10

The geometry of D-branes

As we have seen, branes of various sorts are solutions of string theory which are localised to some extent, and have well-defined mass and charge per unit volume. Since these masses and charges are measured at infinity, meaning that the branes are sources of fields from the massless sector, we might expect that they must be actually be solutions of the low energy equations of motion: the gravity sector and other fields such as the various antisymmetric tensor fields, and possibly the dilaton. These field configurations can be thought of as representing interesting backgrounds in which the string can propagate. It has become increasingly important in many recent research areas to consider the details of such solutions, and we shall begin exploring this highly developed technology in the present chapter.

10.1 A look at black holes in four dimensions

Before we launch into a description of the solutions associated to branes, it is a good idea to start with something more familiar in order to gain some intuition about how the solutions work. We will start in four dimensions with a familiar system: Einstein's gravity coupled to Maxwell's electromagnetism. The more advanced reader may wish to skip directly to section 10.2 if the following is too elementary, but beware, since we shall be uncovering and emphasising probably less familiar features in order to prepare for analogous properties of branes in higher dimensions.

10.1.1 A brief study of the Einstein–Maxwell system

Let us consider the Einstein–Hilbert action for gravity coupled to the Maxwell system:

$$S = \frac{1}{16\pi G}\int d^4x\ (-g)^{1/2}\,[R - GF_{\mu\nu}F^{\mu\nu}], \tag{10.1}$$

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where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The equations of motion for this system resulting from varying with respect to $g_{\mu\nu}$ are of course:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{10.2}$$

where

$$T_{\mu\nu} = \frac{1}{4\pi}\left(g^{\gamma\delta}F_{\mu\gamma}F_{\nu\delta} - \frac{1}{4}g_{\mu\nu}F_{\gamma\delta}F^{\gamma\delta}\right). \tag{10.3}$$

A particularly interesting spherically symmetric solution of this system, (see insert 10.1) representing a source of mass M and electric charge Q is, for the metric:

$$ds^2 = -\left(1 - \frac{2MG}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2MG}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_2^2, \tag{10.4}$$

where $d\Omega_2^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$, is the metric on a round S^2 in standard polar coordinates, and

$$F_{tr} = E_r(r) = -F_{rt}, \quad E_r(r) = \frac{Q}{r^2} \quad \text{or } F = \frac{Q}{r^2}dt\wedge dr.$$

Let us note some of the key properties of these solutions.

10.1.2 Basic properties of Schwarzschild

We begin with the case $Q = 0$, an empty-space solution (i.e. a solution of pure Einstein gravity), which is the Schwarzschild solution. The first thing to take note of is that the solution has various obvious symmetries. Notice that the metric components do not depend on t or ϕ. So there is a pair of symmetries coming from invariance under translations in these coordinates. In other words, the solution is static, and symmetric about the ϕ axis. Well, of course it is manifestly spherically symmetric as well. In a more sophisticated language, we would say that there are 'Killing vectors' $\mathbf{k}$, of this solution satisfying

$$\nabla_\mu \mathbf{k}_\nu + \nabla_\nu \mathbf{k}_\mu = 0, \tag{10.7}$$

where ∇_μ is the covariant derivative. Our two obvious ones are:

$$\xi^\mu = \left(\frac{\partial}{\partial t}\right)^\mu = (1,0,0,0); \tag{10.8}$$

$$\eta^\mu = \left(\frac{\partial}{\partial \phi}\right)^\mu = (0,0,0,1), \tag{10.9}$$

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Insert 10.1. Checking the Reissner–Nordström solution

It is worthwhile listing some of the objects that the diligent reader would have computed if checking by hand that equation (10.4) is a solution. They will be useful later. The non-vanishing components of the 'affine' or 'metric' connection are:

$$\begin{aligned}
\Gamma^t_{tr} &= \frac{M\,r - Q^2}{r\,(r^2 - 2\,M\,r + Q^2)}; \quad \Gamma^r_{tt} = \frac{(r^2 - 2\,M\,r + Q^2)\,(M\,r - Q^2)}{r^5}; \\
\Gamma^r_{rr} &= -\frac{M\,r - Q^2}{r\,(r^2 - 2\,M\,r + Q^2)}; \quad \Gamma^r_{\theta\theta} = -\frac{r^2 - 2\,M\,r + Q^2}{r}; \\
\Gamma^r_{\phi\phi} &= -\frac{(r^2 - 2\,M\,r + Q^2)\sin^2\theta}{r}; \\
\Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta; \quad \Gamma^\phi_{r\phi} = \frac{1}{r}; \quad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}; \quad \Gamma^\theta_{\phi\theta} = \frac{1}{r},
\end{aligned} \tag{10.5}$$

remembering that it is symmetric in its lower components. Taking some more derivatives to make the Riemann–Christoffel tensor, and then contracting gives the non-vanishing components of the Ricci tensor:

$$\begin{aligned}
R_{tt} &= \frac{(r^2 - 2\,M\,r + Q^2)\,Q^2}{r^6}; \quad R_{rr} = -\frac{Q^2}{r^2\,(r^2 - 2\,M\,r + Q^2)}; \\
R_{\theta\theta} &= \frac{Q^2}{r^2}; \quad R_{\phi\phi} = \frac{\sin^2\theta\,Q^2}{r^2},
\end{aligned} \tag{10.6}$$

from which it is easy to see that its trace, the Ricci scalar R, actually vanishes. Computing the stress tensor gives the result that $T_{\mu\nu} = R_{\mu\nu}/8\pi$, proving that it is a solution.

in an obvious notation*. To see the full spherical symmetry, it is in fact better to change variables to the 'isotropic coordinates', so called because it makes the spatial part of the metric conformal to flat space, which means that all distances measured are rescaled by an overall factor, but the locally measured angles between vectors are preserved. Changing to

* Here, and in many other places, we will use the fact that in curved spacetime it is very useful to define vectors as differential operators.

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a new coordinate ρ defined by

$$r = \rho\left(1 + \frac{M}{2\rho}\right)^2 ,$$

the metric becomes†

$$ds^2 = -\frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2}dt^2 + \left(1+\frac{M}{2\rho}\right)^4\left(dx^2+dy^2+dz^2\right), \qquad (10.10)$$

where $\rho^2 = x^2+y^2+z^2$. Then the Killing vectors corresponding to spherical symmetry are

$$\mathbf{L}_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad \mathbf{L}_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad \mathbf{L}_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

One can check that they satisfy the A_1 (i.e. $SO(3)$) Lie algebra: $[\mathbf{L}_i, \mathbf{L}_j] = \epsilon_{ijk}\mathbf{L}_k$. It is worth knowing that the existence of Killing vectors guarantees certain important properties of the solutions, helping to exhibit certain conserved physical quantities. For example, $\partial/\partial t$, being timelike, ensures that the geometry is static, since Killing's equation results in $\partial g_{\mu\nu}/\partial t = 0$.

Recall that a vector (or more properly a vector field in curved spacetime) define a curve, by being the tangent to it at every point. In fact, along a curve generated by a Killing vector $\mathbf{k}$, the combination $\mathbf{u}\cdot\mathbf{k}$ is a conserved quantity, which will be useful later on. Notice that ξ and η, as defined above, define for us (respectively) a conserved energy and angular momentum per unit rest mass.

