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A SUFFICIENT CONDITION UNDER WHICH A SEMIGROUP IS NONFINITELY BASED

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Abstract

We give a sufficient condition under which a semigroup is nonfinitely based. As an application, we show that a certain variety is nonfinitely based, and we indicate the additional analysis (to be presented in a forthcoming paper), which shows that this example is a new limit variety of aperiodic monoids.

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1. Introduction

A variety of algebras is *finitely based* if all of its identities can be derived from a finite subset of identities; otherwise, the variety is said to be *nonfinitely based*. An algebra is *finitely based* if it generates a finitely based variety; otherwise, the algebra is said to be *nonfinitely based*. There are many finitely based and many nonfinitely based finite semigroups, and the finite basis property for finite semigroups and for finite algebras has been studied extensively. We refer to the survey of Volkov [18] for further information on the finite basis problem for semigroups.

A variety is *hereditarily finitely based* if all of its subvarieties are finitely based. A variety that is minimal with respect to being nonfinitely based is commonly known as a *limit variety*. It follows from Zorn's lemma that each nonfinitely based variety contains some limit subvariety; and so a variety is hereditarily finitely based if and only if it contains no limit subvarieties. Therefore classifying hereditarily finitely based varieties, in a certain sense, reduces to classifying limit varieties. However, classifying limit varieties appears to be quite intricate, and even finding any concrete limit variety is nontrivial. For example, no concrete limit variety of groups is known, even though there are an infinite number of them [4, 6, 13]. In contrast, for semigroup varieties, the first concrete example of a limit variety was found by Volkov in 1982 [17]. Pollák

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found a limit variety consisting of left nilsemigroups [14] and, a short time later, Sapir [15] constructed a countably infinite series of limit varieties. Lee and Volkov constructed an infinite series of limit varieties, each of which is generated by a finite 0-simple semigroup with Abelian subgroups [11]. No further explicit examples of limit semigroup varieties have been published.

A monoid is *aperiodic* if all of its subgroups are trivial. The class of aperiodic monoids is denoted by $\mathfrak{A}$. A result of Kozhevnikov implies that there are continuum many limit varieties of monoids consisting of groups [6]. This makes classification of limit varieties of monoids infeasible unless restrictions are placed on the groups lying in the variety. The class $\mathfrak{A}$ is arguably the most obvious natural candidate for attention. In [5, Proposition 5.1], Jackson proved that the variety $\mathsf{var}\{J_1\}$ generated by the monoid

$$J_1 = \langle a, b, s, t \mid xy = 0 \text{ if } xy \text{ is not a factor of } asabtb \rangle \cup \{1\}$$

of order 21 and the variety $\mathsf{var}\{J_2\}$ generated by the monoid

$$J_2 = \langle a, b, s, t \mid xy = 0 \text{ if } xy \text{ is not a factor of either } absatb \text{ or } asbtab \rangle \cup \{1\}$$

of order 35 are limit subvarieties of $\mathfrak{A}$. Jackson commented that no other similar examples could be found and posed the following questions [5, Question 1].

Question 1.1.

(1) Are $\mathsf{var}\{J_1\}$ and $\mathsf{var}\{J_2\}$ the only limit varieties generated by finite aperiodic monoids with central idempotents?

(2) Are there any finitely generated, nonfinitely based aperiodic monoid varieties that contain neither $\mathsf{var}\{J_1\}$ nor $\mathsf{var}\{J_2\}$?

In [7], Lee proved that the only finitely generated limit subvarieties of $\mathfrak{A}$ with central idempotents are $\mathsf{var}\{J_1\}$ and $\mathsf{var}\{J_2\}$, which gives an affirmative answer to Question 1.1(1). Later, in [8], he generalised this result and showed that $\mathsf{var}\{J_1\}$ and $\mathsf{var}\{J_2\}$ are the only limit subvarieties of $\mathfrak{A}$ with central idempotents. In [19], Zhang showed that a certain aperiodic monoid of order seven is nonfinitely based, and thus there exists a limit subvariety of aperiodic monoids that is different from $\mathsf{var}\{J_1\}$ and $\mathsf{var}\{J_2\}$, which gives an affirmative answer to Question 1.1(2).

The main goal of this paper and its sequel is to give an explicit example of a limit variety of $\mathfrak{A}$. Let A^1 denote the monoid obtained by adjoining an identity element to the semigroup $A = \{0, a, b, c, d, e\}$ given by the multiplication table

A	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	a	0	c	0
d	0	0	b	0	d	0
e	0	a	a	c	c	e

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This semigroup was first investigated by Lee and Zhang [12, Section 19], where it was shown to be finitely based. Let B^1 be the semigroup that is dual to A^1. In this paper, we will present a sufficient condition under which a semigroup is nonfinitely based. By using this condition, we will show that the semigroup $A^1 \times B^1$ is nonfinitely based. In the second paper [20], we show that all proper monoid subvarieties of the variety generated by $A^1 \times B^1$ are finitely based, so the variety generated by $A^1 \times B^1$ is a limit variety.

Throughout, $n \geq 2$ is a fixed integer. The following theorem is our main result, giving a sufficient condition for the nonfinite basis property of a semigroup. (For more information on sufficient conditions under which a semigroup is nonfinitely based, refer to Sapir [16], Lee and Li [10] and Lee and Zhang [12]. However, none of the previous sufficient conditions can be applied to the semigroup $A^1 \times B^1$.)

Theorem 1.2. *Suppose that S is any semigroup that satisfies the identities*

$$x^{2n} \approx x^n, \quad x^{n+1}yx^{n+1} \approx xyx, \tag{1.1a}$$

$$xy_1^n y_2^n \cdots y_k^n x \approx xy_1^n x^n y_2^n x^n \cdots x^n y_k^n x, \quad k \in \{2, 3, \ldots\}, \tag{1.1b}$$

but violates all of the identities

$$(x^n y^n)^{n+1} \approx x^n y^n, \tag{1.2a}$$

$$x^n (yx^n)^{n+1} \approx x^n yx^n, \tag{1.2b}$$

$$h^n (x^n y^n)^2 h^n zh^n x^n h^n \approx h^n x^n y^n h^n zh^n x^n h^n, \tag{1.2c}$$

$$h^n x^n h^n zh^n (y^n x^n)^2 h^n \approx h^n x^n h^n zh^n y^n x^n h^n. \tag{1.2d}$$

Then S is nonfinitely based.

Let S be any semigroup that satisfies the identities (1.1) but violates the identities (1.2). In Section 3, we show that S has a basis that consists of identities with certain special properties. In Section 4, some restrictions on identities satisfied by S are established and the proof of Theorem 1.2 is given in Section 5. The application of Theorem 1.2 to show that the semigroup $A^1 \times B^1$ is nonfinitely based is in Section 6.

