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AN ISOPERIMETRIC INEQUALITY FOR THE THREAD PROBLEM

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Given a fixed curve C_0 in $\mathbf{R}^n$ of length L_0 and a variable curve C of fixed length $L \leqslant L_0$, the thread problem seeks a least-area surface bounded by $C_0 + C$. We show that an extreme case is a circular arc and its chord. We provide some counterexamples and generalisations to higher dimensions.

1. INTRODUCTION

Given a fixed curve C_0 in $\mathbf{R}^n$ of length L_0 and a variable curve C of fixed length $L \leqslant L_0$, the classical thread problem seeks a least-area surface bounded by $C_0 + C$. The desired isoperimetric inequality for the least area A should take the scale-invariant form

$$A \leqslant L_0^2 f(L/L_0) \tag{1}$$

for a continuous function f which vanishes when $L = L_0$ The classical isoperimetric inequality says only that $A \leqslant (L_0 + L)^2/4\pi = L_0^2(1 + L/L_0)^2/4\pi$, which fails to vanish when $L = L_0$. Theorem 2.3 provides a sharp isoperimetric inequality by proving the extreme case to be a circular arc of arc length L_0 and chordal distance L.

If the curve C_0 is allowed several components, no isoperimetric inequality of form (1) holds (see Section 2.4).

The analogous question in higher dimensions (see Ecker [**5**]) replaces curves and discs by the rectifiable currents (generalised k-dimensional oriented surfaces) of geometric measure theory (see [**6**]). Theorem 3.1 shows that if the fixed boundary surface C_0 is closed, the extreme case is two concentric spheres. Without the closure hypothesis, no general isoperimetric inequality holds (see Section 3.2).

For more information on the thread problem see Dierkes, Hildebrandt, Küster, and Wohlrab [**4**, Chapter 10] and Nitsche [**7**].

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I would like to thank Klaus Ecker for posing the problem and for his hospitality on my visit to Melbourne University in February, 1995. Mike Gage, Bob Connelly, and Charlie Epstein identified Proposition 2.2 as a special case of Schur's Lemma and provided references. The final inequality in Theorem 2.3 is due to two Bennington College undergraduate students of Professor Jerry Bope: Wenhuan Zhao and Jing Lu. This work was partially supported by a National Science Foundation grant.

2. An isoperimetric inequality for the thread problem for curves in $\mathbf{R}^n$

Theorem 2.3 provides our main isoperimetric inequality for the thread problem for curves in $\mathbf{R}^n$. It depends on an extremal property of circular arcs, Proposition 2.2, a special case of Schur's Lemma (see [**3**, p.36]), which follows easily from Lemma 2.1.

Lemma 2.1. *For $0 < a < \pi$, consider a map*

$$\gamma\colon [-a, a] \to S^{n-1},$$

mapping 0 to the south pole, of Lipschitz constant 1. Then $\left|\int \gamma\right|$ is uniquely minimised when γ is an arc of a great circle.

Proof: Let $\int \gamma = (a_1, \ldots, a_n)$. Then $a_n < 0$, and a_n is maximised by an arc of a great circle, for which $a_1 = \ldots = a_{n-1} = 0$. Conversely if a_n is maximised, γ must consist of two arcs of great circles; if further $a_1 = \ldots = a_{n-1} = 0$, together they form an arc of a single great circle. □

Proposition 2.2. *A circular arc in $\mathbf{R}^n$ uniquely minimises the distance between its endpoints among all curves with the same length and no greater curvature.*

(A differentiable curve γ parametrised by arc length has curvature less than or equal to κ if $|\dot\gamma(t_2) - \dot\gamma(t_1)| \leqslant \kappa\,|t_2 - t_1|$.)

Proof: Apply Lemma 2.1 to $\dot\gamma$. □

Theorem 2.3. *Let $C_0\colon [0, L_0] \to \mathbf{R}^n$ be a rectifiable curve parametrised by arc length. Choose another such C of prescribed length L,*

$$|C_0(L_0) - C_0(0)| \leqslant L \leqslant L_0,$$

to minimise the area A of an area-minimising disc D bounded by $C_0 + C$. Then A is less than or equal to the area A_0 bounded by a circular arc of length L_0 and its chord of length L:

$$A \leqslant A_0 \leqslant L_0^2\sqrt{1 - L/L_0}/\sqrt{6}.$$

The final inequality is asymptotically sharp as $L/L_0 \to 1$.

