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Weak solutions of complex Hessian equations on compact Hermitian manifolds

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Abstract

We prove the existence of weak solutions of complex m-Hessian equations on compact Hermitian manifolds for the non-negative right-hand side belonging to L^p, $p > n/m$ (n is the dimension of the manifold). For smooth, positive data the equation has recently been solved by Székelyhidi and Zhang. We also give a stability result for such solutions.

1. Introduction

Yau [Yau78] confirmed the Calabi conjecture solving the complex Monge–Ampère equation on compact Kähler manifolds. This fundamental result has been extended in several directions. One can consider weak solutions for a possibly degenerate non-smooth right-hand side (see [Koł98]). Then one can generalize the equation, and here the Hessian equations are a natural choice. The solutions were obtained by Dinew and the first author [DK12b, DK14]. One can also drop the Kähler condition and consider just Hermitian manifolds. The Monge–Ampère equation on compact Hermitian manifolds was solved by Tosatti and Weinkove [TW10] for smooth non-degenerate data and by the authors [KN15a] for the non-negative right-hand side in L^p, $p > 1$. Very recently Székelyhidi [Szé15] and Zhang [Zha15] showed the counterpart of Calabi–Yau theorem for Hessian equations on compact Hermitian manifolds.

As in the real case, geometrically meaningful Hessian equations appear in some 'twisted' non-standard form. Thus, for the Kähler manifolds the Fu–Yau equation [FY08] related to a Strominger system for dimension higher than two becomes the Hessian (two) equation with an extra linear term involving the gradient of the solution. It has recently been studied by Phong *et al.* [PPZ15]. Another form of the Hessian equation is shown to be equivalent to the quaternionic Monge–Ampère equation on HKT manifolds in the paper of Alesker and Verbitsky [AV10]. Some related equations are solved by Székelyhidi, Tosatti and Weinkove in their work on the Gauduchon conjecture [STW15].

The main result of this paper extends the Székelyhidi–Zhang [Szé15, Zha15] theorem as follows.

Theorem. *Let (X,ω) be a compact n-dimensional Hermitian manifold and let m be an integer, $1 \leqslant m < n$. Let $0 \leqslant f \in L^p(X,\omega^n)$, $p > n/m$, and $\int_X f\omega^n > 0$. There exist a continuous (ω, m)-subharmonic function u and a constant $c > 0$ satisfying*

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = cf\omega^n.$$

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We also obtain a stability theorem (Proposition 3.16), which for the Monge–Ampère equation was proven in [KN15b]. To obtain those results we need to adapt the methods of pluripotential theory to Hessian equations and the Hermitian setting. One of the key points, which required a different proof was the counterpart of the Chern–Levine–Nirenberg (CLN) inequality. Another stumbling block is the lack of a natural method of monotone approximation of an (ω, m)-subharmonic function by smooth functions from this class. For plurisubharmonic functions (that is, the case $m = n$) this is possible (see, for example, [BK07, DP04]). On Kähler manifolds Lu and Nguyen [LN15] employed the method of Berman [Ber13] and Eyssidieux *et al.* [EGZ15] to construct smooth approximants of an (ω, m)-subharmonic function. However this method requires the existence theorem for Hessian type equations, so it is far more complicated than the ones starting from convolutions with a smoothing kernel. In the final section we carry out a similar construction to the one in [LN15] on Hermitian manifolds.

2. Estimates in $\mathbb{C}^n$

In this section we wish to develop tools, which correspond to results in pluripotential theory, to study the Hessian equations with respect to a Hermitian form. Some of those analogues, notably the CLN inequalities, do not carry over trivially and they require a careful examination of the properties of positive cones associated with elementary symmetric functions. The difficulty is to control the negative values of a vector belonging to such a cone. First we prove pointwise estimates for the cone in $\mathbb{R}^n$, and then we express them in the language of differential forms which live in the cone associated with a Hermitian metric ω in $\mathbb{C}^n$. Next, we use these results to prove basic 'pluripotential' estimates for (ω, m)-subharmonic functions such as the CLN inequality, the Bedford–Taylor convergence theorem, the weak comparison principle and the like. We refer to [Går59, Ivo83, LT94, Wan77] for the properties of elementary symmetric functions which are used here.

2.1 Properties of elementary positive cones

Let $1 \leqslant m < n$ be two integers. We denote by

$$\Gamma_m = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : S_1(\lambda) > 0, \ldots, S_m(\lambda) > 0\}$$

the symmetric positive cone associated with polynomials

$$S_k(\lambda) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

We use the conventions

$$\begin{aligned} &S_0(\lambda) = 1, \\ &S_k(\lambda) = 0, \quad \text{for } k > n \text{ or } k < 0. \end{aligned}$$

For any fixed t-tuple $\{i_1, \ldots, i_t\} \subseteq \{1, \ldots, n\}$, we write

$$S_{k;i_1 i_2 \cdots i_t}(\lambda) := S_k|_{\lambda_{i_1} = \cdots = \lambda_{i_t} = 0}.$$

So $S_{k;i_1 i_2 \cdots i_t}$ is the kth-order elementary symmetric function of $(n - t)$ variables $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_t\}$. A property that we frequently use in the sequel is

$$S_m(\lambda) \leqslant S_m(\lambda + \mu), \quad \text{for every } \lambda, \mu \in \Gamma_m \tag{2.1}$$