2. Preliminaries

Throughout, $\mathcal{X}$ is a countably infinite alphabet. For any subset $\mathcal{Y}$ of $\mathcal{X}$, let $\mathcal{Y}^+$ and $\mathcal{Y}^*$ denote the free semigroup and free monoid over $\mathcal{Y}$, respectively. Elements of $\mathcal{X}$ are called *letters* and elements of $\mathcal{X}^*$ are called *words*. For any word $\mathbf{w}$:

- the *head* of $\mathbf{w}$, denoted by $\mathsf{h}(\mathbf{w})$, is the first letter occurring in $\mathbf{w}$;
- the *tail* of $\mathbf{w}$, denoted by $\mathsf{t}(\mathbf{w})$, is the last letter occurring in $\mathbf{w}$;
- the *content* of $\mathbf{w}$, denoted by $\mathsf{con}(\mathbf{w})$, is the set of letters occurring in $\mathbf{w}$;
- the number of occurrences of a letter x in $\mathbf{w}$ is denoted by $\mathsf{occ}(x, \mathbf{w})$;
- a letter x is *simple* in $\mathbf{w}$ if $\mathsf{occ}(x, \mathbf{w}) = 1$;
- the set of simple letters of $\mathbf{w}$ is denoted by $\mathsf{sim}(\mathbf{w})$; and
- the set of nonsimple letters of $\mathbf{w}$ is denoted by $\mathsf{non}(\mathbf{w})$.

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Note that $\mathsf{con}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}) \cup \mathsf{non}(\mathbf{w})$ and $\mathsf{sim}(\mathbf{w}) \cap \mathsf{non}(\mathbf{w}) = \emptyset$. Two words $\mathbf{w}$ and $\mathbf{w}'$ are *disjoint* if $\mathsf{con}(\mathbf{w}) \cap \mathsf{con}(\mathbf{w}') = \emptyset$.

An identity is written as $\mathbf{w} \approx \mathbf{w}'$, where $\mathbf{w}, \mathbf{w}' \in \mathcal{X}^+$. An identity $\mathbf{w} \approx \mathbf{w}'$ is *nontrivial* if $\mathbf{w} \neq \mathbf{w}'$. A semigroup S *satisfies* an identity $\mathbf{w} \approx \mathbf{w}'$ if, for any substitution $\varphi : \mathcal{X} \to S$, the elements $\mathbf{w}\varphi$ and $\mathbf{w}'\varphi$ of S are equal. An identity $\mathbf{w} \approx \mathbf{w}'$ such that $|\mathsf{con}(\mathbf{w}\mathbf{w}')| \leq k$ is said to be *k-limited*. For any semigroup S, $\mathsf{id}S$ is the set of all identities satisfied by S and $\mathsf{id}_k S$ is the set of all k-limited identities satisfied by S.

An identity $\mathbf{w} \approx \mathbf{w}'$ is *directly provable* from an identity $\mathbf{u} \approx \mathbf{u}'$ if there exist words $\mathbf{e}, \mathbf{f} \in \mathcal{X}^*$ and a substitution $\varphi : \mathcal{X} \to \mathcal{X}^+$ such that $\mathbf{w} = \mathbf{e}(\mathbf{u}\varphi)\mathbf{f}$ and $\mathbf{w}' = \mathbf{e}(\mathbf{u}'\varphi)\mathbf{f}$. By Birkhoff's completeness theorem of equational logic [2], an identity $\mathbf{w} \approx \mathbf{w}'$ is *provable* from some set Σ of identities if there exists a sequence

$$\mathbf{w} = \mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m = \mathbf{w}'$$

of words such that each identity $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ is directly provable from some identity in Σ. A subset $\Sigma \subseteq \mathsf{id}S$ is a *basis* for S if every identity in $\mathsf{id}S$ is provable from Σ. A semigroup is *finitely based* if it has a finite basis. The *variety* generated by a semigroup S, denoted by $\mathsf{var}\{S\}$, is the class of all semigroups that satisfy all identities from $\mathsf{id}S$. We refer to the survey by Volkov [18] for more information on the finite basis problem for finite semigroups, and to the monograph of Burris and Sankappanavar [2] for more information on universal algebra in general.

Lemma 2.1. *Let Σ be any basis for a finitely based semigroup S. Then a finite basis for S can be chosen from the identities in Σ.*

Proof. This is a well known (and immediate) consequence of Birkhoff's completeness theorem for equational logic (see [1, Corollary 1.4.7] or [2, Exercise 10 of Section 14]). □

Lemma 2.2. *Let S be any semigroup that satisfies the identities (1.1a) but violates the identity (1.2b). Suppose that the identity $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by S. Then $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$ and $\mathsf{non}(\mathbf{w}) = \mathsf{non}(\mathbf{w}')$.*

Proof. By Lee and Zhang [12, Lemma 2.9], the variety $\mathrm{var}\{S\}$ contains the monoid $N_2^1 = \{0, a, 1\}$, where $a^2 = 0$. It is well known that N_2^1 satisfies an identity $\mathbf{w} \approx \mathbf{w}'$ if and only if $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$ and $\mathsf{non}(\mathbf{w}) = \mathsf{non}(\mathbf{w}')$ (see [1, Lemma 6.1.4]). □

3. Connected identities

A *singleton* is a simple word that consists of only one letter. A nonsimple word is *connected* if it cannot be written as a product of two disjoint nonempty words. An identity $\mathbf{w} \approx \mathbf{w}'$ is *connected* if the words $\mathbf{w}$ and $\mathbf{w}'$ are connected. Any word $\mathbf{w}$ can be written in *natural form*, that is,

$$\mathbf{w} = \mathbf{w}_1 \mathbf{w}_2 \cdots \mathbf{w}_r,$$

where $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r \in \mathcal{X}^+$ are pairwise disjoint words, each of which is either a singleton or connected.

