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RESEARCH ARTICLE

A cone conjecture for log Calabi-Yau surfaces

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Abstract
We consider log Calabi-Yau surfaces (Y, D) with singular boundary. In each deformation type, there is a distinguished surface (Y_e, D_e) such that the mixed Hodge structure on $H_2(Y \setminus D)$ is split. We prove that (1) the action of the automorphism group of (Y_e, D_e) on its nef effective cone admits a rational polyhedral fundamental domain; and (2) the action of the monodromy group on the nef effective cone of a very general surface in the deformation type admits a rational polyhedral fundamental domain. These statements can be viewed as versions of the Morrison cone conjecture for log Calabi–Yau surfaces. In addition, if the number of components of D is no greater than six, we show that the nef cone of Y_e is rational polyhedral and describe it explicitly. This provides infinite series of new examples of Mori Dream Spaces.

Contents

1. Introduction

Given a smooth projective variety Y over $\mathbb{C}$, the closed cone of curves of Y is the closure of the set of all nonnegative linear combinations of classes of irreducible curves in $H_2(Y, \mathbb{R})$. The cone of curves of any Fano variety is rational polyhedral, meaning it has finitely many rational generators (see Theorem 1.24 on p.22 of [KM98]). But this is not true in general for Calabi-Yau varieties – if Y is Calabi-Yau, the cone of curves of Y could be round, for example. The nef cone is the dual of the cone of curves.

The Morrison cone conjecture states that if Y is a Calabi-Yau variety, then there exists a rational polyhedral cone which is a fundamental domain for the action of the automorphism group of Y on the nef cone. This can be pictured in dimension two using hyperbolic geometry (see, for example, [T11]).

The conjecture is known to be true in dimension two, but for higher dimensions, it is an open question. In [T10], Totaro has shown that a generalization of this conjecture is true in dimension two: if (Y, Δ) is a klt Calabi-Yau pair, then the automorphism group of Y acts on the nef cone with a rational polyhedral fundamental domain.

We study a cone conjecture for log Calabi-Yau surfaces that is similar to, but different from, the conjecture proved by Totaro (see Remark 1.3). Let Y be a smooth projective surface and D a reduced normal crossing divisor on Y such that $K_Y + D = 0$. We call (Y, D) a *log Calabi-Yau surface*. Additionally, we require D to be singular, and write $D = D_1 + \cdots + D_n$ for the irreducible components of D. By the Gross-Hacking-Keel Torelli theorem for log Calabi-Yau surfaces ([GHK15b], Theorem 1.8), in each deformation type of log Calabi-Yau surfaces, there exists a unique pair $(Y, D) = (Y_e, D_e)$ such that the mixed Hodge structure on $Y \setminus D$ is split. The main result of this paper is the proof of the following statement (see Theorem 5.1 and Theorem 5.2):

Theorem 1.1. *Consider a deformation type of log Calabi-Yau surfaces (Y, D) with singular boundary.*

1. *Let (Y_e, D_e) be the unique surface in this deformation type with split mixed Hodge structure. Let K be the kernel of the action of the automorphism group of the pair on $H^2(Y, \mathbb{Z})$. Then $Aut(Y_e, D_e)/K$ acts on the nef effective cone* $\mathrm{Nef}^{\,e}(Y_e)$ *with a rational polyhedral fundamental domain.*
2. *Let (Y_{gen}, D_{gen}) be a very general surface in this deformation type. Then the monodromy group Adm acts on the nef effective cone* $\mathrm{Nef}^{\,e}(Y_{gen})$ *with a rational polyhedral fundamental domain.*

Remark 1.2. In (1) above, it is possible for $\mathrm{Aut}(Y_e, D_e)$ to be infinite. For instance, consider when the negative definite or negative semidefinite boundary D_e has $n = 7$ components and suppose that D does not contain any (-1)-curves. Then $\mathrm{Aut}(Y_e, D_e)$ is infinite (see Example 5.3 in [GHK15b]). For $n \leq 6$, the group $\mathrm{Aut}(Y_e, D_e)$ is trivial (see Section 6).

The Morrison cone conjecture, stated in 1993, is originally inspired by mirror symmetry. The log Calabi-Yau surface version of this conjecture is also related to mirror symmetry through the deformation theory of cusp singularities of surfaces.

Given a log Calabi-Yau surface (Y, D) such that the intersection matrix $(D_i \cdot D_j)$ is negative definite, we may contract the boundary D to obtain a normal surface Y with a cusp singularity $p \in Y'$ (see Grauert [G62] and Definition 3.13). Cusp singularities come in dual pairs such that the links are diffeomorphic but have opposite orientations. If (Y', p) is obtained by contracting the boundary of a log Calabi-Yau surface (Y, D) to a cusp singularity $p \in Y'$, then, conjecturally, (Y, D) corresponds to an irreducible component of the deformation space of the dual cusp ([GHK15a], [E15], [EF16]). This is expected as a consequence of mirror symmetry: $Y \backslash D$ is mirror to the Milnor fiber of the corresponding smoothing of the dual cusp ([Ke15], [HKe21]). Again conjecturally, the component of the deformation space of the dual cusp can be described in terms of the action of the monodromy group Adm on $\mathrm{Nef}(Y')$, by a construction of Looijenga ([L03], §4). However, to use this construction, the group Adm must act with a rational polyhedral fundamental domain on the effective nef cone of Y', and this is the original motivation for our conjecture, cf. [M93].

Remark 1.3 (A brief remark on some differences from Totaro's work in [T10]). Totaro considers klt log Calabi-Yau pairs (Y, Δ) of dimension two, meaning that Y is a normal projective surface and $\Delta \subset Y$ is an effective $\mathbb{Q}$-divisor such that $K_Y + \Delta$ is numerically equivalent to zero. Totaro proved the following theorem:

Theorem 1.4 (Totaro). *Let (Y, Δ) be a klt log Calabi-Yau surface with $K_Y + \Delta$ numerically equivalent to zero. Then the automorphism group of (Y, Δ) acts on* $\mathrm{Nef}^{\,e}(Y)$ *with a rational polyhedral fundamental domain.*

Suppose (Y, Δ) is a klt log Calabi-Yau pair and that Δ is contractible. Then there exists a birational morphism $Y \to Y'$ where Y' is a normal surface and $\mathrm{Supp}(\Delta)$ is the exceptional locus. This the case where the intersection matrix of the components of Δ is negative definite, or equivalently, where the Iitaka dimension of $-K_Y$ is zero. We note that the negative definite case is the essential case – the other possibilities are when the Iitaka dimension is one or two, the former reducing to Mori's cone theorem, and the latter to the study of the Mordell-Weil group of an elliptic fibration. The resulting surface Y' has log terminal singularities, and $K_{Y'}$ is numerically equivalent to zero. There exists some $N \in \mathbb{N}$ such that $NK_{Y'}$ is linearly equivalent to zero, and we choose the minimal such N. Then there exists a finite

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morphism $Z \to Y'$ (the canonical covering), which is a $\mathbb{Z}/N\mathbb{Z}$ covering where K_Z is linearly equivalent to zero and Z has canonical singularities. The minimal resolution $\tilde{Z}$ of Z has $K_{\tilde{Z}}$ linearly equivalent to zero, and by classification of surfaces, we conclude that $\tilde{Z}$ is either a $K3$ surface or an Abelian surface (and thus, the rank of $\mathrm{Pic}(Y')$ cannot exceed 20). Theorem 1.4 holds for Y' by the cone conjecture for $K3$ and Abelian surfaces, and Totaro gives an argument ([T10], p.257-259) showing that Theorem 1.4 holds for (Y, Δ).

In this paper, we consider a log Calabi-Yau surface (Y, D) such that D is contractible: there exists a contraction $Y \to Y'$, where Y' is a normal surface containing a cusp singularity p and $K_{Y'}$ is linearly equivalent to zero. Let (Y_e, D_e) be the distinguished pair in the deformation type of (Y, D) with a split mixed Hodge structure. Then Y'_e is projective. Moreover, $\mathrm{Pic}(Y'_e)$, which is identified via pullback with the orthogonal complement of the irreducible components of D_e in $\mathrm{Pic}(Y_e)$, has arbitrarily large rank. Theorem 1.1 (1) may be deduced from the analogous statement for Y'_e. However, the case of Y'_e is not related to the case of $K3$ nor to the case of Abelian surfaces.

Finally, we make a remark on the contribution to new examples. Looijenga studied the following: let L be a free Abelian group of finite rank and C an open convex cone in $V := L \otimes_{\mathbb{Z}} \mathbb{R}$. Let Γ be a group of automorphisms of L that preserve C, such that Γ acts with a rational polyhedral fundamental domain on $C_+ := \mathrm{Conv}(\bar{C} \cap L)$. In this setting, Looijenga constructed a complex analytic compactification of the tube $(V + iC)/\Gamma$. Some examples of such compactification include the cases where

1. C is a homogeneous self-dual cone;
2. C is the Tits cone of a Coxeter group, which is not of finite nor of affine type; and
3. C is the nef or movable cone of a (log) Calabi-Yau variety for which the Morrison cone conjecture holds.

Theorem 1.1 contributes many new examples. In particular, some examples from Theorem 1.1 (2) (that is, when D has at most five components; see [L81]) are dual to Tits cones of Weyl groups, but this is not the case in general. Many known cases of the Morrison cone conjecture (e.g., $K3$ surfaces [S85]; hyperkähler manifolds [M11], [M15], [AV17]) are closely related to the homogeneous self-dual case via the description of the nef or movable cone in terms of a Coxeter group acting on a homogeneous self-dual cone (see, for example, [BHPV04] and [M11]).

This paper is organized as follows. Section 2 contains the motivations of our project. Sections 3–5 contain the proof of the main theorem. In Section 6, we give explicit descriptions of certain cones of curves, which provide infinite series of new examples of Mori Dream Spaces. Here, I emphasize that this section considers cases $n \leq 6$, where n is the length of the negative definite or negative semidefinite boundary D – these are **specific cases** of Theorem 5.4 described more precisely.

2. Motivations

Let $(q \in X)$ be a cusp singularity. Cusp singularities come in dual pairs such that the links are diffeomorphic but have opposite orientations (cf. [L81], §III.2.1). There is the following conjecture (cf. [E15], Conjecture 6.2.5):

1. The smoothing components of the deformation space of $(q \in X)$, up to isomorphism, are in bijective correspondence with deformation types of log Calabi-Yau surfaces (Y, D) such that D does not contain any (-1)-curves and contracts to the dual cusp p.
2. The smoothing component of $(q \in X)$ associated to (Y, D) is the *Looijenga space* determined by the action of Adm on the nef effective cone $\mathrm{Nef}^{\,e}(Y'_{gen})$, which is contained in $\langle D_1, \ldots, D_n \rangle^{\perp} \otimes_{\mathbb{Z}} \mathbb{R}$. (Looijenga's construction is described in [L03], Section 4).

Looijenga's construction requires that Adm acts on $\mathrm{Nef}^{\,e}(Y'_{gen})$ with a rational polyhedral fundamental domain (Theorem 5.1), and this is one motivation for the cone conjecture for log Calabi-Yau surfaces. This is analogous to the original motivation for the Morrison cone conjecture for Calabi-Yau

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threefolds Y ([M93]): Morrison argues that the Looijenga space determined by the action of $\mathrm{Aut}(Y)$ on $\mathrm{Nef}^{\,e}(Y)$ is identified with a neighborhood of a boundary point of the moduli of the mirror of Y.