(see [Går59]). Furthermore, a characterization of the cone Γ_m (see, for example, [Ivo83, Lemma 8]) tells that if $\lambda \in \Gamma_m$, then

$$S_{k;i_1,\dots,i_t}(\lambda) > 0 \tag{2.2}$$

for all $\{i_1, \dots, i_t\} \subseteq \{1, \dots, n\}$, $k+t \leqslant m$. In particular, if $\lambda \in \Gamma_m$, then at least m of the numbers $\lambda_1, \dots, \lambda_n$ are positive. Hence, throughout this paper we shall write the entries of $\lambda \in \Gamma_m$ in the decreasing order

$$\lambda_1 \geqslant \cdots \geqslant \lambda_m \geqslant \cdots \lambda_p > 0 \geqslant \lambda_{p+1} \cdots \geqslant \lambda_n \tag{2.3}$$

(with $p \geqslant m$ by the remark above). It is clear that

$$S_k(\lambda) = S_{k;i} + \lambda_i S_{k-1;i}(\lambda). \tag{2.4}$$

Therefore we have the expansion

$$\begin{aligned} S_{k-1}(\lambda) &= S_{k-1;1} + \lambda_1 S_{k-2;1} \\ &= S_{k-1;1} + \lambda_1 S_{k-2;12} + \lambda_1\lambda_2 S_{k-3;12} \\ &= S_{k-1;1} + \lambda_1 S_{k-2;12} + \cdots + \lambda_1 \cdots \lambda_{k-2} S_{1;12\cdots(k-1)} + \lambda_1 \cdots \lambda_{k-1}. \end{aligned} \tag{2.5}$$

It follows from (2.2) that for $\lambda \in \Gamma_m$,

$$S_{m-1}(\lambda) \geqslant \lambda_1 \cdots \lambda_{m-1}. \tag{2.6}$$

A more general statement is also true.

Lemma 2.1. *Let $1 \leqslant k \leqslant m-1$ and $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$. Then, for every $\lambda \in \Gamma_m$,*

$$|\lambda_{i_1} \cdots \lambda_{i_k}| \leqslant C_{n,k} S_k(\lambda),$$

where $C_{n,k}$ depends only on n, k.

Proof. Since $k \leqslant m-1$ and $\lambda \in \Gamma_m \subset \Gamma_{k+1}$, the expansion formula (2.5) gives that

$$S_k \geqslant \lambda_1 \cdots \lambda_k.$$

Therefore, if $\{i_1, \dots, i_k\} \subseteq \{1, \dots, p\}$, that is, $\lambda_{i_t} > 0$ for all $t = 1, \dots, k$, then we are done by the arrangement (2.3). Otherwise, without loss of generality, we may assume that

$$\lambda_{i_1} \geqslant \cdots \geqslant \lambda_{i_s} > 0 > \lambda_{i_{s+1}} \cdots \geqslant \lambda_{i_k}.$$

For brevity we write

$$A = \lambda_{i_1} \cdots \lambda_{i_k}.$$

Consequently,

$$\begin{aligned} |A| &= (\lambda_{i_1} \cdots \lambda_{i_s}) |\lambda_{i_{s+1}} \cdots \lambda_{i_k}| \\ &\leqslant (\lambda_{i_1} \cdots \lambda_{i_s}) |\lambda_{i_k}|^{k-s}. \end{aligned}$$

By (2.2) we have that the sum of any $n-k$ of entries λ_i is positive and hence

$$|\lambda_{i_k}| \leqslant (p-k)\lambda_{k+1}.$$

Note that $p \geqslant m \geqslant k+1$. It follows from the lower bound for S_k that

$$\begin{aligned} |A| &\leqslant (p-k)^{k-s} \lambda_{i_1} \cdots \lambda_{i_s} (\lambda_{k+1})^{k-s} \\ &\leqslant (n-k)^k \lambda_1 \cdots \lambda_k \\ &\leqslant (n-k)^k S_k(\lambda). \end{aligned}$$

Thus, the lemma is proven. □

We also get an upper bound for S_m in terms of $S_{m-1;j}$ as follows. There exists $\theta = \theta(n, m) > 0$ such that for any $j \leqslant m$,

$$\lambda_j S_{m-1;j}(\lambda) \geqslant \theta S_m(\lambda) \quad \text{if } \lambda \in \Gamma_m. \tag{2.7}$$

Indeed, by

$$S_m = S_{m;j} + \lambda_j S_{m-1;j},$$

we see that (2.7) is automatically true if $S_{m;j} \leqslant 0$. Otherwise, $S_{m;j}(\lambda) > 0$, and we can estimate as follows:

$$\begin{aligned} S_m &\leqslant C_{n,m}\lambda_1 \cdots \lambda_m \\ &\leqslant C_{n,m}\lambda_j S_{m-1;j}, \end{aligned}$$

where the second inequality uses (2.2) and (2.5). Inequality (2.7) thus follows.

If $m = n$, then the following result is just a simple consequence of the Cauchy–Schwarz inequality.

Lemma 2.2. *Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\lambda \in \Gamma_m$. Then*

$$\frac{nS_1(\lambda)}{S_m(\lambda)} \cdot \left(\sum_{i=1}^n |a_i|^2 S_{m-1;i}(\lambda) \right) \geqslant \theta \sum_{i=1}^n |a_i|^2,$$

where $\theta = \theta(n, m) > 0$ is the constant in (2.7).