Now, it is of course a familiar feature of the solution that the spherical surface $r = r_{\rm H} = 2M$ is an horizon, since we can see that, for example, g_{tt} vanishes there. While looking at the vanishing of g_{tt} is a quick way of reading off the location horizon, for the general geometry (10.4), it is misleading in general. We should characterise it as follows:

The spherical surface at radius $r = R$ has a unit normal vector to it, $\mathbf{n}$, given by (see insert 10.2)

$$n^\mu = \frac{1}{\sqrt{|g_{rr}|}}\left(\frac{\partial}{\partial r}\right)^\mu . \qquad (10.11)$$

In fact, the norm $n^2 = n^\mu n_\mu$ takes the value $+1$ for $r > r_{\rm H}$ and -1 for $r < r_{\rm H}$, while for $r = r_{\rm H}$, it is zero. So the spherical surface corresponding to the horizon is a 'null hypersurface'. For $r > r_{\rm H}$, had we

† It is worth checking that this can be done for non-zero Q also, solving for the new radial coordinate via $(r^2 - 2Mr + Q^2)^{-1/2}dr = \rho^{-1}d\rho$. More generally, any spherically symmetric solution can be written in isotropic form, with sufficient effort.

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approached this spacetime in a spaceship, we can blast our rockets and avoid the horizon if we choose, so any hypersurface this side of it is timelike, while any hypersurface the other side of it is spacelike, since we have to encounter them. Why do we have to encounter them? Well, looking back at the metric we see that in fact the role of t and r have exchanged roles for $r < r_{\rm H}$. This is because it is now the coefficient of dr^2 which is negative, and so it is really a *time* coordinate. So once we are in the region $r < r_{\rm H}$, all smaller values of r are in the inevitable future. The 'singularity' at $r = 0$ is a special case of one the inevitable spacelike hypersurfaces, so it is in our future as soon as we cross the horizon. In other words, Schwarzschild has a spacelike singularity, which is an important fact.

10.1.3 Basic properties of Reissner–Nordstrom

Let us consider the case of $Q \neq 0$, the charged black hole geometry. The set of spacelike Killing vectors representing spherical symmetry is similar to the case we had before, and there is again a timelike Killing vector arising form the t-invariance of the metric components, showing that the solution is static. When we come to look at the horizon structure, things get interesting. There are two, since there are two places where the hypersurface normal in equation (10.11) can go null:

$$r_\pm = M \pm \sqrt{M^2 - Q^2}.$$

It should be clear that there is a singularity at $r = 0$ again. Very interestingly, we can can see by looking at $\mathbf{n}$ that the singularity is *timelike*, and so it is in fact avoidable with sufficient effort, if one were moving in the geometry.

We have tacitly assumed that $M \geq Q$, or there will be no horizons, and the singularity at $r = 0$ will be a 'naked singularity', which is not allowed by the cosmic censors, it is believed[292]. That this is a strict and physical bound makes a lot of sense when we study this solution further, especially in a supersymmetric context, which we should do next.

10.1.4 Extremality, supersymmetry, and the BPS condition

There is a very important special case arising when we saturate the lower bound on M, making it equal to Q. Then we see that both horizons coincide at $r = Q$. Let us change coordinates to $R = r - Q$, giving:

$$ds^2 = -\frac{R^2}{(R+Q)^2}dt^2 + \frac{(R+Q)^2}{R^2}[dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (10.15)$$

Insert 10.2. A little hypersurface technology

Let us formulate the idea of hypersurfaces within the parent geometry a bit more generally. This is a natural thing to consider in a text emphasising branes as hypersurfaces, and it shall be very useful to us later. Our spacetime M has coordinates x^μ, and a metric $G_{\mu\nu}$. A general hypersurface Σ within M deserves its own coordinates ξ^a, and so it is specified by an equation of the form $f(x^\mu(\xi^a)) = 0$. We have already met that there is natural metric induced on Σ, which is the 'pull-back' of the spacetime metric:

$$G_{ab} = \frac{\partial x^\mu}{\partial \xi^a}\frac{\partial x^\nu}{\partial \xi^b} G_{\mu\nu},$$

and we can define other useful quantities too. For example, the unit vector normal to this hypersurface is then specified as

$$n_\mu^\pm = \pm\sigma\frac{\partial f}{\partial x^\mu}, \quad \text{where} \quad \sigma = \left| G^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu}\right|^{-1/2}. \qquad (10.12)$$

In the simple case where Σ is, say, a spherical hypersurface of radius R, of one dimension fewer than M (with radial coordinate r), the equation specifying Σ is just $f = r - R = 0$. We can use the remaining angular coordinates of M as coordinates on Σ. Now, $\partial f/\partial r = 1$, giving (note the contravariant index):

$$n^\mu = \pm\frac{1}{\sqrt{|G_{rr}|}}\left(\frac{\partial}{\partial r}\right)^\mu.$$

A final useful thing we shall need is the extrinsic curvature or 'second fundamental form' of the surface, which is given by the pull-back of the covariant derivative of the normal vector:

$$K_{ab} = \frac{\partial x^\mu}{\partial \xi^a}\frac{\partial x^\nu}{\partial \xi^b}\nabla_\mu n_\nu = -n_\mu\left(\frac{\partial^2 x^\mu}{\partial \xi^a \partial \xi^b} + \Gamma^\mu_{\nu\rho}\frac{\partial x^\nu}{\partial \xi^a}\frac{\partial x^\rho}{\partial \xi^b}\right). \qquad (10.13)$$

Like the induced metric, this is a tensor in the spacetime Σ. This might seem to be a daunting expression, but (like many things) it simplifies a lot in simple symmetric cases. So in our spherical example, using $r = R$, and the coordinates $\xi^a = x^a$, we get the simple expression:

$$K_{ab}^\pm = \frac{1}{2}n_\pm^\mu\frac{\partial G_{ab}}{\partial x^\mu}. \qquad (10.14)$$

and the reader should notice that the metric is in a very special isotropic form. It is worth emphasising that the whole solution has a nice form, and can be written as:

$$\begin{aligned} ds^2 &= -e^{2U}dt^2 + e^{-2U}(dR^2 + R^2 d\Omega_2^2); \\ A &= -\left(e^{-U} - 1\right) dt, \quad \text{where} \quad e^{-U} = 1 + \frac{Q}{R}. \end{aligned} \tag{10.16}$$

This special form and generalisations of it (involving higher dimensions, extended objects, and the presence of other fields) will appear many times in what we study later, and so this is a good place to admire it properly before things get more complicated.

A very important reason why the extremal Reissner–Nordström solution is quite special is because it behaves very much like a BPS object, where $M \geq Q$ is the BPS bound. This is worth looking at very carefully, since it is an important theme that we have already visited, and we shall see many times again. To see the BPS properties, we can think of our Einstein–Maxwell action as the bosonic part of an $\mathcal{N} = 2$ supersymmetric theory of gravity. $\mathcal{N} = 2$ supergravity in four dimensions has three important types of massless multiplet. The gravity multiplet itself contains the graviton, two gravitinos and a vector called the graviphoton. So the bosonic content of our Einstein–Maxwell theory matches this nicely. We need only include a pair of spin $\frac{3}{2}$ (*'Rarita–Schwinger'*) fields Ψ to play the role of the gravitino. The other two multiplets are the massless vector multiplet which contains a vector, a scalar and two spin $\frac{1}{2}$ particles, and the hypermultiplet which contains two spin $\frac{1}{2}$ particles and four scalars. The supersymmetry variations take bosonic fields into fermionic ones and vice versa, and the algebra can be written as:

$$\begin{aligned} \{Q^i_\alpha, Q^{\dagger j}_\beta\} &= 2\gamma^\mu_{\alpha\beta} P_\mu \delta^{ij}, \\ \{Q^i_\alpha, Q^j_\beta\} &= 2\epsilon_{\alpha\beta} Z^{ij}, \end{aligned} \tag{10.17}$$