The log Calabi-Yau cone conjecture can provide insight into the original Morrison cone conjecture because it includes more accessible cases. For instance, in every dimension, there are many log Calabi-Yau pairs (Y, D) such that the variety Y is rational. In addition, the cone conjecture is related to the abundance conjecture ([T11]), which is a long-standing open question of the minimal model program.

Remark 2.1. The cone conjecture for log Calabi-Yau surfaces suggests that the Morrison cone conjecture is false in general, because it is the monodromy group Adm that acts with a rational polyhedral fundamental domain on $\mathrm{Nef}^{\,e}(Y_{gen})$, and not the automorphism group.

Remark 2.2. The explicit description of $\mathrm{Nef}(Y_e)$ can be used to verify the conjecture (1) stated above. For $n \leq 5$, this follows from work of Looijenga [L81], and for $n = 6$, we expect that it can be verified using work of Brohme ([B95]). The deformation theory for $n > 6$ is not known.

3. Background

Let Y be a smooth projective variety. We define $N^1(Y)$ to be the space of divisors with real coefficients modulo numerical equivalence, and the space $N_1(Y)$ to be the space of 1-cycles with real coefficients modulo numerical equivalence. Because Y is a rational surface in our setting,

$$N^1(Y) = H^2(Y, \mathbb{R}) = \mathrm{Pic}(Y) \otimes \mathbb{R}, \tag{3.1}$$

$$N_1(Y) = H_2(Y, \mathbb{R}) = \mathrm{Cl}(Y) \otimes \mathbb{R}, \tag{3.2}$$

and $N^1(Y) = N_1(Y)$. We define the *nef cone of* Y to be

$$\mathrm{Nef}(Y) = \{L \in N^1(Y) \mid L \cdot C \geq 0 \text{ for all irreducible curves } C \subset Y\}.$$

The *effective cone of* Y is

$$\mathrm{Eff}(Y) = \left\{\sum a_i[D_i] \in N^1(Y) \mid a_i \in \mathbb{R}_{\geq 0} \text{ and } D_i \subset Y \text{ are codimension one subvarieties}\right\}.$$

Following Kawamata in [K97], we define the *nef effective cone of* Y to be

$$\mathrm{Nef}^{\,\mathrm{e}}(Y) = \mathrm{Nef}(Y) \bigcap \mathrm{Eff}(Y).$$

We denote the *convex hull* of the set S by $\mathrm{Conv}(S)$, where S is subset of a real vector space. If Y is a surface, then

$$\mathrm{Nef}^{\,\mathrm{e}}(Y) = \mathrm{Conv}\left(\{[L] \in N^1(Y) \mid L \in \mathrm{Pic}(Y) \text{ is nef and } h^0(L) \neq 0\}\right). \tag{3.3}$$

Definition 3.4. The *cone of curves of* Y is defined as follows:

$$\mathrm{Curv}(Y) = \left\{\sum a_i[C_i] \in N_1(Y) \mid a_i \in \mathbb{R}_{\geq 0} \text{ and each } C_i \subset Y \text{ an irreducible curve}\right\}.$$

We write $\overline{\mathrm{Curv}}(Y)$ to mean the closure of the cone of curves.

Definition 3.5. Let L be a finitely generated free Abelian group (i.e., $L \simeq \mathbb{Z}^\rho$ for some $\rho \geq 0$). A cone $C \subset L \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^\rho$ is said to be *rational polyhedral* if

$$C = \langle v_1, \ldots, v_r \rangle_{\mathbb{R}_{\geq 0}} = \{a_1 v_1 + \cdots + a_r v_r \mid a_i \in \mathbb{R}_{\geq 0}\},$$

for some $v_1, \ldots, v_r \in L$. That is, the cone C is generated by finitely many integral vectors $v_1, \ldots, v_r \in L$.

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Definition 3.6. A *log Calabi-Yau surface* is a pair (Y, D) where Y is a smooth complex projective surface and $D \subset Y$ is a reduced normal crossing divisor such that $K_Y + D = 0$. We say that (Y, D) has *maximal boundary* if D is singular. We write $D = D_1 + \cdots + D_n$, where n is the number of irreducible components or the *length* of D.

In this paper, we always assume that (Y, D) has maximal boundary. If (Y, D) is a log Calabi-Yau surface with maximal boundary, then Y is a rational surface ([GHK15b], p.2).

Remark 3.7. The boundary D is either a rational curve of arithmetic genus one with a single node (i.e., a copy of $\mathbb{P}^1$ with two points identified to form a node), or it is a cycle of smooth rational curves (i.e., a cycle of n copies of $\mathbb{P}^1$). This follows from the adjunction formula. We fix a cyclic ordering $D = D_1 + \cdots + D_n$ of the components of D and a compatible orientation (an isomorphism $H_1(D, \mathbb{Z}) \simeq \mathbb{Z}$). This orientation is uniquely determined by the cyclic ordering for $n > 2$.

Definition 3.8. We say that a log Calabi-Yau surface (Y, D) is *generic* if there are no (-2)-curves C contained in $Y \setminus D$. We sometimes write (Y_{gen}, D_{gen}) to denote one such log Calabi-Yau surface in a given deformation type.

Definition 3.9. Two log Calabi-Yau surfaces (Y^1, D^1) and (Y^2, D^2) are said to be *deformation equivalent* if there exists a flat family $(\mathcal{Y}, \mathcal{D}) = (\mathcal{Y}, \mathcal{D}_1 + \cdots + \mathcal{D}_n)$ of log Calabi-Yau surfaces over a connected base S such that there are points $p, q \in S$ with fibers $f^{-1}(p) = (Y^1, D^1)$ and $f^{-1}(q) = (Y^2, D^2)$. In this case, we say that (Y^1, D^1) and (Y^2, D^2) are of the same *deformation type*.

By the Torelli Theorem in [GHK15b], given a log Calabi-Yau surface (Y, D), the moduli space $\mathcal{M}$ of log Calabi-Yau surfaces that are deformation equivalent to (Y, D) can be described explicitly, and the locus of generic surfaces is the complement of a countable union of divisors in $\mathcal{M}$ (see [GHK15b], Section 6). For any two generic surfaces of the same deformation type, the nef cones of the two surfaces are the same. This cone for Y_{gen} is described after the following definition:

Definition 3.10. For a log Calabi-Yau surface (Y, D), an *interior* (-1)-*curve* is a smooth rational curve of self-intersection -1 that is not contained in the boundary D. By the adjunction formula, such a curve must intersect the boundary transversely at a single point.

Proposition 3.11 ([GHK15b], Lemma 2.15).

$$\begin{aligned}\operatorname{Nef}(Y_{gen}) = \{L \in \operatorname{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R} \mid L^2 \geq 0 \textit{ and } L \cdot D_i \geq 0 \textit{ for all } i \textit{ and} \\ L \cdot C \geq 0 \textit{ for any interior (-1)-curve } C\}.\end{aligned}$$

Lemma 3.12. *Let (Y, D) be a log Calabi-Yau surface. If $L \in \operatorname{Pic}(Y)$ is nef, then L is effective.*

Proof. Let $L \in \operatorname{Pic}(Y)$ be nef. By Riemann-Roch, we have

$$\begin{aligned}\chi(L) &= \chi(\mathcal{O}_Y) + \frac{1}{2}L(L - K_Y) \\ &= 1 + \frac{1}{2}(L^2 + L \cdot D) \\ &\geq 1,\end{aligned}$$

since L being nef and D being effective give $L \cdot D \geq 0$ and L nef gives $L^2 \geq 0$. However, we have

$$\begin{aligned}\chi(L) &= h^0(L) - h^1(L) + h^2(L). \\ &\leq h^0(L) + h^2(L)\end{aligned}$$

Next, we show that $h^2(L) = 0$. By Serre Duality, we have $h^2(L) = h^0(K_Y - L) = h^0(-D - L)$. If H is ample and L is nef and D is effective, then we have $H \cdot D > 0$ and $H \cdot L \geq 0$. Then $H \cdot (-D - L) < 0$,

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so $h^0(-D-L) = 0$. Thus, $h^0(L) \geq \chi(L) \geq 1$, and therefore, L is linearly equivalent to an effective divisor. □

Definition 3.13. A *cusp singularity* is a surface singularity whose minimal resolution is a cycle of smooth rational curves that meet transversally. That is, the exceptional locus of the minimal resolution of a cusp singularity is a union of copies of $\mathbb{P}^1$ with nodal singularities such that the dual graph is a cycle.

Given a log Calabi-Yau surface (Y, D) with D having a negative definite intersection matrix $(D_i \cdot D_j)$, it is possible to contract D to a cusp singularity p (by a theorem of Grauert on the contractibility of a negative definite configuration of curves on a smooth complex surface in the analytic category, [G62]). Let $f : Y \to Y'$ be the morphism contracting D to a point. Then we have the induced isomorphism

$$Y \setminus D \cong Y' \setminus \{p\},$$

and $f^{-1}(p) = D$. In addition, the surface Y' is normal and compact (for the usual Euclidean topology). We note that although Y is a projective variety, the new surface Y' is in general no longer a projective variety, but a normal, analytic space. We make the following definitions.

Definition 3.14. We define the nef effective cone of Y' in the following way:

$$\mathrm{Nef}^{\,\mathrm{e}}(Y') := \mathrm{Nef}^{\,\mathrm{e}}(Y) \cap \langle D_1, \ldots, D_n\rangle^{\perp}.$$

Remark 3.15. Equivalently, $\mathrm{Nef}^{\,e}(Y') = \mathrm{Nef}(Y') \cap \mathrm{Eff}(Y')$, where

$$\mathrm{Nef}(Y') := \{L \in \mathrm{Cl}(Y') \otimes \mathbb{R} \mid L \cdot C \geq 0 \text{ for all curves } C \subset Y'\},$$

and we use Mumford's intersection product on a normal surface Y' ([M61], p.17). Note that Y' is not $\mathbb{Q}$-factorial in general (i.e., there may exist divisors which are not $\mathbb{Q}$-Cartier).

Definition 3.16. An *isomorphism of log Calabi-Yau surfaces* (Y^1, D^1) and (Y^2, D^2) is an isomorphism $\theta : Y^1 \to Y^2$, with the property that $\theta(D_i^1) = \theta(D_i^2)$ for each boundary component D_i^k of D^k for $k = 1, 2$, and θ respects the orientations of D^1 and D^2 (automatic for $n \geq 3$).

Definition 3.17. Given any log Calabi-Yau surface (Y, D), the *admissible group* of Y is defined as follows:

$$\begin{aligned}\mathrm{Adm} = \{\theta \in \mathrm{Aut}(\mathrm{Pic}(Y)) \mid \theta([D_i]) = [D_i] \text{ for all } i = 1, \ldots, n \text{ and}\\ \theta(\mathrm{Nef}(Y_{gen})) = \mathrm{Nef}(Y_{gen})\}.\end{aligned}$$

Remark 3.18. Adm is identified with the monodromy group for (Y, D) ([GHK15b], Theorem 5.15).

