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SHARP FUNCTION ESTIMATES FOR FRACTIONAL INTEGRALS AND RELATED OPERATORS

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Abstract

This paper considers analogs of results on integral operators studied by Hörmander. Using the sharp function introduced by Fefferman and Stein, we prove weighted norm inequalities on kernel operators which map an L^p space into an L^q space, with q not equal to p. The techniques recover known results about fractional integral operators and apply to multiplier operators which satisfy a generalization of the Hörmander multiplier condition.

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1. Introduction

Let K be a locally integrable function on $\mathbb{R}^n$ and consider the convolution operator $Tf = K * f$. Hörmander [5] proved that if T defines a bounded operator on some L^p space and K satisfies the condition

$$(1.1) \qquad \int_{\{x:|x|>2|y|\}} |K(x-y) - K(x)|\,dx \le C, \quad \text{for } y \neq 0,$$

then T defines a bounded operator on all L^p spaces, $1 < p < \infty$.

More recently, stronger versions of (1.1) have been studied using the sharp

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function introduced by Fefferman and Stein [2]. In particular, the estimate

$$(1.2)\qquad \left(\int_{\{x:R<|x|<2R\}} |K(x-y)-K(x)|^r dx\right)^{1/r} \le S\left(\frac{|y|}{R}\right) R^{-n/r'}$$

with $\sum_{j=1}^{\infty} S(2^{-j}) < +\infty$, can be used to prove $(Tf)^{\#}(x) \le CM_{r'}f(x)$, where $M_{r'}$ is a variant of the Hardy–Littlewood maximal function. This technique was first introduced by Cordoba and Fefferman [1] and Kurtz and Wheeden [6]. See also the article of Rubio de Francia, Ruiz and Torrea [10], where all of these ideas are carried over to the vector-valued setting.

Hörmander also studied operators which are bounded from L^p to L^q with $p < q$. Let $a \ge 1$. We say K is in K^a if

$$(1.3)\qquad \int_{\{x:|x|>2|y|\}} |K(x-y)-K(x)|^a dx \le C, \quad \text{for } y \ne 0.$$

If $K \in K^a$ and T maps L^p to L^q for some p and $1/q = 1/p - 1/a'$, he proved that T maps L^p to L^q for all pairs p and q with $1 < p \le q < \infty$ and $1/q = 1/p - 1/a'$.

By Hölder's inequality, (1.2) for a particular value r_0 implies (1.2) for every r, $1 \le r \le r_0$, and hence (1.1) holds. Clearly, the sharpest estimates are obtained by the largest values of r satisfying (1.2) while the relationship between p and q depends on the values of a satisfying (1.3).

In this paper, we consider kernels which satisfy analogs of (1.2) which relate to (1.3) for $a > 1$. Let $0 < \alpha < n$. We use estimates of the form

$$(1.4)\qquad \left(\int_{\{x:R<|x|<2R\}} |K(x-y)-K(x)|^r dx\right)^{1/r} \le S\left(\frac{|y|}{R}\right) R^{\alpha-n/r'}$$

to obtain inequalities relating $(Tf)^{\#}$ to variants of the fractional maximal function of order α. Such an estimate is useful only when $\alpha - n/r' \le 0$ or $r \ge (n/\alpha)'$. From (1.4), we derive weighted results from L^p to L^q with $1/q = 1/p - \alpha/n$. One easily sees that the kernel of the Riesz potential, I_α, satisfies (1.4) for all $r \ge (n/\alpha)'$. We thus obtain the results in [7]. These ideas can also be applied to multipliers which satisfy

$$\int_{\{x:R<|x|<2R\}} |D^\beta m(x)|^s\, dx \le BR^{n-s|\beta|-s\alpha}$$

where $0 < \alpha < n$. This is a generalization of the Hörmander multiplier condition, which is obtained by setting $s = 2$ and $\alpha = 0$.

