Glasgow Math. J. **47** (2005) 529–547. © 2005 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089505002764. Printed in the United Kingdom

A UNIFORM DISTRIBUTION RESULT FOR A BILLIARD IN A UNIT SQUARE

E. ALKAN, T. V. SELVAKUMARAN and A. ZAHARESCU
Department of Mathematics, University of Illinois at Urbana-Champaign,
1409 W. Green Street, Urbana, IL, 61801, USA
e-mail: alkan@math.uiuc.edu, selvkmrn@math.uiuc.edu, zaharesc@math.uiuc.edu

(Received 31 March, 2005; accepted 28 April, 2005)

Abstract. In this paper we establish a uniform distribution result for the time spent by a billiard particle in the unit square having vertices (0, 0), (1, 0), (1, 1), (0, 1), with small triangular pockets of size ϵ removed from its corners.

2000 *Mathematics Subject Classification.* 11B57, 11K36.

1. Introduction. A variety of ergodic and statistical properties of the periodic Lorentz gas were investigated during the last decades. Such systems were introduced by Lorentz in 1905 ([**5**]) to study the dynamics of electrons in metals. In [**1**], [**2**] a related problem of Sinai on the length of the trajectory of a billiard in a unit square with small pockets of size ϵ removed at the four corners was considered. In this paper we study a further question about billiards that is meaningful from a physical point of view. We consider a billiard table in the form of the unit square having vertices (0, 0), (1, 0), (1, 1), (0, 1), with small triangular pockets of size ϵ removed from its corners. We look at trajectories of particles that start from the bottom left corner, with constant speed, and angle θ in a small interval $(\alpha - \delta, \alpha + \delta)$. The particle reflects in the side cushions of the table and the trajectory between two such reflections is rectilinear. The particle leaves the table when it reaches one of the four pockets. Let us fix a region Ω inside the unit square. We are interested to see, if θ is chosen randomly, then how much time the particle spends in the region Ω before it leaves the table. Are there any regions inside the unit square where the particle spends more time than in other regions? Or does the particle share its time uniformly over the unit square? As in [**1**], [**2**] we will let $\epsilon \longrightarrow 0$. We will also allow δ to tend to zero while keeping α fixed. Note that if $\tan\alpha$ is rational and δ is as small as ϵ, the particle may be biased towards staying in some prescribed regions of the square. For example, if $\delta = \frac{\epsilon}{3}$ and $\alpha = \arctan\left(\frac{1}{2}\right)$ then all trajectories with $\theta \in (\alpha - \delta, \alpha + \delta)$ bounce exactly once, near the middle point of the right side, before proceeding to fall in the top left pocket. In order to avoid such situations in what follows we will assume that $\tan\alpha$ is irrational and $\epsilon/\delta \to 0$ as $\delta \to 0$. Under these assumptions, we are able to show that one has the following uniform distribution result. For any $\theta \in [0, \pi/2]$ and any region Ω inside the unit square, we denote by $\rho_{\Omega,\epsilon}(\theta)$ the proportion of time that a particle starting its trajectory at an angle θ with the horizontal axis spends in Ω. Here by the proportion of time spent in Ω we mean the ratio between the amount of time spent by the particle inside Ω and the total amount of time spent by the particle in the unit square before reaching one of the pockets. Thus $\rho_{\Omega,\epsilon}(\theta)$ represents the probability that at a randomly chosen moment in time, the particle lies inside Ω. We will denote by $P_{\alpha,\epsilon,\delta}(\Omega)$, the average value of $\rho_{\Omega,\epsilon}(\theta)$

over the short interval $\theta \in (\alpha - \delta, \alpha + \delta)$, i.e.,

$$P_{\alpha,\epsilon,\delta}(\Omega) = \frac{1}{2\delta} \int_{\alpha-\delta}^{\alpha+\delta} \rho_{\Omega,\epsilon}(\theta)\, d\theta. \tag{1}$$

THEOREM 1. *For any* $\alpha \in (0, \pi/2)$ *with* $\tan \alpha$ *irrational and any domain* $\Omega \subseteq [0, 1] \times [0, 1]$ *with piecewise smooth boundary,*

$$\lim_{\delta \to 0, \epsilon/\delta \to 0} P_{\alpha,\epsilon,\delta}(\Omega) = Area(\Omega). \tag{2}$$

2. Two reduction steps. We start the proof of the theorem with two reduction steps. In the first reduction step, we note that it is enough to prove the theorem for the case that Ω is a rectangle. Indeed, let us assume that (2) holds true whenever Ω is a rectangle.

Take now a general domain $\Omega \subseteq [0, 1] \times [0, 1]$ with piecewise smooth boundary. Let us partition the unit square into N^2 smaller squares $T_{n,m}$ each of side length $\frac{1}{N}$,

$$T_{n,m} = \left\{ (x, y) \,\middle|\, \frac{n}{N} \le x \le \frac{n+1}{N}, \frac{m}{N} \le y \le \frac{m+1}{N} \right\}, \quad 0 \le m, n \le N - 1.$$

We denote by E_N the union of those squares that are entirely contained in Ω and by F_N the union of those squares that have non-empty intersection with Ω. Note that $E_N \subseteq \Omega \subseteq F_N$. Under our assumption we know that (2) holds for each square in E_N and in F_N. Note also that for any two disjoint regions Ω_1, Ω_2 we have

$$\rho_{\Omega_1 \cup \Omega_2,\epsilon}(\theta) = \rho_{\Omega_1,\epsilon}(\theta) + \rho_{\Omega_2,\epsilon}(\theta) \quad \text{for any } \theta.$$

Therefore by integration it follows that

$$P_{\alpha,\delta,\epsilon}(\Omega_1 \cup \Omega_2) = P_{\alpha,\delta,\epsilon}(\Omega_1) + P_{\alpha,\delta,\epsilon}(\Omega_2). \tag{3}$$

From this equality we get that if (2) holds for all rectangles, it holds for all finite unions of rectangles. In particular it holds for E_N and F_N. That is,

$$\lim_{\delta \to 0, \epsilon/\delta \to 0} P_{\alpha,\delta,\epsilon}(E_N) = \text{Area}(E_N)$$
$$\lim_{\delta \to 0, \epsilon/\delta \to 0} P_{\alpha,\delta,\epsilon}(F_N) = \text{Area}(F_N).$$

Obviously, $P_{\alpha,\delta,\epsilon}(E_N) \le P_{\alpha,\delta,\epsilon}(\Omega) \le P_{\alpha,\delta,\epsilon}(F_N)$ for any ϵ and δ. Hence it follows that

$$\begin{aligned} \text{Area}(E_N) &= \lim_{\delta \to 0, \epsilon/\delta \to 0} P_{\alpha,\delta,\epsilon}(E_N) \\ &\le \liminf_{\delta \to 0, \epsilon/\delta \to 0} P_{\alpha,\delta,\epsilon}(\Omega) \\ &\le \limsup_{\delta \to 0, \epsilon/\delta \to 0} P_{\alpha,\delta,\epsilon}(\Omega) \\ &\le \lim_{\delta \to 0, \epsilon/\delta \to 0} P_{\alpha,\delta,\epsilon}(F_N) = \text{Area}(F_N). \end{aligned} \tag{4}$$

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We now let $N \longrightarrow \infty$. By the Lipschitz principle on the number of integer points in an r-dimensional domain (see Davenport [**3**]), it follows that

$$\text{Area}\,(F_N \setminus E_N) = O_\Omega\,(1/N)\,, \qquad \text{as } N \longrightarrow \infty.$$