Definition 3.19. Let Γ be a group and X a topological space. Suppose that Γ acts on X by homeomorphisms. We say that a closed subset $D \subset X$ is a *fundamental domain* for the action of Γ on X if the following are true:

1. for all $x \in X$, there exists $d \in D$ and $\gamma \in \Gamma$ such that $\gamma(d) = x$; and
2. for all $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 \neq \gamma_2$, the intersection $\gamma_1 D \cap \gamma_2 D$ has empty interior.

Definition 3.20. Given a log Calabi-Yau surface (Y, D), the *period point* is defined to be the homomorphism $\phi : \langle D_1, \ldots, D_n\rangle^{\perp} \to \mathbb{C}^*$, where a line bundle $L \in \langle D_1, \ldots, D_n\rangle^{\perp}$ is sent to $\theta([L|_D]) \in \mathbb{C}^*$, where $\theta : \mathrm{Pic}^0(D) \xrightarrow{\sim} \mathbb{C}^*$ is the isomorphism determined by the given orientation of D (as explained in [GHK15b], Lemma 2.1). Here, $\mathrm{Pic}^0(D)$ is the kernel of the map $c_1 : \mathrm{Pic}(D) \to H^2(D, \mathbb{Z}) = \mathbb{Z}^n$, given by $L \mapsto (\deg L|_{D_i})_{i=1}^n$.

By [F15], Proposition 3.12, the homomorphism ϕ is the extension class of the mixed Hodge structure on $H_2(U, \mathbb{C})$, where we take $U = Y \setminus D$. There is an exact sequence ([L81], Chapter I, Section 5.1):

$$0 \to \mathbb{Z} \to H_2(U) \to \langle D_1, \ldots, D_n\rangle^{\perp} \to 0.$$

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There exists a unique log Calabi-Yau surface in each deformation type such that $\phi(\alpha) = 1$ for all $\alpha \in \langle D_1, \ldots, D_n\rangle^\perp$ (i.e., the mixed Hodge structure on $H_2(U)$ is split). This follows from the Torelli theorem ([GHK15b], §5; and [F15], Corollary 9.7). We denote this log Calabi-Yau surface by (Y_e, D_e).

Definition 3.21. Given a log Calabi-Yau surface (Y, D), the associated *root system* is the subset of $\mathrm{Pic}(Y)$ defined by

$$\Phi = \{\alpha \in \langle D_1, \ldots, D_n\rangle^\perp \mid \alpha^\perp \cap \mathrm{Int}(\mathrm{Nef}(Y_{gen})) \neq \emptyset \text{ and } \alpha^2 = -2\}.$$

Definition 3.22. We define the *Weyl group* of the root system $\Phi \subset \mathrm{Pic}(Y)$ as follows:

$$W = \langle s_\alpha \mid \alpha \in \Phi\rangle \subset \mathrm{Aut}(\mathrm{Pic}(Y), \cdot),$$

where the generators $s_\alpha(\beta) = \beta + (\alpha\cdot\beta)\alpha$ are the reflections in the hyperplanes $\alpha^\perp$ for $\alpha \in \Phi$.

Definition 3.23. Given (Y_e, D_e), we define the *simple roots* as the set

$$\Delta = \{[C] \mid C \subset Y_e \setminus D_e \text{ is a } (-2)\text{-curve}\}.$$

Proposition 3.24 ([GHK15b], Proposition 3.4). *The set Δ is contained in Φ, and the Weyl group W is generated by the reflections s_δ for $\delta \in \Delta$ – that is,*

$$W = \langle s_\delta \mid \delta \subset \Delta\rangle.$$

By [GHK15b], Lemma 2.15,

$$\mathrm{Nef}(Y_e) = \mathrm{Nef}(Y_{gen}) \bigcap (\delta \geq 0 \text{ for all } \delta \in \Delta).$$

Remark 3.25. The Weyl group is a normal subgroup of Adm. (Proof: it follows from Definitions 3.17 and 3.21 that Adm preserves Φ. If $g \in \mathrm{Adm}$ and $\alpha \in \Phi$, then $g s_\alpha g^{-1} = s_{g(\alpha)}$, which implies that $W \triangleleft \mathrm{Adm}$).

By [GHK15b], Theorem 3.2, the group W acts on $\mathrm{Nef}^{\,e}(Y_{gen})$ with fundamental domain $\mathrm{Nef}^{\,e}(Y_e)$. This is called the *fundamental chamber* in $\mathrm{Nef}(Y_{gen})$. By [GHK15b], Theorem 5.1, there is an exact sequence

$$1 \longrightarrow K \longrightarrow \mathrm{Aut}(Y_e, D_e) \longrightarrow \mathrm{Adm}/W \longrightarrow 1,$$

where K is the kernel of the action of $\mathrm{Aut}(Y_e, D_e)$ on $\mathrm{Pic}(Y)$.

4. Tools

Here, we include some main results that we use in the proof of our results.

Theorem 4.1 ([GHK15b], Theorem 1.8, The global Torelli theorem for log Calabi-Yau surfaces). *Suppose that (Y^1, D^1) and (Y^2, D^2) are log Calabi-Yau surfaces. Consider the following three statements:*

1. $\theta : \mathrm{Pic}(Y^1) \longrightarrow \mathrm{Pic}(Y^2)$ *is an isometry such that* $\theta([D_i^1]) = [D_i^2]$ *for* $i = 1, \ldots, n$.
2. $\theta(L)$ *is ample for some ample* L *on* Y^1.
3. $\phi_{Y^2} \circ \theta = \phi_{Y^1}$, *where* $\phi_Y : \langle D_1, \ldots, D_n\rangle^\perp \longrightarrow \mathbb{C}^*$ *is the period point of* Y.

(1), (2) *and* (3) *hold if and only if* $\theta = f^*$ *for some isomorphism* $f : (Y^2, D^2) \longrightarrow (Y^1, D^1)$.

Proposition 4.2 ([EF16], Proposition 1.5). *Let (Y_{gen}, D_{gen}) be a generic log Calabi-Yau surface, where D_{gen} has at least three boundary components. If B is a divisor on Y_{gen} with nonnegative integer* *coefficients, then B is linearly equivalent to a divisor of the form*

$$\sum a_i D_i + \sum b_j E_j,$$

where the E_j's are disjoint interior (-1)-curves and a_i, b_j are nonnegative integers.

Remark 4.3. Although the Engel-Friedman Proposition 4.2 is stated for B with nonnegative *integer* coefficients, the statement also holds for B with nonnegative *real* coefficients. There is a sketch of the proof in the paper ([EF16], p.55), which uses a continuity argument and the assertion that the collection of subsets

$$\left\{\sum a_j D_j + \sum b_i E_i \mid a_j, b_i \in \mathbb{R}_{\geq 0}\right\},$$

where the E_i's are disjoint interior (-1)-curves, is locally finite in $\mathrm{Nef}^{\,e}(Y_{gen})$ in a sense that is made precise below. Since this is important for our results, we give a complete proof (Proposition 4.10).

Friedman showed in [F15] that Adm acts with finitely many orbits on the set of faces of $\mathrm{Nef}^{\,e}(Y_{gen})$ corresponding to interior (-1)-curves. This is stated in Theorem 4.4.

Theorem 4.4 (Friedman [F15], Theorem 9.8). *Let (Y, D) be a generic log Calabi-Yau surface. Let $\mathcal{E}(Y, D)$ be the set of all interior (-1)-curves of Y. Then the admissible group Adm acts on $\mathcal{E}(Y, D)$, and there are finitely many Adm-orbits for this action.*

The following Corollary 4.5 by Friedman is similar to the statement above. Specifically, it is a statement about the action of Adm on the set of collections of disjoint interior (-1)-curves.

Corollary 4.5 (Friedman [F15], Corollary 9.10). *Given a generic log Calabi-Yau surface (Y, D), let $\mathcal{E}_k(Y, D)$ be the set of collections $\{E_1, \ldots, E_k\}$, for any $k \in \mathbb{N}$, where the curves E_i are disjoint, interior (-1)-curves. Then the admissible group Adm acts on $\mathcal{E}_k(Y, D)$, and the number of Adm orbits for this action is finite.*

Theorem 4.6 (Looijenga [L14], Proposition-Definition 4.1; and Application 4.14; and Proposition 4.7). *Let Γ be a group and L be a lattice (i.e., a finitely generated free abelian group), and let $C \subset L \otimes_{\mathbb{Z}} \mathbb{R}$ be an open nondegenerate convex cone. Define*

$$C_+ := Conv(\bar{C} \cap L).$$

Assume that Γ acts on L faithfully, preserving the cone C. If there exists a polyhedral cone $\Pi \subset C_+$ such that $\Gamma \cdot \Pi = C_+$, then there exists a rational polyhedral fundamental domain for the action of Γ on C_+. Moreover, in this case, the group $N_\Gamma F/Z_\Gamma F$ acts on any face F of C_+ with a rational polyhedral fundamental domain.

Remark 4.7. In Theorem 4.6, following the notation of Looijenga, we use $N_\Gamma F$ to mean the normalizer of F in Γ and $Z_\Gamma F$ to mean the centralizer (i.e., elements of Γ that fix F pointwise). The last statement is a special case of Proposition 4.7 in [L14].

Proposition 4.8. *Let (Y_{gen}, D_{gen}) be a generic log Calabi-Yau surface. For a collection $\{E_1, \ldots, E_k\}$ of disjoint (-1)-curves, define*

$$C'(E_1, \ldots, E_k) := \langle D_1, \ldots, D_n, E_1, \ldots, E_k\rangle_{\mathbb{R}_{\geq 0}} \cap \mathrm{Nef}^{\,e}(Y_{gen}).$$

Then

1. *$C'(E_1, \ldots, E_k)$ is a rational polyhedral cone; and*
2. *If D_{gen} consists of at least three components, then the set of cones $C'(E_1, \ldots, E_k)$ covers* $\mathrm{Nef}^{\,e}(Y'_{gen})$.

Proof. Let C be the cone defined by

$$C = C(E_1, \dots, E_k) := \langle D_1, \dots, D_n, E_1, \dots, E_k \rangle_{\mathbb{R}_{\geq 0}},$$

so that $C'(E_1, \dots, E_k)$ can be expressed as $C \cap \text{Nef}^{\,e}(Y_{gen})$. The cone $C'(E_1, \dots, E_k)$ is rational polyhedral because C is rational polyhedral by definition, and the intersection with the nef cone is given by finitely many inequalities $L \cdot D_i \geq 0$ and $L \cdot E_j \geq 0$ for all $1 \leq i \leq n$ and all $1 \leq j \leq k$. This shows that $C'(E_1, \dots, E_k)$ is rational polyhedral. Now assume that D_{gen} has at least three components. Then, by Proposition 4.10,

$$\text{Nef}^{\,e}(Y_{gen}) = \bigcup C'(E_1, \dots, E_k),$$

where the union is over the set $\bigcup_k \mathcal{E}_k(Y, D)$ of collections $\{E_1, \dots, E_k\}$ of disjoint interior (-1)-curves. □

Theorem 4.9 (The Siegel Property, as stated in [L14], Theorem 3.8). *Let L be a lattice and $V = L \otimes \mathbb{R}$. Let $C \subset V$ be an open convex nondegenerate cone. We denote the convex hull $Conv(\bar{C} \cap L)$ by C_+. Let Γ be a subgroup of $GL(V)$ such that Γ leaves the cone C and the lattice L invariant. Then Γ has the Siegel Property in C_+; that is, if Π_1 and Π_2 are polyhedral cones in C_+, then the collection $\{\gamma\Pi_1 \bigcap \Pi_2\}_{\gamma \in \Gamma}$ is finite.*

Next, we give the precise statement of Engel-Friedman's Proposition 4.2 for real coefficients, followed by a careful proof, which uses the Siegel Property 4.9 and the result 4.5 of Friedman.