In Section 2, we collect known results to be used in the sequel. Results about kernels satisfying (1.4) are found in Section 3. The last section contains applications. We only consider the case $\alpha > 0$ since when $\alpha = 0$ these are known. In general, the proofs are adaptations of ones found in [6] and [7].

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2. Preliminary results

We assume f is a Lebesgue measurable function which is locally integrable. Given a set E, $|E|$ denotes its Lebesgue measure. We use Q to denote a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

DEFINITION 2.1. Let $0 < \alpha < n$ and $1 \le r < n/\alpha$. Define $M_{\alpha,r}$ by

$$M_{\alpha,r}f(x) = \sup_Q \left(|Q|^{(\alpha r/n)-1} \int_Q |f(y)|^r dy \right)^{1/r},$$

where the supremum is taken over all cubes Q which contain x.

Note that $M_{\alpha,1}$ is the standard fractional maximal function.

Let w be a nonnegative, measurable function on $\mathbb{R}^n$ and, for $1 \le p < \infty$, set $\|f\|_{p,w} = (\int_{\mathbb{R}^n} |f(x)|^p w(x) dx)^{1/p}$. Given a measurable set E, let $w(E) = \int_E w(x)\, dx$.

DEFINITION 2.2. We say $w \in A_p$ if there is a constant C so that for every cube Q

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1} \le C, \qquad 1 < p < \infty,$$

or

$$\frac{1}{|Q|}\int_Q w(x)dx \le c \operatorname*{ess\,inf}_{x\in Q} w(x), \qquad p = 1.$$

We say $w \in A_\infty$ if given $\varepsilon > 0$ there is a $\delta > 0$ so that for any cube Q and measurable set $E \subset Q$, $|E| < \delta|Q|$ implies $w(E) < \varepsilon w(Q)$.

We assume the reader is familiar with this condition. See, for example, [3].

Let $1 \le r \le p < \infty$ and $1 \le q < \infty$. Write $w \in A(p, q; r)$ if $w \in A_{1+q(p-r)/pr}$. In particular, $A(p, p; 1) = A_p$ and $A(r, q; r) = A_1$. Since $M_{\alpha,r}f = \{M_{\alpha r,1}|f|^r\}^{1/r}$, by Theorem 3 of [7] with $w = V^q$, we have

THEOREM 2.3. *Let $0 < \alpha < n$, $1 \le r < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w \in A(p, q; r)$. There is a constant C, independent of f, so that*

$$\|M_{\alpha,r}f\|_{q,w} \le C\|f\|_{p,w^{p/q}}.$$

Given a locally integrable function f, set

$$f_Q = \frac{1}{|Q|}\int_Q f(y)dy.$$

DEFINITION 2.4. Define the sharp function of f by

$$f^{\#}(x) = \sup_{Q} \frac{1}{|Q|} \int_{Q} |f(y) - f_{Q}| dy,$$

where the supremum is taken over all cubes Q containing x.

The following result can be found in [1].

LEMMA 2.5. *Let $w \in A_{\infty}$ and $0 < p < \infty$. There is a constant C, independent of f, so that* $\|f\|_{p,w} \leq C\|f^{\#}\|_{p,w}$.

3. Sharp function estimates

Let K be a locally integrable function on $\mathbb{R}^{n}$. We consider kernels which satisfy the following definition.

DEFINITION 3.1. Let $0 < \alpha < n$ and $n/(n-\alpha) \leq r < \infty$. We say $K \in K(r, \alpha)$ if there is a non-decreasing function S on $(0, 1)$ such that

(i)

$$\left(\int_{\{x: R<|x|<2R\}} |K(x-y) - K(x)|^{r} dx \right)^{1/r} \leq S\left(\frac{|y|}{R}\right) R^{\alpha - n/r'}, \qquad |y| < \frac{R}{2},$$

(ii)

$$\|Tf\|_{q} \leq C\|f\|_{r'}, \quad \frac{1}{q} = \frac{1}{r'}, -\frac{\alpha}{n},$$