But from (4) we see that

$$\begin{aligned} 0 &\leq \limsup_{\delta\to 0,\epsilon/\delta\to 0} P_{\alpha,\delta,\epsilon}\,(\Omega) - \liminf_{\delta\to 0,\epsilon/\delta\to 0} P_{\alpha,\delta,\epsilon}\,(\Omega) \\ &\leq \text{Area}\,(F_N) - \text{Area}\,(E_N) \\ &= \text{Area}\,(F_N \setminus E_N) = O_\Omega\,(1/N)\,. \end{aligned}$$

Letting $N \longrightarrow \infty$, it follows that

$$\limsup_{\delta\to 0,\epsilon/\delta\to 0} P_{\alpha,\delta,\epsilon}\,(\Omega) = \liminf_{\delta\to 0,\epsilon/\delta\to 0} P_{\alpha,\delta,\epsilon}\,(\Omega).$$

Hence $\lim_{\delta\to 0,\epsilon/\delta\to 0} P_{\alpha,\delta,\epsilon}\,(\Omega)$ exists. Moreover

$$\begin{aligned} \text{Area}\,(E_N) \leq &\qquad \text{Area}\,(\Omega) &\leq \text{Area}\,(F_N) \\ \text{Area}\,(E_N) \leq &\lim_{\delta\to 0,\epsilon/\delta\to 0} P_{\alpha,\delta,\epsilon}\,(\Omega) &\leq \text{Area}\,(F_N) \end{aligned}$$

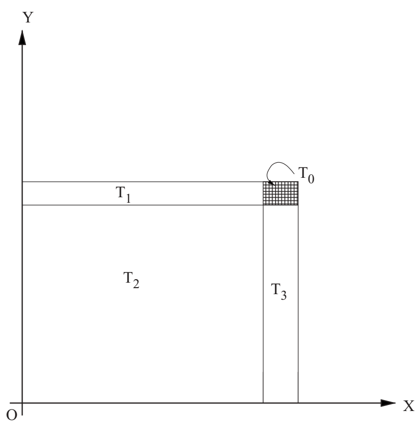
for all N. So,

$$\left|\lim_{\delta\to 0,\epsilon/\delta\to 0} P_{\alpha,\delta,\epsilon}\,(\Omega) - \text{Area}\,(\Omega)\right| \leq \text{Area}\,(E_N \setminus F_N).$$

Again letting $N \longrightarrow \infty$ we find that

$$\lim_{\delta\to 0,\epsilon/\delta\to 0} P_{\alpha,\delta,\epsilon}\,(\Omega) = \text{Area}\,(\Omega).$$

In the second reduction step, we can assume that Ω is a rectangle with two sides as the coordinate axes.



Referring to the figure above we let $T = T_0 \cup T_1 \cup T_2 \cup T_3$ be the rectangle enclosing the four smaller rectangles T_0, T_1, T_2 and T_3. By the hypothesis that (2) holds when Ω

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is a rectangle with two sides as the coordinate axes, we see that

$$P_{\alpha,\delta,\epsilon}(T_2) \longrightarrow \text{Area}(T_2)$$
$$P_{\alpha,\delta,\epsilon}(T_1 \cup T_2) \longrightarrow \text{Area}(T_1 \cup T_2)$$

as $\delta \to 0$, $\epsilon/\delta \to 0$. It follows from equation (3) that

$$P_{\alpha,\delta,\epsilon}(T_1) \longrightarrow \text{Area}(T_1)$$

as $\delta \to 0$, $\epsilon/\delta \to 0$. Similarly,

$$P_{\alpha,\delta,\epsilon}(T_3) \longrightarrow \text{Area}(T_3)$$

as $\delta \to 0$, $\epsilon/\delta \to 0$. But now, applying the hypothesis to T shows that

$$P_{\alpha,\delta,\epsilon}(T) \longrightarrow \text{Area}(T)$$

as $\delta \to 0$, $\epsilon/\delta \to 0$. Applying equation (3) on T we see that

$$P_{\alpha,\delta,\epsilon}(T_0) \longrightarrow \text{Area}(T_0) \tag{5}$$

as $\delta \to 0$, $\epsilon/\delta \to 0$, establishing the second reduction step.

3. Visible points. From now on, Ω will be a fixed rectangle $[0, 2u] \times [0, 2v]$, where the factors of 2 here are inserted to simplify some later computations. By the symmetry of the billiards table with respect to the diagonal $y = x$, we may assume that $(\alpha - \delta, \alpha + \delta) \subseteq [0, \pi/4]$. As in [**1**], [**2**], we work with the equivalent formulation of the billiard problem in the plane. More precisely, we take the unit lattice in the plane, construct around each integer point (n, m) the square with vertices $(n + \epsilon, m)$, $(n, m + \epsilon)$, $(n - \epsilon, m)$, and $(n, m - \epsilon)$, then consider the union of all these squares. Next, for each angle θ consider the linear trajectory that starts at the origin, making an angle θ with the horizontal axis, with the convention that the trajectory ends when it reaches one of these squares. The length of this trajectory equals the length of the trajectory in the original billiard problem. Let us remark that the trajectory never ends in a square around a point (n, m) that is not visible from the origin. Moreover, as was proved in Lemma 3.1 from [**1**], for any $0 \le \theta \le \pi/4$ the corresponding trajectory always ends on the square around a (visible) point (n, m) that lies inside the triangle with vertices $(0, 0)$, $(Q, 0)$, (Q, Q) where $Q = [\frac{1}{\epsilon}]$. This naturally brings Farey fractions into the problem, since the slopes of the straight lines from the origin through the visible points inside the above triangle are exactly the Farey fractions of order Q. We denote by $\mathcal{F}_Q$ the set of Farey fractions a/q with $q \le Q$. For an exposition of the classical properties of Farey fractions, the reader is referred to [**4**], Ch. 3. Now, given an arbitrary angle $\theta \in [0, \pi/4]$, the corresponding slope $\lambda = \tan\theta \in [0, 1]$ will lie in between two consecutive Farey fractions in $\mathcal{F}_Q$, a/q and a'/q' say. Then the trajectory from the origin at angle θ ends at the boundary of the square around one of the visible points (q, a) or (q', a'). More precisely, as shown in [**1**], there is a $\lambda^\star \in [a/q, a'/q']$, whose exact value is given below, with the following property. For any θ for which the corresponding λ lies inside the interval $(a/q, \lambda^\star)$, the trajectory at angle θ ends on the square around the point (q, a), and for any θ for which λ lies inside $(\lambda^\star, a'/q')$, the trajectory at angle θ ends on the square around the point (q', a'). The value of $\lambda^\star$ is

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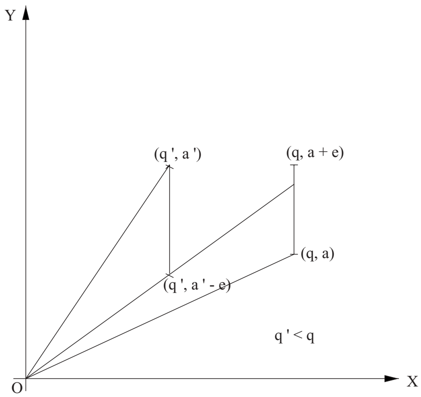
given by

$$\lambda^\star = \begin{cases} \dfrac{a' - \epsilon}{q'}, & \text{if } q' < q \\ \dfrac{a + \epsilon}{q}, & \text{if } q < q'. \end{cases}$$