Proposition 4.10 (Proposition 4.2 for B with real coefficients). *Let (Y_{gen}, D_{gen}) be a generic log Calabi-Yau surface, where D_{gen} has at least three boundary components. If B is a divisor on Y_{gen} with nonnegative real coefficients, then B is $\mathbb{R}$-linearly equivalent to a divisor of the form*

$$\sum a_i D_i + \sum b_j E_j,$$

where the E_j's are disjoint interior (-1)-curves and a_i, b_j are nonnegative real numbers.

Proof. We use the same notation introduced in Proposition 4.8 above; that is, for a log Calabi-Yau surface (Y_{gen}, D_{gen}) where D_{gen} is of length at least three, we let

$$C := \langle D_1, \dots, D_n, E_1, \dots, E_k \rangle_{\mathbb{R} \geq 0}.$$

We want to show that

$$\text{Curv}(Y_{gen}) = \bigcup C(E_1, \dots, E_k), \tag{4.11}$$

where $\text{Curv}(Y) := \{\sum a_i [C_j] \mid a_i \in \mathbb{R}_{\geq 0} \text{ and } C_i \subset Y \text{ are irreducible curves}\}$ (Note, because $\dim(Y_{gen}) = 2$, the cones $\text{Eff}(Y_{gen})$ and $\text{Curv}(Y_{gen})$ coincide). Clearly,

$$\bigcup C(E_1, \dots, E_k) \subseteq \text{Curv}(Y_{gen}),$$

where the union is taken over all $C(E_1, \dots, E_k)$ where $\{E_1, \dots, E_k\}$ are collections of disjoint interior (-1)-curves.

Let $x \in \text{Curv}(Y_{gen})$ be an arbitrary point. A convex cone is the disjoint union of the relative interiors of its faces. This follows from the supporting hyperplane theorem ([S11], Proposition 8.5).

So $x \in \text{relInt}(F)$, where F is some face of $\text{Curv}(Y_{gen})$ (possibly $F = \text{Curv}(Y_{gen})$). Since $\text{Curv}(Y_{gen})$ is generated by rational points, the same is true for any face of $\text{Curv}(Y_{gen})$. In particular, the face F is the convex hull of its rational points, so the rational points are dense in F. Thus, we may choose a sequence of points $x_n \in F \bigcap (\text{Pic}(Y) \otimes \mathbb{Q})$ that converge to x as n approaches infinity. The original

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Engel-Friedman statement (see Proposition 4.2) was stated for integer coefficients, but this implies that the statement for rational coefficeints is also true. So for every n, the point x_n belongs to some cone $C(E_1, \ldots, E_k)$, as defined above.

Since $x \in \mathrm{relInt}(F)$ (i.e., the interior of F regarded as a subset of $\langle F\rangle_{\mathbb{R}}$), there exists a rational polyhedral cone $\Pi \subset F$ such that $x \in \mathrm{relInt}(\Pi)$ and $\dim(\Pi) = \dim(F)$. Then $\mathrm{relInt}(\Pi)$ is an open subset of F. Since the points $\{x_n\}$ converge to x in the face F, there exists some number $N \in \mathbb{N}$ such that $x_n \in \Pi$ for all $n \geq N$. Friedman's results tells us that Adm acts on the cones $C(E_1, \ldots, E_k)$ with finitely many orbits. Say we choose a representative C_i from each orbit, so we have finitely many representatives $C_1, \ldots, C_r$ (note that we drop the $\{E_1, \ldots, E_k\}$ part here to keep the notation simpler). By the Siegel property 4.9, there exists finitely many elements $g \in \mathrm{Adm}$ such that $g(C_i) \bigcap \Pi \neq \emptyset$. Suppose these elements are $g_{i,1}, \ldots, g_{i,m_i}$ for $i = 1, \ldots, r$. Then the following cones intersect Π:

$$\begin{aligned}&g_{1,1}C_1, \ldots, g_{1,m_1}C_1\\ &g_{2,1}C_2, \ldots, g_{2,m_2}C_2\\ &\vdots\\ &g_{r,1}C_r, \ldots, g_{r,m_r}C_r.\end{aligned}$$

As a result, we have a (finite) total of $m = m_1 + \cdots + m_r$ cones σ_l of the form $C(E_1, \ldots, E_k)$ intersecting the cone Π. Since each cone $C(E_1, \ldots, E_k)$ is closed, the finite union

$$\bigcup_{l=1}^{m} \sigma_l$$

is also closed. Recall that each x_n is contained in some cone in the union above, so their limit point x must also lie in the union; that is,

$$x \in \bigcup_{l=1}^{m} \sigma_l \subset \bigcup C(E_1, \ldots, E_k).$$

Now we have shown that $\mathrm{Curv}(Y_{gen}) \subseteq \bigcup C(E_1, \ldots, E_k)$. Therefore,

$$\mathrm{Curv}(Y_{gen}) = \bigcup C(E_1, \ldots, E_k) = \bigcup \langle D_1, \ldots, D_n, E_1, \ldots, E_k\rangle_{\mathbb{R}\geq 0},$$

proving Proposition 4.10. □

Theorem 4.12. *Let (Y_e, D_e) be a log Calabi-Yau surface with split mixed Hodge structure. If L is a nef divisor on Y, then L is semiample.*

Proof. Let L be nef on Y. Then $L^2 \geq 0$. If $L^2 > 0$, then by Theorem 4.8 of [F15], the divisor L is semiample. For the remainder of this proof, we suppose that $L^2 = 0$ and $L \neq 0$. Using Riemann-Roch, we obtain

$$\begin{aligned}\chi(L) &= \chi(\mathcal{O}) + \frac{1}{2}L \cdot (L - K_Y)\\ &= 1 + \frac{1}{2}L^2 + \frac{1}{2}(L \cdot D) \quad \text{since } Y \text{ is rational and } K_Y + D = 0\\ &= 1 + \frac{1}{2}(L \cdot D) \quad \text{using } L^2 = 0.\end{aligned}$$

Here, we note that $\chi(L) \geq 1$, since L nef and D effective imply that $L \cdot D \geq 0$. However, the Euler characteristic of L may also be expressed as

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$$\chi(L) = h^0(L) - h^1(L) + h^2(L).$$

By the last paragraph of the proof of Lemma 3.12, since L is nef, we have $h^2(L) = 0$. Now $\chi(L) = h^0(L) - h^1(L) = 1 + \frac{1}{2}(L \cdot D)$. Next we split the inequality $L \cdot D \geq 0$ (which comes from L being nef) into two subcases, and in each situation, we prove that $h^0(L) \geq 2$.

Subcase (i). Suppose that $L \cdot D > 0$, or $L \cdot D \geq 1$. Then,

$$1 + \frac{1}{2}(L \cdot D) = \chi(L) = h^0(L) - h^1(L) \leq h^0(L).$$

Since $\chi(L) \in \mathbb{Z}$, we have $h^0(L) \geq 2$.

Subcase (ii). Suppose that $L \cdot D = 0$. We have $\chi(L) = h^0(L) - h^1(L) = 1$. We show that $h^1(L) \geq 1$, so $h^0(L) \geq 2$. Since $L \cdot D = 0$ and L is nef, we have $L \cdot D_i = 0$ for all i. Then because (Y, D) has split mixed Hodge structure, it follows that $\mathcal{O}_D(L|_D) \simeq \mathcal{O}_D$ (Section 3). From the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(L - D) \longrightarrow \mathcal{O}_Y(L) \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

we obtain

$$\begin{array}{ccccc} H^1(\mathcal{O}_Y(L)) & \xrightarrow{\delta} & H^1(\mathcal{O}_D) & \longrightarrow & H^2(\mathcal{O}_Y(L-D)) \\ & & \wr\Vert & & \\ & & \mathbb{C}. & & \end{array}$$

By Serre Duality, we have

$$\begin{aligned} h^2(\mathcal{O}_Y(L-D)) &= h^0(\mathcal{O}_Y(K_Y - (L-D))) \\ &= h^0(\mathcal{O}_Y(-L)) \quad \text{since } K_Y + D = 0 \\ &= 0. \end{aligned}$$

Then the map δ in the exact sequence above is surjective, so $h^1(L) \geq 1$.

Therefore, $h^0(L) \geq 2$. This means that in the linear system $|L|$, there is a moving part. Writing $L = M + F$, where M is the moving part and F is the fixed part, we have

$$\begin{aligned} L^2 &= L \cdot (M + F) \\ &= L \cdot M + L \cdot F, \end{aligned}$$

and L nef gives $L \cdot M \geq 0$ and $L \cdot F \geq 0$. Since $L^2 = 0$ by assumption, we obtain $L \cdot M = 0 = L \cdot F$. Now we have

$$\begin{aligned} L \cdot M &= (M + F) \cdot M \\ &= M^2 + M \cdot F \\ &= 0, \end{aligned}$$

and M is nef (since it is moving) so $M^2 \geq 0$ and $M \cdot F \geq 0$, so $M^2 = M \cdot F = 0$. Now $L \cdot F = |M + F| \cdot F = 0$, and thus, $F^2 = 0$. We make two conclusions from the computations above.

(a) The linear system $|M|$ has no fixed part, so $|M|$ is basepoint free: there exists $M' \sim M$ such that M and M' have no common components (since M is moving). Then $M \cdot M' = M^2 = 0$, so $\mathrm{Supp}(M) \cap \mathrm{Supp}(M') = \emptyset$, and therefore, $|M|$ is basepoint free. It follows that there exists a map $\phi_{|M|} : Y \to C$, where $C \subset \mathbb{P}^N$ is a curve. By Stein factorization (see Hartshorne [H77], Chapter III

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(11.5)), replacing $L = M + F$ by $kL = kM + kF$ for sufficiently large k, we may assume that C is a smooth curve and ϕ has connected fibers.

(b) Secondly, we conclude that L is semiample, using the results that $F^2 = 0$ and $F \cdot M = 0$. Since $F \cdot M = 0$, the divisor F is contained in a union of fibers of the map $\phi : Y \to C$. A fiber has negative semidefinite intersection matrix with kernel generated over $\mathbb{Q}$ by the class of the fiber. Therefore, kF is a sum of fibers for some $k > 0$. Then $k'F$ is basepoint free for some $k' > 0$ such that $k|k'$ by Riemann-Roch on the curve C. Now $k' \cdot L = k' \cdot M + k' \cdot F$ is basepoint free, so L is semiample. □

5. Proof of the conjecture

Theorem 5.1. *The cone conjecture for Y_{gen} holds. That is, the group Adm acts on* $\mathrm{Nef}^e(Y_{gen})$ *with a rational polyhedral fundamental domain.*

Proof. First, assume $n \geq 3$. By Friedman's result (Corollary 4.5), the group Adm acts on the set of finite collections of disjoint interior (-1)-curves with finitely many orbits. Let $C_1', \ldots, C_r'$ be representatives for the finitely many orbits of Adm on the set of cones C' of Proposition 4.8. Let $\Pi = \mathrm{Conv}(C_1', \ldots, C_r')$. Then Π is rational polyhedral because the cones C_i' are, by Proposition 4.8 (1). Moreover, $\mathrm{Adm} \cdot \Pi = \mathrm{Nef}^e(Y_{gen})$ by Proposition 4.8 (2). Therefore, Adm acts on $\mathrm{Nef}^e(Y_{gen})$ with a rational polyhedral fundamental domain by Theorem 4.6 of Looijenga: in our setting, the lattice L is $\mathrm{Pic}(Y_{gen})$ and C is the ample cone of Y_{gen} (which is the interior of $\mathrm{Nef}(Y_{gen})$). Its closure $\bar{C}$ is $\mathrm{Nef}(Y_{gen})$. The group Γ acting on L is Adm. By (3.3), $C_+ = \mathrm{Nef}^e(Y_{gen})$. This proves the cone conjecture for Y_{gen} in the case when D_{gen} has at least three components.