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Lemma 3.1. *Let S be any semigroup that satisfies the identities (1.1a) but violates the identity (1.2a). Suppose that S satisfies an identity $\mathbf{w} \approx \mathbf{w}'$, where $\mathbf{w} = \mathbf{w}_1\mathbf{w}_2 \cdots \mathbf{w}_r$ and $\mathbf{w}' = \mathbf{w}'_1\mathbf{w}'_2 \cdots \mathbf{w}'_{r'}$ are words written in natural form. Then $r = r'$ and $\mathsf{con}(\mathbf{w}_i) = \mathsf{con}(\mathbf{w}'_i)$ for all i.*

Proof. The variety $\mathrm{var}\{S\}$ contains the semigroup $A_0 = \langle a, b \mid a^2 = a, b^2 = b, ba = 0\rangle$ ([12, Lemma 2.8]). It follows from Edmunds [3, proof of part 4 of the first proposition] that if A_0 satisfies the identity $\mathbf{w} \approx \mathbf{w}'$, then $r = r'$ and $\mathsf{con}(\mathbf{w}_i) = \mathsf{con}(\mathbf{w}'_i)$ for all i. □

Lemma 3.2. *Let S be any semigroup that satisfies the identities (1.1a) but violates the identities (1.2a) and (1.2b). Then S has a basis that contains only connected identities.*

Proof. Let $\mathbf{w} \approx \mathbf{w}'$ be any nontrivial identity satisfied by the semigroup S and write

$$\mathbf{w} = \mathbf{w}_1\mathbf{w}_2 \cdots \mathbf{w}_r \quad \text{and} \quad \mathbf{w}' = \mathbf{w}'_1\mathbf{w}'_2 \cdots \mathbf{w}'_{r'},$$

in natural form, where, by Lemma 3.1, $r = r'$ and $\mathsf{con}(\mathbf{w}_i) = \mathsf{con}(\mathbf{w}'_i)$ for all i. It suffices to show that S satisfies $\mathbf{w}_i \approx \mathbf{w}'_i$ for all i. If either $\mathbf{w}_i$ or $\mathbf{w}'_i$ is a singleton, then it follows from Lemma 2.2 that $\mathbf{w}_i = \mathbf{w}'_i$, and hence S clearly satisfies $\mathbf{w}_i \approx \mathbf{w}'_i$. If $\mathbf{w}_i$ and $\mathbf{w}'_i$ are connected, then S satisfies $\mathbf{w}_i \approx \mathbf{w}'_i$, by the arguments in [9, Proof of Lemma 12]. □

For any letters x and y of a word $\mathbf{w}$, $x \prec_{\mathbf{w}} y$ indicates that within $\mathbf{w}$, each occurrence of x precedes every occurrence of y. In other words, if $x \prec_{\mathbf{w}} y$ with $p = \mathsf{occ}(x, \mathbf{w})$ and $q = \mathsf{occ}(y, \mathbf{w})$, then retaining only the letters x and y in $\mathbf{w}$ results in the word $x^p y^q$.

For any word $\mathbf{w}$, let $\mathsf{F}_{\mathsf{SS}}(\mathbf{w})$ denote the set of factors of $\mathbf{w}$ of length two that are formed by simple letters, that is,

$$\mathsf{F}_{\mathsf{SS}}(\mathbf{w}) = \{xy \mid x, y \in \mathsf{sim}(\mathbf{w}), \mathbf{w} \in \mathcal{X}^* xy \mathcal{X}^*\}.$$

For example, if $\mathbf{w} = x^3abcyxdy^2efx$, then $\mathsf{F}_{\mathsf{SS}}(\mathbf{w}) = \{ab, bc, ef\}$.

Lemma 3.3. *Let S be any semigroup that satisfies the identities (1.1) but violates the identities (1.2c) and (1.2d). Suppose that S satisfies the identity $\mathbf{w} \approx \mathbf{w}'$. Then*

(i) *for any $x, y \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$, $x \prec_{\mathbf{w}} y$ and $x \prec_{\mathbf{w}'} y$ are equivalent; and*
(ii) $\mathsf{F}_{\mathsf{SS}}(\mathbf{w}) = \mathsf{F}_{\mathsf{SS}}(\mathbf{w}')$.

Proof. (i) Suppose that $x \prec_{\mathbf{w}} y$ and $x \nprec_{\mathbf{w}'} y$ for some $x, y \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$. Then some y occurs before some x in $\mathbf{w}'$. Let $\varphi : \mathcal{X} \to \mathcal{X}^+$ be the substitution given by

$$t \mapsto \begin{cases} x^n & \text{if } t = x, \\ y^n & \text{if } t = y, \\ h^n & \text{otherwise.} \end{cases}$$

Then $h^n x^n (\mathbf{w}\varphi) y^n h^n \overset{(1.1)}{\approx} h^n x^n y^n h^n$ and $h^n x^n (\mathbf{w}'\varphi) y^n h^n \overset{(1.1)}{\approx} h^n (x^n y^n)^2 h^n$, so S satisfies the identity $h^n (x^n y^n)^2 h^n \approx h^n x^n y^n h^n$. But then S also satisfies the identity (1.2c).

(ii) Suppose that $xy \in \mathsf{F}_{\mathsf{SS}}(\mathbf{w}) \backslash \mathsf{F}_{\mathsf{SS}}(\mathbf{w}')$. Let $\varphi : \mathcal{X} \to \mathcal{X}^+$ be the substitution

$$t \mapsto \begin{cases} h^n x & \text{if } t = x, \\ y h^n & \text{if } t = y, \\ h^n & \text{otherwise.} \end{cases}$$

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Then the deduction $h^n(\mathbf{w}\varphi)h^n \overset{(1.1a)}{\approx} h^n xyh^n$ holds. Since $x, y \in \mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$, by Lemma 2.2, and $x \prec_{\mathbf{w}'} y$, by part (i), the deduction $h^n(\mathbf{w}'\varphi)h^n \overset{(1.1a)}{\approx} h^n xh^n yh^n$ holds. Therefore S satisfies the identity $\sigma : h^n xh^n yh^n \approx h^n xyh^n$. Since

$$h^n x^n y^n h^n zh^n x^n h^n \overset{(1.1a)}{\approx} h^n(x^n y^n y^n h^n zh^n x^n)h^n \overset{\sigma}{\approx} h^n x^n y^n x^n y^n h^n zh^n x^n h^n,$$

the semigroup S satisfies the identity (1.2c), which is a contradiction. □

Lemma 3.4. *Let S be any semigroup that satisfies the identities (1.1) but violates all of the identities in (1.2). Suppose that S satisfies the identity $\mathbf{w} \approx \mathbf{w}'$, where*

$$\mathbf{w} = \mathbf{a}\mathbf{w}_1\mathbf{w}_2 \cdots \mathbf{w}_r\mathbf{b}$$

for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r \in \mathcal{X}^+$ are such that

(i) *each $\mathbf{w}_i$ is either a singleton or connected;*
(ii) *at least one of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ is a singleton;*
(iii) *$\mathbf{a}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ are pairwise disjoint; and*
(iv) *$\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r, \mathbf{b}$ are pairwise disjoint.*

