A STRUCTURE THEOREM FOR TOPOLOGICAL LATTICES

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In the study of connected partially ordered spaces a problem of fundamental interest is to determine sufficient conditions to ensure the existence of chains (i.e., simply ordered subsets) which are connected. Recently [5] R. J. Koch proved that, if X is a compact Hausdorff space with continuous partial order (i.e., the partial order has a closed graph), if $L(x) = \{y : y \leqq x\}$ is connected for each $x \in X$, and if X has a zero (i.e., an element 0 such that $0 \leqq x$ for all $x \in X$), then each element of X lies in a connected chain containing zero. It is easy to find simple examples which show that this result is false if X is assumed only to be locally compact. However, if it is assumed that the partial order is that of a topological lattice then the existence of such chains can be shown by elementary methods. This solves a problem which was proposed in [3].

Recall that a *topological semilattice* can be defined to be a partially ordered Hausdorff space $(S, \leqq)$ such that the operation $x \wedge y =$ g.l.b.(x, y) is defined and continuous on $S \times S$. If, in addition, the operation $x \vee y =$ l.u.b.(x, y) is defined and continuous, then $(S, \leqq)$ is a *topological lattice*. It is known [**1**, **4**] that the partial order is continuous in a topological semilattice. Moreover, if S is connected then so is $x \wedge S = L(x)$ for each $x \in S$.

Let a and b be elements of a partially ordered space with $a \leqq b$. We say that *a is chained to b* provided that the space contains a connected chain C such that $a = \inf C$ and $b = \sup C$. In addition, C is said to be a *chain from a to b*. It follows from [**6**] that if the space is locally compact then such a chain is compact.

Finally, we recall that a subset C of a partially ordered set S is *convex* if, whenever $x < y$ and $y < z$ with x and z elements of C and $y \in S$, it follows that $y \in C$. A partially ordered space is *locally convex* provided that the topology possesses a base consisting of convex sets. A subset K of a partially ordered set is *order-dense* if, whenever a and b are elements of K and $a < b$, there exists an element c of K such that $a < c$ and $c < b$.

LEMMA. *Let S be a connected locally compact semilattice, let U be an open subset of S, and let $x \in U$. If x has arbitrarily small closed order-dense neighbourhoods, then there exists an open set V, with $x \in V \subset U$, such that if y and z are elements of V then $y \wedge z$ is chained to z.*

Proof. Let W be an open set such that $x \in W \subset U$ and $\overline{W}$ is order-dense and compact. Since $\wedge$ is continuous there exists an open set V such that

$$x \in V \subset V \wedge V \subset W.$$

If y and z are elements of V then $y \wedge z \in W$. Let C be a chain in $\overline{W}$ which is maximal with respect to containing $y \wedge z$ and z. Since $\overline{W}$ is a compact order-dense partially ordered space, each of its maximal chains is compact [**6**, Lemma 4] and order-dense, and hence C is connected [**6**, Theorem 4]. The set $C \cap \{p : y \wedge z \leqq p \leqq z\}$ is clearly a connected chain from $y \wedge z$ to z.

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THEOREM 1. *Let S be a connected locally compact semilattice with zero and suppose that each element of S has arbitrarily small closed order-dense neighbourhoods. Then zero is chained to each* $x \in S$.

Proof. Let P denote the set of all $x \in S$ such that 0 (zero) is chained to x. Obviously $0 \in P$ so that it is sufficient to prove that P is open and closed. Let $x \in P$ and let U and V be chosen as in the lemma. If $y \in V$, then there is a connected chain $C(x \wedge y, y)$ from $x \wedge y$ to y. If C is a connected chain from 0 to x then $(C \wedge y) \cup C(x \wedge y, y)$ is a connected chain from 0 to y. Hence $y \in P$ and P is open.

To see that P is closed let $x \in \bar{P}$ and again choose U and V as in the lemma. Let $y \in V \cap P$, let C be a connected chain from 0 to y and $C(x \wedge y, x)$ a connected chain from $x \wedge y$ to x. Then $(C \wedge x) \cup C(x \wedge y, x)$ is a connected chain from 0 to x so that $x \in P$, i.e., P is closed.

We do not know whether a connected and locally compact locally order-dense semilattice necessarily satisfies the hypothesis of Theorem 1. However, for lattices the situation is simpler.