The square root expresses the fact that varying a straight line sweeps out area to first order but changes length only to second order.

Remark. A variational argument shows that C is a $C^{1,1}$ curve, with constant curvature away from C_0 and no greater curvature along C_0, unless it is completely contained in C_0.

Proof: We shall actually show that given C_0 of length L_0 and

$$|C_0(L_0) - C_0(0)| \leqslant L_1 \leqslant L_0,$$

for all $L_1 \leqslant L \leqslant L_0$, A is less than or equal to the area bounded by circular arcs of lengths L_0, L with common chord of length L_1. The special case $L = L_1$ proves the theorem.

If $L = L_0$, then the conclusion holds trivially with $C = C_0$ and $D = \emptyset$. Compactness arguments shows that the set of L for which the conclusion holds is closed. Therefore it suffices to show that if the conclusion holds for any $L > L_1$, it holds for slightly smaller L.

If C is not a circular arc, then its curvature is not weakly bounded by that of a circular arc of length L and chordal length L_1 by Proposition 2.2, because $L_1 \leqslant |C_0(L_0) - C_0(0)|$. Consequently, whether or not C is a circular arc, there are small smooth variations of C reducing length by ε and sweeping out a piece of surface S_ε of area no greater than the area between circular arcs of lengths L and $L - \varepsilon$.

To verify the final inequality, note that in terms of the radian measure θ and radius of curvature r of the arc, its arc length $L_0 = r\theta$, its chord length $L = 2r\sin(\theta/2)$, and its area $A_0 = (\theta - \sin\theta)r^2/2$. (Since $A_0/L_0^2 \sim \theta/12$ and $1 - L/L_0 \sim \theta^2/24$, therefore the inequality is asymptotically sharp.) After squaring both sides, the desired inequality becomes

$$f(\theta) = \theta^4 + 3\theta\sin\theta - (3/2)\theta^2 - (3/2)\sin^2\theta - 2\theta^3\sin(\theta/2) \geqslant 0.$$

The estimates

(1) $\sin\theta \geqslant \theta - \theta^3/3!$,
(2) $\sin^2\theta \leqslant \theta^2$, and
(3) $\sin(\theta/2) \leqslant \theta/2 - \theta^3/2^3 3! + \theta^5/2^5 5!$

imply that

$$f(\theta) \geqslant (\theta^4/2^4 5!)(-\theta^4 + 80\theta^2 - 960) \geqslant 0$$

if $\theta^2 \geqslant 40 - 8\sqrt{10} \approx 14.7$. The estimates

(1′) $\sin\theta \geqslant \theta - \theta^3/3! + \theta^5/5! - \theta^7/7!$,
(2′) $\sin^2\theta \leqslant \theta^2 - 2^3\theta^4/4! + 2^5\theta^6/6! - 2^7\theta^8/8! + 2^9\theta^{10}/10!$

and (3) imply that

$$f(\theta) \geqslant \theta^8(a - b\theta^2) \geqslant 0,$$

with

$$a = (3/2)(2^7/8!) - 3/7! - 2/2^5 5! \approx .00365,$$
$$b = (3/2)(2^9/10!) \approx .000212,$$

if $\theta^2 \leqslant a/b \approx 17.2$. ∎

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2.4 SEVERAL COMPONENTS: If C_0 and C are allowed several components, the least area need not even approach 0 as $L \to L_0$. Let C_0 be the top and bottom of a rectangle of length 1 and height $1-\varepsilon$ and take $L=2$. Then C must be the two sides and the enclosed area equals $1-\varepsilon$, which does not go to 0 as $\varepsilon \to 0$.

One can, however, allow C_0 and C to have additional closed curves and allow surfaces of higher genus. The same proof applies.

3. AN ISOPERIMETRIC INEQUALITY FOR THE THREAD PROBLEM FOR CLOSED SURFACES IN GENERAL DIMENSIONS

Theorem 3.1 provides an isoperimetric inequality for the thread problem for closed k-dimensional surfaces in $\mathbf{R}^n$. As surfaces we use the oriented rectifiable currents of geometric measure theory (see [**6**]), for which length or area (counting multiplicity) is called mass $\mathbf{M}$.