Next, for each angle θ, we need to take the trajectory at angle θ, which is a segment in the plane, and identify the set of points on this segment which correspond, in the initial formulation of the billiard problem, to points in the unit square that lie inside the rectangle Ω. This set of points is a union of segments in the trajectory, and we need to estimate the ratio between the sum of lengths of these segments and the total length of the trajectory. Then we will need to integrate this ratio with respect to θ over the short interval $(\alpha - \delta, \alpha + \delta)$ in order to obtain the desired probability $\rho_{\Omega,\epsilon}(\theta)$. Now, by symmetry of the unit square with vertices (0, 0), (1, 0), (1, 1), (0, 1) with respect to one of its sides, the rectangle Ω will correspond to a symmetric rectangle inside that next unit square. By repeating this, we obtain replicates of the rectangle Ω inside each unit square with integer vertices in the plane. These replicates fall into four categories. More precisely, they consist of points (x, y) in the plane satisfying one of the following pair of conditions:

I. $n \le x \le n + 2u$ and $m \le y \le m + 2v$, for some even integers n, m,

II. $n \le x \le n + 2u$ and $m + 1 - 2v \le y \le m + 1$, for some even n and odd m,

III. $n + 1 - 2u \le x \le n + 1$ and $m \le y \le m + 2v$, for some odd n and even m,

IV. $n + 1 - 2u \le x \le n + 1$ and $m + 1 - 2v \le y \le m + 1$, for some odd n, m.

In what follows we will only study the contribution to $\rho_{\Omega,\epsilon}(\theta)$ of those points satisfying I above, and will show that this contribution is asymptotic to $uv = \frac{1}{4}Area(\Omega)$. Similarly one can show that each of the remaining three cases contributes asymptotically $\frac{1}{4}Area(\Omega)$, and this would complete the proof of the theorem. Hence, we restrict to the case I, and denote its contribution to $\rho_{\Omega,\epsilon}(\theta)$ and $P_{\alpha,\epsilon,\delta}(\Omega)$ by $\rho_{I,\Omega,\epsilon}(\theta)$ and $P_{I,\alpha,\epsilon,\delta}(\Omega)$ respectively. Thus, in what follows we need to show that $P_{I,\alpha,\epsilon,\delta}(\Omega)$ is asymptotic to $\frac{1}{4}Area(\Omega)$. To proceed, we first make a homothetic transformation of the plane with ratio 1/2. This will not change the ratios of the lengths of segments, and consequently it will leave $P_{I,\alpha,\epsilon,\delta}(\Omega)$ unchanged. The set of points from case I above will be reduced to the set, call it $\mathcal{B}$, of points (x, y) in the plane for which $n \le x \le n + u$ and $m \le y \le m + v$, for some integer numbers m, n. Note that $\mathcal{B}$ is a countable union of rectangles. Now, take a trajectory from the origin, with angle θ, and denote its end point by (X_T, Y_T). Thus, using the notation from above, if the corresponding slope λ lies in the interval $(\frac{a}{q}, \lambda^\star)$, then (X_T, Y_T) is the point of intersection between the line through the origin at angle θ and the boundary of the square with vertices $\frac{1}{2}(q+\epsilon, a)$, $\frac{1}{2}(q, a+\epsilon)$, $\frac{1}{2}(q-\epsilon, a)$, $\frac{1}{2}(q, a-\epsilon)$. Note that $\frac{1}{2}(q-\epsilon) \le X_T \le \frac{1}{2}q$. Let L be the line segment joining the origin and (X_T, Y_T). Then $\mathcal{B} \cap L$ is a finite union of line segments, $\mathcal{B} \cap L = \cup_{j=1}^{r} L_j$. Therefore

$$\rho_{I,\Omega,\epsilon}(\theta) = \sum_{j=1}^{r} \frac{\text{length}\,(L_j)}{\text{length}\,(L)}.$$

In the case when $\lambda \in (\lambda^\star, a'/q')$, one has a similar formula for $\rho_{I,\Omega,\epsilon}(\theta)$ where now (X_T, Y_T) is the intersection between the line through the origin at angle θ and the boundary of the square with vertices $\frac{1}{2}(q'+\epsilon, a')$, $\frac{1}{2}(q', a'+\epsilon)$, $\frac{1}{2}(q'-\epsilon, a')$, $\frac{1}{2}(q', a'-\epsilon)$. Note that $\frac{1}{2}(q'-\epsilon) \le X_T \le \frac{1}{2}q'$. Let L_j' with $1 \le j \le r$ and L' be the projections of L_j and L on the horizontal axis. Then

$$\text{length}\,(L_j') = \text{length}\,(L_j)\cos\theta$$
$$\text{length}\,(L') = \text{length}\,(L)\cos\theta$$

Hence

$$\rho_{I,\Omega,\epsilon}(\theta) = \frac{\sum_{j=1}^{r} \text{length}\,(L_j')}{\text{length}\,(L')}.$$

Here $L' = [0, X_T]$. So length $(L') = X_T$. On the other hand

$$\begin{aligned} \cup_{j=1}^{r} L_j' &= \{x \in [0, X_T] \,|\, \{x\} \in [0, u], \{\lambda x\} \in [0, v]\} \\ &= \{x \in [0, X_T] \,|\, (x, \lambda x) \in \mathcal{B} \cap L\}. \end{aligned}$$

Thus $\rho_{I,\Omega,\epsilon}(\theta) = \frac{1}{X_T}\mu(\mathcal{A})$ where μ denotes the Lebesgue measure on the horizontal line and $\mathcal{A} = \{x \in [0, X_T] \,|\, \{x\} \in [0, u], \{\lambda x\} \in [0, v]\}$. Now we assume that $q < q'$. Then $\frac{1}{2}(q-\epsilon) \le X_T \le \frac{1}{2}q$ gives

$$\frac{2}{q} \le \frac{1}{X_T} \le \frac{2}{q-\epsilon} \le \frac{4}{q},$$

for $0 < \epsilon < 1$. So,

$$\begin{aligned} \frac{1}{X_T} &= \frac{2}{q} + \left(\frac{1}{X_T} - \frac{2}{q}\right) \\ &\le \frac{2}{q} + \frac{4\epsilon}{q^2} \\ &= \frac{2}{q} + O\left(\frac{\epsilon}{q^2}\right). \end{aligned}$$

Now, $0 \le \mu(\mathcal{A}) \le X_T \le q/2$. So,

$$\begin{aligned} \rho_{I,\Omega,\epsilon}(\theta) &= \frac{1}{X_T}\mu(\mathcal{A}) \\ &= \frac{2\mu(\mathcal{A})}{q} + O\left(\frac{\epsilon}{q}\right). \end{aligned}$$

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Therefore, $\rho_{I,\Omega,\epsilon}(\theta) = 2\mu(\mathcal{A})/q + O(\epsilon/q)$, where the term $O(\epsilon/q)$ is independent of θ. Next we define

$$\mathcal{B} = \{x \in [0, q/2) \,|\, \{x\} \leq u, \{\lambda x\} \leq v\}.$$

Note that $\mathcal{A} \subseteq \mathcal{B}$ and

$$\mu(\mathcal{B}) - \epsilon \leq \mu(\mathcal{A}) \leq \mu(\mathcal{B}).$$

So we can replace $\mathcal{A}$ with $\mathcal{B}$ in the computation of $\rho_{I,\Omega,\epsilon}$ with an error term $O(\epsilon/q)$,

$$\rho_{I,\Omega,\epsilon}(\theta) = \frac{2}{q}\mu(\mathcal{B}) + O\left(\frac{\epsilon}{q}\right)$$