(iii)

$$\sum_{j=1}^{\infty} S(2^{-j}) < +\infty.$$

It is not necessary to use the same r in (i) and (ii), so we could allow $r = \infty$. We will need neither generalization here. Moreover, from Hörmander's result we get

COROLLARY 3.2. *Suppose $0 < \alpha < n$, $n/(n-\alpha) < r_{0} < \infty$, and $K \in K(r_{0}, \alpha)$. Then $K \in K(r, \alpha)$ for $n/(n-\alpha) < r \leq r_{0}$.*

One merely observes that by Hölder's inequality, $K \in K(r_{0}, \alpha)$ implies K satisfies (3.1)(i) for $n/(n-\alpha) \leq r \leq r_{0}$. Therefore, $K \in K^{n/(n-\alpha)}$. Since (3.1)(ii) is satisfied for r_{0}', (3.1)(ii) is true for all pairs r' and q with $1 < r' \leq q < \infty$ and $1/q = 1/r' - \alpha/n$.

Our basic estimate is contained in the following theorem. The proof, which can be found in [6], is included for completeness.

THEOREM 3.3. *Let $0 < \alpha < n$ and $n/(n-\alpha) < r < \infty$. There is a constant A, depending only on n, r, and α, so that for $K \in K(r, \alpha)$ and $f \in L^{r'}$,*

$$(Tf)^{\#}(x) \le A \left\{ C + \sum_{j=1}^{\infty} S(2^{-j}) \right\} M_{\alpha, r'} f(x),$$

where C is the constant in (3.1)(ii).

PROOF. Fix $x \in \mathbb{R}^n$ and let Q be a cube centered at x and having diameter δ. Set $g(y) = f(y)\chi\{y: |x-y| \le 2\delta\}$ and $h(y) = f(y) - g(y)$. By (3.1)(ii),

$$\begin{aligned}\frac{1}{|Q|}\int_Q |Tg(y)|dy &\le \left(\frac{1}{|Q|}\int_Q |Tg(y)|^q dy\right)^{1/q} \le C|Q|^{-1/q}\|g\|_{r'} \\ &= C\left(|Q|^{(\alpha r'/n)-1}\int_{\{y:|x-y|\le 2\delta\}} |f(y)|^{r'} dy\right)^{1/r'} \\ &\le C_1(n, r, \alpha)\cdot CM_{\alpha, r'} f(x),\end{aligned}$$

since g is supported in a ball whose size is comparable to $|Q|$.

Since $h \in L^{r'}$, $|(Th)(y)| < +\infty$ a.e. Without loss of generality, we may assume $|Th(x)| < +\infty$. For $y \in Q$

$$Th(y) = Th(x) + \int \{K(y-z) - K(x-z)\}h(z)dz = C_Q + \varepsilon,$$

where C_Q is independent of y. Let $\chi_j(y) = \chi\{y: 2^j\delta < |x-y| \le 2^{j+1}\delta\}$. Then

$$\begin{aligned}|\varepsilon| &\le \sum_{j=1}^{\infty}\int \chi_j(z)|K(y-z) - K(x-z)||f(z)|dz \\ &\le \sum_{j=1}^{\infty}\left(\int \chi_j(z)|\,|K(y-z) - K(x-z)|^r dz\right)^{1/r} \\ &\qquad \times \left(\int_{\{z:|x-z|\le 2^{j+1}\delta\}} |f(z)|^{r'} dz\right)^{1/r'}.\end{aligned}$$

Since $|x-y| < \delta$, by (3.1)(i)

$$\begin{aligned}|\varepsilon| &\le \sum_{j=1}^{\infty} S\left(\frac{|y-x|}{2^j\delta}\right)(2^j\delta)^{\alpha-(n/r')}\left(\int_{\{z:|x-z|\le 2^{j+1}\delta\}} |f(z)|^{r'} dz\right)^{1/r'} \\ &\le C_2(n, r, \alpha)\sum_{j=1}^{\infty} S(2^{-j})M_{\alpha, r'} f(x).\end{aligned}$$