If the number of components n of D_{gen} is one or two, then we show below in Section 6 that the nef cone is rational polyhedral for Y_e. Moreover, in these cases, the groups Adm and the Weyl group W are equal ([L81], Proposition 4.7 or [F15], Theorem 9.13). Because the action of W on $\mathrm{Nef}^e(Y_{gen})$ has fundamental domain $\mathrm{Nef}^e(Y_e)$ ([GHK15b], Theorem 3.2), we conclude that $\mathrm{Adm} = W$ acts on $\mathrm{Nef}^e(Y_{gen})$ with the *rational polyhedral* fundamental domain $\mathrm{Nef}(Y_e)$, proving the cone conjecture. □

Theorem 5.2. *The cone conjecture for Y'_{gen} holds. That is, the group Adm acts on* $\mathrm{Nef}^e(Y'_{gen})$ *with a rational polyhedral fundamental domain.*

Remark 5.3. We use the definition of $\mathrm{Nef}^e(Y'_{gen})$ as given in 3.15.

Proof. By Theorem 5.1, we know that the cone conjecture holds for Y_{gen}. Since $\mathrm{Nef}^e(Y'_{gen})$ is a face F of $\mathrm{Nef}^e(Y_{gen})$, by Looijenga's result (see the last statement of Theorem 4.6), the cone conjecture also holds for Y'_{gen}. In our setting, the normalizer $N_\Gamma F$ is Adm and the centralizer $Z_\Gamma F$ is $\{e\}$. □

Theorem 5.4. *Let K be the kernel of the action of $Aut(Y_e.D_e)$ on* $\mathrm{Pic}(Y_e)$. *Then $Aut(Y_e, D_e)/K$ acts on* $\mathrm{Nef}^e(Y_e)$ *with a rational polyhedral fundamental domain.*

Remark 5.5. The proof of Theorem 5.4 is similar to the argument of Sterk for $K3$ surfaces (see [S85]).

Proof of Theorem 5.4. By Theorem 5.1, the group Adm acts on $\mathrm{Nef}^e(Y_{gen})$ with a rational polyhedral fundamental domain. Choose a rational point $y \in \mathrm{Int}(\mathrm{Nef}^e(Y_{gen}))$ such that y has trivial stabilizer in Adm. Then by [L81] Application 4.14, we obtain a rational polyhedral fundamental domain $\sigma(y)$ defined as follows:

$$\sigma(y) = \sigma := \{x \in \mathrm{Nef}^{\mathrm{e}}(Y_{gen}) \mid \gamma x \cdot y \geq x \cdot y \text{ for all } \gamma \in \mathrm{Adm}\}.$$

Let $\gamma = s_\alpha$, the reflection associated to a simple root $\alpha = [C]$ where $C \subset Y_e \setminus D_e$ is a (-2)-curve. Because $s_\alpha(x) = x + (x \cdot \alpha)\alpha$, the condition

$$\gamma x \cdot y \geq x \cdot y \tag{5.6}$$

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is equivalent to

$$\begin{aligned} s_\alpha(x)\cdot y \geq x\cdot y &\iff (x+(x\cdot\alpha)\alpha)\cdot y \geq x\cdot y \\ &\iff x\cdot y + (x\cdot\alpha)(\alpha\cdot y) \geq x\cdot y \\ &\iff (x\cdot\alpha)(\alpha\cdot y) \geq 0. \end{aligned}$$

Because α is effective and y is ample (since $y \in \mathrm{Int}(\mathrm{Nef}(Y_e))$, which is the ample cone), the intersection $(\alpha\cdot y)$ is positive. Then $(x\cdot\alpha)(\alpha\cdot y) \geq 0$ if and only if $(x\cdot\alpha) \geq 0$. In particular, this shows the following:

$$\sigma \subset \mathrm{Nef}^{\,\mathrm{e}}(Y_{gen}) \cap (\alpha \geq 0\ \forall\ \alpha \in \Delta) = \mathrm{Nef}^{\,\mathrm{e}}(Y_e),$$

where Δ above denotes the simple roots (see Definition 3.23), and the equality follows from the description of the nef cone in [GHK15b], Lemma 2.15.

The following statements are true:

1. σ is rational polyhedral ([L14], Application 4.14) and $\sigma \subset \mathrm{Nef}^{\,e}(Y_e)$, as shown above;
2. $\mathrm{Adm} = W \rtimes \mathrm{Aut}(Y_e, D_e)/K$ ([GHK15b], Theorem 5.1 or [F15], Theorem 9.6);
3. $\mathrm{Nef}^{\,e}(Y_e)$ is a fundamental domain for the action of W on $\mathrm{Nef}^{\,e}(Y_{gen})$ ([GHK15b], Theorem 3.2), and by the Torelli theorem ([GHK15b], Theorem 1.8), $\mathrm{Aut}(Y_e, D_e)/K \leq \mathrm{Adm}$ is the normalizer of $\mathrm{Nef}^{\,e}(Y_e)$.

Next, we show how the three statements above imply that σ is a rational polyhedral fundamental domain for the action of $\mathrm{Aut}(Y_e, D_e)/K$ on $\mathrm{Nef}^{\,e}(Y_e)$. Let g be an element of Adm. By (2) above, there exist unique $w \in W$ and $\theta \in \mathrm{Aut}(Y_e, D_e)/K$ such that $g = w\theta$. We claim that the following inclusion holds:

$$(g\sigma) \bigcap \mathrm{Nef}^{\,\mathrm{e}}(Y_e) \subset \theta\sigma. \tag{5.7}$$

To see why, let $\mathcal{C}$ be the cone

$$\mathcal{C} = (\alpha \geq 0 \text{ for } \alpha \in \Delta),$$

which is the fundamental chamber for the action of W on $\mathrm{Pic}(Y)\otimes_{\mathbb{Z}}\mathbb{R}$, and we have $\sigma \subset \mathrm{Nef}^{\,e}(Y_e) \subset \mathcal{C}$. From above, $g = w\theta$. Let $x \in (g\sigma) \bigcap \mathrm{Nef}^{\,e}(Y_e)$. The group $\mathrm{Aut}(Y_e, D_e)/K$ acts on $\mathrm{Nef}^{\,e}(Y_e)$. Then $x \in \mathcal{C}$ and $x = gu = w\theta u$, where $u \in \sigma \subset \mathrm{Nef}^{\,e}(Y_e)$ and so $\theta u \in \mathrm{Nef}^{\,e}(Y_e) \subset \mathcal{C}$. Thus, θu and $w\theta u$ are in $\mathcal{C}$. So $\theta u \in w\mathcal{C} \bigcap \mathcal{C} \subset \mathrm{Fix}(w)$ by Sterk ([S85], Lemma 1.2), which means that $w\theta u = \theta u$, i.e., $x = \theta u$. Then $x = \theta u \in \theta\sigma$, proving the inclusion 5.7.

Finally, we want to show that σ is a rational polyhedral fundamental domain for the action of $\mathrm{Aut}(Y_e, D_e)/K$ on $\mathrm{Nef}^{\,e}(Y_e)$. By (1), σ is rational polyhedral, so it remains to show that it is a fundamental domain. Because σ is a fundamental domain for the action of Adm on $\mathrm{Nef}^{\,e}(Y_{gen})$,

$$\mathrm{Nef}^{\,\mathrm{e}}(Y_{gen}) = \bigcup_{g\in\mathrm{Adm}} g\sigma.$$

We can write

$$\begin{aligned} \mathrm{Nef}^{\,\mathrm{e}}(Y_{gen}) \bigcap \mathrm{Nef}^{\,\mathrm{e}}(Y_e) &= \Big(\bigcup_{g\in\mathrm{Adm}} g\sigma\Big) \cap \mathrm{Nef}^{\,\mathrm{e}}(Y_e) \\ &= \bigcup_{g\in\mathrm{Adm}} (g\sigma \cap \mathrm{Nef}^{\,\mathrm{e}}(Y_e)) \\ &= \bigcup_{\theta\in\mathrm{Aut}(Y_e,D_e)/K} \theta\sigma. \end{aligned}$$

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Here is why the last equality holds: if $g = w\theta$, then we showed above that $(g\sigma) \bigcap \text{Nef}^{\,e}(Y_e) \subset \theta\sigma$. If $w = 1$, then $g = \theta \in \text{Aut}(Y_e, D_e)/K$ and

$$(g\sigma) \bigcap \text{Nef}^{\,e}(Y_e) = (\theta\sigma) \bigcap \text{Nef}^{\,e}(Y_e) = \theta\sigma,$$

because θ preserves $\text{Nef}^{\,e}(Y_e)$ and $\sigma \subset \text{Nef}^{\,e}(Y_e)$. Moreover, because

$$\text{Int}\big(g_1\sigma \bigcap g_2\sigma\big) = \emptyset \;\forall\, g_1, g_2 \in \text{Adm},$$

it follows that the same statement holds for g_1, g_2 in the smaller group $\text{Aut}(Y_e, D_e)/K$. That is,

$$\text{Int}\big(g_1\sigma \bigcap g_2\sigma\big) = \emptyset \;\forall\, g_1, g_2 \in \text{Aut}(Y_e, D_e)/K,$$

which is the second property in the definition of a fundamental domain. Therefore, we have shown, using conditions (1), (2) and (3), that σ is a rational polyhedral fundamental domain for the action of $\text{Aut}(Y_e, D_e)/K$ on $\text{Nef}^{\,e}(Y_e)$. □

6. New examples of Mori Dream Spaces

Theorem 6.1. *A log Calabi-Yau surface (Y_e, D_e) with split mixed Hodge structure in which the boundary D_e consists of no more than six components has a rational polyhedral cone of curves. Moreover, if the intersection matrix of $(D_i.D_j)$ is negative definite or negative semidefinite, then the automorphism group $Aut(Y_e, D_e)$ is trivial for $n = 6$ (in fact, this holds for all $n \le 6$). In addition, for each such surface, we give an explicit description of the cone of curves.*

Corollary 6.2. *A log Calabi-Yau surface (Y_e, D_e) with split mixed Hodge structure which has boundary D_e consisting of no more than six components is an example of a Mori Dream Space.*

Proof. This follows from 3.1, Theorem 6.1, Theorem 4.12 and [HK00] Definition 1.10. □

We note that Looijenga has done similar studies for the cases where $n \le 5$ in [L81]. To keep the notation simple, we will use (Y, D) to mean (Y_e, D_e) from here on in this section, unless otherwise specified.