Proof. If $m = 1$, then the statement is obvious. So we may assume that $m \geqslant 2$. Therefore, from (2.3) and (2.6), we have that

$$S_1 \geqslant \lambda_1, \quad S_{m-1,n} \geqslant S_{m-1;n-1} \geqslant \cdots \geqslant S_{m-1;1} > 0.$$

Moreover, by (2.7),

$$\theta S_m \leqslant \lambda_1 S_{m-1;1}.$$

Hence, for $m \geqslant 2$,

$$0 < \frac{S_m}{S_{m-1;n}} \leqslant \cdots \leqslant \frac{S_m}{S_{m-1;1}} \leqslant \frac{\lambda_1}{\theta} \leqslant \frac{S_1}{\theta},$$

and therefore

$$\frac{nS_1}{\theta S_m} \cdot \left(\sum_{i=1}^n |a_i|^2 S_{m-1;i} \right) \geqslant \left(\sum_{i=1}^n \frac{1}{S_{m-1;i}} \right) \left(\sum_{i=1}^n |a_i|^2 S_{m-1;i} \right).$$

The lemma now follows by an application of the Cauchy–Schwarz inequality to the right-hand side of the above inequality. □

2.2 The positive cones associated with a Hermitian metric

Let ω be a Hermitian metric on $\mathbb{C}^n$ and let Ω be a bounded open set in $\mathbb{C}^n$. Given a smooth Hermitian $(1, 1)$-form γ in Ω, we say that γ is (ω, m)-positive if at any point $z \in \Omega$ it satisfies

$$\gamma^k \wedge \omega^{n-k}(z) > 0 \quad \text{for every } k = 1, \ldots, m.$$

Equivalently, in the normal coordinates with respect to ω at z, diagonalizing $\gamma = \sqrt{-1}\sum_i \lambda_i dz_i \wedge d\bar{z}_i$, we have

$$\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_m.$$

This correspondence allows the estimates from § 2.1 to be expressed in the language of differential forms. The first one can be found in [Bło05]. We denote the set of all (ω, m)-positive smooth Hermitian $(1,1)$-forms by $\Gamma_m(\omega, \Omega)$, or $\Gamma_m(\omega)$ when the domain Ω is clear from the context.

Inequality (2.1) is equivalent to

$$(\gamma+\eta)^m \wedge \omega^{n-m} \geqslant \gamma^m \wedge \omega^{n-m} \quad \text{for every } \gamma, \eta \in \Gamma_m(\omega). \tag{2.8}$$

Lemma 2.1 gives a statement which is important for our applications.

Lemma 2.3. *Let $\gamma \in \Gamma_m(\omega)$ and T be a smooth $(n-k, n-k)$-form with $1 \leqslant k \leqslant m-1$. Then*

$$|\gamma^k \wedge T/\omega^n| \leqslant C_{n,k,\|T\|}\, \gamma^k \wedge \omega^{n-k}/\omega^n,$$

where $C_{n,k,\|T\|}$ is a uniform constant depending only on n, k and the sup norm of coefficients of T.

Proof. Fix a point $P \in \Omega$. Choose a local coordinate system at P such that

$$\omega = \sum_{j=1}^n \sqrt{-1} dz_j \wedge d\bar{z}_j \quad \text{and} \quad \gamma = \sum_{j=1}^n \lambda_j \sqrt{-1} dz_j \wedge d\bar{z}_j.$$

In those coordinates we write

$$T = \sum_{|J|=|K|=n-k} T_{JK} dz_J \wedge d\bar{z}_K.$$

In what follows, the computation is performed at P. We first have

$$\gamma^k = k! \sum_{|I|=k, I \subseteq \{1,\dots,n\}} \prod_{i_s \in I} \lambda_{i_s} dz_I \wedge d\bar{z}_I.$$

The non-zero contribution in $\gamma^k \wedge T$ give only triplets of multi-indices $I, J, K \subseteq \{1, \dots, n\}$ such that

$$I \cup J = I \cup K = \{1, \dots, n\},$$

and $|I| = k$. For such sets I, J, K, we have

$$\frac{n!}{k!} (\sqrt{-1})^{(n-k)^2} \gamma^k \wedge dz_J \wedge d\bar{z}_K / \omega^n = \prod_{i_s \in I, |I|=k} \lambda_{i_s}.$$

By Lemma 2.1,

$$\begin{aligned} \prod_{i_s \in I, |I|=k} |\lambda_{i_s}| &\leqslant C_{n,k} S_k(\lambda) \\ &= C_{n,k} \binom{n}{k} \gamma^k \wedge \omega^{n-k}/\omega^n, \end{aligned}$$

where the constant $C_{n,k}$ depends only on n, k. Taking into account the coefficients T_{JK}, we get that each term in

$$|\gamma^k \wedge T/\omega^n|$$

is bounded from above by

$$\gamma^k \wedge \omega^{n-k}/\omega^n,$$

modulo a uniform constant $C_{n,k,\|T\|} = C_{n,k} \sup_{J,K} \|T_{JK}\|_\infty$, where $C_{n,k}$ may differ from the one above. Thus, the lemma follows. ∎

