cdot E/K}(\mathbf{G}_{m,L\cdot E})$$

and

$$\widetilde{\mathbf{T}}' = \mathbf{G}_{m,K} \times R_{L'\cdot E/K}(\mathbf{G}_{m,L'\cdot E}),$$

respectively. Next we need the following.

Lemma 5.2. *The torus $\widetilde{\mathbf{T}}$ (respectively $\widetilde{\mathbf{T}}'$) is the centraliser of $\mathbf{T}$ (respectively $\mathbf{T}'$) in $\mathbf{G}_{m,K} \times R_{E/K}(\mathbf{GL}(1,B))$.*

Proof. We consider the case of the torus $\mathbf{T}$ only. The torus $\mathbf{T}'$ is handled analogously. It suffices to show that

$$C_{R_{E/K}(\mathbf{GL}(1,B))}(\mathbf{T}) = R_{L\cdot E/K}(\mathbf{G}_{m,L\cdot E}).$$

Without loss of generality, we may assume that the base field K is algebraically closed. Then B may be identified with the algebra $M_3 \times M_3$, the involution σ switches the components and L with diagonal matrices fixed by σ. With this identification we have

$$R_{E/K}(\mathbf{GL}(1,B)) = \mathbf{GL}_3 \times \mathbf{GL}_3$$

and

$$R_{L\cdot E/K}(\mathbf{G}_{m,L\cdot E}) = \{(t_1,t_2) \mid t_1, t_2 \text{ are diagonal matrices}\}$$

and, moreover,

$$\mathbf{T} = \{(t_1, ut_1) \mid t_1 \text{ is a diagonal matrix and } u \in K^\times\}.$$

It follows that

$$C_{\mathbf{GL}_3\times\mathbf{GL}_3}(\mathbf{T}) = \{(t_1,t_2) \mid t_1, t_2 \text{ are diagonal matrices}\}$$

and we are done. □

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

We now return to proving the proposition. By the above lemma the equality $b\mathbf{T}b^{-1} = \mathbf{T}'$ implies $b\widetilde{\mathbf{T}}b^{-1} = \widetilde{\mathbf{T}}'$ and

$$\lambda_2(b^{-\tau+1}) = (\lambda_2(b))^{-\tau+1} = (\nu(b^{-1})^{-\tau+1}, b^{-\tau+1}) \in \widetilde{\mathbf{T}}(K^{sep}).$$

Since $H^1(K, R_{L\cdot E/K}(\mathbf{G}_{m,L\cdot E})) = 1$, we can pick an element

$$a \in R_{L\cdot E/K}(\mathbf{G}_{m,L\cdot E})(K^{sep}) = ((L \cdot E) \otimes_K K^{sep})^{\times}$$

such that $b^{-\tau+1} = a^{-\tau+1}$. Clearly, $c := ba^{-1}$ is $\mathrm{Gal}(K^{sep}/K)$-invariant, hence, $c \in B^{\times}$, and, defining s as the map $x \mapsto cx\sigma(c)$, the following claim completes the proof of the proposition.

Claim. $cL\sigma(c) = L'$.

The claim is equivalent to $c(L \cdot E)\sigma(c) = L' \cdot E$. Moreover, it suffices to show that

$$c((L \cdot E) \otimes_K K^{sep})\sigma(c) = (L' \cdot E) \otimes_K K^{sep}.$$

But $\sigma(c) = \sigma(a^{-1})\sigma(b)$ and both a^{-1} and $\sigma(a^{-1})$ are in $(L \cdot E) \otimes_K K^{sep}$. Therefore, it suffices to show that

$$b((L \cdot E) \otimes_K K^{sep})\sigma(b) = (L' \cdot E) \otimes_K K^{sep}.$$

Recall that, by construction, b is a similitude; hence, $\sigma(b) = \nu(b)b^{-1}$, where $\nu(b)$ is the multiplier of b. Since $b\mathbf{T}b^{-1} = \mathbf{T}'$, the claim follows upon noting that the centraliser of $\mathbf{T}$ (respectively $\mathbf{T}'$) in $B \otimes_K K^{sep}$ is $(L \cdot E) \otimes_K K^{sep}$ (respectively $(L' \cdot E) \otimes_K K^{sep}$). □

The proposition in conjunction with Corollary 4.3 yields the following.

Theorem 5.3. *Let A be a division Albert algebra over K and let $L \subset A$ and $L' \subset A$ be isomorphic separable cubic field extensions of K. Then there exists a 9-dimensional subalgebra B_σ^+ of A and an element $s \in \mathrm{Str}(A, B_\sigma^+)$ such that $s(L) = L' = f(L)$.*

Proof. If $L' = L$, we can take any any 9-dimensional subalgebra B_σ^+ containing L and choose $s = \mathrm{Id}$. Otherwise, the cubic subfields L, L' generate a subalgebra in A of the form B_σ^+ and we can take any lift of the element s constructed in the above proposition. □

Remark 5.4. To be able to approach the Skolem–Noether isomorphic embedding problem using torsor techniques is of independent interest. It seems plausible that this approach can shed some light on the still open and much more difficult Skolem–Noether problem for isotopic embeddings.

6. Reduction to F_4

Throughout this section, we let A be a division Albert algebra over K. We will show that an arbitrary element in $\mathrm{Str}(A)$ can be written as a product of R-trivial elements and elements in $\mathbf{H}(K)$, thereby reducing the problem to subgroups of type F_4. To begin with, we recall from [1] how to associate, to any $a \in A^{\times}$, a subgroup of type D_4 in $\mathbf{G} = [\mathbf{Str}(A), \mathbf{Str}(A)]$ and a 2-dimensional torus. Let $L \subset A$ be the K-subalgebra generated by a if a is not a scalar multiple of the identity of A and by any element that is not such a scalar multiple if a is. Since A is a division algebra, it is known that L is a cubic subfield. Let $\mathbf{G}^L$ be the algebraic K-group whose functor of points is given by

$$\mathbf{G}^L(R) = \{x \in \mathbf{G}(R) \mid x(l \otimes 1) = l \otimes 1 \ \ \forall l \in L\}$$

for any K-ring R. Since $\mathbf{G}^L$ stabilises $1 \in L \subset A$, we have $\mathbf{G}^L \subset \mathbf{H} \subset \mathbf{G}$. It is known that over a separable closure of K the group $\mathbf{G}^L$ is conjugate to the standard subgroup in $\mathbf{G}$ of type D_4 generated by roots $\alpha_2, \alpha_3, \alpha_4, \alpha_5$. The lemma below shows that the centraliser $\mathbf{S}'^L = C_{\mathbf{G}}(\mathbf{G}^L)$ of $\mathbf{G}^L$ in $\mathbf{G}$ is a 2-dimensional torus over K and that

$$\mathbf{Z}^L := \mathbf{S}'^L \cap \mathbf{H} = \mathbf{S}'^L \cap \mathbf{G}^L$$

is the centre of $\mathbf{G}^L$.

