STABILITY OF INTERPOLATION ON AN INFINITE INTERVAL

R.B. Saxena*

(received May 10, 1966)

1. Introduction. In 1958, Egerváry and Turán [3] proposed and solved the problem of finding a stable interpolation process of minimal degree on a finite interval. Later [4] they investigated the same problem for an infinite interval with a suitable modification of the definition of stability. For the interval $(-\infty, \infty)$ their definition naturally differs from the one for the semi-infinite interval. More recently Balázs [1] considered the same problem for the open interval $(-1, 1)$ and his definition of stability differs from that of Egerváry and Turán [3] by a factor $(1 - x^2)^{\alpha+1}$, $\alpha > -1$. In the present work we consider a definition of stability for the infinite interval which differs from the corresponding one in [4] by the introduction of a factor $x^{\alpha+1}$, $\alpha > -1$. We then obtain the "most economical" interpolation process which is stable in the sense of the definition. In § § 4, 5, we take up the problem of convergence of the interpolatory polynomials considered in § 3.

2. Consider a triangular matrix whose n^{th} row is

$$(2.1) \qquad 0 < x_{1n} < x_{2n} < \dots < x_{nn} < \infty,$$

and let $\{y_{\nu n}\}_1^n$, $\{y^*_{\nu n}\}_1^n$ be arbitrary real numbers.

Let $R_n(x)$ and $R^*_n(x)$ be polynomials given by

* This research has been supported by the National Research Council (N.R.C.) Grant MCA-41 to the Department of Mathematics, University of Alberta, Edmonton, Alberta.

$$(2.2) \qquad R_n(x) = \sum_{\nu=1}^{n} y_{\nu n} u_\nu(x), \quad R_n^*(x) = \sum_{\nu=1}^{n} y_{\nu n}^* u_\nu(x),$$

$$(2.3) \qquad u_k(x_{jn}) = \delta_{kj}, \text{ (Kroneker delta)}$$

where $u_k(x)$ are the fundamental polynomials. Then we shall say that the process of interpolation defined on (2.1) by (2.2), (2.3) is stable if for some $\alpha > -1$

$$(2.4) \qquad 0 \leq x^{\alpha+1} e^{-x} |R_n(x) - R_n^*(x)|$$

$$\leq \max_{1 \leq \nu \leq n} |y_{\nu n} - y_{\nu n}^*| x_{\nu n}^{\alpha+1} e^{-x_{\nu n}}, \quad 0 < x < \infty .$$

We shall call the sum of the degrees of the polynomials $u_\nu(x)$ the degree of the process R_n, so that the "most economical" process is the process whose degree is smallest. We shall prove the following theorems.

THEOREM 1. The "most economical" interpolation process $R_n(x)$ which is stable in the sense of (2.4) is obtained if and only if the abscissas $x_{\nu n}$ $(1 \leq \nu \leq n)$ of (2.1) are the zeros of n^{th} Laguerre polynomial $L_n^{(\alpha)}(x)$, $\alpha > -1$ and the minimal degree of $R_n(x)$ is $n(2n-2)$.

The explicit form of $R_n(x)$ is given by (3.5).

THEOREM 2. If $f(x)$ is continuous in $0 \leq x < \infty$, then "most economical" interpolatory polynomial $R_n(x)$ of Theorem 1 interpolating $f(x)$ in the nodes (2.1) converges to $f(x)$ uniformly in $0 < \epsilon \leq x \leq \omega < \infty$.

If in (2.1) we allow $0 \leq x_{1n}$ and if we replace (2.4) by

$$(2.4a) \quad 0 \leq x^{\alpha+1} e^{-x} |R_n(x) - R_n^*(x)| \leq \max_\nu |y_{\nu n} - y_{\nu n}^*| x_{\nu n}^{\alpha+1} e^{-x_{\nu n}},$$

$$0 \leq x < \infty,$$

then for the polynomials $R_n(x)$ of Theorem 1, the abscissas are the zeros of $xL_{n-1}^{(\alpha+1)}(x)$, $\alpha \geq -1$, and the minimal degree is then $(n-1)(2n-1)$. For $\alpha = -1$, we then get the result of Egerváry and Turán [3].