We need to generalize the last result to the case of the wedge product of k smooth Hermitian $(1,1)$-forms in $\Gamma_m(\omega)$ in place of γ^k. To do this, fix $k \leqslant m-2$ and consider vectors $x=(x_1,\dots,x_k)\in[0,1]^k\subset\mathbb{R}^k$. The $\mathbb{R}$-vector space of polynomials in x of degree at most $k+1$ is denoted here by $P_{k+1}(\mathbb{R}^k)$. Its dimension is equal to $d=\binom{2k+1}{k}$. We use multi-indices $\alpha=(\alpha_1,\dots,\alpha_k)\in\mathbb{N}^k$, with the length $|\alpha|:=\alpha_1+\cdots\alpha_k$, and ordered in some fixed fashion. The vector space $P_{k+1}(\mathbb{R}^k)$ has the standard monomial basis

$$\{x^\alpha = x_1^{\alpha_1}\cdots x_k^{\alpha_k} : |\alpha|\leqslant k+1\} =: \{e_1,\dots,e_d\},$$

where $d=\binom{2k+1}{k}$. Choose a set $X=\{X_1,\dots,X_d\}$, with $X_i\in[0,1]^k$, such that the Vandermonde matrix

$$V := \{e_i(X_j)\}_{i,j=\overline{1,d}}$$

is non-singular.

Now, for $x\in[0,1]^k$ and $y=(\gamma_0,\dots,\gamma_k), \gamma_j\in\Gamma_m(\omega)$, consider the polynomial

$$\begin{aligned} P(x,y) &= (\gamma_0+x_1\gamma_1+\cdots+x_k\gamma_k)^{k+1}\wedge T/\omega^n \\ &=: \sum_{|\alpha|\leqslant k+1} b_\alpha(y)x^\alpha, \end{aligned}$$

where T is a smooth $(n-k-1,n-k-1)$-form and

$$b_\alpha(y) = \frac{(k+1)!}{\alpha_1!\cdots\alpha_k!}\gamma_0^{k+1-|\alpha|}\wedge\gamma_1^{\alpha_1}\wedge\cdots\wedge\gamma_k^{\alpha_k}\wedge T/\omega^n.$$

Put $\tau := \gamma_0+x_1\gamma_1+\cdots+x_k\gamma_k$. By Lemma 2.3, we get that for every $x\in[0,1]^k$,

$$|P(x,y)| \leqslant C\tau^{k+1}\wedge\omega^{n-k-1}/\omega^n \leqslant C(\gamma_0+\cdots+\gamma_k)^{k+1}\wedge\omega^{n-k-1}/\omega^n.$$

In particular, the $|P(X_j,y)|$, for $X=\{X_1,\dots,X_d\}$ fixed above, are uniformly bounded by the right-hand side of the last inequality. The coefficients $b_\alpha(y)$ are computed by applying the inverse of V to the column vector consisting of entries $P(X_j,y)$. Since V is a fixed matrix we obtain the desired bound and the following statement.

Corollary 2.4. *Fix $k\leqslant m-2$. Let T be a smooth $(n-k-1,n-k-1)$-form. For $\gamma_0,\dots,\gamma_k\in\Gamma_m(\omega)$ we have*

$$|\gamma_0\wedge\cdots\wedge\gamma_k\wedge T/\omega^n| \leqslant C_{n,m,\|T\|}(\gamma_0+\cdots+\gamma_k)^{k+1}\wedge\omega^{n-k-1}/\omega^n,$$

where $C_{n,k,\|T\|}$ is a uniform constant depending only on n,k and the sup norm of coefficients of T.

We end this subsection with the consequence of Lemma 2.2. This will be used later in the proof of the stability of solutions to the Hessian equations.

Lemma 2.5. *Let ψ be a smooth function and $\gamma\in\Gamma_m(\omega)$. Then*

$$\frac{\sqrt{-1}\partial\psi\wedge\bar\partial\psi\wedge\gamma^{m-1}\wedge\omega^{n-m}}{\gamma^m\wedge\omega^{n-m}}\cdot\frac{\gamma\wedge\omega^{n-1}}{\omega^n} \geqslant \frac{\theta\sqrt{-1}\partial\psi\wedge\bar\partial\psi\wedge\omega^{n-1}}{\omega^n},$$

where $\theta=\theta(n,m)>0$.

Proof. This is an application of Lemma 2.2 in the normal coordinates with respect to ω, where $a=(\psi_1,\dots,\psi_n)$ with $\psi_i:=\partial\psi/\partial z_i$ and λ is the vector of eigenvalues of γ in those coordinates. □

2.3 (ω, m)-subharmonic functions

Let Ω be a bounded open set in $\mathbb{C}^n$. Assume that ω is a Hermitian metric on $\mathbb{C}^n$. Fix an integer $1 \leqslant m < n$. In this subsection we will define the notion of (ω, m)-subharmonicity for non-smooth functions which is adapted from Błocki [Bło05] and Dinew and Kołodziej [DK12b, DK14]. We refer to papers by Lu [Lu13], Lu and Nguyen [LN15] and Dinew and Lu [DL15] for more properties of this class of functions when ω is a Kähler metric. Then we will prove several results which correspond to basic pluripotential theory theorems from [BT76, BT82].

A $C^2(\Omega)$ real-valued function u is called (ω, m)-subharmonic if the associated form $\omega_u := \omega + dd^c u$ belongs to $\overline{\Gamma_m(\omega)}$. This means that

$$\omega_u^k \wedge \omega^{n-k} \geqslant 0 \quad \text{for every } k = 1, \dots, m.$$

Definition 2.6. An upper semicontinuous function $u : \Omega \to [-\infty, +\infty[$ is called (ω, m)-subharmonic if $u \in L^1_{\mathrm{loc}}(\Omega)$ and for any collection of $\gamma_1, \dots, \gamma_{m-1} \in \Gamma_m(\omega)$,

$$(\omega + dd^c u) \wedge \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m} \geqslant 0,$$

with the inequality understood in the sense of currents.