Then

$$\mathbf{w}' = \mathbf{a}'\mathbf{w}_1'\mathbf{w}_2' \cdots \mathbf{w}_r'\mathbf{b}' \tag{3.1}$$

for some $\mathbf{a}', \mathbf{b}' \in \mathcal{X}^*$ and $\mathbf{w}_1', \mathbf{w}_2', \ldots, \mathbf{w}_r' \in \mathcal{X}^+$ such that the following hold:

(i′) $\mathsf{con}(\mathbf{a}) = \mathsf{con}(\mathbf{a}')$ *and* $\mathsf{con}(\mathbf{b}) = \mathsf{con}(\mathbf{b}')$*;*
(ii′) $\mathsf{con}(\mathbf{w}_i) = \mathsf{con}(\mathbf{w}_i')$ *for all i;*
(iii′) *if $\mathbf{w}_i$ is a singleton, then $\mathbf{w}_i'$ is a singleton with $\mathbf{w}_i = \mathbf{w}_i'$; and*
(iv′) *if $\mathbf{w}_i$ is connected, then $\mathbf{w}_i'$ is connected.*

Proof. It is convenient to write $\mathcal{E} = \mathsf{con}(\mathbf{a}) \cap \mathsf{con}(\mathbf{b})$, so that

$$\mathsf{con}(\mathbf{a}) = (\mathsf{con}(\mathbf{a}) \backslash \mathsf{con}(\mathbf{b})) \cup \mathcal{E} \quad \text{and} \quad \mathsf{con}(\mathbf{b}) = (\mathsf{con}(\mathbf{b}) \backslash \mathsf{con}(\mathbf{a})) \cup \mathcal{E}.$$

By (iii), (iv) and Lemma 2.2,

(A.1) $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$ and $\mathsf{non}(\mathbf{w}) = \mathsf{non}(\mathbf{w}')$; and
(A.2) $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ is the disjoint union of the sets

$$\mathsf{con}(\mathbf{a}) \backslash \mathsf{con}(\mathbf{b}), \mathsf{con}(\mathbf{w}_1), \mathsf{con}(\mathbf{w}_2), \ldots, \mathsf{con}(\mathbf{w}_r), \mathsf{con}(\mathbf{b}) \backslash \mathsf{con}(\mathbf{a}), \mathcal{E}.$$

Now, for any $a \in \mathsf{con}(\mathbf{a}) \backslash \mathsf{con}(\mathbf{b})$, $x_i \in \mathsf{con}(\mathbf{w}_i)$ and $b \in \mathsf{con}(\mathbf{b}) \backslash \mathsf{con}(\mathbf{a})$, since

$$a \prec_{\mathbf{w}} x_1 \prec_{\mathbf{w}} x_2 \prec_{\mathbf{w}} \cdots \prec_{\mathbf{w}} x_r \prec_{\mathbf{w}} b$$

by (iii) and (iv), it follows from Lemma 3.3 that

$$a \prec_{\mathbf{w}'} x_1 \prec_{\mathbf{w}'} x_2 \prec_{\mathbf{w}'} \cdots \prec_{\mathbf{w}'} x_r \prec_{\mathbf{w}'} b.$$

Therefore, in view of (A.2), the word $\mathbf{w}'$ can be written in the form

$$\mathbf{w}' = \mathbf{a}'\mathbf{w}_1'\mathbf{e}_1\mathbf{w}_2'\mathbf{e}_2 \cdots \mathbf{w}_{r-1}'\mathbf{e}_{r-1}\mathbf{w}_r'\mathbf{b}',$$

where $\mathbf{a}', \mathbf{w}_1', \mathbf{w}_2', \ldots, \mathbf{w}_r', \mathbf{b}' \in \mathcal{X}^+$ and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{r-1} \in \mathcal{X}^*$ are such that

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(A.3) $\mathsf{con}(\mathbf{a}')$ is all letters from $\mathsf{con}(\mathbf{a})\backslash\mathsf{con}(\mathbf{b})$ and possibly some letters from $\mathcal{E}$;

(A.4) $\mathsf{con}(\mathbf{b}')$ is all letters from $\mathsf{con}(\mathbf{b})\backslash\mathsf{con}(\mathbf{a})$ and possibly some letters from $\mathcal{E}$;

(A.5) $\mathsf{con}(\mathbf{w}_i')$ is formed by all letters from $\mathsf{con}(\mathbf{w}_i)$ and possibly some letters from $\mathcal{E}$, with $\mathsf{h}(\mathbf{w}_i'), \mathsf{t}(\mathbf{w}_i') \in \mathsf{con}(\mathbf{w}_i)$; and

(A.6) $\mathsf{con}(\mathbf{e}_i) \subseteq \mathcal{E}$.

It then follows from (A.1) and (A.5) that (iii′) holds. In particular, by (ii), there exists some j such that $\mathbf{w}_j = t$ is a singleton, so that $\mathbf{w}_j' = \mathbf{w}_j = t \in \mathsf{sim}(\mathbf{w}')$.

Suppose that some letter $x \in \mathcal{E}$ belongs to $\mathbf{w}_1'\mathbf{e}_1\mathbf{w}_2'\mathbf{e}_2 \cdots \mathbf{w}_{r-1}'\mathbf{e}_{r-1}\mathbf{w}_r'$. Since $\mathsf{con}(\mathbf{w}_j) \cap \mathcal{E} = \emptyset$ by (A.2) with $\mathbf{w}_j = t$, the letters x and t are different. Therefore x occurs either before or after $t = \mathbf{w}_j'$. By symmetry, it suffices to assume that

(A.7) $x \in \mathsf{con}(\mathbf{w}_1'\mathbf{e}_1\mathbf{w}_2'\mathbf{e}_2 \cdots \mathbf{w}_{j-1}'\mathbf{e}_{j-1})$.

Let $\varphi : \mathcal{X} \to \mathcal{X}^+$ denote the substitution given by

$$z \mapsto \begin{cases} y^n & \text{if } z \in \mathsf{con}(\mathbf{w}_1\mathbf{w}_2 \cdots \mathbf{w}_{j-1}), \\ y^n t & \text{if } z = t = \mathbf{w}_j, \\ x^n & \text{otherwise.} \end{cases}$$

Then the deduction $x^n(\mathbf{w}\varphi)x^n \overset{(1.1a)}{\approx} x^n y^n t x^n$ holds by (iii) and (iv). For any letter $z \in \mathsf{con}(\mathbf{w}_1\mathbf{w}_2 \cdots \mathbf{w}_{j-1})$, since $z \prec_{\mathbf{w}} t = \mathbf{w}_j$, it follows from Lemma 3.3 that $z \prec_{\mathbf{w}'} t = \mathbf{w}_j'$. Therefore

(A.8) if z is any letter in $\mathbf{w}'$ occurring after $t = \mathbf{w}_j'$, then $z \notin \mathsf{con}(\mathbf{w}_1\mathbf{w}_2 \cdots \mathbf{w}_{j-1})$.