Remark 6.3. When the cone of curves of Y has finitely many generators, it is automatically closed. Because the cones we describe below are all rational polyhedral, we have $\overline{\text{Curv}}(Y) = \text{Curv}(Y)$.

In this next part, we only consider log Calabi-Yau surfaces with the split mixed Hodge structure. We will show that each surface Y described for each $n \le 6$ is the surface Y_e in the given deformation type with the split mixed Hodge structure (that is, the period point ϕ given by $\phi(x) = 1$ for all $x \in \langle D_1, \ldots, D_n\rangle^{\perp}_{\mathbb{Z}}$).

Lemma 6.4. *Let (Y, D) be a log Calabi-Yau surface and suppose that $\langle D_1, \ldots, D_n\rangle^{\perp}_{\mathbb{Z}}$ is generated by classes of curves $C \subset Y \setminus D$. Then $\phi(x) = 1$ for all $x \in \langle D_1, \ldots, D_n\rangle^{\perp}_{\mathbb{Z}}$.*

Proof. Using the notation $\theta : \text{Pic}^0(D) \xrightarrow{\sim} \mathbb{C}^*$ from Definition 3.20, we recall that $\phi([C]) = \theta(\mathcal{O}_Y(C)|_D)$. Because $C \bigcap D = \emptyset$, the restriction $\mathcal{O}_Y(C)|_D = \mathcal{O}_D$ is the trivial line bundle on D. Then $\phi([C]) = 1$. From our assumption, it follows that $\phi(x) = 1$ for all $x \in \langle D_1, \ldots, D_n\rangle^{\perp}_{\mathbb{Z}}$. □

The lemma applies in our situation because for the surfaces we describe in cases $n \le 6$, the lattice $\langle D_1, \ldots, D_n\rangle^{\perp}_{\mathbb{Z}}$ is generated by the classes of (-2)-curves C in $Y \setminus D$.

Remark 6.5. We cover every deformation type for each $n \le 6$ of log Calabi-Yau surfaces such that the intersection matrix $(D_i \cdot D_j)$ is negative definite or negative semidefinite. This follows from results of Looijenga for $n \le 5$ ([L81]) and Simonetti for $n = 6$ ([S21]).

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Remark 6.6. If the intersection matrix $(D_i \cdot D_j)$ is not negative definite nor negative semidefinite, then $\mathrm{Nef}(Y)$ is rational polyhedral by the Cone Theorem ([GHK15a], Lemma 6.9).

Here we prove the statement of Theorem 6.1 about $\overline{\mathrm{Curv}}(Y_e)$ being rational polyhedral for $n \leq 6$. To do this, we consider separately the six cases $n = 1, \ldots, 6$ and use Lemma 6.7 below. We note that in each of the cases considered, the number of boundary components remains the same. Here, we refer to Theorem 1.1 in [L81], which gives the description of $\bar{Y}$ and $\bar{D}$ in each of the cases $n \leq 5$. The general blowup picture can be described as follows: the boundary D is a cycle of components D_i, where $i = 1, \ldots, n$, and each boundary component is linked to a 'chain' of a single (-1)-curve followed by arbitrarily many (-2)-curves.

For all cases $n = 1, \ldots, 6$, we denote the exceptional curves by $E_{i,j}$, where $E_{i,1}$ is the unique (-1)-curve in the chain intersecting the boundary component D_i at one point q, and j ranges from 1 to the total number of blowups at q.

Lemma 6.7. *Let Y be a smooth projective complex surface. Let $\mathcal{B}$ be a basis for $N_1(Y)$ consisting of irreducible curves. Suppose the dual basis may be expressed as effective combinations of a set $\mathcal{C}$ of curves. Then $\mathrm{Curv}(Y) = \langle \mathcal{B} \cup \mathcal{C} \rangle_{\mathbb{R}\geq 0}$.*

Proof. Let $C \subset Y$ be a curve and suppose that $C \notin \mathcal{B}$. Then $C \cdot B_i \geq 0$ for all $B_i \in \mathcal{B}$. Since B_i^* is an effective linear combination of elements in $\mathcal{C}$ and $C \cdot B_i \geq 0$, it follows that $C = \sum (C \cdot B_i) B_i^*$ belongs to $\langle\ \mathcal{C}\ \rangle_{\mathbb{R}_{\geq 0}}$. □

Number of boundary components $n = 1$**.**

Let $\bar{Y} = \mathbb{P}^2$ with a rational nodal curve $\bar{D}_1$ and a flex point q. In coordinates, we may take

$$\bar{D}_1 : (X_0 X_2^2 = X_1^2 (X_1 + X_0)) \subseteq \mathbb{P}^2_{(X_0:X_1:X_2)} \text{ and } q = (0:0:1).$$

We denote the tangent line at point q by $\bar{F}$, and we blow up the point q some number p_1 of times. This results in chain of exceptional curves with self-intersections $-1, -2, \ldots, -2, -2, -2$. We label these curves by $E_{1,1}, E_{1,2}, \ldots, E_{1,p_1-2}, E_{1,p_1-1}, E_{1,p_1}$. The curve E_{1,p_1-2} intersects a (-2)-curve F at one point. A basis for $\mathrm{Pic}(Y)$ is

$$\mathcal{B}_1 = \{E_{1,j}, F \mid 1 \leq j \leq p_1\},$$

and its dual basis $\mathcal{B}_1^*$ consists of the following elements:

$$\begin{aligned}
E_{1,p_1}^* &= D_1 \\
E_{1,p_1-1}^* &= D_1 + E_{1,p_1} \\
E_{1,p_1-2}^* &= D_1 + 2E_{1,p_1} + E_{1,p_1-1} \\
E_{1,p_1-3}^* &= D_1 + 3E_{1,p_1} + 2E_{1,p_1-1} + E_{1,p_1-2} \\
&\vdots \\
E_{1,j}^* &= D_1 + (p_1 - j)E_{1,p_1} + (p_1 - j - 1)E_{1,p_1-1} + \cdots + 2E_{1,j+2} + E_{1,j+1} \quad \text{for } 3 \leq j \leq p_1;
\end{aligned}$$

and

$$\begin{aligned}
E_{1,2}^* &= 4E_{1,p_1} + 4E_{1,p_1-1} + \cdots + 4E_{1,4} + 4E_{1,3} + 2E_{1,2} + E_{1,1} + 2F \\
E_{1,1}^* &= 2E_{1,p_1} + 2E_{1,p_1-1} + \cdots + 2E_{1,4} + 2E_{1,3} + E_{1,2} + F \\
F^* &= 3E_{1,p_1} + 3E_{1,p_1-1} + \cdots + 3E_{1,4} + 3E_{1,3} + 2E_{1,2} + E_{1,1} + F.
\end{aligned}$$

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By Lemma 6.7, we can describe the cone of curves as follows:

$$\mathrm{Curv}(Y) = \langle D_1, E_{1,j}, F \mid 1 \le j \le p_1 \rangle_{\mathbb{R}_{\ge 0}}.$$

Remark 6.8. The formulas above can be obtained in the following way. Here, we use the notation

$$Y_m \longrightarrow Y_{m-1} \longrightarrow \cdots \longrightarrow Y_2 \longrightarrow Y_1 \longrightarrow Y_0,$$

and we label the maps by $\pi_i : Y_i \longrightarrow Y_{i-1}$. We let $D = D_m \subset Y_m = Y$ and $\bar{D} = D_0 \subset \bar{Y} = Y_0$, and $D_i \subset Y_i$ for $i = 1, \ldots m-1$. Let $f = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m$ and let $f_i : Y \longrightarrow Y_i$ be the map $f_i := \pi_i \circ \cdots \circ \pi_m$. Since there is only one component of D in this case ($n = 1$ so $D = D_1$), instead of writing $E_{1,j}$, for simplicity we will just keep track of the second subscript and write E_j. Then,

$$E_i^* = f_i^*(D_i) = (\pi_i \circ \cdots \circ \pi_m)^*(D_i),$$

so that $E_i^* \cdot E_j = \delta_{ij}$. In addition, if we let $F^* := \pi^*(H)$, where H is a hyperplane class on $\mathbb{P}^2$ and π is the composition of maps π_i, then $F^* \cdot E_i = 0$ and $F^* \cdot F = 1$. This deriviation results in the same expressions as listed above, and we omit similar arguments which give the expressions of dual basis elements for the remaining n values.

Number of boundary components $n = 2$*.*

Let $\bar{Y}$ be the Hirzebruch surface $\mathbb{F}_2$ with two smooth curves $\bar{D}_1$ and $\bar{D}_2$ in the linear system $|B + 2A|$. Here, B denotes the negative section of the $\mathbb{P}^1$ fibration $\mathbb{F}_2 \longrightarrow \mathbb{P}^1$, and A denotes the fiber. We may assume that the curves $\bar{D}_1$ and $\bar{D}_2$ intersect transversely. We fix two points $q_i \in \bar{D}_i$ where $i = 1, 2$, such that the points lie on a common fiber $\bar{F}_1$, and let $\bar{F}_2$ be the second (-2)-curve meeting $\bar{F}_1$ at a single point. Then blow up at the points q_i some number of times (say we blow up a total of p_i times at points q_i for $i = 1, 2$). The curves F_i are the strict transforms of $\bar{F}_i$ for $i = 1, 2$.

A basis $\mathcal{B}_2$ for $\mathrm{Pic}(Y)$ is given by

$$\mathcal{B}_2 = \{E_{i,j}, F_i \mid i = 1, 2 \text{ and } 1 \le j \le p_i\}.$$

The dual basis $\mathcal{B}_2^*$ consists of the following elements:

$$\mathcal{B}_2^* = \{E_{i,j}^*, F_1^*, F_2^* \mid i = 1, 2 \text{ and } 1 \le j \le p_i\},$$

where for $i = 1, 2$,

$$\begin{aligned}
E_{i,p_i}^* &= D_i \\
E_{i,p_i-1}^* &= D_i + E_{i,p_i} \\
E_{i,p_i-2}^* &= D_i + 2E_{i,p_i} + E_{i,p_i-1} \\
&\vdots \\
E_{i,1}^* &= D_i + (p_i - 1)E_{i,p_i} + (p_i - 2)E_{i,p_i-1} + \cdots + 2E_{i,3} + E_{i,2}
\end{aligned}$$

and

$$F_1^* = D_i + p_i E_{i,p_i} + (p_i - 1)E_{i,p_i-1} + \cdots + 2E_{i,2} + E_{i,1} \quad \text{for } i = 1 \text{ or } 2,$$

and

$$F_2^* = D_i + (p_i + 1)E_{i,p_i} + p_i E_{i,p_i-1} + \cdots + 2E_{i,1} + F_1 \quad \text{for } i = 1 \text{ or } 2.$$

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By Lemma 6.7, we can describe the cone of curves as follows:

$$\mathrm{Curv}(Y) = \langle D_i, E_{i,j}, F \mid i = 1, 2 \text{ and } 1 \le j \le p_i \rangle_{\mathbb{R}_{\ge 0}}.$$

Number of boundary components $n = 3$**.**

Let $\bar{Y} = \mathbb{P}^2$ with $\bar{D} = \bar{D}_1 + \bar{D}_2 + \bar{D}_3$ its toric boundary, which is the union of three lines. Fix three collinear points $q_i \in \bar{D}_i$ where $i = 1, 2, 3$ and blow them up some number of times. Let F be the strict transform of the line $\bar{F}$ passing through the three points of blowup, one on each component $\bar{D}_i$.