Next, we write $\mathcal{B} = \cup_{n=0}^{q^*-1}\mathcal{B}_n$, where $\mathcal{B}_n = \{n \leq x \leq n+u \,|\, \{\lambda x\} \leq v\}$ and q^* denotes the smallest integer larger or equal to $q/2$. So, $\mu(\mathcal{B}) = \sum_{n=0}^{q^*-1}\mu(B_n)$ and

$$\rho_{I,\Omega,\epsilon}(\theta) = \frac{2}{q}\sum_{n=0}^{q^*-1}\mu(\mathcal{B}_n) + O\left(\frac{\epsilon}{q}\right).$$

For any real number β, we define the set

$$\mathcal{C}_\beta = \{0 \leq t \leq u \,|\, \{\beta + \lambda t\} \leq v\}.$$

We note that $\mathcal{B}_n = \mathcal{C}_{\lambda n}$. So we may write

$$\rho_{I,\Omega,\epsilon}(\theta) = \frac{2}{q}\sum_{n=0}^{q^*-1}\mu(\mathcal{C}_{\lambda n}) + O\left(\frac{\epsilon}{q}\right). \tag{6}$$

4. The weight function *h*. We consider the following function $h : \mathbf{R} \longrightarrow [0, \infty)$ periodic with period 1 and which is defined on the interval [0, 1] as follows:

if $u \geq v/\lambda$, $v \leq 1 - \lambda u$, then we set

$$h(\beta) = \begin{cases} \dfrac{v-\beta}{\lambda}, & \text{if } 0 \leq \beta \leq v \\ 0, & \text{if } v \leq \beta \leq 1-\lambda u \\ \dfrac{\beta-(1-\lambda u)}{\lambda}, & \text{if } 1-\lambda u \leq \beta \leq 1-\lambda u+v \\ \dfrac{v}{\lambda}, & \text{if } 1-\lambda u+v \leq \beta \leq 1 \end{cases}$$

if $u \geq v/\lambda$, $v \geq 1 - \lambda u$, then we set

$$h(\beta) = \begin{cases} \dfrac{v-\beta}{\lambda}, & \text{if } 0 \leq \beta \leq 1-\lambda u \\ \dfrac{v+\lambda u-1}{\lambda}, & \text{if } 1-\lambda u \leq \beta \leq v \\ \dfrac{\beta-(1-\lambda u)}{\lambda}, & \text{if } v \leq \beta \leq 1-\lambda u+v \\ \dfrac{v}{\lambda}, & \text{if } 1-\lambda u+v \leq \beta \leq 1. \end{cases}$$

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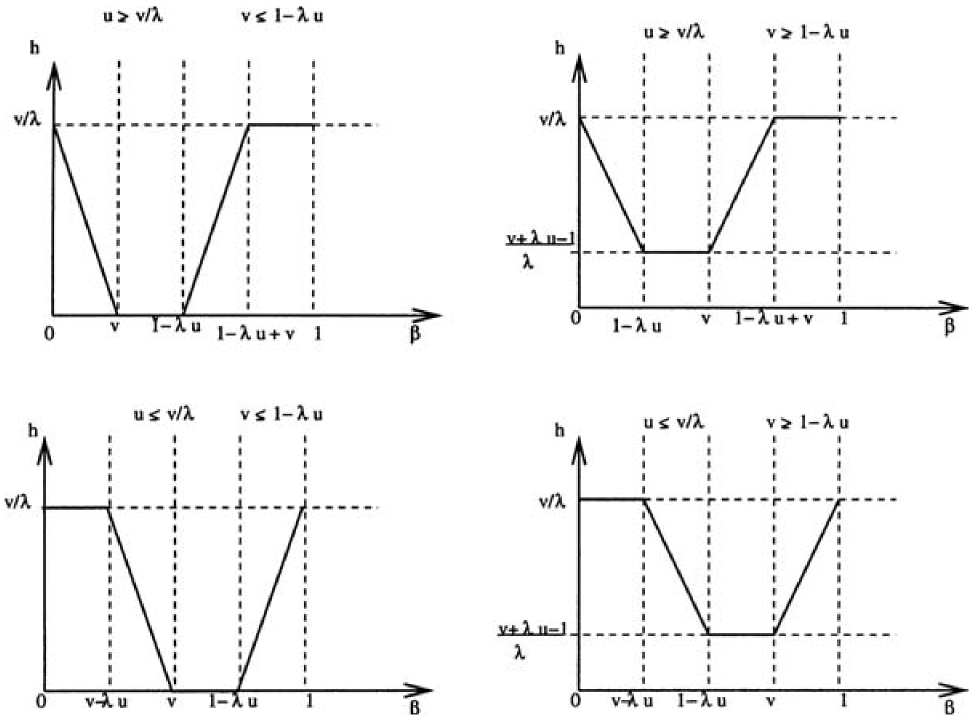
if $u \leq v/\lambda$, $v \leq 1-\lambda u$, then we set

$$h(\beta)=\begin{cases} u, & \text{if } 0 \leq \beta \leq v-\lambda u \\ \dfrac{v-\beta}{\lambda}, & \text{if } v-\lambda u \leq \beta \leq v \\ 0, & \text{if } v \leq \beta \leq 1-\lambda u \\ u-\dfrac{1-\beta}{\lambda}, & \text{if } 1-\lambda u \leq \beta \leq 1. \end{cases}$$

if $u \leq v/\lambda$, $v \geq 1-\lambda u$, then we set

$$h(\beta)=\begin{cases} u, & \text{if } 0 \leq \beta \leq v-\lambda u \\ \dfrac{v-\beta}{\lambda}, & \text{if } v-\lambda u \leq \beta \leq 1-\lambda u \\ \dfrac{v+\lambda u-1}{\lambda}, & \text{if } 1-\lambda u \leq \beta \leq v \\ u-\dfrac{1-\beta}{\lambda}, & \text{if } v \leq \beta \leq 1. \end{cases}$$



We will prove the following lemma.