Hence by (A.5), (A.7), and (A.8),

$$\begin{aligned} x^n(\mathbf{w}'\varphi)x^n &= x^n(\cdots \mathsf{h}(\mathbf{w}_1')\varphi \cdots x\varphi \cdots t\varphi \cdots)x^n \\ &\in x^n\{x^n, y^n\}^* y^n \{x^n, y^n\}^* x^n \{x^n, y^n\}^* y^n t \{x^n\}^* x^n \overset{(1.1)}{\approx} x^n y^n x^n y^n t x^n. \end{aligned}$$

The semigroup S thus satisfies the identity $x^n y^n t x^n \approx (x^n y^n)^2 t x^n$ and so also the identity (1.2c), which contradicts the assumption. Therefore no letter $x \in \mathcal{E}$ can belong to $\mathbf{w}_1'\mathbf{e}_1\mathbf{w}_2'\mathbf{e}_2 \cdots \mathbf{w}_{r-1}'\mathbf{e}_{r-1}\mathbf{w}_r'$, and hence $\mathbf{e}_1 = \mathbf{e}_2 = \cdots = \mathbf{e}_{r-1} = \emptyset$ by (A.6) and

(A.9) $\mathsf{con}(\mathbf{w}_i) = \mathsf{con}(\mathbf{w}_i')$ for all i

by (A.5). Thus $\mathbf{w}'$ is a word of the form (3.1) that satisfies (ii′) and (iii′).

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Suppose that for some i, the word $\mathbf{w}_i$ is connected but $\mathbf{w}_i'$ is not connected, say $\mathbf{w}_i' = \mathbf{cd}$ for some disjoint $\mathbf{c}, \mathbf{d} \in \mathcal{X}^+$. Let $\chi : \mathcal{X} \to \mathcal{X}^+$ denote the substitution given by

$$z \mapsto \begin{cases} x^n & \text{if } z \in \mathsf{con}(\mathbf{c}), \\ y^n & \text{if } z \in \mathsf{con}(\mathbf{d}), \\ h^n & \text{otherwise.} \end{cases}$$

Since $\mathsf{con}(\mathbf{w}_i') \cap \mathsf{con}(\mathbf{a}'\mathbf{w}_1' \cdots \mathbf{w}_{i-1}'\mathbf{w}_{i+1}' \cdots \mathbf{w}_r'\mathbf{b}') = \emptyset$ by (iii), (iv), (A.3), (A.4) and (A.9), the deduction $h^n x^n(\mathbf{w}'\chi)y^n h^n \stackrel{(1.1a)}{\approx} h^n x^n y^n h^n$ holds. On the other hand, note that $\mathsf{con}(\mathbf{w}_i\chi) = \{x, y\}$ because $\mathsf{con}(\mathbf{w}_i) = \mathsf{con}(\mathbf{w}_i')$. However, since $\mathbf{w}_i$ is connected, $\mathbf{w}_i\chi \notin \{x^n\}^+\{y^n\}^+$. Therefore $\mathbf{w}_i\chi \in \{x^n, y^n\}^* y^n x^n \{x^n, y^n\}^*$, so that

$$h^n x^n(\mathbf{w}\chi)y^n h^n = h^n x^n(\cdots y^n x^n \cdots)y^n h^n \stackrel{(1.1)}{\approx} h^n(x^n y^n)^2 h^n.$$

It follows that the semigroup S satisfies the identity $h^n x^n y^n h^n \approx h^n(x^n y^n)^2 h^n$ and so also the identity (1.2c), which contradicts the assumption. Hence (iv′) is satisfied.

It remains to verify that (i′) is satisfied by $\mathbf{w}'$. By (A.3),

$$\text{(A.10)} \quad \mathsf{con}(\mathbf{a})\backslash\mathsf{con}(\mathbf{b}) \subseteq \mathsf{con}(\mathbf{a}') \subseteq (\mathsf{con}(\mathbf{a})\backslash\mathsf{con}(\mathbf{b})) \cup \mathcal{E} = \mathsf{con}(\mathbf{a}).$$

Suppose that $x \in \mathcal{E}$. Then $x \not\prec_{\mathbf{w}} y \not\prec_{\mathbf{w}} x$ for all $y \in \mathsf{con}(\mathbf{w}_1\mathbf{w}_2 \cdots \mathbf{w}_r)$. Hence (iii), (iv), (A.9) and Lemma 3.3 imply that $x \not\prec_{\mathbf{w}'} y \not\prec_{\mathbf{w}'} x$ for all $y \in \mathsf{con}(\mathbf{w}_1'\mathbf{w}_2' \cdots \mathbf{w}_r')$. It follows that $x \in \mathsf{con}(\mathbf{a}') \cap \mathsf{con}(\mathbf{b}')$. Therefore $\mathcal{E} \subseteq \mathsf{con}(\mathbf{a}') \cap \mathsf{con}(\mathbf{b}')$, so that

$$\begin{aligned} \mathsf{con}(\mathbf{a}) &= (\mathsf{con}(\mathbf{a})\backslash\mathsf{con}(\mathbf{b})) \cup \mathcal{E} \\ &\subseteq \mathsf{con}(\mathbf{a}') \cup (\mathsf{con}(\mathbf{a}') \cap \mathsf{con}(\mathbf{b}')) \quad \text{by (A.10)} \\ &= \mathsf{con}(\mathbf{a}') \subseteq \mathsf{con}(\mathbf{a}) \quad \text{by (A.10).} \end{aligned}$$

Hence $\mathsf{con}(\mathbf{a}) = \mathsf{con}(\mathbf{a}')$. By a symmetrical argument, $\mathsf{con}(\mathbf{b}) = \mathsf{con}(\mathbf{b}')$. □

4. Some restrictions on identities

For each $k \geq 2$, define the sets of words

$$\mathcal{P}_k = \{x^{r_1} y_1^{s_1} y_2^{s_2} \cdots y_k^{s_k} x^{r_2} \mid r_1, r_2 \geq 1, s_1, s_2, \ldots, s_k \geq 2\},$$

and

$$\mathcal{Q}_k = \left\{ \mathbf{w} \in \{x, y_1\}^*\{x, y_2\}^* \cdots \{x, y_k\}^* \;\middle|\; \begin{array}{l} \mathsf{non}(\mathbf{w}) = \{x, y_1, y_2, \ldots, y_k\}, \\ x \not\prec_{\mathbf{w}} y_i \not\prec_{\mathbf{w}} x \text{ for all } i, \\ \mathbf{w} \in \mathcal{X}^* y_1 \mathcal{X}^* x \mathcal{X}^* y_k \mathcal{X}^* \end{array} \right\}.$$