Lemma 6.1. *Let K be an arbitrary field. $\mathbf{G}$ be a split simple simply connected K-group of type E_6. Let $\mathbf{T} \subset \mathbf{G}$ be a maximal split torus. Denote by $\Pi = \{\alpha_1, \ldots, \alpha_6\}$ a basis of the root system $\Sigma(\mathbf{G}, \mathbf{T})$ and by Σ_1 the root subsystem in Σ generated by $\alpha_2, \alpha_3, \alpha_4, \alpha_5$. Then $\mathbf{S} = C_{\mathbf{G}}(\mathbf{G}_{\Sigma_1})$ is a 2-dimensional subtorus in $\mathbf{T}$ and $\mathbf{S} \cap \mathbf{G}_{\Sigma_1}$ is the centre of $\mathbf{G}_{\Sigma_1}$.*

Proof. Let $\mathbf{T}_{\Sigma_1} = \mathbf{G}_{\Sigma_1} \cap \mathbf{T}$. This is a maximal split torus of $\mathbf{G}_{\Sigma_1}$. Clearly, one has $C_{\mathbf{G}}(\mathbf{G}_{\Sigma_1}) \subset C_{\mathbf{G}}(\mathbf{T}_{\Sigma_1})$. By properties of reductive groups, $C_{\mathbf{G}}(\mathbf{T}_{\Sigma_1})$ is a reductive group whose derived subgroup $[C_{\mathbf{G}}(\mathbf{T}_{\Sigma_1}), C_{\mathbf{G}}(\mathbf{T}_{\Sigma_1})]$ is generated by roots in Σ orthogonal to $\alpha_2, \alpha_3, \alpha_4, \alpha_5$. Since Σ is of type E_6 there are no such roots, and this implies that $C_{\mathbf{G}}(\mathbf{T}_{\Sigma_1}) = \mathbf{T}$. Thus, $S = C_{\mathbf{G}}(\mathbf{G}_{\Sigma_1})$ is a subtorus in $\mathbf{T}$. To describe it explicitly we will use the canonical cocharacters $\check{\alpha}_i : \mathbf{G}_m \to \mathbf{T}$.

Since $\mathbf{G}$ is simply connected, every element in $t \in \mathbf{T}(K)$ can be written uniquely in the form $t = \prod_i \check{\alpha}_i(t_i)$ where $t_i \in K$. For a root $\alpha \in \Sigma$ let $\mathbf{U}_\alpha$ be the associated root subgroup in $\mathbf{G}$. Using Steinberg's notation (introduced in [21]), every element in $\mathbf{U}_\alpha$ can be written as $x_\alpha(u)$ where $u \in K$. According to the Steinberg relations [21], we have

$$\check{\alpha}_i(t_i) x_\alpha(u) \check{\alpha}_i(t_i)^{-1} = x_\alpha(t_i^{<\alpha, \alpha_i>} u).$$

Using this relation, it is straightforward to verify that $\mathbf{S}$ consists of the elements in $\mathbf{T}$ of the form

$$\check{\alpha}_1\left(\frac{b^2}{a^2}\right) \check{\alpha}_3(b) \check{\alpha}_4(a^2) \check{\alpha}_2(a) \check{\alpha}_5\left(\frac{a^3}{b}\right) \check{\alpha}_6\left(\frac{a^4}{b^2}\right)$$

where $a, b \in K^\times$. Lastly, we note that the centre of $\mathbf{G}_{\Sigma_1}$ is generated by the two elements $\check{\alpha}_3(-1)\check{\alpha}_2(-1)$ and $\check{\alpha}_5(-1)\check{\alpha}_2(-1)$ which are contained in $\mathbf{S}$. □

Let $\mathbf{S}^L \subset \mathbf{Str}(A)$ be the 3-dimensional torus in $\mathbf{Str}(A)$ generated by $\mathbf{S}'^L$ and $\mathbf{G}_m$, where the latter is embedded in $\mathbf{Str}(A)$ as the subgroup of homotheties. The following was proved in [1].

Lemma 6.2. *With the notation above, $\mathbf{S}^L \simeq R_{L/K}(\mathbf{G}_{m,L})$ and $\mathbf{S}'^L \simeq R^{(1)}_{L/K}(\mathbf{G}_{m,L})$. Moreover,*

$$\{a \in A \mid x(a) = a \ \ \forall x \in \mathbf{G}^L(K)\} = L,$$

and the natural action of $\mathbf{Str}(A)$ on A induces an action of $\mathbf{S}^L$ on L, which is transitive on the level of the K^{sep}-points of the open subset

$$L^\times = \{x \in L \mid N(x) \neq 0\}$$

of L and gives rise to an exact sequence

$$1 \longrightarrow \mathbf{Z}^L \longrightarrow \mathbf{S}^L \longrightarrow \boldsymbol{L}^\times \longrightarrow 1.$$

From the exact sequence in cohomology associated to the sequence in the lemma we have a map $L^\times \to H^1(K, \mathbf{Z}^L)$, which is surjective since $H^1(K, \mathbf{S}^L)$ is trivial. Using this map we can attach, to any $a \in L^\times$, a class $[\xi_a] = (a_\tau) \in H^1(K, \mathbf{Z}^L)$. From [1] we moreover know that the image of this class in $H^1(K, \mathbf{H})$ is trivial if a is in the $\mathrm{Str}(A)$-orbit of 1.