LEMMA 7. *For any real number β,*

$$\mu\left(C_\beta\right)=h(\beta).$$

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Proof. h is by definition periodic with period 1. We also observe that $\mu\left(C_\beta\right)$ is periodic with period 1. So it is enough to check the equality for $0 \le \beta \le 1$. We rewrite C_β as

$$\begin{aligned} C_\beta &= \{0 \le t \le u \,|\, \beta + t\lambda \in [0, v] \cap [1, 1+v]\} \\ &= \left\{0 \le t \le u \,\middle|\, t \in \left[-\frac{\beta}{\lambda}, \frac{v-\beta}{\lambda}\right] \cup \left[\frac{1-\beta}{\lambda}, \frac{1+v-\beta}{\lambda}\right]\right\} \\ &= \left\{0 \le t \le u \,\middle|\, t \in \left[0, \frac{v-\beta}{\lambda}\right] \cup \left[\frac{1-\beta}{\lambda}, \frac{1+v-\beta}{\lambda}\right]\right\} \\ &= \left([0, u] \cap \left[0, \frac{v-\beta}{\lambda}\right]\right) \cap \left([0, u] \cup \left[\frac{1-\beta}{\lambda}, \frac{1+v-\beta}{\lambda}\right]\right) \end{aligned}$$

Hence

$$\begin{aligned} \mu\left(C_\beta\right) &= \mu\left([0, u] \cap \left[0, \frac{v-\beta}{\lambda}\right]\right) + \mu\left([0, u] \cap \left[\frac{1-\beta}{\lambda}, \frac{1+v-\beta}{\lambda}\right]\right) \\ &= h_1(\beta) + h_2(\beta) \\ &= h(\beta), \end{aligned}$$

where

$$\begin{aligned} h_1(\beta) &= \mu\left([0, u] \cap \left[0, \frac{v-\beta}{\lambda}\right]\right) = \begin{cases} 0 & \text{if } \beta \ge v \\ \min\left(u, \frac{v-\beta}{\lambda}\right) & \text{otherwise,} \end{cases} \\ h_2(\beta) &= \mu\left([0, u] \cap \left[\frac{1-\beta}{\lambda}, \frac{1+v-\beta}{\lambda}\right]\right) = \begin{cases} 0 & \text{if } \beta + u\lambda \le 1 \\ \min\left(u, \frac{1+v-\beta}{\lambda}\right) & \text{otherwise.} \end{cases} \end{aligned}$$

□

Using the previous lemma and equation (6) we obtain

$$\rho_{I,\Omega,\epsilon}(\theta) = \frac{2}{q} \sum_{n=0}^{q^*-1} h(\lambda n) + O\left(\frac{\epsilon}{q}\right).$$

5. Computing the Fourier coefficients. Next, we compute the Fourier coefficients of the function h. Let us first compute the Fourier coefficients of a triangular function of the form

$$g_{a,b}(t) = \begin{cases} 0 & \text{if } t < a, \\ t-a & \text{if } a \le t \le \frac{a+b}{2}, \\ b-t & \text{if } \frac{a+b}{2} \le t \le b, \\ 0 & \text{if } b < t. \end{cases}$$

Let the Fourier expansion of $g_{a,b}$ be

$$g_{a,b}(t) = \sum_{m \in Z} c_{a,b,m} e(mt),$$

where $e(x) = e^{2\pi ix}$. So

$$\begin{aligned} c_{a,b,m} &= \int_0^1 e(-mt)\, g_{a,b}(t)\, dt \\ &= \int_a^{\frac{a+b}{2}} e(-mt)(t-a)\, dt + \int_{\frac{a+b}{2}}^{b} e(-mt)(b-t)\, dt \end{aligned} \tag{8}$$

To compute the first integral in (8), set $u = t - a$, then

$$\begin{aligned} \int_a^{\frac{a+b}{2}} e(-mt)(t-a)\, dt &= \int_0^{\frac{b-a}{2}} e(-m(t+a))\, u\, du \\ &= \int_0^{\frac{b-a}{2}} e^{-2\pi im(u+a)} u\, du \\ &= e^{-2\pi ima}\left[\frac{ue^{-2\pi imu}}{-2\pi im}\Bigg|_0^{\frac{b-a}{2}} - \int_0^{\frac{b-a}{2}} \frac{e^{-2\pi imu}}{-2\pi im} du \right] \\ &= e^{-2\pi ima}\left[\frac{ue^{-2\pi imu}}{-2\pi im} - \frac{e^{-2\pi imu}}{(2\pi im)^2} \right]_0^{\frac{b-a}{2}} \\ &= e^{-2\pi ima}\left[\frac{(b-a)\, e^{-\pi im(b-a)}}{-4\pi im} + \frac{1 - e^{-\pi im(b-a)}}{(2\pi im)^2} \right] \\ &= \frac{(b-a)\, e^{-\pi im(b+a)}}{-4\pi im} + \frac{e^{-2\pi ima} i}{(2\pi im)^2} \left[1 - e^{-\pi im(b-a)}\right] \\ &= \frac{(b-a)}{-4\pi im} e^{-\pi i(b+a)} + \frac{e^{-\pi im(b+a)}}{(2\pi im)^2} \left[e^{\pi im(b-a)} - 1\right]. \end{aligned}$$

Similarly, setting $u = b - t$, we obtain

$$\begin{aligned} \int_{\frac{a+b}{2}}^{b} e(-mt)\, t\, dt &= \int_0^{\frac{b-a}{2}} e(-m(b-u))\, u\, du \\ &= \int_0^{\frac{b-a}{2}} e^{-2\pi im(b-u)} u\, du \\ &= e^{-2\pi imb}\left[\frac{ue^{2\pi imu}}{2\pi im}\Bigg|_0^{\frac{b-a}{2}} - \int_0^{\frac{b-a}{2}} \frac{e^{2\pi imu}}{2\pi im} du \right] \\ &= e^{-2\pi imb}\left[\frac{ue^{-2\pi imu}}{2\pi im} - \frac{e^{2\pi imu}}{(2\pi im)^2} \right]_0^{\frac{b-a}{2}} \\ &= e^{-2\pi imb}\left[\frac{\frac{b-a}{2} e^{2\pi im\frac{(b-a)}{2}}}{2\pi im} - \frac{e^{2\pi im\frac{b-a}{2}} - 1}{(2\pi im)^2} \right] \\ &= \frac{(b-a)\, e^{-\pi i(a+b)}}{4\pi im} + \frac{e^{-\pi im(a+b)}}{(2\pi im)^2} \left(e^{-\pi im(b-a)} - 1\right). \end{aligned}$$

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Therefore,

$$\begin{aligned} c_{a,b,m} &= \frac{e^{-\pi i m(a+b)}}{(2\pi i m)^2}\left[e^{\pi i m(b-a)} + e^{-\pi i m(b-a)} - 2\right] \\ &= \frac{2e^{-\pi i m(a+b)}}{(2\pi i m)^2}\left[\cos\left(\pi m\,(b-a)\right) - 1\right] \\ &= \frac{-4e^{-\pi i m(a+b)}}{(2\pi i m)^2}\sin^2\left(\frac{\pi}{2}m\,(b-a)\right) \\ &= \frac{e^{-\pi i m(a+b)}}{\pi^2 m^2}\sin^2\left(\frac{\pi}{2}m\,(b-a)\right). \end{aligned}$$

We note that

$$|c_{a,b,m}| = \frac{\sin^2\left(\frac{\pi}{2}m\,(b-a)\right)}{\pi^2 m^2} \le \frac{1}{\pi^2 m^2}. \tag{9}$$

Since the graph of h is a trapezoid in all four cases, we see that in each case h can be expressed as a linear combination of $g_{a,b}$'s for suitable values of a, b. More precisely,

$$h(\beta) = \begin{cases} -\frac{1}{\lambda}g_{0,1-\lambda u+v}(\beta) + \frac{1}{\lambda}g_{v,1-\lambda u}(\beta) + \frac{v}{\lambda}, & \text{if } u \le v/\lambda,\ v \le 1-\lambda u \\ -\frac{1}{\lambda}g_{0,1-\lambda u+v}(\beta) + \frac{1}{\lambda}g_{1-\lambda u,v}(\beta) + \frac{v}{\lambda} & \text{if } u \ge v/\lambda,\ v \ge 1-\lambda u \\ -\frac{1}{\lambda}g_{v-\lambda u,1}(\beta) + \frac{1}{\lambda}g_{v,1-\lambda u}(\beta) + u & \text{if } u \le v/\lambda,\ v \ge 1-\lambda u \\ -\frac{1}{\lambda}g_{v-\lambda u,1}(\beta) + \frac{1}{\lambda}g_{1-\lambda u,v}(\beta) + u & \text{if } u \le v/\lambda,\ v \ge 1-\lambda u \end{cases} \tag{10}$$