Lemma 4.1. *Let S be any semigroup that satisfies the identities (1.1) but violates all of the identities in (1.2). Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by S such that $\mathbf{w} \in \mathcal{P}_k$. Then $\mathbf{w}' \in \mathcal{P}_k \cup \mathcal{Q}_k$.*

Proof. Since $\mathbf{w} \in \mathcal{P}_k$ implies that $\mathsf{sim}(\mathbf{w}) = \emptyset$, $\mathsf{non}(\mathbf{w}) = \{x, y_1, y_2, \ldots, y_k\}$ and $y_1 \prec_{\mathbf{w}} y_2 \prec_{\mathbf{w}} \cdots \prec_{\mathbf{w}} y_k$, it follows from Lemmas 2.2 and 3.3(i) that $\mathsf{sim}(\mathbf{w}') = \emptyset$, $\mathsf{non}(\mathbf{w}') = \{x, y_1, y_2, \ldots, y_k\}$ and $y_1 \prec_{\mathbf{w}'} y_2 \prec_{\mathbf{w}'} \cdots \prec_{\mathbf{w}'} y_k$. Consequently, we have $\mathbf{w}' \in \{x, y_1\}^*\{x, y_2\}^* \cdots \{x, y_k\}^*$. Further, since $x \not\prec_{\mathbf{w}} y_i \not\prec_{\mathbf{w}} x$ for all i, it follows from Lemma 3.3(i) that $x \not\prec_{\mathbf{w}'} y_i \not\prec_{\mathbf{w}'} x$ for all i. Now if $\mathbf{w}' \in \mathcal{X}^* y_1 \mathcal{X}^* x \mathcal{X}^* y_k \mathcal{X}^*$, then $\mathbf{w}' \in \mathcal{Q}_k$. If $\mathbf{w}' \notin \mathcal{X}^* y_1 \mathcal{X}^* x \mathcal{X}^* y_k \mathcal{X}^*$, then $\mathbf{w}' \in \mathcal{P}_k$. □

Lemma 4.2. *Let S be any semigroup that satisfies the identities (1.1) but violates all of the identities in (1.2). Suppose that an identity $\mathbf{x} \approx \mathbf{y}$ is directly provable from some connected identity in $\mathsf{id}_k S$ with $\mathbf{x} \in \mathcal{P}_k$. Then $\mathbf{y} \in \mathcal{P}_k$.*

Proof. Let $\mathbf{w} \approx \mathbf{w}'$ be a connected identity in $\mathsf{id}_k S$ from which the identity $\mathbf{x} \approx \mathbf{y}$ is directly provable. Then there exist words $\mathbf{e}, \mathbf{f} \in \mathcal{X}^*$ and a substitution $\varphi : \mathcal{X} \to \mathcal{X}^+$ such that $\mathbf{x} = \mathbf{e}(\mathbf{w}\varphi)\mathbf{f}$ and $\mathbf{y} = \mathbf{e}(\mathbf{w}'\varphi)\mathbf{f}$. By assumption,

$$\mathbf{x} = x^{r_1} y_1^{s_1} y_2^{s_2} \cdots y_k^{s_k} x^{r_2} \in \mathcal{P}_k$$

for some $r_1, r_2 \geq 1$ and $s_1, s_2, \ldots, s_k \geq 2$. Since the word $\mathbf{w}$ is connected, the image $\mathbf{w}\varphi$ is a connected factor of $\mathbf{x}$. The connected factors of $\mathbf{x}$ are exhausted by the following:

(†) nonsimple factors of $x^{r_1}, y_1^{s_1}, y_2^{s_2}, \ldots, y_k^{s_k}, x^{r_2}$; and
(‡) $x^{q_1} y_1^{s_1} y_2^{s_2} \cdots y_k^{s_k} x^{q_2}$, where $1 \leq q_1 \leq r_1$ and $1 \leq q_2 \leq r_2$.

Case 1. $\mathbf{w}\varphi$ belongs to (†). It suffices to assume that $\mathbf{w}\varphi$ is a nonsimple factor of some $y_i^{s_i}$, say $\mathbf{w}\varphi = y_i^p$ with $2 \leq p \leq s_i$, since the argument is very similar when $\mathbf{w}\varphi$ is a nonsimple factor of either x^{r_1} or x^{r_2}. Hence

$$\mathbf{x} = \underbrace{x^{r_1} y_1^{s_1} y_2^{s_2} \cdots y_{i-1}^{s_{i-1}} y_i^{p'}}_{\mathbf{e}} \cdot \underbrace{y_i^p}_{\mathbf{w}\varphi} \cdot \underbrace{y_i^{p''} y_{i+1}^{s_{i+1}} \cdots y_k^{s_k} x^{r_2}}_{\mathbf{f}}$$

for some $p', p'' \geq 0$ such that $p' + p + p'' = s_i$. The assumption $\mathbf{w} \approx \mathbf{w}' \in \mathsf{id}_k S$ and Lemma 2.2 imply that $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$ and $\mathsf{non}(\mathbf{w}) = \mathsf{non}(\mathbf{w}')$. It follows that $\mathbf{w}'\varphi = y_i^\ell$ for some $\ell \geq 2$, and hence

$$\mathbf{y} = \mathbf{e}(\mathbf{w}'\varphi)\mathbf{f} = x^{r_1} y_1^{s_1} y_2^{s_2} \cdots y_{i-1}^{s_{i-1}} y_i^{p'} \cdot y_i^\ell \cdot y_i^{p''} y_{i+1}^{s_{i+1}} \cdots y_k^{s_k} x^{r_2} \in \mathcal{P}_k.$$

Case 2. $\mathbf{w}\varphi$ belongs to (‡). Then

$$\mathbf{x} = \underbrace{x^{r_1 - q_1}}_{\mathbf{e}} \cdot \underbrace{x^{q_1} y_1^{s_1} y_2^{s_2} \cdots y_k^{s_k} x^{q_2}}_{\mathbf{w}\varphi} \cdot \underbrace{x^{r_2 - q_2}}_{\mathbf{f}}.$$

Write $\mathbf{w} = \mathbf{a}\mathbf{u}\mathbf{b}$, where $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$ and $\mathbf{u} \in \mathcal{X}^+$ satisfy the following:

(B.1) $\mathbf{a}$ is the longest prefix of $\mathbf{w}$ such that $\mathbf{a}\varphi$ is a prefix of x^{q_1}, say $\mathbf{a}\varphi = x^{p_1}$ for some $p_1 \in \{0, 1, \ldots, q_1\}$; and
(B.2) $\mathbf{b}$ is the longest suffix of $\mathbf{w}$ such that $\mathbf{b}\varphi$ is a suffix of x^{q_2}, say $\mathbf{b}\varphi = x^{p_2}$ for some $p_2 \in \{0, 1, \ldots, q_2\}$.