Let now $g \in \mathrm{Str}(A)$ be an arbitrary element and set $a := g(1)$. As detailed above, we attach to a a subfield L in A and with it the closed subgroups $\mathbf{G}^L$, $\mathbf{Z}^L$ and $\mathbf{S}^L$ of $\mathbf{Str}(A)$, as well as the class

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

$[\xi_a] = (a_\tau) \in H^1(K, \mathbf{Z}^L)$. Since the image of this class in $H^1(K, \mathbf{H})$ is trivial, there exists $f \in \mathbf{H}(K^{sep})$ such that $a_\tau = (f^{-1})^\tau f$ for all $\tau \in \mathrm{Gal}(K^{sep}/K)$.

Lemma 6.3. *The subset $f(L) \subset A_{K^{sep}} = A \otimes_K K^{sep}$ is contained in $A = A \otimes_K K$.*

Proof. Let $l \in L$. We need to show that $f(l)$ is $\mathrm{Gal}(K^{sep}/K)$-invariant. Take any $\tau \in \mathrm{Gal}(K^{sep}/K)$. Then

$$\tau(f(l)) = (f^\tau)(\tau(l)) = f^\tau(l) = f(f^{-1}f^\tau(l)) = fa_\tau^{-1}(l) = f(l),$$

since $\mathbf{Z}^L \subset \mathbf{G}^L$ fixes L pointwise and the statement follows. □

Since $\mathbf{H} = \mathbf{Aut}(A)$, the above lemma implies that the map $L \to L' = f(L)$ given by $l \mapsto f(l)$ is a field isomorphism over K. Assume that $L \neq L'$. Let $B^+_\sigma \subset A$ be the 9-dimensional subalgebra generated by L and L'. Recall from [1] that ξ_a is constructed explicitly as follows. Choose $t \in \mathbf{S}^L(K^{sep})$ such that $t(1) = a = g(1)$; this is possible by Lemma 6.2. Then $a_\tau = t^{-\tau+1}$, from which we conclude that ft^{-1} is defined over K and that

$$ft^{-1}(g(1)) = ft^{-1}(a) = f(1) = 1,$$

which implies that $ft^{-1}g \in \mathbf{H}(K)$. Thus, modulo $\mathbf{H}(K)$, we may assume that $g = ft^{-1}$.

Let now $s \in \mathrm{Str}(A, B^+_\sigma)$ be the element constructed in Theorem 5.3. It is R-trivial, since so is $\mathbf{Str}(A, B^+_\sigma)^\circ$ and it satisfies $s(L) = f(L)$. Furthermore,

$$L = s^{-1}(f(L)) = s^{-1}ft^{-1}(L) = s^{-1}g(L),$$

since $t(L) = L$. It follows that modulo R-trivial elements we may assume that $g(L) = L$; that is, $g \in \mathbf{Str}(A, L)(K)$, where $\mathbf{Str}(A, L) \subset \mathbf{Str}(A)$ is the subgroup of all elements stabilising L.

Passing to a separable closure of K one can easily check that the connected component of $\mathbf{Str}(A, L)$ is $\mathbf{S}^L \cdot \mathbf{G}^L$. Hence,

$$\mathbf{Str}(A, L) = N_{\mathbf{Str}(A)}(\mathbf{S}^L \cdot \mathbf{G}^L)$$

and

$$\mathbf{Str}(A, L)/\mathbf{S}^L \cdot \mathbf{G}^L \simeq N_{\mathbf{H}}(\mathbf{G}^L)/\mathbf{G}^L \simeq \mathbf{Out}(\mathbf{G}^L).$$

By Lemma 2.7 we can, if necessary, multiply g by an element from $N_{\mathbf{H}}(\mathbf{G}^L)(K)$ to obtain an element $g' \in (\mathbf{S}^L \cdot \mathbf{G}^L)(K)$. To complete our reduction to subgroups of type F_4, it remains to show that g' is R-trivial modulo elements from $\mathbf{H}(K)$. This is the content of the following result.

Proposition 6.4. *Let $g \in (\mathbf{S}^L \cdot \mathbf{G}^L)(K)$. Then there exists an R-trivial element j in $(\mathbf{S}^L \cdot \mathbf{G}^L)(K)$ such that $gj \in \mathbf{G}^L(K) \subset \mathbf{H}(K)$.*

Proof. Our argument is based on the consideration of the exact sequence

$$1 \longrightarrow \mathbf{Z}^L \longrightarrow \mathbf{S}^L \times \mathbf{G}^L \longrightarrow \mathbf{S}^L \cdot \mathbf{G}^L \longrightarrow 1. \tag{8}$$

In the corresponding exact sequence in cohomology, the element g is mapped to a class $[\eta]$ in $H^1(K, \mathbf{Z}^L)$. Since $H^1(K, \mathbf{S}^L)$ is trivial, this class belongs to

$$\mathrm{Ker}\,[H^1(K, \mathbf{Z}^L) \longrightarrow H^1(K, \mathbf{G}^L)].$$

We first prove that it is R-trivial. Since $\mathbf{Z}^L$ is a group of exponent 2, by the norm principle it suffices to prove that $[\eta]$ becomes R-trivial after extending scalars to L/K. Two cases are possible.

If L/K is a Galois extension, then $\mathbf{G}_L^L$, being a strongly inner form of type ${}^1D_4$, is of the form $\mathbf{G}_L^L \simeq \mathbf{Spin}(h)$, where h is a 3-fold Pfister form. Hence, by Example 2.3, the class $res_K^L([\eta])$ is R-trivial.

If L/K is not a Galois extension, then $\mathbf{G}_L^L \simeq \mathbf{Spin}(h)$ for some h satisfying all conditions in Example 2.4. Therefore, $res_K^L([\eta])$ is also R-trivial.