Now we write the Fourier expansion of h as

$$h(t) = \sum_{m=-\infty}^{\infty} c_m e(-mt)$$

where $c_0 = \int_0^1 h(t)\,dt = uv$. In view of (10) we see that for $m \ne 0$

$$c_m = \begin{cases} -\frac{1}{\lambda}c_{0,1-\lambda u+v,m} + \frac{1}{\lambda}c_{v,1-\lambda u,m} & \text{if } u \ge v/\lambda,\ v \le 1-\lambda u \\ -\frac{1}{\lambda}c_{0,1-\lambda u+v,m} + \frac{1}{\lambda}c_{1-\lambda u,v,m} & \text{if } u \ge v/\lambda,\ v \ge 1-\lambda u \\ -\frac{1}{\lambda}c_{v-\lambda u,1,m} + \frac{1}{\lambda}c_{v,1-\lambda u,m} & \text{if } u \le v/\lambda,\ v \le 1-\lambda u \\ -\frac{1}{\lambda}c_{v-\lambda u,1,m} + \frac{1}{\lambda}c_{1-\lambda u,v,m} & \text{if } u \le v/\lambda,\ v \ge 1-\lambda u \end{cases} \tag{11}$$

From the above formulae for the Fourier coefficients of h it follows that

$$|c_m| \le \frac{2}{\lambda\pi^2 m^2}.$$

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On the other hand,

$$
\begin{aligned}
|c_m| &= \left| \int_0^1 h(t)\, e(-mt)\, dt \right| \\
&\leq \int_0^1 |h(t)|\, dt \\
&= \int_0^1 h(t)\, dt = uv \leq 1.
\end{aligned}
$$

So

$$|c_m| \leq \min\left\{1, \frac{2}{\lambda\pi^2 m^2}\right\}, m \neq 0. \tag{12}$$

6. Proof of Theorem 1. We are now ready to complete the proof of Theorem 1. We set $c_{\Omega,\theta,m} = c_m$ for all $m \in \mathbf{Z}$. We have

$$
\begin{aligned}
\rho_{I,\Omega,\epsilon}(\theta) &= \frac{2}{q} \sum_{n=0}^{q^*-1} h_{\Omega,\theta}(\lambda n) + O\left(\frac{\epsilon}{q}\right) \\
&= \frac{2}{q} \sum_{n=0}^{q^*-1} \sum_{m\in\mathbf{Z}} c_{\Omega,\theta,m} e(m\lambda n) + O\left(\frac{\epsilon}{q}\right) \\
&= \frac{2}{q} \sum_{m\in\mathbf{Z}} c_{\Omega,\theta,m} \sum_{n=0}^{q^*-1} e(nm\lambda) + O\left(\frac{\epsilon}{q}\right)
\end{aligned}
$$

Therefore,

$$\rho_{I,\Omega,\epsilon}(\theta) = \frac{2q^*uv}{q} + \frac{2}{q} \sum_{0\neq m\in\mathbf{Z}} c_{\Omega,\theta,m} \frac{(1 - e(q^* m\lambda))}{1 - e(m\lambda)} + O\left(\frac{\epsilon}{q}\right). \tag{13}$$

Now

$$\left| \sum_{n=0}^{q^*-1} e(mn\lambda) \right| \leq \min\left\{q, \frac{2}{|1 - e(m\lambda)|}\right\} = O\left(\min\left\{q, \frac{1}{\|m\lambda\|}\right\}\right).$$

So

$$
\begin{aligned}
\rho_{I,\Omega,\epsilon}(\theta) &= \frac{2q^*uv}{q} + \frac{2}{q} \sum_{0\neq m\in\mathbf{Z}} c_{\Omega,\theta,m} \sum_{n=0}^{q^*-1} e(nm\lambda) + O\left(\frac{\epsilon}{q}\right) \\
&= \frac{2q^*uv}{q} + O\left(\frac{\epsilon}{q}\right) + O\left(\sum_{m=1}^{\infty} \min\left\{1, \frac{1}{\lambda m^2}\right\} \min\left\{1, \frac{1}{q\|m\lambda\|}\right\}\right).
\end{aligned}
$$

Since $\tan\alpha$ is irrational, if we fix a large positive number V, for δ small enough we will have that for all fractions a/q inside the interval $[\tan(\alpha - \delta), \tan(\alpha + \delta)]$, q will be larger than V. Note also that $\frac{2q^*uv}{q}$ differs from uv by a quantity bounded by $1/q$, therefore

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bounded by $1/V$. It follows that, on average over θ in $[\alpha-\delta,\alpha+\delta]$, this error will be bounded by $1/V$. Thus

$$\begin{aligned} P_{I,\alpha,\epsilon,\delta}(\Omega) &= \frac{1}{2\delta}\int_{\alpha-\delta}^{\alpha+\delta} \rho_{I,\Omega,\epsilon}(\theta)\,d\theta \\ &= uv + O\left(\frac{1}{V}\right) + O(E_{\Omega,\alpha,\delta,\epsilon}) \end{aligned}$$

where

$$E_{\Omega,\alpha,\delta,\epsilon} = \frac{1}{2\delta}\int_{\alpha-\delta}^{\alpha+\delta}\left(\sum_{m=1}^{\infty}\min\left\{1,\frac{1}{\lambda m^2}\right\}\min\left\{1,\frac{1}{q(\theta)\,\|m\lambda\|}\right\}\right)d\theta$$

and $q(\theta)$ depends on θ as described in the beginning of Section 3. Since V may be chosen as large as we please, in order to finish the proof of the theorem it remains to show that

$$\lim_{\delta\to 0,\epsilon/\delta\to 0} E_{\Omega,\alpha,\delta,\epsilon} = 0. \tag{14}$$

Let $\eta_0,\eta_1,\ldots,\eta_r$ be consecutive fractions in $\mathcal{F}_Q$ such that

$$\eta_0 \le (\alpha-\delta) < \eta_1 < \eta_2 < \ldots < \eta_{r-1} < (\alpha+\delta) \le \eta_r$$

where $Q = [\frac{1}{\epsilon}]$. Then

$$E_{\Omega,\alpha,\delta,\epsilon} \le \frac{1}{2\delta}\sum_{m=1}^{\infty}\sum_{j=1}^{r} I_j$$

where

$$I_j = \int_{\arctan\eta_{j-1}}^{\arctan\eta_j}\min\left\{1,\frac{1}{m^2\tan\theta}\right\}\min\left\{1,\frac{1}{q(\theta)\,\|m\tan\theta\|}\right\}d\theta.$$

By the discussion at the beginning of Section 3, we know that for each j we have a $\lambda_j^\star \in [\eta_{j-1},\eta_j]$ such that the following holds: if $\eta_j = a_j/q_j$, $\eta_{j-1} = a_{j-1}/q_{j-1}$, then

$$\lambda_j^\star = \begin{cases} \dfrac{a_j-\epsilon}{q_j} & \text{if } q_j < q_{j-1}, \\ \dfrac{a_{j-1}+\epsilon}{q_{j-1}} & \text{if } q_{j-1} < q_j. \end{cases}$$