We denote by $\mathrm{SH}_m(\Omega, \omega)$ the set of all (ω, m)-subharmonic functions in Ω. We often write $\mathrm{SH}_m(\omega)$ if the domain is clear from the context.

Remark 2.7. By results of Gårding [Går59], if $u \in C^2(\Omega)$, then u is (ω, m)-subharmonic according to Definition 2.6 if and only if $\omega_u \in \overline{\Gamma_m(\omega)}$. In particular, we have that for $\gamma_1, \dots, \gamma_k \in \Gamma_m(\omega)$, $k \leqslant m$,

$$\gamma_1 \wedge \cdots \wedge \gamma_k \wedge \omega^{n-m}$$

is a strictly positive $(n-m+k, n-m+k)$-form.

By [Mic82, § 4, Equation (4.8)], given $\gamma_1, \dots, \gamma_{m-1} \in \Gamma_m(\omega)$, we can find a Hermitian metric $\tilde{\omega}$ such that

$$\tilde{\omega}^{n-1} = \gamma_1 \wedge \cdots \wedge \gamma_{m-1} \wedge \omega^{n-m}.$$

Thus, according to Definition 2.6, checking the (ω, m)-subharmonicity of a given function u can be reduced to verifying that u is $(\tilde{\omega}, 1)$-subharmonic for a collection of Hermitian metrics $\tilde{\omega}$. Therefore, some properties of $(\omega, 1)$-subharmonic functions are preserved by (ω, m)-subharmonic functions. Below we list several of them and refer to [DK14, Lu13] for more (if the Kähler condition does not play a role).

Proposition 2.8. *Let Ω be a bounded open set in $\mathbb{C}^n$.*

(a) *If $u, v \in \mathrm{SH}_m(\omega)$, then $\max\{u, v\} \in \mathrm{SH}_m(\omega)$.*

(b) *Let $\{u_\alpha\}_{\alpha \in I} \subset \mathrm{SH}_m(\omega)$ be a family locally uniformly bounded from above, and $u := \sup_\alpha u_\alpha$. Then, the upper semicontinuous regularization u^* is (ω, m)-subharmonic.*

It follows from Remark 2.7 (see also [Bło05]) that for any collection of $C^2(\Omega)$ (ω, m)-subharmonic functions $u_1, \dots, u_k$ with $1 \leqslant k \leqslant m$,

$$\omega_{u_1} \wedge \cdots \wedge \omega_{u_k} \wedge \omega^{n-m} \tag{2.9}$$

is a positive form. The above properties of (ω, m)-subharmonic functions are the same as in the Kähler case. However, there are differences too. If we replace the exponent $n - m$ by a smaller one, then the positivity of the differential form (2.9) is no longer true in general. This makes computations involving integration by parts more tricky.

Let $u_1, \ldots, u_p \in \mathrm{SH}_m(\omega) \cap C^2(\Omega)$. If we write

$$\omega_{u_{j_1}} \wedge \cdots \wedge \hat{\omega}_{u_j} \wedge \cdots \wedge \omega_{u_{j_q}},$$

where $j_1 \leqslant j \leqslant j_q$, the hat symbol indicates that the term does not appear in the wedge product. Then we have

$$\begin{aligned} d(\omega_{u_1} \wedge \cdots \wedge \omega_{u_p} \wedge \omega^{n-m}) = \sum_{j=1}^{p} & d\omega \wedge \omega_{u_1} \wedge \cdots \wedge \hat{\omega}_{u_j} \wedge \cdots \wedge \omega_{u_p} \wedge \omega^{n-m} \\ & + (n-m) d\omega \wedge \omega_{u_1} \wedge \cdots \wedge \omega_{u_p} \wedge \omega^{n-m-1} \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} & dd^c(\omega_{u_1} \wedge \cdots \wedge \omega_{u_p} \wedge \omega^{n-m}) \\ & \quad = \sum_{1 \leqslant j \leqslant p} dd^c\omega \wedge \omega_{u_1} \wedge \cdots \wedge \hat{\omega}_{u_j} \wedge \cdots \wedge \omega_{u_p} \wedge \omega^{n-m} \\ & \qquad + \sum_{i \neq j; 1 \leqslant i,j \leqslant p} d\omega \wedge d^c\omega \wedge \omega_{u_1} \wedge \cdots \wedge \hat{\omega}_{u_i} \wedge \cdots \wedge \hat{\omega}_{u_j} \wedge \cdots \wedge \omega_{u_p} \wedge \omega^{n-m} \\ & \qquad + 2(n-m) \sum_{1 \leqslant j \leqslant p} d\omega \wedge d^c\omega \wedge \omega_{u_1} \wedge \cdots \wedge \hat{\omega}_{u_j} \wedge \cdots \wedge \omega_{u_p} \wedge \omega^{n-m-1} \\ & \qquad + (n-m) dd^c\omega \wedge \omega_{u_1} \wedge \cdots \wedge \omega_{u_p} \wedge \omega^{n-m-1} \\ & \qquad + (n-m)(n-m-1) d\omega \wedge d^c\omega \wedge \omega_{u_1} \wedge \cdots \wedge \omega_{u_p} \wedge \omega^{n-m-2}. \end{aligned} \tag{2.11}$$