Now the sequence (8) induces the exact sequence

$$(\mathbf{S}^L \times \mathbf{G}^L)(K(x)) \longrightarrow (\mathbf{S}^L \cdot \mathbf{G}^L)(K(x)) \longrightarrow H^1(K(x), \mathbf{Z}^L) \longrightarrow H^1(K(x), \mathbf{G}^L),$$

where x is a variable over K. Let

$$\eta(x) \in \operatorname{Ker}[H^1(K(x), \mathbf{Z}^L) \longrightarrow H^1(K(x), \mathbf{G}^L)]$$

be defined at 0 and 1 and such that $\eta(0) = 1$ and $\eta(1) = \eta$; such an element exists since η is R-trivial. Take any element $g(x) \in (\mathbf{S}^L \cdot \mathbf{G}^L)(K(x))$ that is defined at 0 and 1 and whose image in $H^1(K(x), \mathbf{Z}^L)$ is $\eta(x)$. Note that $g(0)^{-1}g(1)$ is R-trivial in $\mathbf{S}^L \cdot \mathbf{G}^L$. By construction, $g(0) = u_0$ and $g(1) = u_1 g$ for some $u_0, u_1 \in (\mathbf{S}^L \times \mathbf{G}^L)(K)$. Hence,

$$g = u_1^{-1} g(1) = u_1^{-1} g(0) \left(g(0)^{-1} g(1)\right) = u_1^{-1} u_0 \left(g(0)^{-1} g(1)\right).$$

Writing $u_1^{-1} u_0 = us$ for some $u \in \mathbf{G}^L(K)$ and $s \in \mathbf{S}^L(K)$ and noting that $\mathbf{S}^L$ is a rational torus, the proof is complete upon setting $j = \left(s g(0)^{-1} g(1)\right)^{-1}$. □

We have thus altogether proved the following.

Theorem 6.5. *Let $L \subset A$ be a cubic subfield and let $g \in \mathrm{Str}(A)$ be such that $g(L) = L$. Then modulo R-trivial elements, g can be written as a product $g = g_1 g_2$ where $g_1(L) = g_2(L) = L$, $g_1 \in N_{\mathbf{H}}(\mathbf{G}^L)(K)$ and $g_2 \in \mathbf{G}^L(K)$.*

7. End of the Proof

We now finish the proof that $\mathbf{Str}(A)$ is R-trivial. To begin we recall the following known fact.

Theorem 7.1. *Let K be a perfect field. Let $\mathbf{G}$ be a semisimple K-group and $g \in \mathbf{G}(K)$. Then the semisimple and unipotent components g_s, g_u of the Jordan decomposition $g = g_s g_u$ are defined over K. Moreover, if $\mathbf{G}$ is K-anisotropic, then $g_u = 1$.*

Proof. For the first statement see [2, Page 81, Corollary 1] and for the second statement see [3]. □

Next we have the following.

Lemma 7.2. *Let A be a division Albert algebra over K and let $g \in \mathbf{H}(K)$. Then g fixes a cubic subfield in A pointwise.*

Proof. Since $\mathbf{H}$ is K-anisotropic, g is semisimple. Indeed, let $g = g_s g_u$ be its Jordan decomposition over an algebraic closure $\overline{K}$ of K. Let $\widetilde{K} = K^{p^{-\infty}}$. Here p is the characteristic of K. Since $p \neq 2, 3$, the extension $\widetilde{K}/K$ cannot kill the cohomological invariant g_3 attached to $\mathbf{H}$. By [13, Theorem 40.8, part (2)], A is still a division algebra over $\widetilde{K}$, whence $\mathbf{H}_{\widetilde{K}}$ is still anisotropic. It follows from the above theorem that $g_u = 1$.[4]

Hence, g is contained in a maximal K-torus $\mathbf{T} \subset \mathbf{H}$. Using an explicit reduced model of the split Albert algebra $A \otimes_K K^{sep}$, one can easily check that over K^{sep}, every element in $\mathbf{T}$ fixes a commutative subalgebra

$$W \simeq K^{sep} \times K^{sep} \times K^{sep}$$

[4] If the characteristic of K is 2 or 3, then g may be unipotent (see [10]).

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

of $A \otimes_K K^{sep}$ pointwise. It follows that g fixes a vector $v \in A$ which is not a scalar multiple of the identity, whence it fixes the cubic subfield $K(v) \subset A$ generated by v. □

In order to finish our proof we need one final ingredient. Let A be an Albert algebra over K and let $p \in A^\times$. Recall that this is equivalent to $N(p) \neq 0$, where N is the cubic norm of A. The *isotope* $A^{(p)}$ of A is the algebra with underlying linear space A and multiplication

$$x \bullet_p y = x(yp) + y(xp) - (xy)p,$$

where juxtaposition denotes the multiplication of A. It is known that $A^{(p)}$ is an Albert algebra and that the cubic norm of $A^{(p)}$ is νN, where $\nu = N(p)$. From this it follows that $A^{(p)}$ is a division algebra if and only if A is and that $\mathbf{Str}(A) \simeq \mathbf{Str}(A^{(p)})$.

Proposition 7.3 [25]. *Let A be a division Albert algebra over K. Then there exists $p \in A^\times$ such that $A^{(p)}$ contains a cubic cyclic subfield.*

Remark 7.4. If the ground field F contains a cubic root of unity, then due to a result of H. Petersson and M. Racine [16], one can take $p = 1$.

We are now ready to prove our main result.

Theorem 7.5. *Let A be a division Albert algebra over K. Then the algebraic K-group $\mathbf{Str}(A)$ is R-trivial.*

Proof. By the above discussion, we may replace A by any isotope $A^{(p)}$. Thus, by the preceding proposition we may assume that A contains a cubic cyclic subfield L. Let $g \in \mathrm{Str}(A)$. By the results of Section 6, summarised in Theorem 6.5, we may assume that $g \in \mathbf{H}(K)$. Three cases are possible.

Case 1: g fixes L but not pointwise. Let $F \subset A$ be a cubic field extension of K pointwise fixed by g; such a field exists by Lemma 7.2. Since $F \neq L$, L and F generate a 9-dimensional subalgebra $B^+_\sigma \subset A$ that is stabilised by g. In the course of the proof of Corollary 4.3 we saw that

$$\mathbf{Str}(A, B^+_\sigma)(K) = \mathbf{Str}(A, B^+_\sigma)^\circ(K).$$

By Proposition 4.2, the algebraic K-group $\mathbf{Str}(A, B^+_\sigma)^\circ$ is R-trivial and therefore g is R-trivial.

Case 2: g fixes L pointwise. Hence, g is in the group of K-points of the algebraic K-group $\mathbf{Str}(A/L)$ whose functor of points is as follows: if R is a K-ring,

$$\mathbf{Str}(A/L)(R) = \{x \in \mathbf{Str}(A)(R) \mid x(l) = l \text{ for all } l \in L\} = \mathbf{G}^L(R).$$

Since L is cyclic, by Lemma 2.7 there is an element $h_1 \in N_{\mathbf{H}}(\mathbf{G}^L)(K)$ that stabilises L but does not fix it pointwise. The same is true for $h_2 := h_1^{-1} g$, since g fixes L pointwise. From Case (1) we know that h_1 and h_2 are R-trivial and hence so is g.