We know that for any $\theta \in [\eta_{j-1},\eta_j]$,

$$q(\theta) = \begin{cases} q_{j-1} & \text{if } \tan\theta < \lambda_j^\star \\ q_j & \text{if } \tan\theta > \lambda_j^\star \end{cases}$$

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Now

$$I_j = \int_{\eta_{j-1}}^{\lambda_j^\star} \min\left\{1, \frac{1}{m^2\lambda}\right\} \min\left\{1, \frac{1}{q\|m\lambda\|}\right\} \frac{d\lambda}{1+\lambda^2} + \int_{\lambda_j^\star}^{\eta_j} \min\left\{1, \frac{1}{m^2\lambda}\right\} \min\left\{1, \frac{1}{q_j\|m\lambda\|}\right\} \frac{d\lambda}{1+\lambda^2}$$
$$\leq \int_{\eta_{j-1}}^{\lambda_j^\star} \min\left\{1, \frac{1}{m^2\lambda}\right\} \min\left\{1, \frac{1}{q_{j-1}\|m\lambda\|}\right\} d\lambda + \int_{\lambda_j^\star}^{\eta_j} \min\left\{1, \frac{1}{m^2\lambda}\right\} \min\left\{1, \frac{1}{q_j\|m\lambda\|}\right\} d\lambda$$

Since $\alpha > 0$ is fixed and δ is sufficiently small, we can assume that $\lambda \geq \tan(\alpha - \delta) \leq \tan(\alpha/2)$. Let

$$J_j = \int_{\eta_{j-1}}^{\lambda_j^\star} \min\left\{1, \frac{1}{q_{j-1}\|m\lambda\|}\right\} d\lambda + \int_{\lambda_j^\star}^{\eta_j} \min\left\{1, \frac{1}{q_j\|m\lambda\|}\right\} d\lambda$$

Then

$$I_j \leq \frac{1}{m^2 \tan\frac{\alpha}{2}} J_j.$$

We are done provided we show that

$$\lim_{\delta\to 0, \epsilon/\delta \to 0} \frac{1}{\delta} \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{j=1}^{r_{\alpha,\delta}} J_j = 0.$$

First observe that for any j,

$$J_j \leq \int_{\eta_{j-1}}^{\lambda_j^\star} d\lambda + \int_{\lambda_j^\star}^{\eta_j} d\lambda = \eta_j - \eta_{j-1}.$$

Therefore

$$\sum_{j=1}^{r_{\alpha,\delta}} J_j < \eta_r - \eta_0 \leq (\tan(\alpha+\delta) - \tan(\alpha-\delta)) + (\eta_r - \eta_{r-1}) + (\eta_1 - \eta_0).$$

Since η_r, η_{r-1} are consecutive fractions in $\mathcal{F}_Q$ their difference $\eta_r - \eta_{r-1}$ is at most $1/Q < \epsilon$. Similarly the difference $\eta_1 - \eta_0$ is at most ϵ. Moreover by the Mean Value Theorem,

$$\tan(\alpha+\delta) - \tan(\alpha-\delta) = 2\delta \sec^2 \xi,$$

for some $\xi \in [\alpha - \delta, \alpha + \delta]$. Since $\alpha + \delta \leq \pi/4$ for sufficiently small δ we see that $\sec^2 \xi \leq 2$. Hence

$$\tan(\alpha+\delta) - \tan(\alpha-\delta) \leq 4\delta,$$

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so that

$$\sum_{j=1}^{r_{\alpha,\delta}} J_m, j \le O(\delta).$$

Denote

$$S_{\alpha,\delta,\epsilon} = \frac{1}{\delta}\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{j=1}^{r_{\alpha,\delta}} J_{m,j}.$$

We need to show that

$$\lim_{\delta\to 0,\epsilon/\delta\to 0} S_{\alpha,\delta,\epsilon} = 0.$$

Fix a large number M. Then

$$\frac{1}{\delta}\sum_{m>M}\frac{1}{m^2}\sum_{j=1}^{r_{\alpha,\delta}} J_j = O\left(\sum_{m>M}\frac{1}{m^2}\right) = O\left(\frac{1}{M}\right).$$

Thus

$$S_{\alpha,\delta,\epsilon} = S_M + O\left(\frac{1}{M}\right)$$

where

$$S_M = \frac{1}{\delta}\sum_{m=}^{M}\frac{1}{m^2}\sum_{j=1}^{r_{\alpha,\delta}} J_{m,j} + O\left(\frac{1}{M}\right).$$

If $\lambda \in [\lambda_j^\star, \eta_j]$ then $a_j/q_j - 1/Q \le \lambda \le a_j/q_j$ since $q_j + q_{j-1} > Q$. So $\frac{1}{q_j q_{j-1}} \le \frac{1}{Q}$. Hence

$$\frac{ma_j}{q_j} - \frac{M}{Q} \le m\lambda \le \frac{ma_j}{q_j}.$$

Let $R_{m,j}$ and $n_{m,j}$ be integers, such that

$$ma_j = q_j n_{m,j} + R_{m,j}, \qquad |R_{m,j}| \le \frac{q_j}{2}.$$

So

$$\|m\lambda\| = \left\|m\frac{a_j}{q_j}\right\| + O\left(\frac{M}{Q}\right).$$

Now

$$\left\|m\frac{a_j}{q_j}\right\| = \left|\frac{R_{m,j}}{q_j}\right| \le \frac{1}{2}.$$

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If $q_j > M$, then $q_j \not| m$. So $\frac{ma_j}{q_j} \notin \mathbf{Z}$, $R_{m,j} \neq 0$. Then

$$\begin{aligned}\|m\lambda\| &\geq \left\| m\frac{a_j}{q_j} \right\| - \frac{M}{Q} \\ &= \frac{|R_{m,j}|}{q_j} - \frac{M}{Q} \\ &\geq \frac{|R_{m,j}| - M}{q_j}.\end{aligned}$$

Denote as Type I, the pairs $(m, \frac{a_j}{q_j})$ for which $|R_{m,j}| \leq 2M$, and as Type II the pairs $(m, \frac{a_j}{q_j})$ for which $|R_{m,j}| > 2M$. In Type II $\|m\lambda\| \geq \frac{M}{q_j}$. If both $(m, \frac{a_j}{q_j})$ and $(m, \frac{a_{j-1}}{q_{j-1}})$ are of Type II, then $\|m\lambda\| q_j \geq M$ uniformly for $\lambda \in [\lambda_j^\star, \eta_j]$ and $\|m\lambda\| q_{j-1} \geq M$ uniformly for $\lambda \in [\eta_{j-1}, \lambda_j^\star]$. Then

$$\begin{aligned}J_{m,j} &\leq \int_{\eta_{j-1}}^{\lambda_j^\star} \frac{1}{M}\, d\lambda + \int_{\lambda_j^\star}^{\eta_j} \frac{1}{M}\, d\lambda \\ &= \frac{1}{M}\left(\eta_j - \eta_{j-1}\right).\end{aligned}$$