THEOREM 3.1. *Let C_0 be a k-dimensional rectifiable current without boundary in $\mathbf{R}^n$. Choose another such C of prescribed mass*

$$0 \leqslant \mathbf{M}(C) \leqslant \mathbf{M}(C_0)$$

to minimise the mass of a $(k+1)$-dimensional mass-minimising rectifiable current S bounded by $C_0 + C$. Then

$$\mathbf{M}(S) \leqslant \gamma_{k+1}\Big(\mathbf{M}(C_0)^{(k+1)/k} - \mathbf{M}(C)^{(k+1)/k}\Big), \tag{1}$$

where γ_{k+1} is the optimal isoperimetric constant

$$\gamma_{k+1} = \frac{\mathit{vol}\,\mathbf{B}^{k+1}(1)}{(\text{area } \mathbf{S}^k(1))^{(k+1)/k}}.$$

REMARK. The existence of such C and S comes from compactness arguments of geometric measure theory [**6**, 5.5]. (First, for given C, there is a minimiser S. Second, choose C with mass less than or equal to the prescribed mass M_0 to minimise $\mathbf{M}(S)$. Third argue that unless $\mathbf{M}(C) = M_0$, $\mathbf{M}(S)$ could be further reduced.)

A variational argument shows that C has weakly bounded mean curvature (unless it is contained in C_0) and hence is a C^1 embedded manifold on an open dense set [**1**, Section 8]. If $k = n-1$, away from C_0, C is a C^∞ embedded constant-mean-curvature manifold (possibly with multiplicity) except for a singular set of dimension at most $n-8$ [**6**, 8.6].

PROOF: If the prescribed mass $M = \mathbf{M}(C_0)$, then (1) holds trivially with $C = C_0$ and $S = 0$. Compactness arguments show that the set of M for which (1) holds is closed. Therefore it suffices to show that if (1) holds for any M, it holds for slightly smaller M.

If C is not a round sphere, then its mean curvature is not weakly bounded by the mean curvature of a round sphere of area M [**2**, (3), p.452]. Consequently, whether or not C is a round sphere, there are small smooth variations of C reducing mass by ε and sweeping out a piece of $(k+1)$-dimensional surface S_ε of no greater mass than the difference of the volumes of round spheres of area M and $M-\varepsilon$:

$$\mathbf{M}(S_\varepsilon) \leqslant \gamma_{k+1}\left(M^{(k+1)/k} - (M-\varepsilon)^{(k+1)/k}\right).$$

Therefore $S + S_\varepsilon$ satisfies (1). ∎

3.2. Nonclosed surfaces. If the hypothesis that C be closed is omitted from Theorem 3.1, no isoperimetric inequality holds. For example, let C_0 be a portion of a catenoid of area A between two parallel congruent circles which bound discs of total area $A-\varepsilon$. As ε approaches 0, the volume enclosed does not approach 0. The boundary circles may be connected by a thin bridge.

4. A singular example

Example 4.1 (Case b, $L = L_0/2 \leqslant 4\pi$) shows that the thread need not be $C^{1,1}$, even at a point on the support of the area-minimising surface. (The thread is contained in the fixed boundary curve.) We are allowing general oriented threads and surfaces of any number of componments (rectifiable currents).

Example 4.1. Let C_0 be a C^1 "Figure 8" in $\mathbf{R}^2$, consisting of two crossing line segments and two unit circular arcs, of total length L_0, as in Figure 1. For $0 \leqslant L \leqslant L_0$, consider a thread (1-dimensional rectifiable current) C of length L which minimises the area bounded by $C_0 + C$. Then C falls into one of the four cases $a-d$ of Figure 2, consisting of circular arcs of the same curvature and portions of C_0. Each case occurs for some C_0, L, as indicated in Figure 2.

Proof: C_0 and C are boundaries of oriented regions (rectifiable currents) R_0, R. The multiplicity of R_0 is always ± 1. We claim that the multiplicity of R is always ± 1. Otherwise, changing every positive multiplicity of R to $+1$ and every negative multiplicity to -1 would reduce the length $\mathbf{M}(C)$ and the area $\mathbf{M}(R_0 + R)$; but C minimises $\mathbf{M}(R_0 + R)$ among curves with length at most L. Therefore $R = R_1 - R_{-1}$, where R_1 is the region of multiplicity 1 and R_{-1} is the region of multiplicity -1.