Case 3: $g(L) \neq L$. Since $g \in \mathrm{Aut}(A)$, the field $g(L)$ is a cubic cyclic subfield of A isomorphic to L. The subfields L and $g(L)$ generate a 9-dimensional subalgebra B^+_σ of A. By Theorem 5.3 there exists $h_1 \in \mathrm{Str}(A, B^+_\sigma)$ such that $h_1(g(L)) = L$ and by Proposition 4.2, h_1 is R-trivial. Let $h_2 := h_1 g$. By construction, it belongs to

$$\mathbf{Str}(A, L)(K) = \{x \in \mathrm{Str}(A) \mid x(L) = L\}.$$

By Theorem 6.5, h_2 can be written, modulo R-trivial elements, as a product $h_2 = h_3 h_4$ with $h_3 \in \mathbf{G}^L(K)$ and $h_4 \in N_{\mathbf{H}}(\mathbf{G}^L)(K)$. In particular, h_3 and h_4 are in $\mathbf{H}(K)$ and stabilise L. By Cases (1) and (2), the elements h_3 and h_4 are R-trivial and hence so is h_2 and g. This completes the proof. □

Theorem 7.6. *Let $\overline{\mathbf{G}}$ be an adjoint strongly inner form of type E_6. Then $\overline{\mathbf{G}}$ is R-trivial.*

Proof. Indeed, the canonical map $\mathbf{Str}(A) \to \overline{\mathbf{G}}$ has a rational section, since its kernel $\mathbf{G}_m$ has trivial Galois cohomology in dimension 1. We conclude by the above theorem. □

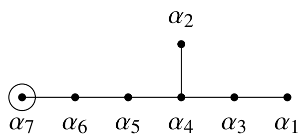
https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

8. Applications

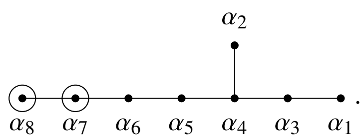
8.1. The Kneser–Tits Problem for $\mathrm{E}_{7,1}^{78}$ *and* $\mathrm{E}_{8,2}^{78}$

Theorem 8.1. *Let K be an arbitrary field. Let $\mathbf{G}$ be a simple simply connected K-group of type $\mathrm{E}_{7,1}^{78}$ or $\mathrm{E}_{8,2}^{78}$. Then the Kneser–Tits conjecture for $\mathbf{G}$ holds.*

Proof. According to [9, Remarque 7.4] we may, without loss of generality, assume that $\mathrm{char}(K) = 0$. The Tits index of $\mathbf{G}$ is of the form



or



In both cases the semisimple anisotropic kernel $\mathbf{H}$ of $\mathbf{G}$ is a strongly inner form of type E_6. If $\mathbf{S} \subset \mathbf{G}$ is a maximal split torus whose centraliser is $\mathbf{H}$, then arguing as in [5] one easily verifies that over an algebraic closure of K the intersection $\mathbf{S} \cap \mathbf{H}$ is the centre of $\mathbf{H}$.

Furthermore, by [9, Théorème 7.2], the Kneser–Tits problem has an affirmative answer if and only if $\mathbf{G}(K)/R = 1$. It follows from the Bruhat–Tits decomposition that

$$\mathbf{G}(K)/R \simeq C_{\mathbf{S}}(\mathbf{G})/R \simeq (\mathbf{H}/(C_{\mathbf{S}}(\mathbf{G}) \cap \mathbf{H}))/R \simeq \overline{\mathbf{H}}/R$$

where $\overline{\mathbf{H}}$ is the corresponding adjoint group. It remains to note that by Theorem 7.6, the group $\overline{\mathbf{H}}$ is R-trivial. □

8.2. The Tits–Weiss Conjecture

Theorem 8.2. *Let A be an Albert algebra defined over an arbitrary field K. Then the group $\mathbf{Str}(A)(K)$ is generated by the U-operators U_a with $a \in A^\times$ and the scalar homotheties.*

Proof. By the main result of the Appendix, the Tits–Weiss conjecture holds if and only if the Kneser–Tits conjecture holds for groups of type $\mathrm{E}_{7,1}^{78}$ and $\mathrm{E}_{8,2}^{78}$. The result follows. □

Remark 8.3. From this theorem, which we have now established in arbitrary characteristic, the R-triviality of $\mathbf{Str}(A)$ is immediate. Thus, Theorem 7.5 holds in arbitrary characteristic as well.

8.3. Properties of the Functor of R-Equivalence Classes for Strongly Inner Forms of Type ${}^1\mathrm{E}_6$

To a reductive K-group $\mathbf{G}$ one can attach the functor of R-equivalence classes

$$\mathcal{G}/R : \mathit{Fields}/K \longrightarrow \mathit{Groups}, \quad F/K \to \mathbf{G}(F)/R$$

where Fields/K is the category of field extensions of K and Groups is the category of abstract groups. The experts expect that the following conjectures hold:

Conjecture 1. *The functor $\mathcal{G}/R$ factors through the subcategory of abelian groups of the category Groups; that is, for all field extensions F/K the group $\mathbf{G}(F)/R$ is abelian.*

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

Conjecture 2. *If F is a finitely generated field over its prime subfield, then $\mathbf{G}(F)/R$ is finite.*

Conjecture 3. *The functor $\mathcal{G}/R$ has transfers; that is, there is a functorial collection of maps tr_K^F : $\mathbf{G}(F)/R \to \mathbf{G}(K)/R$ for all finite field extensions F/K.*

Furthermore, one expect that the norm principle holds for all semisimple K-groups. Of course, all of these conjectures are obviously true for R-trivial K-groups. In particular, they hold for rational K-groups. For instance, this is the case for groups of type G_2. In [1] we investigated the case of groups of type F_4 arising from the first Tits construction. Here we consider the next case of simple simply connected strongly inner forms of type E_6.