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It follows that $\mathsf{h}(\mathbf{u})\varphi$ contains the first occurrence of y_1 in $\mathbf{x}$, and $\mathsf{t}(\mathbf{u})\varphi$ contains the last occurrence of y_k in $\mathbf{x}$, that is,

(B.3) $\mathsf{h}(\mathbf{u})\varphi = x^{q_1-p_1}y_1\cdots$ and $\mathsf{t}(\mathbf{u})\varphi = \cdots y_k x^{q_2-p_2}$.

Suppose that $\mathbf{u} = \mathbf{w}_1\mathbf{w}_2\cdots\mathbf{w}_r$ is in natural form, so that

$$\mathbf{w} = \mathbf{a}\mathbf{w}_1\mathbf{w}_2\cdots\mathbf{w}_r\mathbf{b},$$

where $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r \in \mathcal{X}^+$ are pairwise disjoint words, each of which is either a singleton or connected.

We are going to show that the word $\mathbf{w}$ satisfies the conditions of Lemma 3.4. First, it is clear that the word $\mathbf{w}$ satisfies condition (i) of Lemma 3.4.

Lemma 4.3. *The word* $\mathbf{w}$ *satisfies condition (ii) of Lemma 3.4.*

Proof. The assumption $\mathbf{w} \approx \mathbf{w}' \in \mathsf{id}_k S$ implies that $|\mathsf{con}(\mathbf{w})| < k+1 = |\mathsf{con}(\mathbf{w}\varphi)|$. Hence there exists $z \in \mathsf{con}(\mathbf{w})$ such that the factor $z\varphi$ of $\mathbf{w}\varphi$ contains at least two distinct letters. Therefore the word $z\varphi$ contains at least one of the following as factor: $xy_1, y_1y_2, \ldots, y_{k-1}y_k, y_kx$. Since each of these factors occurs precisely once in $\mathbf{w}\varphi$, the letter z is simple in $\mathbf{w}$. Further, since $|\mathsf{con}(z\varphi)| \geq 2$ and $\mathsf{con}((\mathbf{ab})\varphi) = \{x\}$ by (B.1) and (B.2), it follows that $z \notin \mathsf{con}(\mathbf{ab})$. Hence $z \in \mathsf{con}(\mathbf{w}_i)$ for some i. The proof is complete if $z = \mathbf{w}_i$. Therefore assume that $z \neq \mathbf{w}_i$. Then, since $\mathbf{w}_i$ is connected, the letter z is sandwiched between two occurrences of some letter, say $\mathbf{w}_i = \cdots t \cdots z \cdots t \cdots$ for some $t \in \mathcal{X}$. Hence

$$x^{q_1}y_1^{s_1}y_2^{s_2}\cdots y_k^{s_k}x^{q_2} = \mathbf{w}\varphi = \mathbf{a}\varphi\cdot\mathbf{w}_1\varphi\cdots\underbrace{(\cdots t\varphi\cdots z\varphi\cdots t\varphi\cdots)}_{\mathbf{w}_i\varphi}\cdots\mathbf{w}_r\varphi\cdot\mathbf{b}\varphi,$$

but it is easily seen from (B.3) that this is impossible. □

Lemma 4.4. *The word* $\mathbf{w}$ *satisfies conditions (iii) and (iv) of Lemma 3.4.*

Proof. By symmetry, it suffices to show that $\mathsf{con}(\mathbf{a}) \cap \mathsf{con}(\mathbf{u}) = \emptyset$. The result holds vacuously if $\mathsf{con}(\mathbf{a}) = \emptyset$, so assume that $\mathsf{con}(\mathbf{a}) \neq \emptyset$. Let $z \in \mathsf{con}(\mathbf{a}) \cap \mathsf{con}(\mathbf{u})$. Since $z \in \mathsf{con}(\mathbf{a})$, it follows from (B.1) that $z\varphi \in \mathsf{con}(\mathbf{a}\varphi) = \{x\}$, say $z\varphi = x^\ell$ for some $\ell \geq 1$. Further, $\mathsf{h}(\mathbf{u}) \neq z \neq \mathsf{t}(\mathbf{u})$ by (B.3). Then

$$\begin{aligned} x^{q_1}y_1^{s_1}y_2^{s_2}\cdots y_k^{s_k}x^{q_2} &= \mathbf{w}\varphi = \mathbf{a}\varphi\cdot\underbrace{\mathsf{h}(\mathbf{u})\varphi\cdots z\varphi\cdots\mathsf{t}(\mathbf{u})\varphi}_{\mathbf{u}\varphi}\cdot\mathbf{b}\varphi \\ &= x^{p_1}\cdot x^{q_1-p_1}y_1\cdots x^\ell\cdots y_k x^{q_2-p_2}\cdot x^{p_2} \end{aligned}$$

by (B.1)–(B.3), but this is impossible by simple inspection. □

Therefore, since $\mathbf{w}$ satisfies conditions (i)–(iv) of Lemma 3.4, it follows that $\mathbf{w}'$ is a word of the form (3.1) that satisfies (i′)–(iv′). By (i′), (B.1) and (B.2),

(B.4) $\mathsf{con}(\mathbf{a}'\varphi) = \mathsf{con}(\mathbf{a}\varphi) \subseteq \{x\}$ and $\mathsf{con}(\mathbf{b}'\varphi) = \mathsf{con}(\mathbf{b}\varphi) \subseteq \{x\}$.

It is clear that, by (iii′),

(B.5) if $\mathbf{w}_i$ is singleton, then $\mathsf{con}(\mathbf{w}_i'\varphi) = \mathsf{con}(\mathbf{w}_i\varphi)$.

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Suppose that $\mathbf{w}_i$ is connected, so that the factor $\mathbf{w}_i\varphi$ of $\mathbf{w}\varphi$ is also connected. Then, since $\mathbf{w}\varphi = x^{q_1}y_1^{s_1}y_2^{s_2}\cdots y_k^{s_k}x^{q_2}$, the word $\mathbf{w}_i\varphi$ can be any of the following:

(a) $x^{m_1}y_1^{s_1}y_2^{s_2}\cdots y_k^{s_k}x^{m_2}$, where $1 \le m_1 \le q_1$ and $1 \le m_2 \le q_2$;
(b) nonsimple factors of x^{q_1} and of x^{q_2}; or
(c) nonsimple factors of $y_1^{s_2}, y_2^{s_2}, \ldots, y_k^{s_k}$.

