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# COLOURED GRAPHS: A CORRECTION AND EXTENSION

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Let $M_n = M_n(k)$ be the number of graphs on $n$ labelled nodes, each node being coloured with one of $k$ colours. Every pair of nodes of different colour can be joined or not joined by an edge; no pair of nodes of the same colour can be so joined. We write $F_n = F_n(k)$ for the number of these graphs in which all $k$ colours are used and $f_n = f_n(k)$ for the number of these latter graphs which are connected.

If $r_n$ is the number of those connected graphs on $n$ labelled nodes which have some property P and if $R_n$ is the number of graphs on $n$ labelled nodes each of whose connected components has property P, we have

$$(1) \qquad 1 + \sum_{n=1}^{\infty} \frac{R_n X^n}{n!} = \exp\left(\sum_{n=1}^{\infty} \frac{r_n X^n}{n!}\right)$$

by [**1**]. Hence, if we write $m_n$ for the number of connected graphs on $n$ labelled nodes, each node being coloured with one of $k$ colours, we have

$$(2) \qquad 1 + \sum_{n=1}^{\infty} \frac{M_n X^n}{n!} = \exp\left(\sum_{n=1}^{\infty} \frac{m_n X^n}{n!}\right).$$

But (1) is not true with $r_n = f_n$, $R_n = F_n$, that is, we cannot equate the two expressions

$$1 + \sum_{n=1}^{\infty} \frac{F_n X^n}{n!}, \qquad \exp\left(\sum_{n=1}^{\infty} \frac{f_n X^n}{n!}\right),$$

as we erroneously assumed in [**3**; **4**]. For a graph may use all $k$ colours although some of its connected components use fewer than $k$ (consider, for example, a graph with $k$ nodes, each coloured differently, and no edges). Hence [**3**, (8) and (9)] do not hold, nor does [**4**, (1.3)].

We can however easily find a method of calculating $f_n$. We have, obviously,

$$m_n(k) = \sum_{s=1}^{k} \binom{k}{s} f_n(s),$$

since we may choose $s$ colours out of $k$ in $\binom{k}{s}$ ways and $m_n(k)$ enumerates connected graphs using all of every possible set of $s$ colours for all $s$ such that $1 \leqq s \leqq k$. From this we can deduce that

$$\sum_{s=1}^{k} (-1)^{k-s}\binom{k}{s} m_n(s) = \sum_{s=1}^{k} (-1)^{k-s}\binom{k}{s} \sum_{t=1}^{s} \binom{s}{t} f_n(t)$$
$$= \sum_{t=1}^{k} A_{kt} f_n(t),$$

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where

$$A_{kt} = \sum_{s=t}^{k} (-1)^{k-s}\binom{k}{s}\binom{s}{t} = \binom{k}{t}\sum_{s=t}^{k} (-1)^{k-s}\binom{k-t}{s-t},$$

so that $A_{kk} = 1$ and

$$A_{kt} = \binom{k}{t}(1-1)^{k-t} = 0 \qquad \text{if } t < k.$$

We then have

$$f_n(k) = \sum_{s=1}^{k} (-1)^{k-s}\binom{k}{s} m_n(s). \tag{3}$$

This could also be found by the Exclusion Theorem [**2**, Theorem 260].

From (2) we can deduce that

$$M_n = m_n + \sum_{s=1}^{n-1} \binom{n-1}{s-1} m_s M_{n-s}. \tag{4}$$

From this we can calculate $m_n(k)$ from $M_s(k)$ $(1 \leqq s \leqq n)$ and then, by (3), $f_n(k)$ from $m_n(s)$ $(1 \leqq s \leqq k)$.

We have thus corrected [**3**, § 4], the only section of that paper in error, and have shown how to calculate $f_n(k)$ and, incidentally, the newly introduced $m_n(k)$.

We now turn to correct [**4**]. The proof in that paper, that

$$F_n = M_n\{1 - O(e^{-An^2})\} \tag{5}$$

as $n \to \infty$ is still valid, since it does not involve [**4**, (1.3)]. Again, from (4) of the present paper, we can deduce that

$$m_n = M_n\{1 - O(e^{-An})\}, \tag{6}$$

just as we deduced a similar result for $f_n$, $F_n$ from the erroneous equation [**4**, (1.3)].

Next we remark that $M_n - m_n$ is the number of disconnected coloured graphs on $n$ labelled nodes and $F_n - f_n$ is the number of these graphs which use all $k$ colours. Hence

$$0 \leqq F_n - f_n \leqq M_n - m_n$$

and so

$$\begin{aligned} 0 \leqq M_n - f_n &= M_n - F_n + F_n - f_n \\ &\leqq (M_n - F_n) + (M_n - m_n) = M_n O(e^{-An}) \end{aligned}$$

by (5) and (6). From all this and the results of [**4**] we can deduce the following theorem.

THEOREM. *$M_n$, $m_n$, $F_n$, and $f_n$ each have the same asymptotic expansion, viz.*

$$\left(\frac{k}{n \log 2}\right)^{(k-1)/2} k^n T(Kn^2)\left\{\sum_{h=0}^{H-1} C_h n^{-h} + O(n^{-H})\right\}, \tag{7}$$

*where $T(\theta) = 2^\theta$ and $C_h = C_h(k, a)$ depends on $k$, $h$, and the residue $a$ of $n$ (mod $k$), but not otherwise on $n$.*

We have thus restored (and indeed added to) the results of [**4**]. If we allow any two nodes of different colours to be "joined" in $j$ different ways as in [**5**], i.e. we may not join them, we may join them by a red edge, by a blue edge, and so on, then $M_n$, $m_n$, $F_n$, and $f_n$ still have the same asymptotic expansion, viz. that given in [**5**, Theorem 2], that is (7) above with $\log j$ replacing $\log 2$ and $T(\theta) = j^\theta$.

We add tables of $m_n(k)$ and $f_n(k)$.

*Values of $m_n(k)$*

| k \ n | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 2 | 2 | 6 | 38 | 390 | 6062 | 134526 |
| 3 | 3 | 6 | 42 | 618 | 15990 | 668526 | 43558242 |
| 4 | 4 | 12 | 132 | 3156 | 136980 | 10015092 | 1199364852 |
| 5 | 5 | 20 | 300 | 9980 | 616260 | 65814020 | 11878194300 |
| 6 | 6 | 30 | 570 | 24330 | 1956810 | 277164210 | 67774951650 |
| 7 | 7 | 42 | 966 | 50358 | 4999050 | 885312162 | 274844567886 |
| 8 | 8 | 56 | 1512 | 93128 | 11008200 | 2343695816 | 884716732812 |
| 9 | 9 | 72 | 2232 | 158616 | 21761640 | 5417215272 | 2411955530712 |

*Values of $f_n(k)$*

| k \ n | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 2 | 6 | 38 | 390 | 6062 | 134526 |
| 3 | 0 | 0 | 24 | 504 | 14820 | 650340 | 43154664 |
| 4 | 0 | 0 | 0 | 912 | 75360 | 7377360 | 1025939040 |
| 5 | 0 | 0 | 0 | 0 | 87360 | 22363200 | 6315607200 |
| 6 | 0 | 0 | 0 | 0 | 0 | 19226880 | 13627111680 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 9405930240 |

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