A basis $\mathcal{B}_3$ for $\mathrm{Pic}(Y)$ is given by

$$\mathcal{B}_3 = \{E_{i,j}, F \mid 1 \le i \le 3 \text{ and } 1 \le j \le p_i\}.$$

The dual basis $\mathcal{B}_3^*$ consists of the elements, for $i = 1, 2, 3$:

$$\begin{aligned} E^*_{i,p_i} &= D_i \\ E^*_{i,p_i-1} &= D_i + E_{i,p_i} \\ E^*_{i,p_i-2} &= D_i + 2E_{i,p_i} + E_{i,p_i-1} \\ &\vdots \\ E^*_{i,1} &= D_i + (p_i - 1)E_{i,p_i} + (p_i - 2)E_{i,p_i-1} + \cdots + 2E_{i,3} + E_{i,2} \end{aligned}$$

and

$$F^* = D_i + p_i E_{i,p_i} + (p_i - 1)E_{i,p_i-1} + \cdots + 2E_{i,2} + E_{i,1}.$$

By Lemma 6.7, we can describe the cone of curves as follows:

$$\mathrm{Curv}(Y) = \langle D_i, E_{i,j}, F \mid 1 \le i \le 3 \text{ and } 1 \le j \le p_i \rangle_{\mathbb{R}_{\ge 0}}.$$

Number of boundary components $n = 4$**.**

Let $\bar{Y} = \mathbb{P}^1 \times \mathbb{P}^1$ with its toric boundary, which is the union of two fibers of each of the two projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Fix four points $q_i \in \bar{D}_i$ where $i = 1, \ldots, 4$ such that q_1 and q_3 lie on a fiber $\bar{F}_1$ of the first projection and q_2 and q_4 lie on a fiber $\bar{F}_2$ of the second projection. Then blow them up some number of times.

A basis for $\mathrm{Pic}(Y)$ is

$$\mathcal{B}_4 = \{E_{i,j} F_1, F_2 \mid 1 \le i \le 4 \text{ and } 1 \le j \le p_i\}.$$

The dual basis $\mathcal{B}_4^*$ consists of the following elements:

$$\begin{aligned} E^*_{i,p_i} &= D_i \\ E^*_{i,p_i-1} &= D_i + E_{i,p_i} \\ E^*_{i,p_i-2} &= D_i + 2E_{i,p_i} + E_{i,p_i-1} \\ &\vdots \\ E^*_{i,1} &= D_i + (p_i - 1)E_{i,p_i} + (p_i - 2)E_{i,p_i-1} + \cdots + 2E_{i,3} + E_{i,2}, \end{aligned}$$

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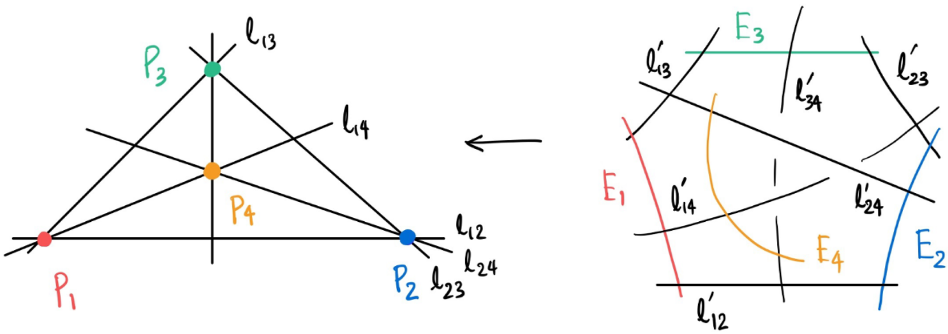

Figure 6.1. *Blowing up once at each point $p_1, \ldots, p_4$ results in the diagram above.*

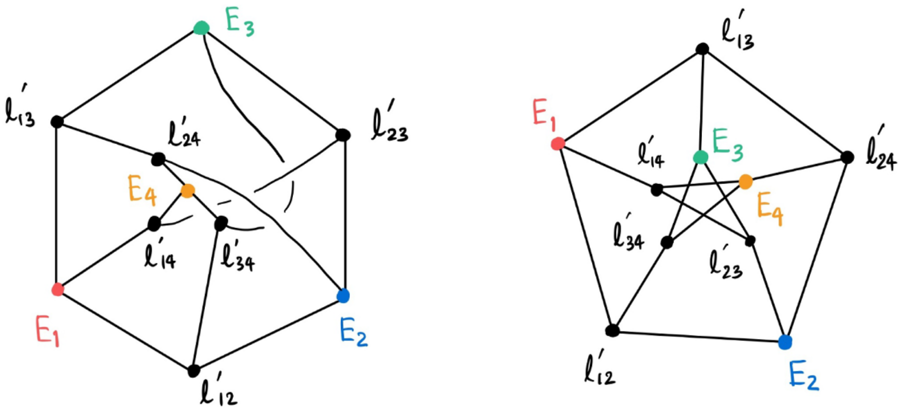

Figure 6.2. *The dual graph of the blowup shown in Figure 6.1 is drawn on the left, and it is equivalent to the Petersen graph shown on the right.*

and for each of F_j where $j = 1, 2$, there are two (linearly equivalent) possibilities:

$$F_j^* = D_i + p_i E_{i,p_i} + (p_i - 1)E_{i,p_i-1} + \cdots + 2E_{i,2} + E_{i,1}$$

with $i = 2$ or 4 for $j = 1$ and $i = 1$ or 3 for $j = 2$. By Lemma 6.7, we can describe the cone of curves as follows:

$$\mathrm{Curv}(Y) = \langle D_i, E_{i,j}, F_1, F_2 \mid 1 \le i \le 4 \text{ and } 1 \le j \le p_i \rangle_{\mathbb{R}_{\ge 0}}.$$

***Number of boundary components** $n = 5$.*

Let $\bar{Y}$ be the blowup of four points in general position in $\mathbb{P}^2$ and let $\bar{D}$ be a cycle of five (-1)-curves. The surface $\bar{Y}$ contains ten (-1)-curves:

1. Four are exceptional curves E_i from blowing up the points p_i, for $i = 1, 2, 3, 4$.
2. Six (obtained by $6 = \binom{4}{2}$) are strict transforms l'_{ij} of lines l_{ij} defined by points p_i and p_j.

The process of blowing up points p_i for $i = 1, \ldots, 4$ on $\mathbb{P}^2$ to obtain a surface with ten curves is shown in Figure 6.1. Taking the dual of this figure (see Figure 6.2), we choose a pentagon inside and rearrange vertices so that this pentagon encloses all other vertices. Then the interior vertices can be rearranged

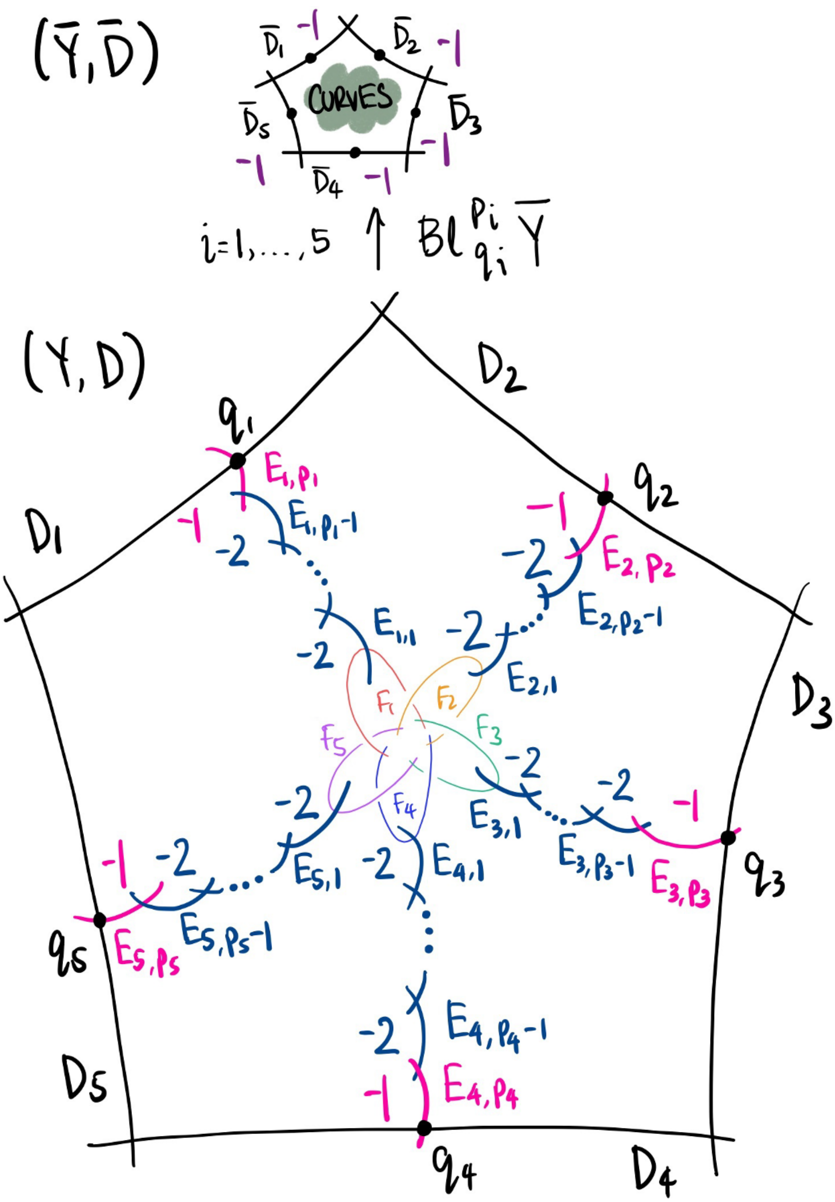

Figure 6.3. *Blowing up once at each point $q_1, \ldots, q_5$ results in the diagram above.*

to form a star, resulting in the Petersen graph. The dual of the interior five-pointed star is a pentagon, and the dual of the outside pentagonal boundary is again a pentagon. Together, these two parts form the configuration of (-1)-curves on the surface $\bar{Y}$. The $\bar{F}_i$'s are (-1)-curves not contained in the boundary in $\bar{Y}$; the $\bar{F}_i$'s correspond to a pentagonal star. Each $\bar{F}_i$ intersects the boundary component $\bar{D}_i$, and we denote their strict transforms by F_i.