where the supercharges are written as Weyl spinors Q^j_α, ($\alpha = 1, 2$, $i = 1, 2$), with $Q^{\dagger j}_\alpha$ being the Hermitian conjugate. The quantity Z^{ij} is antisymmetric, and commutes with everything else in the algebra. It is the *central charge*. Let us consider massive representations of the superalgebra. We can choose a basis in which $P_\mu = (M, 0, 0, 0)$. The little group is $SO(3)$. Writing the Z eigenvalue as simply $Z^{12} = Z$, we get

$$\begin{aligned} \{Q^i_\alpha, Q^{\dagger j}_\beta\} &= M\delta^{ij}, \\ \{Q^i_\alpha, Q^j_\beta\} &= \epsilon_{\alpha\beta}|Z|\epsilon^{ij}, \end{aligned}$$

which, after taking linear combinations, we can write in terms of two families of fermionic creation and annihilation operators, a_α, a^+_α and b_α, b^+_α:

$$\{a_\alpha, a^+_\beta\} = (M + |Z|)\delta^{ij},$$
$$\{b_\alpha, b^+_\beta\} = (M - |Z|)\delta^{ij}.$$

We can build representations of the algebra by starting with a Lorentz representation of some $SO(3)$ spin, s. We can write a ground state $|s\rangle$, which is defined as being annihilated by a^+_α and b^+_α, and then we can proceed make 2^4 states by acting with the a_α and b_α. For example, starting with spin 1, one can make a massive vector multiplet whose content is the sum of the vector and hypermultiplet above. This the generic 'long' massive multiplet[63].

Since we must make unitary representations, the left hand side of the algebra above must be positive, and so we find that there is a bound

$$M \geq |Z|. \tag{10.18}$$

The only way to saturate this bound is if the state is annihilated by the $b^+_\alpha s$, which is to say the state is invariant under half of the supersymmetry algebra. Then we only have the a_αs acting to make our multiplets and they are half the size. These are the special 'short' massive multiplets[63, 64]. There is a vector and a hyper of the same content mentioned above for the massless case, except that these can have any mass M.

The key point about extremal Reissner–Nordström is that it is part of a short hypermultiplet[65, 69]. This comes about in two stages. First, it has no fermion fields, and so the variation of all of the bosonic fields vanish when evaluated on this solution. This would be true for any old bosonic solution, of course. The remaining property is of course that the fermionic variations vanish for some choice of infinitessimal spinor ϵ_α generating the variation. Of course, it must be that only some of the spinors do this, otherwise we would be in a trivial situation. Setting the variation of the gravitino to zero, asks that there exists a spinor which solves:

$$D_\mu \epsilon_\alpha - \frac{1}{4} F^-_{\nu\kappa} \Gamma^\nu \Gamma^\kappa \Gamma_\mu \epsilon_\alpha = 0, \tag{10.19}$$

where $F^\pm_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} \pm i^* F_{\mu\nu})$, and recall from equation (2.125) that the covariant derivative on the spinor involves the spin connection, ${\omega^{ab}}_\mu$

$$D_\mu \epsilon_\alpha \equiv \left(\delta_{\alpha\beta} \partial_\mu + \frac{1}{8} {\omega^{ab}}_\mu [\Gamma_a, \Gamma_b]_{\alpha\beta} \right) \epsilon^\beta.$$

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This is asking for the existence of a *'covariantly constant'*, or *'Killing'*, spinor. It is a useful exercise to show that there are indeed such spinors. In fact, the problem reduces to just one differential equation which is satisfied everywhere by half of the available spinor components, matching the result above that half of the supersymmetries annihilate the solution.

In terms of the mass and the charge, things match as well. The graviphoton embedded in the gravity multiplet is a $U(1)$ gauge field whose charge is in fact the central charge. (There are gauge symmetries associated to the central charge operator which are local symmetries in supergravity[67].) So in fact, Z is the integral of the field strength two-form: $Z = (\int F)/4\pi = Q$, in the normalisation we are using. This matches with the property of our black hole solution.

10.1.5 Multiple black holes and multicentre solutions

It is important to note that there is a simple generalisation of the extremal solution to a case representing N distinct black holes of the same type:

$$\begin{aligned} ds^2 &= -e^{2U}dt^2 + e^{-2U}(dR^2 + R^2 d\Omega_2^2); \\ A &= -\left(e^{-U} - 1\right)dt, \quad \text{where} \quad e^{-U} = 1 + \sum_{i=1}^{N} \frac{q_i}{|\vec{R} - \vec{R}_i|}, \end{aligned} \tag{10.20}$$

where, in this 'multicentre' solution, $\vec{R}_i$ is a three-vector giving the location of the centre of the ith black hole with mass $m_i = q_i$. The total charge sourced by the whole configuration is, by Gauss's Law, simply $Q = \sum_{i=1}^{n} q_i$, which, by the BPS bound, is also equal to the total mass. This implicitly tells us that there is also a no-force condition applying to our black holes, since the total mass-energy is simply the sum of the individual mass-energies – there is no binding energy, coming from work against interaction forces.

The quickest way to see that this form arises as a solution is to rewrite the equation for the present Killing spinor as a condition on the solution written in the form in the first line of equation (10.20). We can do it for the slightly more general form where $dR^2 + R^2 d\Omega_2^2$ is replaced by $d\vec{x} \cdot d\vec{x}$. The resulting equation is simply that the e^{-U} be an *harmonic* function on the transverse space $\mathbb{R}^3$, for which after normalising it to be unity at infinity, we can choose for it to be written in the multicentre form. These are in general known as the Majumdar–Papapetrou solutions[66], and the spherical cases we've been looking at here are a special subclass corresponding to Reissner–Nordström.

10.1.6 Near horizon geometry and an infinite throat

It is particularly interesting to look closely at the horizon of the charged black hole in the extremal limit. Let us look at equation (10.15) in the neighbourhood of $R = 0$, the horizon, where we have

$$ds^2 = -\frac{R^2}{Q^2}dt^2 + \frac{Q^2}{R^2}dR^2 + Q^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (10.21)$$

The spatial part of the solution has degenerated into the product of an infinitely long tube or 'throat' of topology $\mathbb{R} \times S^2$ with fixed radius set by the charge. The whole geometry, called the 'Bertotti–Robinson' universe is actually $\mathrm{AdS}_2 \times S^2$, a two dimensional 'anti-de Sitter' spacetime being the (t, R) part. Anti-de Sitter spacetime is the most symmetric 'vacuum' solution to *two dimensional* Einstein's equations with a negative cosmological constant. This pleasingly simple near-horizon geometry is a sign of something more general which will occur in all its glory in chapter 18 and so it is worthwhile understanding the toy example presented here, and also worthwhile digressing on solutions of Einstein's equations in the presence of cosmological constant, for later use.

This has special meaning for the supersymmetric discussion above as well. At infinity, the solution is of course flat space, which has all eight of the maximum set of available Killing spinors. At arbitrary radius, there are four, as mentioned above. It turns out that the Bertotti–Robinson geometry also has eight Killing spinors, and so is also a maximally suppersymmetric vacuum of the theory, just like flat space. In this sense we see that the extremal Reissner–Nordström solution is akin to a soliton[65], since it behaves as an interpolating solution between two vacua (see insert 1.4). Much the same thing will be true for some of the extremal brane solutions which we shall encounter later[68].