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Therefore, if $A = \max\{C_1, C_2\}$,

$$\frac{1}{|Q|}\int_Q |Tf(y) - C_Q|dy \le \frac{1}{|Q|}\int_Q |Tg(y)|dy + \frac{1}{|Q|}\int_Q |Th(y) - C_Q|dy$$
$$\le A\left\{C + \sum_{j=1}^{\infty} S(2^{-j})\right\} M_{\alpha, r'} f(x).$$

From this, one has

$$(Tf)^{\#}(x) \le 2A\left\{C + \sum_{j=1}^{\infty} S(2^{-j})\right\} M_{\alpha, r'} f(x).$$

Recall that $w \in A_p$ implies $w \in A_\infty$. Therefore, combining Lemma 2.5 and Theorems 2.3 and 3.3 yields

LEMMA 3.4. *Let $0 < \alpha < n$, $n/(n-\alpha) < r < \infty$ and $K \in K(r, \alpha)$. If $r' < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w \in A(p, q; r')$, then there is a constant C, independent of f, such that*

$$\|Tf\|_{q,w} \le C\|f\|_{p, w^{p/q}}.$$

Often, $K \in K(r, \alpha)$ for all r satisfying $n/(n-\alpha) < r < R' \le \infty$. In that case, we have

THEOREM 3.5. *Let $0 < \alpha < n$, $1 \le R < n/\alpha$, and suppose $K \in K(r, \alpha)$ for all r, $n/(n-\alpha) < r < R'$. If $R < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w \in A(p, q; R)$, then there is a constant C, independent of f, such that*

$$\|Tf\|_{q,w} \le C\|f\|_{p, w^{p/q}}.$$

PROOF. Suppose p, q and w satisfy the hypothesis. By Lemma 3.4, we need only show that there is an r, $n/(n-\alpha) < r < R'$, such that $r' < p$ and $w \in A(p, q; r')$. We can choose such an r by observing that if $1 \le r < p < \infty$ and $w \in A(p, q; r)$, then $w \in A(p, q; s)$ for some $s > r$.

By duality, we get

COROLLARY 3.6. *Suppose K satisfies the hypothesis of Theorem 3.5. If $1 < p < (\frac{Rn}{n-\alpha R})$, $1/q = 1/p - \alpha/n$, and $w^{-p'/q} \in A(q', p'; R)$, then*

$$\|Tf\|_{q,w} \le C\|f\|_{p, w^{p/q}}.$$

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Note that For $R = 1$, $w^{-p'/q} \in A(q', p'; 1)$ if and only if $w \in A(p, q; 1)$ and the corollary gives no new information.

4. Applications

Let $0 < \alpha < n$ and suppose K is a measurable function which satisfies

$$(4.1)\qquad \begin{aligned} &\text{(i)} \quad |\{x \in \mathbb{R}^n : |K(x)| > \lambda\}| \le C\lambda^{-n/(n-\alpha)} \\ &\text{(ii)} \quad |K(x-y) - K(x)| \le \frac{C|y|}{|x|^{n+1-\alpha}}, \qquad |x| > 2|y|. \end{aligned}$$

By (4.1)(i), $Tf = K * f$ maps L^p into L^q, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ [8, page 121]. By (4.1)(ii), K satisfies (3.1)(i) with $S(t) = t$ for any $r > n/(n-\alpha)$. Thus, K satisfies Theorem 3.5 with $R = 1$ and we have

THEOREM 4.2. *Suppose K satisfies* (4.1) *with* $0 < \alpha < n$. *Let* $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, *and* $w \in A(p, q; 1)$. *Then, there is a constant* C, *independent of* f, *such that*

$$\|Tf\|_{q,w} \le C\|f\|_{p,w^{p/q}}.$$

Since $K(x) = |x|^{\alpha-n}$ clearly satisfies (4.1), Theorem 4.2 includes the result of Muckenhoupt and Wheeden [7] for the fractional integral operators $I_\alpha f(x) = \frac{1}{\delta(\alpha)} \int_{\mathbb{R}^n} f(y)|x-y|^{\alpha-n} dy$, $\delta(\alpha) = \Pi^{n/2} 2^\alpha \Gamma(n/2)/\Gamma(n/2 - \alpha/2)$. (See also [4, 11].)