In those formulas forms of three types appear:

$$\begin{gathered} \omega_{u_1} \wedge \cdots \wedge \hat{\omega}_{u_j} \wedge \omega_{u_p} \wedge \omega^{n-m-1}, \\ \omega_{u_1} \wedge \cdots \wedge \omega_{u_p} \wedge \omega^{n-m-1}, \\ \omega_{u_1} \wedge \cdots \wedge \omega_{u_p} \wedge \omega^{n-m-2}. \end{gathered}$$

As ω_{u_i} is not a positive $(1,1)$-form, these forms are not necessarily positive (the exponent of ω is less than $n-m$). Therefore, in the estimates that follow, we cannot directly apply the bounds for $dd^c\omega$ or $d\omega \wedge d^c\omega$ in terms of ω^2 or ω^3 as in the case of the Monge–Ampère equation. Fortunately, the results from previous subsections make the important estimates go through if $p \leqslant m-1$ (see Corollary 2.4).

We are ready to prove the CLN inequality which guarantees the compactness of a sequence of Hessian measures provided that (ω, m)-subharmonic potentials are uniformly bounded.

Proposition 2.9 (CLN inequality). *Let $K \subset\subset U \subset\subset \Omega$, where K is compact and U is open. Let $u_1, \ldots, u_k \in \mathrm{SH}_m(\omega) \cap C^2(\Omega)$, $1 \leqslant k \leqslant m$. Then, there exists a constant $C_{K,U,\omega} > 0$ such that*

$$\int_K \omega_{u_1} \wedge \cdots \wedge \omega_{u_k} \wedge \omega^{n-k} \leqslant C_{K,U,\omega} \left(1 + \sum_{j=1}^{k} \|u_j\|_{L^\infty(U)}\right)^k.$$

Proof. Observe that, by (2.8),

$$\omega_{u_1} \wedge \cdots \wedge \omega_{u_k} \wedge \omega^{n-k} \leqslant k^k \left(\omega + dd^c \frac{u_1 + \cdots + u_k}{k}\right)^k \wedge \omega^{n-k}.$$

Set $u := (u_1 + \cdots + u_k)/k$. Thus we are reduced to estimating $\int_K \omega_u^k \wedge \omega^{n-k}$, where $\omega_u \in \Gamma_m(\omega)$.

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Our proof is by induction on k. For $k = 1$, let χ be a cut-off function such that $\chi = 1$ on K and $\operatorname{supp}\chi \subset\subset U$. Then

$$\int_K \omega_u \wedge \omega^{n-1} \leqslant \int \chi\omega_u \wedge \omega^{n-1} = \int \chi\omega^n + \int \chi dd^c u \wedge \omega^{n-1}.$$

It is clear that $\int \chi\omega^n \leqslant C_{K,U,\omega}$, and by integration by parts we have

$$\int \chi dd^c u \wedge \omega^{n-1} = \int u dd^c(\chi\omega^{n-1}) \leqslant C_{K,U,\omega}\|u\|_{L^\infty(U)}.$$

Thus, the CLN inequality holds for $k = 1$. Suppose now that

$$\int_K \omega_u^{k-1} \wedge \omega^{n-k+1} \leqslant C_{K,U,\omega}(1 + \|u\|_{L^\infty(U)})^{k-1}.$$

We need to infer the inequality

$$\int_K \omega_u^k \wedge \omega^{n-k} \leqslant C_{K,U,\omega}(1 + \|u\|_{L^\infty(U)})^k.$$

Indeed, as

$$\begin{aligned}\int_K \omega_u^k \wedge \omega^{n-k} &\leqslant \int \chi\omega_u^k \wedge \omega^{n-k}\\ &= \int \chi\omega_u^{k-1} \wedge \omega^{n-k+1} + \int \chi dd^c u \wedge \omega_u^{k-1} \wedge \omega^{n-k},\end{aligned}$$

using the induction hypothesis, it is enough to estimate the second term on the right-hand side. Integration by parts gives

$$\int \chi dd^c u \wedge \omega_u^{k-1} \wedge \omega^{n-k} = \int u dd^c(\omega_u^{k-1} \wedge \chi\omega^{n-k}).$$

An elementary computation yields

$$\begin{aligned}dd^c(\omega_u^{k-1} \wedge \chi\omega^{n-k}) = {}& (k-1)(k-2)\omega_u^{k-3} \wedge d\omega \wedge d^c\omega \wedge \chi\omega^{n-k}\\ &+ (k-1)\omega_u^{k-2} \wedge dd^c\omega \wedge \chi\omega^{n-k}\\ &- (k-1)\omega_u^{k-2} \wedge d^c\omega \wedge d(\chi\omega^{n-k})\\ &+ (k-1)\omega_u^{k-2} \wedge d\omega \wedge d^c(\chi\omega^{n-k})\\ &+ \omega_u^{k-1} \wedge dd^c(\chi\omega^{n-k}).\end{aligned}$$

Since $k \leqslant m$, applying Lemma 2.3 for $\gamma = \omega_u$, we get that

$$|u dd^c(\omega_u^{k-1} \wedge \chi\omega^{n-k})| \leqslant C_{K,U,\omega}\|u\|_{L^\infty(U)}(\omega_u^{k-1} \wedge \omega^{n-k+1} + \omega_u^{k-2} \wedge \omega^{n-k+2} + \omega_u^{k-3} \wedge \omega^{n-k+3}).$$