10.1.7 Cosmological constant; de Sitter and anti-de Sitter

In General Relativity, the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is arrived at by asking that the field equations be written in terms of the unique symmetric, rank two covariantly conserved object constructed out of the metric and its derivatives which has Minkowski space as a vacuum solution. If we wish to relax that final condition somewhat, we have a slightly more general choice. Of course, the metric itself is a symmetric rank two tensor, and since $\nabla_\mu g^{\mu\nu} = 0$, so it is also a candidate. We can add it in with an arbitrary constant, to give

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \qquad (10.22)$$

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for which the generalisation of the Einstein–Hilbert Lagrangian is

$$\mathcal{L} = (-g)^{1/2}(R - 2\Lambda).$$

Recall from General Relativity[292] the form of the stress tensor for a perfect fluid of scalar density and pressure ρ and p:

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}. \tag{10.23}$$

We see that the 'cosmological constant' Λ acts like an intrinsic universal pressure. $\Lambda > 0$ is a cosmological repulsion, while $\Lambda < 0$ is an attraction.

While Minkowski space is no longer a solution, there are highly symmetric solutions analogous to it in the presence of non-zero Λ. Actually, the type of solutions we are looking for are called 'maximally symmetric' and satisfy the condition

$$R_{\lambda\mu\kappa\nu} = \mp\frac{2}{\ell^2}\left(g_{\lambda\kappa}g_{\mu\nu} - g_{\lambda\nu}g_{\kappa\mu}\right), \quad \text{or } R_{\mu\nu} = \pm\frac{2\Lambda}{(D-2)}g_{\mu\nu}$$
$$\text{where} \quad \ell^2 = -\frac{(D-1)(D-2)}{2\Lambda}. \tag{10.24}$$

Already familiar are the signature $(+++\cdots)$ spaces which satisfy equation (10.24) with the plus sign, the round spheres S^D. In fact, for signature $(-++\cdots)$ the spaces of interest here may be written as:

$$ds^2 = -\left(1 - \pm\frac{r^2}{\ell^2}\right)dt^2 + \left(1 - \pm\frac{r^2}{\ell^2}\right)^{-1}dr^2 + r^2 d\Omega^2_{D-2}, \tag{10.25}$$

where $d\Omega^2_{D-2}$ is the metric on a unit round $D-2$ sphere.

The cosmological constant sets a length scale, ℓ. The larger the cosmological constant, the smaller the scale. The limit $r \ll \ell$ therefore returns us locally to Minkowski space, since if we fall below the length scale set by Λ, we simply do not notice, locally, that we have a cosmological constant, Λ. For $r \simeq \ell$ or greater we cannot ignore the effect of the cosmological constant.

10.1.8 de-Sitter spacetime and the sphere

For instance, notice that for the case of the plus sign, de Sitter space, there is an horizon at $r = \ell$. Since r cannot exceed ℓ, we might as well write $r = \ell \sin\theta$. A little algebra shows that, if we analytically continue time via $it = \ell\psi$, we get the metric

$$ds^2 = \ell^2(d\theta^2 + \cos^2\theta\, d\psi^2 + \sin^2\theta\, d\Omega^2_{D-2}), \tag{10.26}$$

which is the metric on a round sphere S^D, with radius ℓ, if ψ and θ have the appropriate periodicities.

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10.1.9 Anti-de Sitter in various coordinate systems

The case of anti-de Sitter, the minus sign, we can instead take $r = \ell \sinh \rho$, and get

$$ds^2 = -\cosh^2\rho\, dt^2 + \ell^2 d\rho^2 + \ell^2 \sinh^2 \rho d\Omega^2_{D-2}, \tag{10.27}$$

which is a useful form which we will see much later. Notice that we can view this as an analytic continuation of the metric of the sphere S^D, given in equation (10.26).

There is a useful form of the metric to present which can be thought of as the $r \gg \ell$ limit. In this case, drop the 1 from $(1+r^2/\ell^2)$, and work with local coordinates. So write $\ell^2 d\Omega^2_{D-2}$, the metric on the S^{D-2} of radius ℓ embedded in $\mathbb{R}^{D-1}$ in Cartesian coordinates

$$\ell^2 d\Omega^2_{D-2} = dx_1^2 + dx_2^2 + \cdots + dx^2_{D-2} + \frac{(x_1 dx_1 + x_2 dx_2 + \cdots + x_{D-2} dx_{D-2})^2}{x^2_{D-1}},$$

where $x^2_{D-1} = \ell^2 - \sum_{i=1}^{D-1} x_i^2$. Then we can work in the local neighbourhood of $x_i \sim 0$, $x_{D-1} \sim \ell$, giving

$$\ell^2 d\Omega^2_{D-2} \simeq dx_1^2 + dx_2^2 + \cdots + dx^2_{D-2}.$$

Choosing these local coordinates is equivalent to the large radius limit of the sphere, and the rest of the geometry therefore takes the form:

$$ds^2 = \frac{r^2}{\ell^2}\left(-dt^2 + dx_1^2 + \cdots + dx^2_{D-2}\right) + \ell^2 \frac{dr^2}{r^2}, \tag{10.28}$$

which is known as the 'Poincaré' form of the metric, which arose already as part of the throat (10.21) of the Reissner–Nordstrom solution, and it shall arise again later. The radial coordinate R used there should be compared to r here, and the infinite line $\mathbb{R}$ coordinatised by t should be compared to the $\mathbb{R}^{D-1}$ coordinatised by $(t, x_1, \ldots, x_{D-2})$. Notice that the metric on that subspace (obtained by radial slices of constant r) is actually that of $D-1$ dimensional Minkowski, a fact which will be important for us later. The horizon at $R = 0$ compares to an horizon at $r = 0$ here, which is an important clue as to where anti-de Sitter will arise in later sections and chapters.

Actually, we can write another metric for AdS as follows:

$$ds^2 = -\left(-1 + \frac{r^2}{\ell^2}\right) dt^2 + \left(-1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Xi^2_{D-2},$$

where $d\Xi^2_{D-2}$ is the 'unit' metric on a $D-2$ dimensional hyperbolic space $\mathbb{H}^{D-2}$. This metric can be obtained by analytically continuing $d\Omega^2_{D-2}$. For

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this case, the radial slices are $\mathbb{H}^{D-2}\times\mathbb{R}$ instead of $D-1$ Minkowski space for the previous form (10.28) or $S^{D-2}\times\mathbb{R}$ for the form in equation (10.25). Just as before, we can do a hyperbolic change to a new coordinate $r=\ell$ $\cosh\rho$, and get

$$ds^2 = -\sinh^2\rho\, dt^2 + \ell^2 d\rho^2 + \ell^2\cosh^2\rho\, d\Xi^2_{D-2}).$$

In summary, we have AdS_D in the following metrics:

$$ds^2 = -\left(k+\frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{\left(k+\frac{r^2}{\ell^2}\right)} + \frac{r^2}{\ell^2}d\Sigma^2_{k,D-2}, \tag{10.29}$$

where the $(D-2)$-dimensional metric $d\Sigma^2_{k,D-2}$ is

$$d\Sigma^2_{k,D-2} = \begin{cases} \ell^2 d\Omega^2_{D-2} & \text{for } k=+1 \\ \sum_{i=1}^{D-2} dx_i^2 & \text{for } k=0 \\ \ell^2 d\Xi^2_{D-2} & \text{for } k=-1, \end{cases} \tag{10.30}$$

The $k=0$ form can be thought of as the local physics in all three cases.

