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THIN LENS SPACES

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In Theorem 1 below we study the existence of spaces whose cohomology rings are isomorphic (as *ungraded* rings) to those of lens spaces. The case $p=2$ is very simple and instructive, so let us consider it first.

Suppose X is a space such that $H^*(X)\cong Z[x]/\langle 2x, x^3\rangle$ where $\dim x = 2d$ (for example $X=RP^4$ with $d=1$). Then the Atiyah-Hirzebruch spectral sequence collapses, so $\tilde{K}(X)$ has an element y such that either

(i) $\tilde{K}(X)\cong Z_4$ generated by y, and $y^2=2y$, or
(ii) $\tilde{K}(X)\cong Z_2\oplus Z_2$ with generators y and y^2.

But, in case (ii),

$$0=\lambda^2(0)=\lambda^2(2y)=2\lambda^2(y)+y^2=y^2, \quad \text{a contradiction.}$$

Hence we must have case (i). In general

$$\lambda^2(ab)=a^2\lambda^2(b)+b^2\lambda^2(a)-2\lambda^2(a)\lambda^2(b).$$

Thus, since $2\tilde{K}(X)\cdot\tilde{K}(X)=0$,

$$\begin{aligned}2y+2\lambda^2(y)=y^2+2\lambda^2(y)&=\lambda^2(2y)=\lambda^2(y^2)\\&=2y^2\lambda^2(y)-2[\lambda^2(y)]^2=0.\end{aligned}$$

Hence $\lambda^2(y)\equiv -y(\text{mod } y^2)$. But in general, if $y\in K_{2d}$ the dth filtration subgroup, then $\lambda^2(y)\equiv -2^{d-1}y$ (mod K_{2d+2}). Thus we must have $d=1$, so X looks much like RP^4.

The corresponding result for odd primes goes as follows.

THEOREM 1. (a) *If p is a prime and d is a positive integer, then there is a space X with $H^*(X)\cong Z[x]/\langle px\rangle$ where* $\dim x=2d$ *if and only if d is a divisor of $p-1$.*

(b) *More precisely, $Z[x]/\langle px, x^{p+1}\rangle$ is not realizable as an integral cohomology ring unless* $\dim x/2$ *divides* $p-1$.

The proof determines the group extensions in the computation of the K-theory of skeletons of X for certain d. This includes $d=1$, where the lens spaces are examples. It gives a simple algebraic determination of their K-theory, normally done by exhibiting bundles, or better, representations of cyclic groups.[1]

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[1] R. Kane has pointed out another proof of half of the theorem using cohomology operations.

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We both wish to thank the Oxford Mathematical Institute for its hospitality during the autumn of 1975. We also thank the referee for several valuable suggestions, particularly the one described in the final paragraph.

A. Proof that d divides $p-1$. The symmetric polynomial $p^{-1}[\sum x_i^p-(\sum x_i)^p]$ can be expressed as $f_p(\sigma_1,\ldots,\sigma_p)$, where f_p is a polynomial with integer coefficients and σ_i is the ith elementary symmetric function. If R is a special λ-ring [4], define $\Theta^p:R\to R$ as in [3]:

$$\Theta^p(x)=f_p[\lambda^1(x),\ldots,\lambda^p(x)].$$

LEMMA 2. *For $x\in R$ and positive integers n, we have*

(a) $\Theta^p(nx)=n\Theta^p(x)+p^{-1}(n-n^p)x^p$.

(b) $\Theta^p(x^n)=\sum_{j=1}^{n} p^{j-1}\binom{n}{j}\Theta^p(x)^j x^{p(n-j)}$.

Proof. Since the free λ-ring on two generators is torsion free, these follow immediately from the properties of the Adams operations ψ^p and the equation $\Theta^p(x)=p^{-1}[\psi^p(x)-x^p]$. Alternatively, they follow from the Verification Principle [4; 3.2]. Two more proofs are by induction on n, first working out the formulae for $\Theta^p(x+y)$ and $\Theta^p(xy)$ by either method above.

NOTE. There are two reasons why Θ^p is easier to use than λ^p in this context: Theorem 3 below, and the fact that the formulae above involve sums, products and Θ^p, but no other operations.

THEOREM 3. (*Atiyah*) *Suppose Y is a finite complex with $H^k(Y)=0$ for all odd k. Then for $y\in K_{2d}(Y)$, the image of $\Theta^p(y)$ in $K(Y)/K_{2d+2t+2}(Y)$ is divisible by $p^{d-[t/p-1]-1}$, where $[s]$ denotes the integer part of s.*

This follows immediately from [3, Theorem 5.3].