This implies that $|\int \chi dd^c u \wedge \omega_u^{k-1} \wedge \omega^{n-k}|$ is bounded by

$$C_{K,U,\omega}\|u\|_{L^\infty(U)} \int_{\operatorname{supp}\chi} (\omega_u^{k-1} \wedge \omega^{n-k+1} + \omega_u^{k-2} \wedge \omega^{n-k+2} + \omega_u^{k-3} \wedge \omega^{n-k+3}).$$

Combined with the induction hypothesis, this finishes the proof. □

For general $1 < m < n$ and Hermitian metrics ω, it is not known yet that any (ω, m)-subharmonic function is approximable by a decreasing sequence of smooth (ω, m)-subharmonic functions. Therefore we need the following definition.

Definition 2.10 (Smoothly approximable functions). Let u be an (ω, m)-subharmonic function in Ω. We say that u belongs to $\mathcal{A}_m(\omega)$ if at each point $z \in \Omega$ there exist a ball $B(z,r) \subset\subset \Omega$ and smooth (ω, m)-subharmonic functions u_j in $B(z,r)$ decreasing to u as j goes to ∞.

We shall now develop 'pluripotential theory' for (ω, m)-subharmonic functions in the class $\mathcal{A}_m(\omega)$.

Proposition 2.11 (Wedge product). *Fix a ball $B(z_0,r) \subset\subset \Omega$. Let $u_1, \ldots, u_k \in \mathrm{SH}_m(\omega) \cap C(\bar{B}(z_0,r))$, $1 \leqslant k \leqslant m$. Assume that there exists a sequence of smooth (ω, m)-subharmonic functions $u_1^j, \ldots, u_k^j$ decreasing to $u_1, \ldots, u_k$ in $\bar{B}(z_0,r)$, respectively. Then the sequence*

$$(\omega + dd^c u_1^j) \wedge \cdots \wedge (\omega + dd^c u_k^j) \wedge \omega^{n-m}$$

converges weakly to a unique positive current, in $B(z_0,r)$, as j goes to $+\infty$.

Proof. Thanks to Corollary 2.4 and the CLN inequality (Proposition 2.9), the proof is a standard modification of the Bedford and Taylor convergence theorem [BT76, BT82]. For notational simplicity we only give it in the case $k = m$, $u_1 = \cdots = u_m = u$ and $u_1^j \equiv \cdots \equiv u_m^j \equiv u^j =: u_j$. The general case follows by the same method. Set $B := B(z_0,r)$. Since u is continuous on $\overline{B}$, it follows that $u_j \to u$ uniformly on that set. Hence, $\|u_j\|_\infty$ is uniformly bounded, where we denote here and below

$$\|\cdot\|_\infty := \sup_{\bar{B}} |\cdot|.$$

For any compact set $K \subset B$ we have

$$\int_K \omega_{u_j}^m \wedge \omega^{n-m} \leqslant C_{K,B,\omega}(1 + \|u_j\|_\infty)^m$$

by the CLN inequality (Proposition 2.9). Therefore, the sequence

$$\omega_{u_j}^m \wedge \omega^{n-m}, \quad j \geqslant 1,$$

is weakly compact in B. This implies that there exists a weak limit μ upon passing to a subsequence.

It remains to check that every weak limit is equal to μ. Suppose that $\{v_j\}_{j=1}^\infty$ and $\{w_j\}_{j=1}^\infty$ are two decreasing sequences of smooth (ω, m)-subharmonic functions converging to u. Since the statement is local we may assume that all functions are equal near the boundary of B (see [BT82, Koł05]). We need to show that for any test function $\chi \in C_c^\infty(B)$,

$$\left| \int_B \chi \omega_{v_j}^m \wedge \omega^{n-m} - \int_B \chi \omega_{w_j}^m \wedge \omega^{n-m} \right| \longrightarrow 0$$

as $j \to +\infty$. Since u is continuous on $\overline{B}$, it follows that both $\{v_j\}$ and $\{w_j\}$ converge uniformly to u on that set. Hence, $\|v_j\|_\infty$, $\|w_j\|_\infty$ are uniformly bounded. By integration by parts we have

$$A_j := \int_B \chi dd^c(v_j - w_j) \wedge T_j = \int_B (v_j - w_j) dd^c(\chi T_j),$$

where $T_j = \sum_{s=0}^{m-1} \omega_{v_j}^s \wedge \omega_{w_j}^{m-1-s} \wedge \omega^{n-m}$. From Corollary 2.4 and the above proof of the CLN inequality we get that

$$A_j \leqslant \|v_j - w_j\|_\infty \int_{\mathrm{supp}\,\chi} \|dd^c(\chi T_j)\|,$$

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where the last integral is controlled by

$$C(1+\|v_j\|_\infty)^{m-1}(1+\|w_j\|_\infty)^{m-1}.$$

Therefore, we can conclude that $\lim_{j\to+\infty} A_j = 0$, and thus the result follows. $\square$

Corollary 2.12. *Let $u_1,\ldots,u_k \in \mathcal{A}_m(\omega)\cap C(\Omega)$, $1 \leqslant k \leqslant m$. Then the wedge product*

$$\omega_{u_1}\wedge\cdots\wedge\omega_{u_k}\wedge\omega^{n-m}$$

is a well-defined positive current of bidegree $(n-m+k, n-m+k)$. In particular, for $u \in \mathcal{A}_m(\omega)\cap C(\Omega)$, the current

$$\omega_u^m\wedge\omega^{n-m}$$

is the complex Hessian operator of u, which is a positive Radon measure in Ω.