Anti-de Sitter space in D dimensions has an $SO(2,D-1)$ isometry, of which a subgroup $SO(1,1)\times ISO(1,D-2)$ is manifest as

$$(t,u,x_1,\ldots,x_{D-2}) \longrightarrow (\lambda t, \lambda^{-1}u, \lambda x_1,\ldots,\lambda x_{D-2}),$$

for the first factor, and the Poincaré group (i.e. Lorentz boosts and translations) acting on the Minkowski part. The group $SO(2,D-1)$ is the conformal group in $D-1$-dimensional Minkowski space, and the $SO(1,1)$ is the dilation part of it. The reader may recall that we met this group all the way back in chapter 3, and its appearance here will be given physical significance in terms of a duality in chapter 18.

10.1.10 Anti-de Sitter as a hyperbolic slice

It is worth noting that AdS_D has a very natural geometrical representation. Start with the $(D+1)$-dimensional spacetime with signature $(-,-,+,+,\cdots)$, with metric:

$$ds^2 = -dX_0^2 - dX_D^2 + \sum_{i=1}^{D-1} dX_i^2. \tag{10.31}$$

Notice that the isometry group of this homogeneous and isotropic spacetime is $SO(2,D-1)$. Now consider the hyperboloid within this spacetime, given by the equation

$$X_0^2 + X_D^2 - \sum_{i=1}^{D-1} X_i^2 = \ell^2.$$

A solution of this equation is

$$X_0 = \ell \cosh\rho \cos\tau/\ell, \quad X_D = \ell\cosh\rho\sin\tau/\ell, \quad X_i = \ell\Omega_i \sinh\rho,$$

where the angles Ω_i are chosen such that $\sum_{i=1}^{D-1}\Omega_i = 1$. We can substitute this solution into the metric (10.31) in order to find the metric on this hyperboloid, and we find the global AdS_D metric given in equation (10.27). With $0 \leq \tau \leq 2\pi$ and $0 \leq \rho$, our solution covers the entire hyperboloid once, and this is why these are called the 'global' coordinates on AdS. The time τ is usually taken not as a circle (which gives closed timelike curves) but on the real line, $-\infty \leq \tau + \infty$ giving the universal cover of the hyperboloid.

Another solution to the hyperboloid equation is:

$$\begin{aligned} X_0 &= \frac{1}{2r}\left(1 + r^2(\ell^2 + \vec{x}^2 - t^2)\right), \quad X_D = rt \\ X_{D-1} &= \frac{1}{2r}\left(1 - r^2(\ell^2 - \vec{x}^2 + t^2)\right), \quad X_i = rx_i, \end{aligned}$$

which defines coordinates which cover a half of the hyperboloid. The resulting metric after substitution into equation (10.31) is the Poincaré form exhibited in equation (10.28). These are the 'local' coordinates.

10.1.11 Revisiting the extremal solution

How did constant curvature spaces, and negative cosmological constant become relevant to the Reissner–Nordström solution near the horizon at extremality? Well, it is worth examining the Ricci tensor in the extremal limit, in the coordinate $R = r - Q$, in the neighbourhood of the horizon $r = Q$:

$$R_{tt} = \frac{R^2}{Q^4}; \quad R_{rr} = -\frac{1}{R^2}; \quad R_{\theta\theta} = 1; \quad R_{\phi\phi} = \sin^2\theta, \tag{10.32}$$

and so we see that, upon comparing to equation (10.21):

$$\begin{aligned} R_{\mu\nu} &= -\frac{1}{Q^2}g_{\mu\nu}; \qquad \text{for } \mu,\nu = t \text{ or } r; \\ R_{\mu\nu} &= +\frac{1}{Q^2}g_{\mu\nu}; \qquad \text{for } \mu,\nu = \theta \text{ or } \phi. \end{aligned} \tag{10.33}$$

Since the Maxwell stress tensor essentially obeys the same relations, giving something proportional to the metric tensor, it can be seen that the flux due to the charge carried by the hole is what is responsible for supplying the effective cosmological constant. It is worth noting that we could have formulated the same sort of features in terms of magnetic fields. In that case, we would have traded in the electric two form components for magnetic components $F = Q\epsilon_2$, where $\epsilon_2 = \sin\theta d\theta \wedge d\phi$ is the volume form of S^2. In this form, the decomposition of the throat solution by dualising the electric source into a magnetic source will generalise into something called the 'Freund–Rubin' ansatz in higher dimensional supergravity[19].

10.2 The geometry of D-branes

Now let us return to the full ten dimensional equations of motion of the type IIA and type IIB supergravity equations (7.41) and (7.42), where we have additional fields coming from the R–R sector and the NS–NS sector.

10.2.1 A family of 'p-brane' solutions

There is an interesting family of ten dimensional solutions, which source gravity, the dilaton, and the R–R potentials, and can be written as follows[94, 95]:

$$dS^2 = Z_p^{-1/2}(r)\left(-K(r)dt^2 + \sum_{i=1}^{p} dx_i^2\right) + Z_p^{1/2}(r)\left(\frac{dr^2}{K(r)} + r^2 d\Omega_{8-p}^2\right), \tag{10.34}$$

where $d\Omega_{8-p}^2$ is the metric on a unit round S^{8-p} sphere, and

$$\begin{aligned} Z_p(r) &= 1 + \alpha_p\left(\frac{r_p}{r}\right)^{7-p}, \\ K(r) &= 1 - \left(\frac{r_H}{r}\right)^{7-p}, \\ e^{2\Phi} &= g_s^2 Z_p(r)^{\frac{(3-p)}{2}}, \\ C_{(p+1)} &= g_\mathrm{s}^{-1}\left[Z_p(r)^{-1} - 1\right] dx^0 \wedge \cdots \wedge dx^p. \end{aligned} \tag{10.35}$$

In the above

$$\begin{aligned} r_p^{7-p} &= d_p(2\pi)^{p-2} g_\mathrm{s} N \alpha'^{(7-p)/2}, \quad d_p = 2^{7-2p}\pi^{\frac{9-3p}{2}}\Gamma\left(\frac{7-p}{2}\right), \\ \alpha_p &= \sqrt{1 + \left(\frac{r_\mathrm{H}^{7-p}}{2r_p^{7-p}}\right)^2} - \frac{r_\mathrm{H}^{7-p}}{2r_p^{7-p}}. \end{aligned} \tag{10.36}$$

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One should not be intimidated by the form of these solutions. They represent p-dimensional extended objects called 'p-branes', and as such, are localised in the $9-p$ directions transverse to them. Since we have rotational symmetry in those directions, we can use polar coordinates with a radial coordinate r, and the angles on an $(8-p)$-sphere. The branes are aligned along the $(x^1, x^2, \ldots, x^p)$ directions, and move in time, so they have a $(p+1)$ dimensional world volume, with geometry $\mathbb{R}^{p+1}$, generalising the worldline of the black hole solutions we studied earlier. It is useful to observe how the solution is split between the transverse and parallel coordinates and then look at, say, the Schwarzschild or Reissner–Nordström solution (10.4) and see that the analogue of this is happening in that solution too. There, the world-volume is replaced by a simple world-line, the space $\mathbb{R}$ coordinatised by t. The rest of the solution concerns the transverse part of the spacetime. Since there is rotational symmetry it has a simple presentation in terms of the radius r and the two angles on the round S^2. From our analysis of the black hole solutions, it should be clear that these solutions have an horizon at radius $r = r_{\rm H}$, and a singularity at $r = 0$.