The remaining (-1)-curves on $\bar{Y}$ intersect $\bar{D}$ transversely in five points q_i where $i = 1, \ldots, 5$. Blow up these points some number of times to obtain Figure 6.3. A basis $\mathcal{B}_5$ for $\mathrm{Pic}(Y)$ is the collection

$$\mathcal{B}_5 = \{E_{i,j}, F_i \mid 1 \leq i \leq 5 \text{ and } 1 \leq j \leq p_i\}.$$

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The dual elements $E^*_{i,j}$ and F_i, where $1 \leq i \leq 5$ and $1 \leq j \leq p_i$, are defined as follows:

$$\begin{aligned}
E^*_{i,p_i} &= D_i \\
E^*_{i,p_i-1} &= D_i + E_{i,p_i} \\
E^*_{i,p_i-2} &= D_i + 2E_{i,p_i} + E_{i,p_i-1} \\
&\vdots \\
E^*_{i,1} &= D_i + (p_i-1)E_{i,p_i} + (p_i-2)E_{i,p_i-1} + \cdots + 2E_{i,3} + E_{i,2} \\
F^*_i &= D_i + p_iE_{i,p_i} + (p_i-1)E_{i,p_i-1} + \cdots + 2E_{i,2} + E_{i,1}.
\end{aligned}$$

By Lemma 6.7, we can describe the cone of curves as follows:

$$\mathrm{Curv}(Y) = \langle D_i, E_{i,j}, F_i \mid 1 \leq i \leq 5 \text{ and } 1 \leq j \leq p_i \rangle_{\mathbb{R}_{\geq 0}}.$$

Number of boundary components $n = 6$**.**

In this case, by [S21], (Y, D) is obtained as a blowup of the toric surface $\bar{Y}$ with toric boundary $\bar{D} = \bar{D}_1 + \cdots + \bar{D}_6$, a hexagon of (-1)-curves (shown in Figure 6.4). Take q_i where $i = 1 \ldots, 6$ to be the points $(-1) \in \mathbb{C}^* \subset \mathbb{P}^1 = \bar{D}_i$ for some choice of toric coordinates on $\bar{Y}$, and Y the blowup of $\bar{Y}$ some number of times at each q_i. Define an index set as follows:

$$K = \{\{1,4\}, \{2,5\}, \{3,6\}, \{1,3,5\}, \{2,4,6\}\}. \tag{6.9}$$

Let F_k be the strict transform of $\bar{F}_k$ on Y. We consider the following five curves $\bar{F}_k$ on $\bar{Y}$ $(k \in K)$, such that $\bar{F}_k \cap \bar{D} = \{q_i \mid i \in k\}$.

1. $\bar{F}_{1,3,5}$ and $\bar{F}_{2,4,6}$ are pullbacks of a line in $\mathbb{P}^2$ for two different birational morphisms $\bar{Y} \to \mathbb{P}^2$; and
2. $\bar{F}_{1,4}$ and $\bar{F}_{2,5}$ and $\bar{F}_{3,6}$ are fibers of three different morphisms $\bar{Y} \to \mathbb{P}^1$.

The classes of the curves $\{\bar{F}_k\}$ span $\mathrm{Pic}(\bar{Y})$ with one relation:

$$\bar{F}_{1,4} + \bar{F}_{2,5} + \bar{F}_{3,6} = \bar{F}_{1,3,5} + \bar{F}_{2,4,6}.$$

For $i = 1, \ldots, 6$ and $j = 0, \ldots, p_i$, define a divisor

$$A_{i,j} = D_i + (p_i - j)E_{i,p_i} + (p_i - j - 1)E_{i,p_i-1} + \ldots, +E_{i,j+1}.$$

Then for $j > 0$, we have

$$A_{i,j} \cdot E_{s,t} = \begin{cases} 1 & \text{if } i = s \text{ and } j = t; \\ 0 & \text{otherwise.} \end{cases} \tag{6.10}$$

The set $S = \{E_{i,j} \mid i = 1, \ldots, 6\} \cup \{F_k \mid k \in K\}$ spans $\mathrm{Pic}(Y)$ because the set $\{\bar{F}_k \mid k \in K\}$ spans $\mathrm{Pic}(\bar{Y})$.

Let $C \subset Y$ be an irreducible curve. Suppose that $C \neq D_i, E_{i,j}$ for all i, j. Then $C \cdot A_{i,j} \geq 0$ for all i, j. We can write

$$C = \sum a_{i,j}E_{i,j} + \sum b_kF_k \in \mathrm{Pic}(Y).$$

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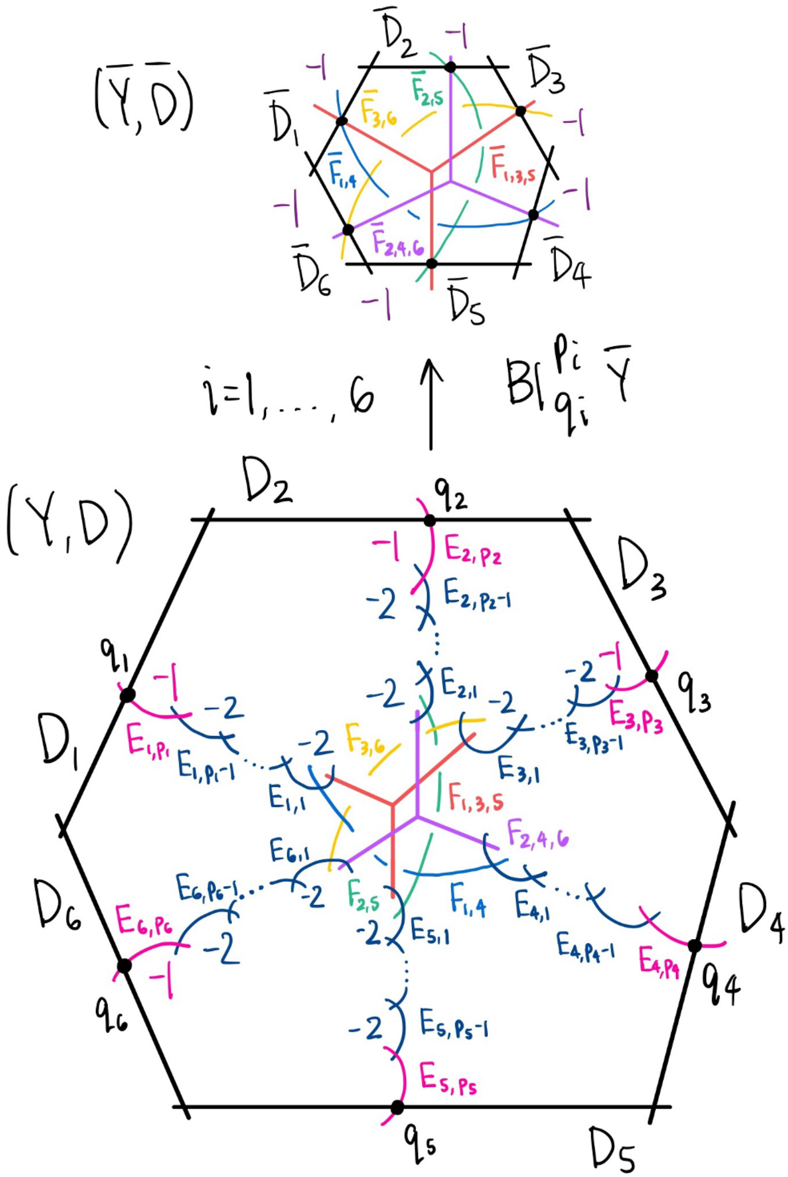

Figure 6.4. *This diagram shows the surface* $(\bar{Y}, \bar{D})$ *blown up at points* q_i*, each a total of* p_i *times, for when* $n = 6$*.*

Computing the intersection numbers $A_{i,j} \cdot E_{s,t}$ and $A_{i,j} \cdot F_k$ results in the following inequalities:

$$
\begin{aligned}
&a_{i,j} \geq 0 \text{ for all } i, j;\\
&b_{1,4} + b_{1,3,5} \geq 0;\\
&b_{1,4} + b_{2,4,6} \geq 0;\\
&b_{2,5} + b_{1,3,5} \geq 0;\\
&b_{2,5} + b_{2,4,6} \geq 0;\\
&b_{3,6} + b_{1,3,5} \geq 0; \text{ and}\\
&b_{3,6} + b_{2,4,6} \geq 0;
\end{aligned}
$$

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The last six inequalities define the cone

$$\sigma := \langle [F_k] \mid k \in K \rangle_{\mathbb{R}_{\geq 0}} \subset V,$$

where $V := \langle [F_k] \mid k \in K \rangle_{\mathbb{R}}$. Using the spanning set

$$\{[F_k]\} \text{ where } k \in K = \{\{1,4\},\{2,5\},\{3,6\},\{1,3,5\},\{2,4,6\}\}$$

of V, we can identify σ with the cone

$$\langle \bar{e}_1, \ldots, \bar{e}_5 \rangle_{\mathbb{R}_{\geq 0}} \subset \mathbb{R}^5 / \langle (1,1,1,-1,-1) \rangle_{\mathbb{R}}.$$

Then, we may assume that $b_k \geq 0$ for all $k \in K$, so that C lies in the cone generated by the $E_{i,j}$ and the F_k. Therefore, $\overline{\mathrm{Curv}}(Y) = \langle D_i, E_{i,j}, F_k \mid i = 1, \ldots, 6 \text{ and } j = 1, \ldots, p_i \text{ and } k \in K \rangle_{\mathbb{R}_{\geq 0}}$.

To finish the proof of Theorem 6.1, we first explain why $\mathrm{Aut}(Y, D)$ is trivial for $n = 6$. We note that the cases where $n \leq 5$ is shown by Looijenga in [L81]. Consider the case where $n = 6$. We denote by $(\bar{Y}, \bar{D})$ the toric pair (a log Calabi-Yau surface where $\bar{Y}$ is a toric surface and $\bar{D}$ its toric boundary) and (Y, D) is the blowup of $(\bar{Y}, \bar{D})$. Here, D is either negative definite or negative semidefinite, and D does not contain any (-1)-curves. We note that $\mathrm{Aut}(\bar{Y}, \bar{D}) \cong (\mathbb{C}^*)^2$. Moreover, $\mathrm{Aut}(Y, D) \subset \mathrm{Aut}(\bar{Y}, \bar{D})$: this is because there exists a unique chain consisting of a single (-1)-curve (which intersects a boundary component), and the (-1)-curve intersects a chain of (-2)-curves (which are disjoint from the boundary). In fact, the list of (-2)-curves in $Y \setminus D$ and (-1)-curve in Y (not contained in D but intersecting D at a single point) is complete. This follows from the description of $\overline{\mathrm{Curv}}(Y)$ using Lemma 6.7. To be more precise, if S is a surface and $C \subset S$ a curve with $C^2 < 0$, then C is an extremal ray of $\overline{\mathrm{Curv}}(Y)$. Thus, any (-1) or (-2)-curve is an extremal ray. Above, we have the list of generators of $\overline{\mathrm{Curv}}(Y)$ for $n = 6$, so any (-1) or (-2)-curve must be one of those generators. Any automorphism of (Y, D) will permute this unique chain of (-1) and (-2)-curves, and in particular, it must fix the unique (-1)-curve in this chain. The image must fix the points which are blown up because it fixes the exceptional divisors which are mapped to it. By a short affine coordinates computation, it follows that $\mathrm{Aut}(Y, D)$ is trivial.

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Data availability statement. Replication data and code can be found in Harvard Dataverse: https://doi.org/link.

Ethical standards. The research meets all ethical guidelines, including adherence to the legal requirements of the study country.

Author contributions. A.A. and A.B.C. designed the study, abstracted the data wrote the first draft, and approved the final version of the manuscript. A.R.E.J., M.R.L., K.L.S. and A.D.P. revised the manuscript and approved the final version.

Competing interest. The authors have no competing interest to declare.

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