Now we note that R_1 is contained in the top half of the Figure 8. Otherwise, replacing R_1 by its restriction to the top half would reduce $M(R_0 + R)$ without increasing the length of C, a contradiction. Similarly, R_{-1} is contained in the bottom half of the Figure 8.

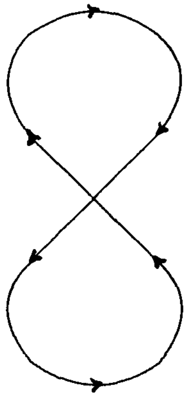

Figure 1 The fixed boundary curve C_0 is a "Figure 8."

By a variational argument, away from the vertex, ∂R is a $C^{1,1}$ curve consisting of circular arcs, all of the same curvature, and pieces of C_0, of no greater curvature. If it contained two circles, expanding one and shrinking the other could maintain area and reduce length (unless neither could be expanded as in degenerate Case c. Therefore, all possibilities are represented by Cases a–d.

It remains to be shown that each case occurs as claimed. Case a must occur if $L \leqslant 2\pi$, because the Cases b–d have $L > 2\pi$. Similarly Cases c, d must occur as claimed. We now show that Case b occurs if $2\pi < L \leqslant \min\{L_0/2, 4\pi\}$ for the equivalent problem of minimising L for fixed area $A \leqslant A_0$, where A_0 is the area of half the Figure 8. We must rule out Case c. Consider generalisations of Case c, in which the circular arc has radius of curvature r_1 and the circle has radius r_2, so that the upper area is of the form $B_0 r_1^2$ and the lower area equals $A_0 - \pi r_2^2$. For a minimiser, $r_1 = r_2 \equiv r_0$; since $B_0 r_0^2 + A_0 - \pi r_0^2 \leqslant A_0$, therefore $B_0 \leqslant \pi$. If we decrease r_1 and r_2 keeping area fixed, $r_1 \geqslant r_2$. Hence the rate of increase of length ∂R_1 is less than the rate of decrease of length ∂R_{-1} and total length decreases as we approach Case b, which must therefore provide the minimiser. ∎

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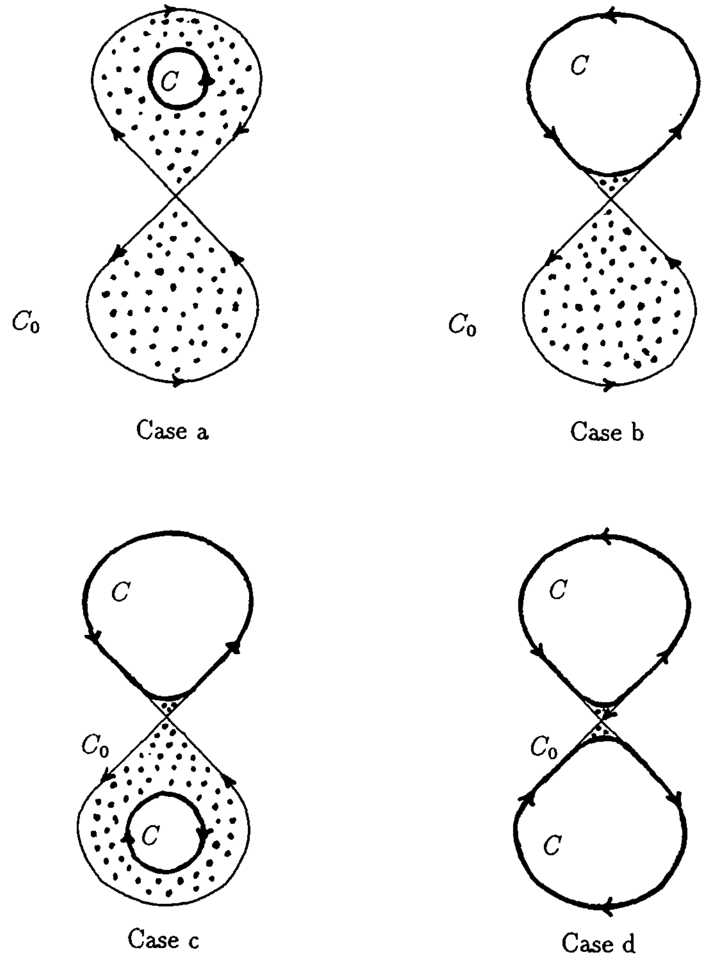

Figure 2 The solution to the thread problem falls into four cases, all of which occur.

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