10.2.2 The boost form of solution

Actually there is another way of writing the solution which is instructive and useful for later. We could instead write:

$$Z_p(r) = 1 + \alpha_p \left(\frac{r_p}{r_{\rm H}}\right)^{7-p} \left(\frac{r_{\rm H}}{r}\right)^{7-p} = 1 + \sinh^2\beta_p \left(\frac{r_{\rm H}}{r}\right)^{7-p},$$

where, given the nice form of α_p in equation (10.36), we can write

$$\alpha_p \left(\frac{r_p}{r_{\rm H}}\right)^{7-p} = \sqrt{\frac{1}{4} + \left(\frac{r_p^{7-p}}{r_{\rm H}^{7-p}}\right)^2} - \frac{1}{2} = \sinh^2\beta_p,$$

and hence

$$\alpha_p = \tanh\beta_p; \qquad \cosh^2\beta_p = \frac{1}{2} + \sqrt{\frac{1}{4} + \left(\frac{r_p^{7-p}}{r_{\rm H}^{7-p}}\right)^2}.$$

The tension and charge can be written in terms of these nicely as:

$$\begin{aligned} \tau_p &= \left(\frac{r_{\rm H}}{r_p}\right)^{7-p} \frac{N}{g_{\rm s}} \left(\frac{1}{7-p} + \cosh^2\beta_p\right), \\ Q_p &= N \left(\frac{r_{\rm H}}{r_p}\right)^{7-p} \sinh\beta_p \cosh\beta_p = N. \end{aligned} \qquad (10.37)$$

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So we see that in fact the solutions above are normalised such that they carry N units of the basic D-brane R–R charge μ_p, where N is an integer. Observe that the mass is larger than the charge, in a manner analogous to the Reissner–Nordström solution.

Notice that when the parameter β_p goes to zero, the solution simplifies drastically, becoming uncharged. The function Z_p becomes unity, the dilaton becomes constant, and the solution simply becomes a $(10-p)$-dimensional Schwarzschild black hole, with horizon at $r = r_{\rm H}$, times the space $\mathbb{R}^p$.

10.2.3 The extremal limit and coincident D-branes

Just like in the case of the charged black hole solution, there are extremal limits of these solutions. The extremal cases are BPS solutions of the ten dimensional supersymmetry algebra, as we shall see. For now, the similarity with the detailed case study of Reissner–Nordström black holes in earlier sections should be borne in mind, although there are differences which will become apparent shortly. The extremal limit is simply $\alpha_p = 1$, where the solutions are:

$$\begin{aligned} ds^2 &= H_p^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H_p^{1/2}dx^i dx^i, \\ e^{2\Phi} &= g_{\rm s}^2 H_p^{\frac{(3-p)}{2}}, \\ C_{(p+1)} &= -(H_p{}^{-1}-1)g_{\rm s}^{-1}dx^0\wedge\cdots\wedge dx^p, \end{aligned} \qquad (10.38)$$

where $\mu = 0,\ldots,p$, and $i = p+1,\ldots,9$, and the harmonic function H_p is

$$H_p = 1 + \left(\frac{r_p}{r}\right)^{7-p}, \qquad (10.39)$$

where r_p is still given in equation (10.36). In the boost form mentioned at the end of the last subsection, it is the limit of infinite boost, $\beta_p \to \infty$, combined with sending the horizon parameter $r_{\rm H}$ to zero while holding fixed the combination $r_{\rm H}^{(7-p)}e^{2\beta_p}/4 = r_p^{7-p}$.

It is worth comparing this to the form in equation (10.16), where the extremal black hole is written in isotropic form analogous to what we have here. Furthermore, it should be clear that there is a multicentre generalisation of this solution, where we write for the harmonic function

$$H_p = 1 + \sum_{i=1}^{N}\frac{r_p^{7-p}}{|\vec{r}-\vec{r}_i|^{7-p}}. \qquad (10.40)$$

This represents N different branes located at arbitrary positions given by the vectors $\vec{r}_i$. A clear sign that the solution is a BPS object made of lots of smaller such objects is the fact that the mass computed for this solution is just the sum of the individual masses and is equal to the total charge. There is no binding energy since the interaction forces are zero.

It is clear that in all cases (except $p = 3$) the horizon, located at $r = 0$, is a singular place of zero area, since the radius of the S^{8-p} vanishes there. In the $p = 3$ case, however, the inverse quartic power of r appearing in the harmonic function means that the square root yields a cancellation between the vanishing of the horizon size and the divergence of the metric, leaving an horizon of finite size $r_3^{1/2} = \alpha'(4\pi g_s N)^{1/2}$. Some simple algebra shows that the geometry is simply $\mathrm{AdS}_5 \times S^5$, with the sizes of each factor set by $r_3^{1/2}$. The dilaton is constant, and the R–R field is $F_{(5)} = dC_{(4)} + *dC_{(4)}$, where $dC_{(4)} = r_3\epsilon_{(5)}$ where $\epsilon_{(5)}$ is the volume form on S^5.

Note again the sharp analogy with the case of Reissner–Nordström. The appearance of this simple smooth near-horizon geometry is interesting, and we will explore this much later, in chapter 18.

More complicated supergravity solutions preserving fewer supersymmetries (in the extremal case) can be made by combining these simple solutions in various ways, by intersecting them with each other, boosting them to finite momentum, and by wrapping, and/or warping them on compact geometries. This allows for the construction of finite area horizon solutions, corresponding to R–R charged Reissner–Nordström black holes, and generalisations thereof. We shall in fact do this in chapter 17.

These solutions are R–R charged with N units of Dp-brane charge, but we have already established to all orders in string perturbation theory that Dp-branes actually are the *basic sources* of the R–R fields. It is natural to suppose that there is a connection between these two families of objects: perhaps the solution (10.38) is 'made of D-branes' in the sense that it is actually the field due to N Dp-branes, all located at $r = 0$. This is precisely how we are to make sense of this solution as a supergravity soliton solution. We *must* do so, since (except for $p = 3$ as we have seen) the solution is actually singular at $r = 0$, and so one might have simply discarded them as pathological, since solitons 'ought to be smooth'. However, string duality, which we shall encounter in chapter 12, *forces* us to consider them, since smooth NS–NS solitons of various extended sizes (which can be made by wrapping or warping NS5-branes (see section 12.3 for their entry into our story) in an arbitrary compactification) are mapped[165] into these R–R solitons under it, generalising what we have already seen in ten dimensions. With the understanding that there are D-branes 'at their core', which fits with the fact that they are R–R charged, they make sense of the whole spectrum of extended solitons in string theory.

Let us build up the logic of how they can be related to D-branes. Recall that the form of the action of the ten dimensional supergravity with NS–NS and R–R field strengths H and G respectively is, roughly:

$$S = \int d^{10}x\,\sqrt{-g}\,\Big[e^{-2\Phi}\Big(R - H^2\Big) - G^2\Big]. \tag{10.41}$$

There is a balance between the dilaton dependence of the NS–NS and gravitational parts, and so the mass of a soliton solution[95] carrying NS–NS charge (like the NS5-brane) scales like the action: $T_{\rm NS} \sim e^{-2\Phi} \sim g_{\rm s}^{-2}$. An R–R charged soliton has, on the other hand, a mass which goes like the geometric mean of the dilaton dependence of the R–R and gravitational parts: $T_{\rm R} \sim e^{-\Phi} \sim g_{\rm s}^{-1}$. This is just the behaviour we saw for the tension of the Dp-brane, computed in string perturbation theory, treating them as boundary conditions.

We have so far treated Dp-branes as point-like (in their transverse dimensions) in an otherwise flat spacetime. We were able to study an arbitrary number of them by placing the appropriate Chan–Paton factors into amplitudes. However, the solutions (10.38) have non-trivial spacetime curvature, and is only asymptotically flat. How are these two descriptions related?