COROLLARY 1. *If L is a connected and locally compact topological lattice with zero, then zero is chained to each element of L.*

Proof. It suffices, in view of Theorem 1, to show that each point of L has arbitrarily small closed order-dense neighbourhoods. Let $x \in U$, an open set in L. It is known [2] that L is locally convex and hence $x \in V \subset U$, where V is some open convex set. Let W be open and $x \in W \subset \overline{W} \subset V$; if $C(\overline{W})$ denotes the smallest convex set containing $\overline{W}$ then $C(\overline{W}) \subset C(V) = V$. From [2] $C(\overline{W})$ is closed; hence x has arbitrarily small closed convex neighbourhoods. To see that $C(\overline{W})$ is order-dense, let a and b be elements of $C(\overline{W})$ with $a < b$; then $b \wedge (a \vee L)$ is a connected subset of $C(\overline{W})$ and hence $C(\overline{W})$ contains an element c such that $a < c < b$.

COROLLARY 2. *If L is a connected and locally compact topological lattice and if* $a \leqq b$ *in L, then a is chained to b in L.*

Proof. Apply Corollary 1 to the lattice $a \vee L$.

There exists a connected and locally compact topological semilattice with zero such that zero is not chained to each point. In the cartesian plane let

$$A_{-1} = \{(1, y) : 0 \leqq y \leqq 1\},$$

$$A_n = \{(1-2^{-n}, y) : 0 \leqq y \leqq 1\} \quad (n = 0, 1, \ldots),$$

$$B = \{(x, 0) : 0 \leqq x \leqq 1\},$$

$$L' = B \cup \bigcup_{n=-1}^{\infty} \{A_n\}.$$

If $L = L' - \{(1, 0)\}$ is partially ordered by $(a, b) \leqq (c, d)$ if and only if $a \leqq c$ and $b \leqq d$, then it is easy to verify that L is a connected and locally compact topological semilattice with zero. However, there is no connected chain from zero to (1, 1).

If the topological semi-lattice is also locally connected, then it is not known whether zero is chained to each point. However, there exists a locally compact and locally connected partially ordered space X satisfying these conditions: the partial order is continuous and there exists a zero, $L(x) = \{y : y \leqq x\}$ is connected for each $x \in X$, and there is a point $p \in X$ such that zero is not chained to p. To see this, let X be the product of the closed unit interval with itself, with the point $(1, 0)$ deleted. Define $(a, b) \leqq (c, d)$ if and only if the following condition is satisfied: if $c < 1$ then either $a = c$ and $b \leqq d$ or $a \leqq c$ and $b = 0$; if $c = 1$ then either $a = 1$ and $b \leqq d$, or $a \leqq 1$ and $b = 0$, or $a = (n-1)/n$ for some positive integer n and $b \leqq d$. It is a tedious but elementary exercise to verify that this relation is a continuous partial order, that $L(x)$ is connected for each $x \in X$, and that $(0, 0)$ is the zero of X. Moreover, there is no connected chain from $(0, 0)$ to $(1, 1)$.

Let I denote the closed unit interval of real numbers. An arcwise connected space X is said to be simply connected if, given a point $a \in X$ and a continuous function $f\colon I \to X$ with $f(0) = f(1)$, there is a homotopy $g\colon I \times I \to X$ such that $g(t, 0) = f(t)$, $g(t, 1) = a$, and $g(0, r) = g(1, r) = a$ for each $r \in I$.

THEOREM 2. *If S is an arcwise connected topological semilattice with zero, then S is simply connected.*

Proof. Let $f\colon I \to S$ be continuous with $f(0) = f(1) = 0$. Define $g\colon I \times I \to S$ by $g(t, r) = f(t) \wedge f(t - tr)$.

COROLLARY 3. *If S is a connected and locally compact metric topological semilattice with zero and if each element of S has arbitrarily small closed order-dense neighbourhoods, then S is simply connected.*

Proof. By Theorem 1, zero is chained to each point of S. It is well-known that a compact connected metric chain is an arc (see, for example, [7, p. 30]) and hence S is arcwise connected.

COROLLARY 4. *If L is a connected and locally compact metric topological lattice with zero, then L is simply connected.*

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