In the remainder of section A, we assume Y is a finite complex with $H^*(Y)\cong Z[x]/\langle px, x^{p+1}\rangle$, where $\dim x=2d$, and we prove d divides $p-1$.

LEMMA 4. *There exist $y\in\tilde{K}(Y)$ and integers h, with $2\le h\le p+1$, and a, prime to p, such that*

(i) $K(Y)=Z[y]/\langle py-ay^h, y^{p+1}\rangle$;

(ii) *$K_{2td}(X)$ is the group generated by $\{y^i \mid t\le i\le p\}$ for all t, and $K_{2s}(Y)=K_{2s+2}(Y)$ if $s\nmid d$;*

(iii) *$p-1$ is a divisor of $d(h-1)$, if $h\ne p+1$.*

Proof. Since $H^{2i+1}(Y)=0$ for all i, the Atiyah-Hirzebruch spectral sequence collapses. Thus we can find y' such that (ii) holds with y' in place of y and such that

$$K(Y)=Z[y']/\left\langle py'-\sum_{i=h}^{p} a_i y'^i,\ y'^{p+1}\right\rangle$$

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where $2 \le h \le p+1$ and a_h is prime to p. By [1, Cor. 8], $\tilde{K}(Y)$ splits as a direct sum $G^{(0)} \oplus G^{(1)} \oplus \cdots \oplus G^{(p-2)}$ of filtered groups, where for each index $\alpha \in Z_{p-1}$, we have $G^{\alpha}_{2i} = G^{\alpha}_{2i+2}$ if $i \notin \alpha$. Since $y' \in K_{2d}(Y)$, by projecting into $G^{(d)}$, we may take $y' \in G^{(d)}$ without affecting anything but the exact values of the coefficients a_i. To verify (iii), note that if $h \ne p+1$, we have

$$py' = \sum_{i=h}^{p} a_i y'^i \in G^{(d)}_{2dh} - G^{(d)}_{2dh+2}.$$

Hence $dh \in (d)$ i.e. $dh \equiv d \bmod (p-1)$, proving (iii). Now, without affecting (ii), we can alter y' to make $a_{h+1} = 0$ by setting $y'' = y' + a_{h+1}\bar{a}_h(\overline{h-1})y'^2$, where $\bar{c}c \equiv 1 \pmod p$. Continuing inductively we can eliminate $a_{h+2}, \ldots, a_p$ arriving at an element y such that (i) holds.

Lemma 5. $d(h-1)^2 \le (p-1)^2$.

Proof. By (2a) and (b) we have

$$\Theta^p(py) = p\Theta^p(y) + y^p$$

and

$$\Theta^p(ay^h) = a\Theta^p(y^h) + \frac{a-a^p}{p} y^{ph} = a\Theta^p(y^h)$$

$$= a\sum_{j=1}^{h} p^{j-1}\binom{h}{j}[\Theta^p(y)]^j y^{p(h-j)} = ap^{h-1}[\Theta^p(y)]^h$$

since $y^{p(h-j)} = 0$ unless $j = h$ or $h-1$, and since $[\Theta^p(y)]^{h-1}y^p = 0$. Since $py = ay^h$, we obtain $p\Theta^p(y) + y^p = ap^{h-1}[\Theta^p(y)]^h$. Since $h \ge 2$, the filtration of $ap^{h-1}[\Theta^p(y)]^h$ is strictly larger than that of $p\Theta^p(y)$, a non-zero element since $y^p \ne 0$. But y^p is in the last non-zero filtration subgroup, so we must have $p\Theta^p(y) + y^p = 0$. This implies $p\tilde{K}(Y) \ne 0$ i.e. $h \ne p+1$, and that $\Theta^p(y) \equiv by^{p-h+1} \bmod K_{2d(p-h+1)+2}(Y)$ for some b prime to p. In $K(Y)/K_{2d(p-h+1)+2}(Y)$, y^{p-h+1} is divisible by exactly $p^{[(p-1)/(h-1)]-1}$. By Theorem 3,

$$d - \left[\frac{d(p-h)}{p-1}\right] - 1 \le \left[\frac{p-1}{h-1}\right] - 1$$

But

$$\frac{d(p-h)}{p-1} = d - \frac{d(h-1)}{p-1}$$

is an integer by 4(iii), so we obtain

$$\frac{d(h-1)}{p-1} \le \left[\frac{p-1}{h-1}\right] \quad \text{or} \quad d \le \left[\frac{p-1}{h-1}\right]\left(\frac{p-1}{h-1}\right) \le \left(\frac{p-1}{h-1}\right)^2,$$

as required.