For the proof of the Theorem 1, we shall need the following

LEMMA 1. If $\{x_\nu\}_{\nu=1}^{n}$ denote the zeros of $L_n^{(\alpha)}(x)$, then for $0 < x < \infty$, we have

$$(2.5) \qquad v(x) \overset{\text{def}}{=} x^{-\alpha-1} e^{x} - \sum_{\nu=1}^{n} x_\nu^{-\alpha-1} e^{x_\nu} \left[\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)'}(x_\nu)(x-x_\nu)} \right]^2 \geq 0 .$$

Proof. For x_ν $(\nu = 1, 2, \ldots, n)$ we have $v(x_\nu) = 0$,

$$V'(x_\nu) = x_\nu^{-\alpha-1} e^{x_\nu} - (\alpha+1) x_\nu^{-\alpha-2} e^{x_\nu} - x_\nu^{-\alpha-1} e^{x_\nu} \frac{L_n^{(\alpha)''}(x_\nu)}{L_n^{(\alpha)'}(x_\nu)} = 0 ,$$

owing to the differential equation

$$(2.6) \qquad x\, w_n''(x) + (\alpha + 1 - x)\, w_n'(x) + n w_n(x) = 0$$

satisfied by $w_n(x) = L_n^{(\alpha)}(x)$
i.e., $v(x)$ has at least $2n$ real zeros at the x_ν's. If for a ξ we had $v(\xi) < 0$, then owing to $v(0) = +\infty$, $v(x)$ had at least one more real zero. Then according to Rolles theorem $v^{(2n)}(x)$ would have at least one real zero. But this contradicts the fact, because

$$v(x)^{(2n)} = (x^{-\alpha-1} e^{x})^{(2n)} > 0 ,$$

and this completes the proof of the lemma.

We shall also make use of the weaker inequality

$$(2.7) \qquad \sum_{\nu=1}^{n} e^{-x} \left(\frac{x}{x_\nu}\right)^{\alpha+1} l_\nu^2(x) \leq 1, \quad x > 0,$$

which is obvious from (2.5).

3. Proof of Theorem 1. We shall henceforth write x_ν for $x_{\nu n}$ and y_ν for $y_{\nu n}$, $1 \leq \nu \leq n$. We first prove the necessity of the condition in Theorem 1. Choosing $y_\nu = 1$ for $\nu = k$ and $y_\nu = 0$, $\nu \neq k$ and $y_\nu^* = 0$, $1 \leq \nu \leq n$, we obtain from (2.3) and (2.4), the inequality

$$(3.1) \qquad 0 \leq F_k(x) \leq 1 \quad (x \geq 0, \ 1 \leq k \leq n)$$

where we set

$$F_k(x) \equiv \left(\frac{x}{x_k}\right)^{\alpha+1} e^{x_k - x} u_k(x),$$

whence from (2.1) and (2.3) we have

$$(3.2) \qquad u_k(x_\nu) = u_k'(x_\nu) = 0, \quad \nu \neq k.$$

Also from (2.3) and (3.1), we have $F_k(x_k) = 1$, and $F_k'(x_k) = 0$. From the last condition we have owing to (2.3)

$$(3.3) \qquad u_k'(x_k) = -\frac{\alpha+1}{x_k} + 1.$$

If
$$w_n(x) = \prod_{\nu=1}^{n} (x - x_\nu),$$

then (3.2) shows that $u_k(x)$ is divisible by $\left(\frac{w_n(x)}{x - x_k}\right)^2$ so that the degree of R_n is $\geq n(2n-2)$. If the process R_n is "most economical", then it is enough to take

$$(3.4) \qquad u_k(x) = \left[\frac{w_n(x)}{(x - x_k) w_n'(x_k)}\right]^2.$$

https://doi.org/10.4153/CMB-1966-079-x Published online by Cambridge University Press

From (3.3), we now obtain on differentiating (3.4)

$$\frac{w_n''(x_k)}{w_n'(x_k)} = -\frac{\alpha+1-x_k}{x_k}, \quad 1 \le k \le n.$$

Whence $x_k\, w_n''(x_k) + (\alpha+1-x_k)\, w_n'(x_k) = 0, \; 1 \le k \le n$

so that $x\, w_n''(x) + (\alpha+1-x)\, w_n'(x) = C\, w_n(x) =$ for some constant C.

Thus $w_n(x) = L_n^{(\alpha)}(x)$, with $C = -n$. Hence

$$(3.5) \qquad R_n(x) = \sum_{\nu=1}^{n} y_\nu\, 1_\nu^2(x), \quad 1_\nu(x) = \frac{L_n^{(\alpha)}(x)}{(x-x_\nu)\, L_n^{(\alpha)\prime}(x_\nu)}$$

and this proves the necessary part.