Let $\widehat{g}$ be the Fourier transform of the function g. Given a function $m(x)$ on $\mathbb{R}^n$, define the multiplier operator $T = T_m$ by $(Tf)^\wedge(x) = m(x)\widehat{f}(x)$. Let $\beta = (\beta_1, \dots, \beta_n)$ be a multi-index of non-negative integers and $|\beta| = \beta_1 + \cdots + \beta_n$. Consider multipliers, m, which satisfy the following definition.

DEFINITION 4.3. Let $1 \le s < \infty$, $\alpha \in \mathbb{R}$, and $l \in \mathbb{N}$. We say $m \in M(s, l, \alpha)$ if there is a constant B such that $|m(x)| \le B|x|^{-\alpha}$ and

$$\left(R^{s|\beta| + s\alpha - n} \int_{\{x : R < |x| < 2R\}} |D^\beta m(x)|^s dx\right)^{1/s} \le B, \quad \text{for } R > 0,\ |\beta| \le l.$$

The condition $M(s, l, 0)$ was studied in [6].

Suppose $1 < s \le 2$, $n/s < l \le n$, $0 < \alpha < n$, and $m \in M(s, l, \alpha)$. Let $m_\alpha(x) = |x|^\alpha m(x)$ and T_α be the operator with multiplier m_α. Then $m_\alpha \in M(s, l, 0)$ so that T_α is a bounded operator on L^p, $1 < p < \infty$. Further, $|x|^{-\alpha}$ is the multiplier for I_α. Since $m(x) = |x|^{-\alpha} m_\alpha(x)$, $T_m = T_\alpha I_\alpha$ which implies T_m maps L^p to L^q, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Let K be the inverse Fourier transform of m. To see if $K \in K(r, \alpha)$, we need only check conditions (3.1)(i) and (3.1)(ii).

The proof of [6, Lemma 1] can be applied to $m \in M(s, l, \alpha)$, so that by Theorem 3.3 and a limiting argument we get

THEOREM 4.4. *Let* $1 < s \leq 2$, $l \in \mathbb{N}$, $n/s < l \leq n$, $0 < \alpha < n$, *and* $m \in M(s, l, \alpha)$. *Set* $R = \max(1, n/(l+\alpha))$. *If* $R < p < n/\alpha$, $1/q = 1/p - \alpha/n$, *and* $w \in A(p, q; R)$, *then there is a constant* C, *independent of* f, *such that*

$$\|T_m f\|_{q,w} \leq C\|f\|_{p, w^{p/q}}.$$

When $R > 1$, we can extend the theorem using Corollary 3.6. Notice that for $R = n/(l+\alpha) > 1$, $(Rn/(n-\alpha R))' = (n/l)' = n/(n-l)$.

COROLLARY 4.6. *Let* $1 < s \leq 2$, $l \in \mathbb{N}$, $n/s < l \leq n$, *and* $m \in M(s, l, \alpha)$. *If* $l + \alpha < n$, $1 < p < n/(n-l)$, $1/q = 1/p - \alpha/n$, *and* $w^{-p/q} \in A(q', p'; n/(l+\alpha))$, *then*

$$\|Tf\|_{q,w} \leq C\|f\|_{p, w^{p/q}}.$$

We note that these two results are equivalent to the ones obtained by writing $T_m = T_{m_\alpha} I_\alpha = I_\alpha T_{m_\alpha}$ and using Theorem 4.2 and results in [6].

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