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Proof that $d \mid p-1$: If $\bar{x} \in H^{2d}(Y; Z_p)$ is the reduction of x, then $\mathcal{P}^d(\bar{x}) = \bar{x}^p \neq 0$. The usual argument using $\mathcal{P}^1\mathcal{P}^{d-1} = d\mathcal{P}^d$ then shows that either $d \mid p-1$ or $p \mid d$. But, if $d = sp$, Lemma 5 gives $sp(h-1)^2 \leq (p-1)^2$. Lemma (4) (iii) gives $p-1 \mid sp(h-1)$, so

$$p-1 \mid s(h-1) \leq (p-1)^2 p^{-1}(h-1)^{-1} < p-1.$$

This contradiction eliminates the possibility that $p \mid d$, and so $d \mid p-1$, as required.

Note. It is immediate from 4 (iii) and 5 that $h = p$ if and only if $d = 1$, giving the group extension for the K-theory of a lens space. W. M. Chan has made some calculations analogous to these for the case when p is not prime. He also noticed a gap in an earlier version of the proof above.

B. Construction of X. In [5], a method is given for realizing in certain cases the image of an induced map as the cohomology of a space. A simple case is the following. Induced maps f^*, s^*, t^* and q^* always refer to Z_p-cohomology.

Theorem 2. *Given $f: W \to Y$, assume*

(a) *$H^k(W)$ and $H^k(Y)$ are finite p-groups for all $k > 0$, and Y is at least 2-connected;*

(b) *there exist maps s and t such that*

(*i*)
$$\begin{array}{ccc} W & \xrightarrow{f} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ W & \xrightarrow{f} & Y \end{array}$$
commutes,

(ii) s^* and t^* are diagonalizable,

(iii) the λ-eigenspace of s^* lies in the image of f^* for any product λ of eigenvalues of t^*.

Then f factors as $W \xrightarrow{g} X \xrightarrow{h} Y$ where

(1) $H^k(X)$ is a finite p-group for all $k > 0$;

(2) g^* is injective;

(3) $\operatorname{Im} g^* = \operatorname{Im} f^*$.

To construct X, a lens space will do for $d = 1$, so we may take $d > 1$ with $d \mid p-1$. Let $W = K(Z_p, 1)$ and $Y = K(Z_p, 2d-1)$ and specify f in Theorem 2 by $f^*(\iota_{2d-1}) = \iota_1(\beta\iota_1)^{d-1}$. Here ι is the fundamental class for the Eilenberg-Maclane space and β is the Bockstein operator. That X has the required cohomology ring is elementary.

We are grateful to the referee for pointing out that such spaces X occur "in nature" as well (see [2] for analogues): The field Z_p has a multiplicative

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subgroup S of order d. Let G be the split extension of Z_p by S, the action being multiplication. Let $\sigma: S \to G$ be a splitting and take $X = B_G \cup_{B_\sigma} CB_S$. It follows that $H^*(X) \simeq \ker B_\sigma^*$. The action of S on $H^*(B_{Z_p})$ as ring automorphisms may, on $H^2(B_{Z_p}) \cong Z_p$, be identified with the multiplication action. From the splitting

$$H^*(B_G) \cong H^*(B_{Z_p})^S \oplus H^*(B_S)$$

it follows that $H^*(X) \cong H^*(B_{Z_p})^S$, the fixed subring. This is what we want, since S acts trivially exactly in the dimensions $2kd$ for $k \geq 0$.

References

1. J. F. Adams: Lecture 4, *Splitting Generalized Cohomology Theories with coefficients, in Category Theory, Homology Theory and their Applications III*, Springer Lecture Notes in Mathematics, Vol. **99**.

2. J. F. Adams: *The Kahn-Priddy Theorem*, Proc. Comb. Phil. Soc. **73** (1973), 45–55.

3. M. F. Atiyah: *Power Operations in K-theory*, Quart. J. Math. Oxford (2), **17** (1966), 165–93.

4. M. F. Atiyah and D. O. Tall: *Group Representations, λ-Rings and the J-Homomorphism*, Topology **8** (1969), 253–97.

5. A. Zabrodsky: *On the category of endomorphisms of finite complexes* (to appear).

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