To prove the sufficiency we show that the interpolatory polynomials $R_n(x)$ in (3.5) satisfy the stability condition (2.4). From the inequality (2.5) proved in lemma 1 we have

$$(3.6) \qquad \sum_{\nu=1}^{n} \left(\frac{x}{x_\nu}\right)^{\alpha+1} e^{x_\nu - x} \left[\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)\prime}(x_\nu)(x-x_\nu)}\right]^2 \le 1, \quad x > 0.$$

Due to the non negativity of $1_\nu^2(x)$ and arbitrary $y_\nu, y_\nu^*(\nu = 1, 2, \ldots, n)$ we have

$$0 < x^{\alpha+1} e^x \left| \sum_{\nu=1}^{n} y_\nu\, 1_\nu^2(x) - \sum_{\nu=1}^{n} y_\nu^*\, 1_\nu^2(x) \right| \le \sum_{\nu=1}^{n} \{ |y_\nu - y_\nu^*|\, x_\nu^{\alpha+1} e^{x_\nu} \}$$

$$\left(\frac{x}{x_\nu}\right)^{\alpha+1} e^{x_\nu - x}\, 1_\nu^2(x).$$

Thus owing to (3.6) the interpolatory polynomials (3.5) satisfy the stability condition (2.4). This proves Theorem 1.

4. To prove Theorem 2, we need the following results. For the Hermite-Fejér interpolation on Laguerre abscissas we have [Szegö 4]

$$(4.1)\quad \sum_{\nu=1}^{n} \left(1 + \frac{\alpha + 1 - x_\nu}{x_\nu}(x - x_\nu)\right) \left[\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)'}(x_\nu)(x - x_\nu)}\right]^2 = 1 .$$

Let $K_{2\omega}$ denote the continuity module of $f(x)$ with respect to the interval $[0, 2\omega]$, then for positive λ and δ with $\lambda\delta \leq 2\omega$ the inequality

$$(4.2)\qquad K_{2\omega}(\lambda\delta) \leq (\lambda + 1) K_{2\omega}(\delta)$$

holds. Further we shall need the following lemmas:

LEMMA 2. For $\epsilon \leq x \leq \omega$, we have

$$\sum_{\nu=1}^{n} |x - x_\nu| \, l_\nu^2(x) \leq c_3 n^{-1/4} .$$

Proof. We write

$$\sum_{\nu=1}^{n} |x - x_\nu| \, l_\nu^2(x) = \sum_{|x-x_\nu| \leq n^{-1/4}} + \sum_{|x-x_\nu| > n^{-1/4}}$$

$$= S_1 + S_2 .$$

Now

$$e^{-x} x^{\alpha+1} S_1 = \sum_{|x-x_\nu| \leq n^{-1/4}} e^{-x} x^{\alpha+1} |x - x_\nu| \, l_\nu^2(x)$$

$$\leq n^{-1/4} \sum_{\nu=1}^{n} e^{-x} x^{\alpha+1} l_\nu^2(x)$$

$$(4.3)\qquad \leq \omega^{\alpha+1} n^{-1/4} \sum_{\nu=1}^{n} e^{-x} \left(\frac{x}{x_\nu}\right)^{\alpha+1} l_\nu^2(x)$$

$$\leq \omega^{\alpha+1} n^{-1/4} .$$

Because of inequality (2.7),

https://doi.org/10.4153/CMB-1966-079-x Published online by Cambridge University Press

$$S_2 = \sum_{|x-x_\nu| > n^{-1/4}} |x - x_\nu| \, l_\nu^2(x)$$

$$= \sum_{|x-x_\nu| > n^{-1/4}} \frac{[L_n^{(\alpha)}(x)]^2}{(x - x_\nu)\,[L_n^{(\alpha)'}(x_\nu)]^2}$$

$$\leq n^{1/4} \, [L_n^{(\alpha)}(x)]^2 \sum_{\nu=1}^{n} \frac{1}{[L_n^{(\alpha)'}(x_\nu)]^2} \, .$$

Now using the following two formulas

$$(4.4) \quad L_n^{(\alpha)}(x) = x^{-\alpha/2 - 1/4} \, 0(n^{\alpha/2 - 1/4}) \quad c_n^{-1} \leq x \leq \omega \, ,$$

[4] p. 175, formula (7.6.8.) and

$$(4.5) \qquad \sum_{\nu=1}^{n} x_\nu^{m-1} \, \{L_n^{(\alpha)}(x_\nu)\}^{-2} = \frac{\Gamma(n+1)\Gamma(m+\alpha+1)}{\Gamma(n+\alpha+1)} ,$$

m a positive integer $\leq 2n - 1$, [4]p. 343, formula (14.7.5.), we have

$$S_2 \leq c_4 \, n^{1/4} \, 0(n^{\alpha - 1/2}) \, 0(n^{-\alpha})$$

$$(4.6) \qquad \leq c_5 \, n^{-1/4} \, .$$

Thus (4.3) and (4.6) complete the proof of lemma 2.