The total contribution to the sum S_M of terms where both $(m, \frac{a_j}{q_j})$ and $(m, \frac{a_{j-1}}{q_{j-1}})$ are of Type II is at most

$$\begin{aligned}\frac{1}{\delta}\sum_{m=1}^{M}\frac{1}{m^2}\sum_{j=1}^{r_{\alpha,\delta}}\frac{1}{M}\left(\eta_j - \eta_{j-1}\right) &= \frac{1}{\delta M}\sum_{m=1}^{M}\frac{1}{m^2}\left(\eta_{r_{\alpha,\delta}} - \eta_0\right) \\ &= \frac{O(\delta)}{\delta M}\sum_{m=1}^{M}\frac{1}{m^2} \\ &\leq \frac{O(\delta)}{\delta M}\sum_{m=1}^{\infty}\frac{1}{m^2} = O\left(\frac{1}{M}\right).\end{aligned}$$

Hence

$$S_{\alpha,\delta,\epsilon} = S'_M + S''_M + O\left(\frac{1}{M}\right),$$

where

$$S'_M = \frac{1}{\delta}\sum_{m=1}^{M}\frac{1}{m^2}\sum_{j}\left\{J_{m,j} \,|\, 1 \leq j \leq r_{\alpha,\delta}, \text{ and } (m, j) \text{ is of Type I}\right\}$$

and

$$S''_M = \frac{1}{\delta}\sum_{m=1}^{M}\frac{1}{m^2}\sum_{j}\left\{J_{m,j} \,|\, 1 \leq j \leq r_{\alpha,\delta},\ \text{ and }\ (m, j)\ \text{ is of Type II}\right\}.$$

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Then

$$J_{m,j} \leq (\eta_j - \eta_{j-1}) = \frac{1}{q_j q_{j-1}}.$$

So

$$S'_M = \frac{1}{\delta} \sum_{m=1}^{M} \frac{1}{m^2} \sum_j \left\{ \frac{1}{q_j q_{j-1}} \mid 1 \leq j \leq r_{\alpha,\delta}, \text{ and } \left(m, \frac{a_j}{q_j}\right) \text{ is of Type I.} \right\}$$

Let $I_\delta = [\tan(\alpha - \delta), \tan(\alpha + \delta)]$. Then since $\frac{1}{q_j q_{j-1}} \leq \frac{1}{Q}$, we see that

$$\begin{aligned} S'_M &\leq \frac{1}{\delta} \sum_{m=1}^{M} \frac{1}{m^2} \# \left\{ j \,\middle|\, \left(m, \frac{a_j}{q_j}\right) \text{ is of Type I} \right\} \\ &= \sum_{r=-2M}^{2M} \sum_{m=1}^{M} \frac{1}{m^2 Q} \# \left\{ j \,\middle|\, \left(m, \frac{a_j}{q_j}\right) \text{ is Type I, } R_{m,j} = r \right\} \\ &= \frac{1}{\delta} \sum_{r=-2M}^{2M} \sum_{m=1}^{M} \frac{1}{m^2 Q} \# \left\{ \frac{a}{q} \in I_\delta \cap \mathcal{F}_Q, ma \equiv r \bmod q \right\}. \end{aligned}$$

For a given r, the number of solutions to

$$mX \equiv r \bmod q, \qquad 1 \leq X \leq q$$

is $(m, q) \leq m \leq M$. Let $ma = r + kq$. Then

$$|k| = \left| \frac{ma - r}{q} \right| \leq m\frac{a}{q} + \frac{|r|}{q} \leq m + 2M \leq 3M.$$

So

$$S'_M \leq \frac{1}{\delta} \sum_{r=-2M}^{2M} \sum_{k=-3M}^{3M} \sum_{m=1}^{M} \frac{1}{m^2 Q} \# \left\{ \frac{a}{q} \in \mathcal{F}_Q \cap I_\delta, ma = r + kq \right\}.$$

Let

$$\tau_{\alpha,\delta,\epsilon,m,r,k} = \left\{ \frac{a}{q} \in \mathcal{F}_Q \cap I_\delta : ma = r + qk \right\}.$$

Then

$$S'_M \leq \frac{1}{\delta} \sum_{r=-2M}^{2M} \sum_{k=-3M}^{3M} \sum_{m=1}^{M} \frac{1}{m^2} \sum_{a/q \in \tau_{\alpha,\delta,m,r,k}} \frac{1}{qq'} \tag{15}$$

where q' denotes the denominator of the fraction a'/q' immediately before a/q in $\mathcal{F}_Q$. Let

$$W_{\alpha,M} = \min \left\{ \left| \tan\alpha - \frac{k}{m} \right| \mid m, k \in \mathbf{Z}, \ 1 \leq m \leq M, \ |k| \leq 3M \right\}.$$

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Recall that $\tan\alpha$ is irrational, so we have $W_{\alpha,M} > 0$. Let $\delta < W_{\alpha,M}/100$. Then if $a/q \in \tau_{\alpha,\delta,\epsilon,m,r,k}$, then we claim $q \leq 3M/W_{\alpha,M}$. Indeed let $a/q \in \tau_{\alpha,\delta,\epsilon,m,r,k}$. Then $ma = qk + r$. So $a/q = k/m + r/mq$. But $a/q \in I_\delta$. So

$$\left|\frac{a}{q} - \tan\alpha\right| \leq 2\delta < \frac{W_{\alpha,M}}{50}$$
$$\left|\tan\alpha - \frac{k}{m}\right| \geq W_{\alpha,M}$$
$$\left|\frac{a}{q} - \frac{k}{m}\right| = \frac{49}{50}W_{\alpha,M}$$
$$\left|\frac{a}{q} - \frac{k}{m}\right| = \left|\frac{r}{mq}\right| < \frac{2M}{q}.$$

So,

$$\frac{2M}{q} \geq \frac{49}{50}W_{\alpha,M}.$$

Hence,

$$q \leq \frac{100M}{49W_{\alpha,M}} \leq \frac{3M}{W_{\alpha,M}}.$$

Thus (15) becomes

$$S'_M \leq \frac{1}{\delta}\sum_{r=-2M}^{2M}\sum_{k=-3M}^{3M}\sum_{m=1}^{M}\frac{1}{m^2}\sum_{a/q\in\tau_{\alpha,\delta,\epsilon,m,r,k}}\frac{1}{qq'}.$$

Now $q + q' > Q = [1/\epsilon]$. Since $q \leq 3M/W_{\alpha,M}$, we have $q' > 1/2\epsilon$. So

$$S'_M \leq \frac{2\epsilon}{\delta}\sum_{r=-2M}^{2M}\sum_{k=-3M}^{3M}\sum_{m=1}^{M}\frac{1}{m^2}\sum_{q=1}^{3M/W_{\alpha,M}}\frac{1}{q}$$
$$= O_M\left(\frac{\epsilon}{q}\right).$$

Now $q + q' > Q = [1/\epsilon]$. Since $q \leq 3M/W_{\alpha,M}$, we have $q' > \frac{1}{2\epsilon}$. So

$$S'_M \leq \frac{2\epsilon}{\delta}\sum_{r=-2M}^{2M}\sum_{k=-3M}^{3M}\sum_{m=1}^{M}\frac{1}{m^2}\sum_{q=1}^{3M/W_{\alpha,M}}\frac{1}{q}$$
$$= O_M\left(\frac{\epsilon}{q}\right).$$

So $S'_m \longrightarrow 0$ as $\epsilon/\delta \to 0$, $\delta \to 0$ with M fixed. We conclude that

$$S_{\alpha,\delta,\epsilon} = O\left(\frac{1}{M}\right).$$

Letting $M \longrightarrow \infty$, we see that $S_{\alpha,\delta,\epsilon} \longrightarrow 0$. This completes the proof of the theorem.

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