Well, for every Dp-brane which is added to a situation, another boundary is added to the problem, and so a typical string diagram has a factor $g_{\rm s}N$ since every boundary brings in a factor $g_{\rm s}$ and there is the trace over the N Chan–Paton factors. So open string perturbation theory is good as long as $g_{\rm s}N < 1$. Notice that this is the regime where the supergravity solution (10.38) fails to be valid, since the typical squared curvature invariant behaves as

$$R^2 \sim \left(\frac{r_p}{r}\right)^{7-p} \sim g_{\rm s}N\left(\frac{\sqrt{\alpha'}}{r}\right)^{7-p}.$$

On the other hand, for $g_{\rm s}N > 1$, the supergravity solution has its curvature weakened, and can be considered as a workable solution. This regime is where the open string perturbation theory, on the other hand, breaks down.

So we have a fruitful complementarity between the two descriptions. In particular, since we only derived the supergravity equations of motion in string perturbation theory, i.e. $g_{\rm s} < 1$, for most computations, we can work with the supergravity solution with the interpretation that N is very large, such that the curvatures are small. Alternatively, if one restricts oneself to studying only the BPS sector, then one can work with arbitrary N, and extrapolate results – computed with the D-brane description for small

g_s – to the large g_s regime (since there are often non-renormalisation theorems which apply), where they can be related to properties of the non-trivial curved solutions. This is the basis of the successful statistical enumeration of the entropy of black holes, for cases where the solutions (10.38) are used to construct R–R charged black holes. We shall do this in chapter 17.

In summary, for a large enough number of coincident D-branes or for strong enough string coupling, one cannot consider them as points in flat space: they deform the spacetime according to the geometry given in equation (10.38). Given that D-branes are also described very well at low energy by gauge theories, this gives plenty of scope for finding a complementarity between descriptions of non-trivially curved geometry and of gauge theory. This is the basis of what might be called 'gauge theory/geometry' correspondences. In some cases, when certain conditions are satisfied, there is a complete decoupling of the supergravity description from that of the gauge theory, signalling a complete *duality* between the two. This is the basis of the AdS/CFT correspondence, which we shall come to in chapter 18.

10.3 Probing p-brane geometry with Dp-branes

In the previous section, we argued that the spacetime geometry given by equations (10.38) represents the spacetime fields produced by N Dp-branes. We noted that as a reliable solution to supergravity, the product $g_s N$ ought be be large enough that the curvatures are small. This corresponds to either having N small and g_s large, or vice versa. Since we are good at studying situations with g_s small, we can safely try to see if it makes sense to make N large.

10.3.1 Thought experiment: building p with Dp

One way to imagine that this spacetime solution came about at weak coupling was that we built it by bringing in N Dp-branes, one by one, from infinity. If this is to be a sensible process, we must study whether it is really possible to do this. Imagine that we have been building the geometry for a while, bringing up one brane at a time from $r = \infty$ to $r = 0$. Let us now imagine bringing the next brane up, in the background fields created by all the other N branes. Since the branes share p common directions where there is no structure to the background fields, we can ignore those directions and see that the problem reduces to the motion of a test particle in the transverse $9 - p$ spatial directions. What is the mass of this particle, and what is the effective potential in which it moves?

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We can answer this sort of question using the toolbox which combines the fact that at low energy we know the world-volume action of the D-brane, describing how it interacts with the background fields with the fact that the probe brane is a heavy object which can examine many distance scales in the theory[106].

10.3.2 Effective Lagrangian from the world-volume action

We can find the answers to all of the above questions by deriving an effective Lagrangian for the problem which results from the world-volume action of the brane. We can exploit the fact that we have spacetime Lorentz transformations and world-volume reparametrisations at our disposal to choose the work in the 'static gauge'. In this gauge, we align the world-volume coordinates, ξ^a, of the brane with the spacetime coordinates such that:

$$\begin{aligned} \xi^0 &= x^0 = t; \\ \xi^i &= x^i; \quad i = 1, \dots, p, \\ \xi^m &= \xi^m(t); \quad m = p+1, \dots, 9. \end{aligned} \tag{10.42}$$

The Dirac–Born–Infeld part of the action (5.21) requires the insertion of the induced metric derived from the metric in question. In static gauge, it is easy to see that the induced metric becomes:

$$[G]_{ab} = \begin{pmatrix} G_{00} + \sum_{mn} G_{mn} v_m v_n & 0 & 0 & \cdots & 0 \\ 0 & G_{11} & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & G_{pp} \end{pmatrix}, \tag{10.43}$$

where $v_m \equiv dx^m/d\xi^0 = \dot{x}^m$.

In our particular case of a simple diagonal metric, the determinant turns out as

$$\det[-G_{ab}] = H_p^{-\frac{(p+1)}{2}} \left(1 - H_p \sum_{m=p+1}^{9} v_m^2\right) = H_p^{-\frac{(p+1)}{2}} \left(1 - H_p v^2\right). \tag{10.44}$$

The Wess–Zumino term representing the electric coupling of the brane is, in this gauge:

$$\begin{aligned} \mu_p \int C_{(p+1)} &= \mu_p \int d^{p+1}\xi \; \varepsilon^{a_0 a_1 \cdots a_p} [C_{(p+1)}]_{\mu_0\mu_1\dots\mu_p} \frac{\partial x^{\mu_0}}{\partial \xi^{a_0}} \frac{\partial x^{\mu_1}}{\partial \xi^{a_1}} \cdots \frac{\partial x^{\mu_p}}{\partial \xi^{a_p}} \\ &= \mu_p V_p \int dt \left[H_p^{-1} - 1\right] g_s^{-1}, \end{aligned} \tag{10.45}$$

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where $V_p = \int d^px$, the spatial world-volume of the brane. Now, we are going to work in the approximation that we bring the branes *slowly* up the the main stack of branes so we keep the velocity small enough such that only terms up to quadratic order in v are kept in our computation. We can therefore the expand the square root of our determinant, and putting it all together (not forgetting the crucial insertion of the background functional dependence of the dilaton from (10.38)) we get that the action is:

$$\begin{aligned} S &= \mu_p V_p \int dt \left(-g_{\rm s}^{-1} H_p^{-1} + \frac{1}{2g_{\rm s}} v^2 + g_{\rm s}^{-1} H_p^{-1} - g_{\rm s}^{-1} \right) \\ &= \int dt \mathcal{L} = \int dt \left(\frac{1}{2} m_p v^2 - m_p \right), \end{aligned} \tag{10.46}$$

which is just a Lagrangian for a free particle moving in a constant potential, (which we can set to zero) where $m_p = \tau_p V_p$ is the mass of the particle.

This result has a number of interesting interpretations. The first is simply that we have successfully demonstrated that our procedure of 'building' our geometry (10.38) by successively bringing branes up from infinity to it, one at a time, makes sense. There is no non-trivial potential in the effective Lagrangian for this process, so there is no force required to do this; correspondingly there is no binding energy needed to make this system.

That there is no force is simply a restatement of the fact that these branes are BPS states, all of the same species. This manifests itself here as the fact that the R–R charge is equal to the tension (with a factor of $1/g_{\rm s}$), saturating the BPS bound. It is this fact which ensured the cancellation between the r-dependent parts in (10.46) which would have otherwise resulted in a non-trivial potential $U(r)$. (Note that the cancellation that we saw only happens at order v^2 – the slow probe limit. Beyond that order, the BPS condition is violated, since it really only applies to statics.)