LEMMA 3. For $\epsilon \leq x \leq \omega$, we have

$$\sum_{\nu=1}^{n} \frac{1}{x_\nu} |x - x_\nu| \, l_\nu^2(x) \leq c_6 \, n^{-1/4} \, .$$

Proof. We again write

https://doi.org/10.4153/CMB-1966-079-x Published online by Cambridge University Press

$$\sum_{\nu=1}^{n} \frac{1}{x_\nu} |x - x_\nu| \; l_\nu^2(x)$$

$$= \sum_{|x-x_\nu| \le n^{-1/4}} + \sum_{|x-x_\nu| > n^{-1/4}} .$$

As before

$$e^{-x} x^{\alpha+1} s_3 = \sum_{|x-x_\nu| \le n^{-1/4}} e^{-x} \frac{x^{\alpha+1}}{x_\nu} |x - x_\nu| \; l_\nu^2(x)$$

$$\le n^{-1/4} \sum_{\nu=1}^{n} e^{-x} \frac{x^{\alpha+1}}{x_\nu} \; l_\nu^2(x)$$

$$\le n^{-1/4} \omega^\alpha \sum_{\nu=1}^{n} e^{-x} \left(\frac{x}{x_\nu}\right)^{\alpha+1} l_\nu^2(x) , \quad \alpha \ge 0$$

$$\le n^{-1/4} \omega^\alpha \sum_{\nu=1}^{n} e^{-x} \left(\frac{x}{x_\nu}\right)^{\alpha+1} l_\nu^2(x), \quad \alpha < 0 .$$

Hence, if $c_7 = \max(\epsilon^\alpha, \omega^\alpha)$, we have

$$e^{-x} x^{\alpha+1} s_3 \le c_7 n^{-1/4} \sum_{\nu=1}^{n} e^{-x} \left(\frac{x}{x_\nu}\right)^{\alpha+1} l_\nu^2(x) \text{ for } \alpha > -1$$

(4.7)

$$\le c_7 n^{-1/4}$$

owing to the inequality (2.7).

For $s_4 = \sum_{|x-x_\nu| > n^{-1/4}} \frac{1}{x_\nu} |x - x_\nu| \; l_\nu^2(x)$

https://doi.org/10.4153/CMB-1966-079-x Published online by Cambridge University Press

$$\leq n^{-1/4} L_n^{(\alpha)}(x)^2 \sum_{\nu=1}^{n} \frac{1}{x_\nu} [L_n'^{(\alpha)}(x_\nu)]^{-2}$$

(4.8)
$$\leq n^{-1/4} L_n^{(\alpha)}(x)^2 \frac{\Gamma(n+1)\Gamma(\alpha+3)}{\Gamma(\alpha+n+1)}$$

$$\leq c_8 n^{-1/4} O(n^{\alpha-1/2}) O(n^{-\alpha})$$

$$\leq c_9 n^{-1/4} .$$

Thus (3.7) and (3.8) prove lemma 3.

From Lemma 2 and Lemma 3 we at once have the

LEMMA 4. For $\epsilon \leq x \leq \omega$ we have

$$\sum_{\nu=1}^{n} l_\nu^2(x) - 1 \leq c_{10} n^{-1/4} .$$

In fact we have from the Hermite-Fejér interpolation [Szegö 4]

$$\sum_{\nu=1}^{n} \left(1 + \frac{\alpha + 1 - x_\nu}{x_\nu}\right)(x - x_\nu) l_\nu^2(x) = 1$$

i.e.

$$\sum_{\nu=1}^{n} l_\nu^2(x) - 1 = \sum_{\nu=1}^{n} (x - x_\nu) l_\nu^2(x) - (\alpha+1) \sum_{\nu=1}^{n} \frac{1}{x_\nu}(x - x_\nu) l_\nu^2(x) .$$

LEMMA 5. We have

$$\sum_{x_\nu > 2\omega} l_\nu^2(x) \leq c_4 n^{-1/2} \qquad \epsilon \leq x \leq \omega .$$