Theorem 8.4. *Let K be an arbitrary field and let $\mathbf{G}$ be a simple simply connected K-group which is a strongly inner form of type E_6. Then*

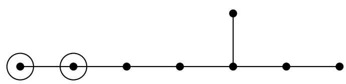
(i) *$\mathbf{G}(K)/R$ is an abelian group.*
(ii) *the functor $\mathcal{G}/R$ has transfers.*
(iii) *if $\mathbf{Z}$ is the centre of $\mathbf{G}$, the norm principle holds for $(\mathbf{Z}, \mathbf{G})$.*

Proof. (i) The group $\mathbf{G}$ is the derived subgroup of the R-trivial K-group $\mathbf{Str}(A)$ for an Albert algebra A. It follows that $[\mathrm{Str}(A), \mathrm{Str}(A)] \subset R\mathbf{G}(K)$ and hence $\mathbf{G}(K)/R$ is abelian.
(ii) Apply Theorem 3.4 in [4] to $\mathbf{G}' = R_{F/K}(\mathbf{G})$.
(iii) Since the corresponding adjoint group $\overline{\mathbf{G}}$ is R-trivial, the result follows from Theorem 2.5. □

A. Appendix by Richard M. Weiss

Department of Mathematics
Tufts University
Medford, MA 02155, USA
rweiss@tufts.edu

In this appendix we examine the connection between groups $G(k)$ with index



(that is to say, $E_{8,2}^{78}$) and anisotropic exceptional cubic norm structures. Our main goal is show the equivalence of Assertions A.3 and A.4.

Notation A.1. Let k be a field, let G be a semi-simple simply connected algebraic group of absolute type E_8 defined over k such that the index of $G(k)$ is $\mathscr{I} := E_{8,2}^{78}$, let S be a maximal k-split torus of G, let T be a maximal torus containing S and defined over k and let Φ be the root system of G with respect to T. The nodes of $\mathscr{I}$ form a root basis of Φ. Let $\tilde{\alpha}$ denote the highest root with respect to this basis.

Notation A.2. Let Φ_k denote the relative root system of G with respect to S; it is a root system of type G_2. For each $\alpha \in \Phi_k$, let $U_{(\alpha)}$ denote the unipotent k-subgroup defined in [1, Section 5.2]. We call the groups $U_{(\alpha)}$ for $\alpha \in \Phi_k$ the *relative root groups* of G.

Here now are the two assertions whose equivalence we want to demonstrate.

Assertion A.3. $G(k) = \langle U_{(\alpha)}(k) \mid \alpha \in \Phi_k \rangle$ for all $G(k)$ as in A.1.

Assertion A.4. Let $\Xi = (J, k, N, \#, T, \times, 1)$ be an exceptional cubic norm structure and suppose that Ξ is anisotropic. Then the structure group $\mathrm{Str}(\Xi)$ of Ξ is generated by the set

$$\{U_a \mid a \in J^*\} \cup \{b \mapsto tb \mid t \in k^*\}, \tag{9}$$

where U_a is as in [4, Section (15.42)].

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

Remark A.5. All anisotropic cubic norm structures arise from one of two constructions of Tits. A proof of this result, due to Racine and Petersson, can be found in [4, Chapters 15 and 30 and Section (17.6)].

Let k, G, $\mathscr{I}$, S, Φ and $\tilde{\alpha}$ be as in A.1. We begin our demonstration.

Proposition A.6. *The following hold:*

1. *The root group $U_{\tilde{\alpha}}$ of G is defined over k.*
2. *The quotient $N_{G(k)}(U_{\tilde{\alpha}}(k))/C_{G(k)}(U_{\tilde{\alpha}}(k))$ acts freely on the set of nontrivial elements of $U_{\tilde{\alpha}}(k)$.*

Proof. The two claims follow from the observation that the root $\tilde{\alpha}$ is orthogonal to the subspace spanned by the nodes in the anisotropic part of $\mathscr{I}$. □

Remark A.7. Let Φ_k and $U_{(\alpha)}$ for $\alpha \in \Phi_k$ be as in A.2. By A.6(i), $U_{\tilde{\alpha}}$ is a relative root group of G for some long root of Φ_k. Thus, in particular, $\dim_k U_{(\alpha)} = 1$ for all long roots α of Φ_k.

Remark A.8. We have

$$\sum_{\alpha \in \Phi_k} \dim_k U_{(\alpha)} = n_8 - n_6,$$

where n_ℓ denotes the cardinality of a root system of type E_ℓ for $\ell = 6$ and 8. It follows that $\dim_k U_{(\alpha)} = 27$ for all of the short roots α of Φ_k.

Now let $\mathrm{BldSc}(G(k))$ be the spherical building attached to $G(k)$ as described in [4, Section 42.3.6] and let X denote the Moufang hexagon associated with $\mathrm{BldSc}(G(k))$. Thus, X is the bipartite graph whose vertices are the nonminimal parabolic subgroups of G defined over k, where two of these parabolic subgroups are adjacent whenever their intersection is also a parabolic subgroup defined over k. Since the cocentre of E_8 is trivial, the centre of G is trivial. Thus, $G(k)$ acts faithfully on $\mathrm{BldSc}(G(k))$ and hence on X. From now on, we identify $G(k)$ with its image in $\mathrm{Aut}(X)$. For each $\alpha \in \Phi_k$, let U_α denote the subgroup $U_{(\alpha)}(k)$ of $G(k)$.

Let $\alpha_0, \alpha_1, \ldots, \alpha_{11}$ be a labelling of the 12 roots in Φ_k with subscripts in $\mathbb{Z}_{12}$ such that the angle between α_{i-1} and α_i is $\pi/6$ for each i and α_i is long if and only if i is even. An apartment of X is a circuit of length 12.

Proposition A.9. *There is a unique apartment Σ of X for which there is a labelling $x_0, x_1, \ldots, x_{11}$ of the vertices of Σ with subscripts in $\mathbb{Z}_{12}$ such that for each i, x_{i-1} is adjacent to x_i and U_{α_i} is the root group of X corresponding to the root $(x_i, x_{i+1}, \ldots, x_{i+6})$ of Σ as defined in [4, Section (4.1)].*

Proof. This holds by [4, Prop. 42.3.6]. □

Proposition A.10. *Let $U_i = U_{\alpha_i}$ for each i. Then there exists an anisotropic cubic norm structure*

$$\Xi = (J, F, N, \#, T, \times, 1)$$

and isomorphisms x_i from the additive group of F to U_i for $i = 2$, 4 and 6 and isomorphisms x_i from the additive group of J to U_i for $i = 1$, 3 and 5 such that the commutator relations in [4, Section (16.8)] hold and $[U_i, U_j] = 1$ for all index pairs (i, j) of indices with $1 \le i < j \le 6$ that do not appear in this list of relations.