10.3.3 A metric on moduli space

All of this is pertinent to the world-volume field theory as well. Recall that there is a $U(N)$ $(p+1)$-dimensional gauge theory on a family of N Dp-branes. Recall furthermore that there is a sector of the theory which consists of a family of $(9-p)$ scalars, Φ^m, in the adjoint. Geometrically, these are the collective coordinates for motions of the branes transverse to their world-volumes. Classical background values for the fields, (defining vacua about which we would then do perturbation theory) are equivalent to data about how the branes are distributed in this transverse space. Well, we have just confirmed that there is a 'moduli space' of inequivalent vacua of the theory corresponding to the fact that one can give a vacuum expectation value to a component of a Φ^m, representing a brane moving away from the clump of N branes. That there is no potential translates into the fact that we can place the brane anywhere in this transverse clump, and it will stay there.

It is also worth noting that this metric on the moduli space is *flat*; treating the fields Φ^m as coordinates on the space $\mathbb{R}^{9-p}$, we see (from the fact that the velocity squared term in (10.46) appears as $v^2 = \delta_{mn}v^m v^n$) that the metric seen by the probe is simply

$$ds^2 \sim \delta_{mn} d\Phi^m d\Phi^n. \tag{10.47}$$

This flatness is a consequence of the high amount of supersymmetry (16 supercharges). For the case of D3-branes (whether or not they are in the $\mathrm{AdS}_5 \times S^5$ limit, to be described later), this result translates into the fact there that there is no running of the gauge coupling g^2_{YM} of the superconformal gauge theory on the brane, (since in this example, and in the case of eight supercharges, supersymmetry relates the coupling to the kinetic term). This is read off from the prefactor $g^{-2}_{\mathrm{YM}} = \tau_3(2\pi\alpha')^2 = (2\pi g_{\mathrm{s}})^{-1}$ in the metric. The supersymmetry ensures that any corrections which could have been generated are zero. We shall later see less trivial versions, where we have nontrivial metrics in the case of eight supercharges and even four supercharges. Before we do that, we have to go back to studying D-branes as boundary conditions, in order to see how to put together multiple D-branes, and branes of different types.

10.4 T-duality and supergravity solutions

In principle, nothing stops us from studying the action of T-duality on the Dp-branes, now starting with their representation as a supergravity solution, and correspondingly using the background field T-duality rules given in equation (5.4) for the NS–NS sector, and equations (8.2) for the R–R sector. One should expect to get the supergravity solution of a D$(p+1)$-brane or D$(p-1)$-brane, depending upon whether one T-dualised in a direction containing the Dp-brane's world-volume or not. This expectation is indeed borne out to some extent, but we must be careful. Let us discuss the subtlety by example.

10.4.1 D$(p+1)$ from Dp

Start with the case of T-dualising in a direction transverse to a Dp-brane, lying in directions $X^1, \ldots, X^p$. What this really means, recall, is that we must place the branes on a circle of radius R, and find an equivalent representation for the system on a dual circle of radius $R' = \alpha'/R$. We can represent this as an infinite array of identical branes on the line with coordinate X^{p+1}, a distance $2\pi R$ apart, identifying $X^{p+1} \sim X^{p+1} + 2\pi R$. We can easily write a supergravity solution for this, since the branes are BPS, and so the multibrane harmonic function in equation (10.40) can be employed here. Let us write the radius in the directions transverse to the Dp-brane in terms of X^{p+1} and a radius in the remaining directions:

$$r^2 = (X^{p+1})^2 + (X^{p+2})^2 + \cdots + (X^{9-p})^2 = \hat{r}^2 + (X^{p+1})^2,$$

in terms of which the appropriate harmonic function including all of the images is:

$$H_p^{\rm array} = 1 + \sum_{n=-\infty}^{+\infty} \frac{r_p^{7-p}}{|\hat{r}^2 + (X^{p+1} - 2\pi nR)^2|^{(7-p)/2}}. \qquad (10.48)$$

If the circle's radius is very small, then the sum in the above can be replaced by an integral, to a good approximation, since the difference between each term in the sum is small. Defining a new variable u via $\hat{r}u = 2n\pi R - X^{p+1}$, we get:

$$H_p^{\rm array} \sim 1 + \frac{r_p^{7-p}}{2\pi R}\frac{1}{\hat{r}^{6-p}} \int_{-\infty}^{\infty} \frac{du}{(1+u^2)^{(7-p)/2}}, \qquad (10.49)$$

where we have used $\hat{r}\delta u = 2\pi R\delta n$ to get the measure right. The integral is:

$$\int_{-\infty}^{\infty} \frac{du}{(1+u^2)^{(7-p)/2}} = \frac{\sqrt{\pi}\,\Gamma\left[\frac{1}{2}(6-p)\right]}{\Gamma\left[\frac{1}{2}(7-p)\right]},$$

and so looking at the definition of the constant r_p^{7-p} given in equation (10.36), we see that

$$H_p^{\rm array} \sim H_{p+1} = 1 + \frac{\sqrt{\alpha'}\, r_{p+1}^{7-(p+1)}}{R} \frac{1}{\hat{r}^{7-(p+1)}}, \qquad (10.50)$$

which is the correct form of the harmonic function for a D$(p+1)$-brane.

We should check normalisations here. If we had started with a single brane on the array, i.e. with $N = 1$, then we get the new number of branes as $\tilde{N} = \sqrt{\alpha'}/R$. So if $R = \sqrt{\alpha'}$, then we have the correct normalisation for a single brane on the dual side also. Better perhaps is to have $N = R/\sqrt{\alpha'}$, giving a single $\tilde{N} = 1$ as the T-dual. This has the interpretation in the original theory as $R/\sqrt{\alpha'}$ for each $2\pi R$ of length, or $2\pi\sqrt{\alpha'}$ branes per unit length.

We can work on the full Dp-brane metric with the T-duality rules (5.4), treating X^{p+1} as the isometry direction. Following the rules through, we see that the transformation will invert the metric function $G_{p+1,p+1}$, which will indeed convert the metric for a p-brane to that of a $(p+1)$-brane. So the new dilaton is, according to the rules in equation (5.4),

$$e^{2\tilde{\Phi}} = \frac{e^{2\Phi}}{G_{p+1,p+1}} = \frac{e^{2\Phi}}{H_p^{1/2}} = g_s^2 H_p^{\frac{(3-(p+1))}{2}},$$

which after replacing H_p by H_p^{array}, which becomes H_{p+1} as we have shown above, gives the dilaton for the D$(p+1)$-brane supergravity solution. Similarly, equations (8.2) give the correct R–R potential.

This works very well because it is easy to soften the power of r which appears in the denominator of the harmonic function, as needed for a larger brane.

10.4.2 D(p − 1) from Dp

Harder to get is the increase of the power of r in the dependence of the harmonic function, which we would need for a D$(p-1)$-brane, if we T-dualised in a world-volume direction, say X^p. Clearly the powers of the harmonic function itself will in the metric, dilaton and R–R potential, using the rules (5.4) and (8.2). The problem is that we would get

$$H_p = 1 + \frac{r_p^{7-p}}{r^{7-p}}. \tag{10.51}$$

This is not really what we want. We can, however, interpret this as the result of 'smearing' the brane in the direction X^p, i.e. the result of integrating a uniform density of branes (with the correct $1/r^{(8-p)}$ behaviour) over X^p. This will indeed yield the behaviour given in (10.51). We shall encounter such smeared solutions, or 'brane distributions' in later chapters.

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