<u>Proof.</u> Since $\epsilon \leq x \leq \omega$ and $x_\nu > 2\omega$, we have $|x - x_\nu| > \omega$, so that using (4.4) and (4.5)

https://doi.org/10.4153/CMB-1966-079-x Published online by Cambridge University Press

$$\sum_{x_\nu > 2\omega} 1_\nu^2(x) \leq \frac{1}{\omega^2} [L_n^{(\alpha)}(x)]^2 \sum_{\nu=1}^{n} \frac{1}{[L_n^{(\alpha)'}(x_\nu)]^2}$$

$$= \frac{1}{\omega^2} [L_n^{(\alpha)}(x)]^2 \frac{\Gamma(n+1)\,\Gamma(\alpha+2)}{\Gamma(n+\alpha+1)}$$

$$\leq c_{12}\ 0(n^{\alpha-1/2})\ 0(n^{-\alpha})$$

$$\leq c_{11}\, n^{-1/2} .$$

5. Proof of Theorem 2. The proof of this theorem runs exactly on the lines of the proof given by J. Balázs and P. Turán [2]. We sketch its proof simply for the sake of completeness. Let $|f(x)| \leq M$ for $x \geq 0$. Then

$$|R_n(f) - f(x)| = |\sum_{\nu=1}^{n} (f(x_\nu) - f(x))\, 1_\nu^2(x) - f(x)\{1 - \sum_{\nu=1}^{n} 1_\nu^2(x)\}|$$

(5.1)
$$\leq \sum_{\nu=1}^{n} |f(x) - f(x_\nu)|\, 1_\nu^2(x) + |f(x)|\ |1 - \sum_{\nu=1}^{n} 1_\nu^2(x)| .$$

Owing to lemma 4, we have for $\epsilon \leq x \leq \omega$,

(5.2)
$$|f(x)|\ |1 - \sum_{\nu=1}^{n} 1_\nu^2(x)| \leq c_{10} M\, n^{-1/4} .$$

Again, for $\epsilon \leq x \leq \omega$,

$$\sum_{\nu=1}^{n} |f(x) - f(x_\nu)|\ 1_\nu^2(x) = \sum_{\epsilon \leq x_\nu \leq 2\omega} + \sum_{x_\nu > 2\omega}$$

$$\leq \sum_{\epsilon \leq x_\nu \leq 2\omega} K_{2\omega}(|x - x_\nu|)\, 1_\nu^2(x) + c_{11}\, n^{-1/2}$$

https://doi.org/10.4153/CMB-1966-079-x Published online by Cambridge University Press

(5.3)
$$\le K_{2\omega}(n^{-1/4})\,[n^{-1/4}\sum_{\nu=1}^{n}|x-x_\nu|\,l_\nu^2(x)$$

$$+\sum_{\nu=1}^{n} l_\nu^2(x)] + c_{11}\, n^{-1/2}$$

$$\le c_3 K_{2\omega}(n^{-1/4}) + c_{12} K_{2\omega}(n^{-1/4}) + c_{11}\, n^{-1/2}$$

using lemmas 2, 4, 5 and the remark (4.2) on the modulus of continuity of $f(x)$. Thus (5.1), (5.2) and (5.3) complete the proof of Theorem 2.

6. Results analogous to theorems 1 and 2 can be obtained if we define stability on $(-\infty, \infty)$ by the requirement

$$0 \le |x|^{2K} e^{-x^2} |R_n(x) - R_n^*(x)|$$

$$\le \max_{\substack{1\le\nu\le n\\ -\infty<x<\infty}} |y_{\nu n} - y_{\nu n}^*|\,|x_{\nu n}|^{2K} e^{-x_{\nu n}^2} .$$

It turns out that for a "most economical" stable interpolation, the x_j's must be the zeros of the polynomials $H_n^{(k)}(x)$ which form a generalization of Hermite polynomials. The important properties of these polynomials are given in [6], p. 377. The proofs are analogous to the proofs of Theorem 1 and 2.

REFERENCES

1. J. Balázs, Remarks on the stability of interpolation, Math. Lapok., 11 (1960), pages 280-293.

2. J. Balázs and P. Turán, Notes on interpolation VII. Acta Math. Acad. Sei. Hung., 10 (1959), pages 63-68.

3. E. Egerváry and P. Turán, Notes on interpolation V,

https://doi.org/10.4153/CMB-1966-079-x Published online by Cambridge University Press

Ibid 9, (1958), pages 259-267.

4. E. Egerváry and P. Turán, Notes on interpolation VI, Ibid 10, (1959), pages 55-62.

5. A. Sharma, Remarks on quasi-Hermite-Fejér Interpolation. Canad. Math. Bull. 7, (1964), pages 101-119.

6. G. Szegö, Orthogonal polynomials. American Math. Soc. Coll. Second Edition, (1959).

University of Alberta, Edmonton, Canada
and
Lucknow University, Lucknow, India

https://doi.org/10.4153/CMB-1966-079-x Published online by Cambridge University Press