Proof. This holds by [4, Sections (29.1)–(29.35)]. (Note that since $[U_{(\alpha)}, U_{(\beta)}] = 1$ for all long roots α, β of Φ_k at an angle of $\pi/3$, the assumption that $V_i \neq 1$ for all even i at the top of [4, page 303] is valid.) □

Remark A.11. By [4, Section (7.5)], X is uniquely determined by Ξ. We can thus set $X = \mathscr{H}(\Xi)$ as in [4, Section 16.8]. By [4, Section (35.13)], $\mathscr{H}(\Xi) \cong \mathscr{H}(\Xi')$ for two anisotropic cubic norm structures Ξ and Ξ' if and only if Ξ and Ξ' are isotopes of each other.

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

Notation A.12. Let $G_0 = \mathrm{Aut}(X)$, let H_0 denote the pointwise stabiliser of Σ in G_0, let $G_0^\dagger = \langle U_i \mid i \in \mathbb{Z}_{12}\rangle$, let $H_0^\dagger = G_0^\dagger \cap H_0$ and let $J = G(k) \cap H_0$.

Since $G_0^\dagger \subset G(k)$, we have $H_0^\dagger \subset J$. In fact, we have the following.

Proposition A.13. *$H_0^\dagger = J$ if and only if $G(k) = G_0^\dagger$.*

Proof. Suppose that $H_0^\dagger = J$ and let $g \in G(k)$. By [4, Section (4.12)], there exists $a \in G_0^\dagger$ such that ga fixes both Σ and the edge $\{x_0, x_1\}$. Hence, $ga \in H_0$. Since $G_0^\dagger \subset G(k)$, it follows that $ga \in J$. Hence, $ga \in H_0^\dagger$ and thus $g \in G_0^\dagger$. Therefore, $G(k) = G_0^\dagger$. □

Proposition A.14. *Let $X_i = \langle \mu_i(a)\mu_i(b) \mid a, b \in U_i^*\rangle$ for $i = 1$ and 6, where μ_i is as defined in [4, Section (6.1)]. Then $H_0^\dagger = X_1X_6$.*

Proof. This holds by [4, Section (33.9)]. □

Proposition A.15. *The image of $H_0^\dagger$ in $\mathrm{Aut}(U_6)$ is $\{x_6(t) \mapsto x_6(st) \mid s \in F^*\}$, where x_6 is as in A.10. In particular, the derived group of $H_0^\dagger$ centralises U_6.*

Proof. This holds by [4, Section (33.16)]. □

Proposition A.16. *The groups J and $H_0^\dagger$ have the same image in* $\mathrm{Aut}(U_6)$.

Proof. By A.15, $H_0^\dagger$ acts transitively on U_6^*. By A.6(ii) and A.7, the image of J in $\mathrm{Aut}(U_6)$ acts freely on U_6^*. Since $H_0^\dagger \subset J$, the claim follows. □

Proposition A.17. *$H_0^\dagger = J$ if and only if $C_J(U_6) = C_{H_0^\dagger}(U_6)$.*

Proof. Suppose that $C_J(U_6) = C_{H_0^\dagger}(U_6)$ and let $g \in J$. By A.16, there exists $h \in H_0^\dagger$ such that $gh \in C_J(U_6)$. Hence, $gh \in H_0^\dagger$ and thus $g \in H_0^\dagger$. Therefore, $H_0^\dagger = J$. □

Proposition A.18. *$C_{H_0}(U_6)$ acts faithfully on U_1.*

Proof. This holds by [4, Section (33.5)]. □

Notation A.19. Let H be the anisotropic kernel of G. Thus, H is the derived group of the centraliser $C(S)$.

Proposition A.20. *$H(k) \subset C_J(U_6)$.*

Proof. The fixed point set of $S(k)$ in X is the apartment Σ. Hence, $C(S)(k) \subset J$. By A.15 and A.16, it follows that $H(k) \subset C_J(U_6)$. □

Proposition A.21. *$F \cong k$ and $\dim_F J = 27$, where F is as in A.10.*

Proof. We give U_1 the structure of a vector space over F by setting $t \cdot x_1(a) = x_1(ta)$ for all $t \in F$ and all $a \in J$. Let X_6 be as in A.14. By [4, Section (29.20)], the image of X_6 in $\mathrm{Aut}(U_1)$ is $\{x_1(a) \mapsto t \cdot x_1(a) \mid t \in F^*\}$. The root groups U_{α_0} and U_{α_6} of G are opposite and $X_6 = \langle U_0, U_6\rangle \cap S(k)$. It follows that there exists an isomorphism ψ from k to F such that $tu = \psi(t) \cdot u$ for all $t \in k$ and all $u \in U_1$. Hence, $\dim_F J = \dim_F U_1 = 27$ by A.8. □

Proposition A.22. *There is an isomorphism φ from $C_J(U_6)$ to $\mathrm{Str}(\Xi)$ and Ξ is exceptional.*

Proof. By A.18 and the equation in the third display in [4, Section (37.41)], there exists an isomorphism φ from $C_{H_0}(U_6)$ to $\mathrm{Str}(\Xi)$ such that $x_1(a)^h = x_1(a^{\varphi(h)})$ for all $a \in J$ and by A.20, we have $H(k) \subset C_J(U_6) \subset C_{H_0}(U_6)$. By A.21, it follows that Ξ is the Jordan k-algebra in [4, Sections 42.5.6(d) and 42.6, Type (8)]. Hence, Ξ is exceptional and φ maps $H(k)$ to $\mathrm{Str}(\Xi)$ surjectively. (See also the remarks that follow [3, Section 3.3.1].) It follows that $H(k) = C_J(U_6) = C_{H_0}(U_6)$. □

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press

Proposition A.23. *$\varphi(C_{H_0^\dagger}(U_6))$ is the subgroup of $\mathrm{Str}(\Xi)$ generated by the set defined in (9), where φ is as in A.22.*

Proof. This holds by [4, Section (33.16)]. □

Proposition A.24. *Every anisotropic exceptional cubic norm structure arises from an application of A.10 to the Moufang hexagon attached to a group $G(k)$ for some G and some k satisfying the conditions in A.1.*

Proof. This holds by [4, Section 42.6, Type (8)]. □

Theorem A.25. *The Assertions A.3 and A.4 are equivalent.*

Proof. By A.13 and A.17, $G(k) = G_0^\dagger$ if and only if $C_{H_0^\dagger}(U_6) = C_J(U_6)$. By A.22 and A.23, $C_{H_0^\dagger}(U_6) = C_J(U_6)$ if and only if $\mathrm{Str}(\Xi)$ is generated by the set defined in (9). The claim holds, therefore, by A.24. □

Remark A.26. In [2, Thm. 6.1], Thakur showed that A.3 holds in the case that the cubic norm structure in A.10 is a first Tits construction and in [2, Thm. 7.2], he showed that the claim in A.4 holds for Ξ a reduced (rather than anisotropic) exceptional cubic norm structures.