Now by (ii), some $\mathbf{w}_j$ is a singleton, so $r \ge 2$. If $\mathbf{w}_i\varphi$ is from (a) and $i > 1$, then φ maps the prefix $\mathbf{a}\mathbf{w}_1\mathbf{w}_2\cdots\mathbf{w}_{i-1}$ of $\mathbf{w}$ to a prefix of x^{q_1}, and hence the maximality of $\mathbf{a}$ in (B.1) is violated. By symmetry, if $\mathbf{w}_i\varphi$ is from (a) and $i < r$, then the maximality of $\mathbf{b}$ in (B.2) is violated. Similarly, if $\mathbf{w}_i\varphi$ is from (b), then either $(\mathbf{a}\mathbf{w}_1\mathbf{w}_2\cdots\mathbf{w}_i)\varphi$ is a prefix of x^{q_1} or $(\mathbf{w}_i\mathbf{w}_{i+1}\cdots\mathbf{w}_r\mathbf{b})\varphi$ is a suffix of x^{q_2}, and hence (B.1) or (B.2) is violated. Therefore the only possibility is for $\mathbf{w}_i\varphi$ to be from (c). By (ii′),

(B.6) if $\mathbf{w}_i$ is connected, then $\mathsf{con}(\mathbf{w}_i'\varphi) = \mathsf{con}(\mathbf{w}_i\varphi) \subseteq \{y_1, y_2, \ldots, y_k\}$.

It is then easily shown by (B.4)–(B.6) that

$$\mathbf{y} = \mathbf{e}(\mathbf{w}'\varphi)\mathbf{f} = \mathbf{e}(\mathbf{a}'\varphi)(\mathbf{w}_1'\varphi)(\mathbf{w}_2'\varphi)\cdots(\mathbf{w}_r'\varphi)(\mathbf{b}'\varphi)\mathbf{f} \notin \mathcal{X}^* y_1 \mathcal{X}^* x \mathcal{X}^* y_k \mathcal{X}^*,$$

so that $\mathbf{y} \notin \mathcal{Q}_k$. Since the identity $\mathbf{x} \approx \mathbf{y}$ is satisfied by the semigroup S with $\mathbf{x} \in \mathcal{P}_k$, it follows from Lemma 4.1 that $\mathbf{y} \in \mathcal{P}_k \cup \mathcal{Q}_k$. Consequently, $\mathbf{y} \in \mathcal{P}_k$. □

5. Proof of Theorem 1.2

Let S be any semigroup that satisfies the identities (1.1) but violates all of the identities from (1.2). By Lemma 3.2, there exists a basis Σ of connected identities for S. Working toward a contradiction, suppose that the semigroup S is finitely based. Then, by Lemma 2.1, a finite basis Σ_{fin} for S can be chosen from the identities in Σ. Hence there exists some fixed integer $k \ge 2$ such that $\Sigma_{\mathsf{fin}} \subseteq \Sigma \cap \mathsf{id}_k S$. Since the semigroup S satisfies the identity $\mathbf{p} \approx \mathbf{q}$ from (1.1b), where

$$\mathbf{p} = xy_1^ny_2^n\cdots y_k^nx \in \mathcal{P}_k \quad \text{and} \quad \mathbf{q} = xy_1^nx^ny_2^nx^n\cdots x^ny_k^nx \in \mathcal{Q}_k,$$

there exists some sequence

$$\mathbf{p} = \mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m = \mathbf{q}$$

of words where each identity $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ is directly provable from some identity in Σ_{fin}. Clearly, $\mathbf{w}_0 = \mathbf{p} \in \mathcal{P}_k$. If $\mathbf{w}_i \in \mathcal{P}_k$ for some $i \ge 0$, then $\mathbf{w}_{i+1} \in \mathcal{P}_k$, by Lemma 4.2. Therefore, by induction, $\mathbf{w}_i \in \mathcal{P}_k$ for all i. But this implies the contradiction $\mathbf{q} = \mathbf{w}_m \in \mathcal{P}_k$. Consequently, the finite basis Σ_{fin} for the semigroup S does not exist.

6. $A^1 \times B^1$ is nonfinitely based

We use Theorem 1.2 to show that the semigroup $A^1 \times B^1$ is nonfinitely based.

Lemma 6.1.

(i) *The semigroup A^1 satisfies the identities (1.1) but violates the identities (1.2a), (1.2b) and (1.2c).*

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(ii) *The semigroup B^1 satisfies the identities (1.1) but violates the identities (1.2a), (1.2b), and (1.2d).*

Proof. By symmetry, it suffices to establish part (i). It is routinely verified that the semigroup A^1 satisfies the identities (1.1a) and

$$xhy^n x \approx xhxy^n x. \tag{6.1}$$

For any $k \geq 2$,

$$\begin{aligned} xy_1^n y_2^n \cdots y_{k-1}^n y_k^n x &\overset{(6.1)}{\approx} xy_1^n y_2^n \cdots y_{k-1}^n x^n y_k^n x \overset{(6.1)}{\approx} xy_1^n y_2^n \cdots x^n y_{k-1}^n x^n y_k^n x \overset{(6.1)}{\approx} \cdots \\ &\overset{(6.1)}{\approx} xy_1^n y_2^n x^n \cdots x^n y_{k-1}^n x^n y_k^n x \overset{(6.1)}{\approx} xy_1^n x^n y_2^n x^n \cdots x^n y_{k-1}^n x^n y_k^n x. \end{aligned}$$

Therefore A^1 also satisfies the identities (1.1b). Since

$$\begin{aligned} (e^n d^n)^{n+1} &= 0 \neq c = e^n d^n, \\ e^n (ae^n)^{n+1} &= 0 \neq a = e^n a e^n, \\ 1^n (e^n d^n)^2 1^n b 1^n e^n 1^n &= 0 \neq a = 1^n e^n d^n 1^n b 1^n e^n 1^n, \end{aligned}$$

the semigroup A^1 violates the identities (1.2a), (1.2b) and (1.2c). □

Theorem 6.2. *The semigroup $A^1 \times B^1$ is non-finitely based.*

Proof. By Lemma 6.1, the semigroup $A^1 \times B^1$ satisfies the identities (1.1) but violates all identities from (1.2). The result thus follows from Theorem 1.2. □

Corollary 6.3. *Let T be any semigroup that satisfies the identities (1.1). Suppose that the semigroups A^1, B^1 belong to the variety generated by T. Then T is nonfinitely based.*

Proof. Since the semigroups A^1, B^1 belong to the variety generated by T, it follows from Lemma 6.1 that the semigroup T violates all of the identities from (1.2). The result now follows from Theorem 1.2. □

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