References

[1] A. Borel and J. Tits, 'Groupes réductifs', *Publ. Math. I.H.E.S.* **27** (1965), 55–151.

[2] M. Thakur, 'Automorphisms of Albert algebras and a conjecture of Tits and Weiss, II', *Trans. A.M.S.* **372**, (2019) 4701–4728.

[3] J. Tits, 'Strongly inner anisotropic forms of simple algebraic groups', *J. Alg.* **131** (1990), 648–677.

[4] J. Tits and R. M. Weiss, *Moufang Polygons* (Springer, Berlin, Heidelberg, New York, 2002).

Acknowledgments. We are grateful to F. Morel, H. Petersson, A. Stavrova and R. Weiss for fruitful discussions and comments. During the preparation of the article, the second author visited the University of Munich; their hospitality and the support of the Alexander von Humboldt Foundation are gratefully acknowledged. S. Alsaody thanks PIMS for their financial support and the Department of Mathematical and Statistical Sciences at the University of Alberta, where he was affiliated during the preparation of the article. V. Chernousov was partially supported by the Alexander von Humboldt Foundation and an NSERC research grant. A. Pianzola thanks NSERC and CONICET for their continuous support.

Conflicts of interest: None.

References

[1] S. Alsaody, V. Chernousov and A. Pianzola, 'R-triviality of groups of type F_4 arising from the first Tits construction', *Bull. London Math. Soc.* **53** (2021), 535–547.

[2] A. Borel, *Linear Algebraic Groups*, Second enlarged edition, Springer, New York-Berlin (1991).

[3] A. Borel and J. Tits, 'Eléments unipotents et sous-groupes paraboliques de groupes réductivefs I', *Invent. Math.* **12** (1971), 95–104.

[4] V. Chernousov and A. Merkurjev, 'R-equivalence and special unitary groups', *J. Algebra* **209** (1998), 175–198.

[5] V. Chernousov and V. Platonov, 'The rationality problem for semisimple group varieties', *J. Reine Angew. Math.* **504** (1998), 1–28.

[6] T. De Medts, B. Mühlherr and A. Stavrova, *Proposal for a Mini-Workshop at the MFO* (2018).

[7] S. Garibaldi and H. P. Petersson, 'Outer automorphisms of algebraic groups and a Skolem–Noether theorem for Albert algebras', *Doc. Math.* **21** (2016), 917–954.

[8] P. Gille, *'La R-équivalence sur les groupes algébriques réductifs définis sur un corps global*', Pub. Math. l'I.H.E.S. **86** (1997), 199–235.

[9] P. Gille, 'Le problème de Kneser–Tits', Séminaire Bourbaki, Exposé no. 983, Astérisque, no. 326 (2009).

[10] P. Gille and A. Quéguiner-Mathieu, 'Exemples de groupes semi-simples simplement connexes anisotropes contenant un sous-groupe unipotent', 2012, hal-00731109v2.

[11] J. E. Humphreys, *Linear Algebraic Groups*, Springer, New York–Berlin (1975).

[12] N. Jacobson, *Finite-Dimensional Division Algebras over Fields* (Springer, Berlin, 1996).

[13] M. Knus, A. Merkurjev, M. Rost and J.-P. Tignol, *The Book of Involutions*, AMS Colloquium Publications, Vol. **44**, USA (1998).

[14] A. Merkurjev, '*R-equivalence and rationality problem for semisimple adjoint classical algebraic groups*', Pub. Math. l'I.H.É.S. **84** (1996), 189–213.

[15] H. P. Petersson, 'A survey on Albert algebras', *Transformation Groups* **24** (2019), 219–278.

[16] H. P. Petersson and M. Racine, 'Cubic subfields of exceptional simple Jordan algebras', *Proc. Amer. Math. Soc.* **91** (1984), 31–36.

[17] V. Platonov, 'On the Tannaka–Artin problem', *Dokl. Akad. Nauk SSSR, Ser. Math.* **221** (1975), 1038–1041; English translation: *Soviet Math. Dokl.* 16 (1975), 468–471.

[18] G. Prasad and M. S. Raghunathan, 'On the Kneser–Tits problem', *Comment. Math. Helv.* **60** (1985), 107–121.

[19] G. Prasad and A. Rapinchuk, '*Weakly commensurable arithmetic groups and isospectral locally symmetric spaces*', Pub. Math. l'I.H.É.S. **109** (2009), 113–184.

[20] T. A. Springer and F. D. Veldkamp, *Octonion Algebras, Jordan Algebras and Exceptional Groups*, *Springer Monographs in Mathematics*, Springer, Berlin (2000).

[21] R. Steinberg, *Lectures on Chevalley Groups* Yale University, unpublished notes (1967).

[22] M. Thakur, 'Automorphisms of Albert algebras and a conjecture of Tits and Weiss', *Trans. AMS* **365** (2013), 3041–3068.

[23] M. Thakur, 'Automorphisms of Albert algebras and a conjecture of Tits and Weiss II', *Trans. AMS* **372** (2019), 4701–4728.

[24] M. Thakur, 'Albert algebras and the Tits–Weiss conjecture', Preprint, 2019, arXiv:1911.04976v1.

[25] M. Thakur, 'The cyclicity problem for Albert algebras', *Israel J. Math.* **241** (2021), 139–145.

[26] J. Tits and R. Weiss, *Moufang Polygons*, *Springer Monographs in Mathematics*, Springer, Berlin, (2002).

https://doi.org/10.1017/fms.2021.65 Published online by Cambridge University Press