2.4 The comparison principle and maximality

Let Ω be a bounded open set in $\mathbb{C}^n$. Given ω a Hermitian metric there exists a constant $B_\omega > 0$, which we fix, satisfying, in $\bar{\Omega}$,

$$-B_\omega\omega^2 \leqslant 2n dd^c\omega \leqslant B_\omega\omega^2, \quad -B_\omega\omega^3 \leqslant 4n^2 d\omega\wedge d^c\omega \leqslant B_\omega\omega^3. \tag{2.12}$$

Thanks to Lemma 2.3 and Corollary 2.4, the proof of [KN15a, Theorem 0.2] can be adapted to Hessian operators and as a consequence we get the following domination principle.

Proposition 2.13. *Let $u, v \in \mathcal{A}_m(\omega)\cap C(\bar{\Omega})$ be such that $u \geqslant v$ on $\partial\Omega$. Assume that $(\omega+dd^c u)^m\wedge\omega^{n-m} \leqslant (\omega+dd^c v)^m\wedge\omega^{n-m}$. Then $u \geqslant v$ on $\bar{\Omega}$.*

Proof. See [KN15a, Corollary 3.4]. We remark here that if u, v belong to $C^2(\Omega)$, then the corollary can be proven simply by using the ellipticity of the Hessian operator [CNS85, Lemma B]. $\square$

The above proposition shows that if $u \in \mathcal{A}_m(\omega)\cap C(\bar{\Omega})$ and $\omega_u^m\wedge\omega^{n-m} = 0$, then it is maximal in $\mathcal{A}_m(\omega)\cap C(\bar{\Omega})$. We shall see that a stronger result is true. First, we recall a couple of facts from classical potential theory. For a general fixed Hermitian metric γ in $\mathbb{C}^n$ and a Borel set $E \subset \Omega$ we define

$$C_\gamma(E) = \sup\left\{\int_E dd^c w\wedge\gamma^{n-1} : w \text{ is } \gamma\text{-subharmonic in } \Omega,\ 0 \leqslant w \leqslant 1\right\}.$$

Proposition 2.14. *Every γ-subharmonic function u is quasi-continuous with respect to the capacity C_γ, that is, for any $\varepsilon > 0$, there exists an open set $U \subset \Omega$ such that $C_\gamma(U) < \varepsilon$ and u restricted to $\Omega\backslash U$ is continuous.*

Lemma 2.15. *Every γ-subharmonic in a neighbourhood of the closure of Ω is the limit of a decreasing sequence of smooth γ-subharmonic functions, in Ω.*

Next, we strengthen the domination principle. It is usually applied locally, so we formulate it for Ω being a ball.

Theorem 2.16 (Maximality). *Let Ω denote a ball and let $v \in \mathrm{SH}_m(\omega)\cap L^\infty(\Omega)$. Let $u \in \mathcal{A}_m(\omega)\cap C(\bar{\Omega})$ be the uniform limit of $\{u_j\}_{j=1}^\infty \subset \mathrm{SH}_m(\omega)\cap C^\infty(\bar{\Omega})$. Suppose that $G := \{u < v\} \subset\subset \Omega$. If $\omega_u^m\wedge\omega^{n-m} = 0$ on G, then G is empty.*

To prove the theorem, we need the following result.

Lemma 2.17. *Fix $0 < \varepsilon < 1$ and the constant B_ω in (2.12). Let $v \in \mathrm{SH}_m(\omega) \cap L^\infty(\Omega)$, with Ω denoting a ball. Assume that $\{u_j\}_{j=1}^\infty \subset \mathrm{SH}_m(\omega) \cap C^\infty(\bar\Omega)$ converges uniformly to u as $j \to \infty$ in $\bar\Omega$. Denote $S(\varepsilon) := \inf_\Omega[u - (1-\varepsilon)v]$ and $U(\varepsilon, t) := \{u < (1-\varepsilon)v + S(\varepsilon) + t\}$ for $t > 0$. Suppose that $U(\varepsilon, t_0) \subset\subset \Omega$ for some $t_0 > 0$. Then, for $0 < t < \min\{\varepsilon^3/16B_\omega, t_0\}$,*

$$\varepsilon \int_{U(\varepsilon,t)} \omega_u^{m-1} \wedge \omega^{n-m+1} \leqslant \left(1 + \frac{Ct}{\varepsilon^m}\right) \int_{U(\varepsilon,t)} \omega_u^m \wedge \omega^{n-m},$$

where C is a uniform constant depending only on n, m, B_ω.

Proof. By Corollary 2.12, it is enough to show that

$$\varepsilon \int_{U_j(\varepsilon,t)} \omega_{u_j}^{m-1} \wedge \omega^{n-m+1} \leqslant \left(1 + \frac{Ct}{\varepsilon^m}\right) \int_{U_j(\varepsilon,t)} \omega_{u_j}^m \wedge \omega^{n-m},$$

where $U_j(\varepsilon, t)$ is the sublevel set corresponding to u_j and v defined as above. In other words, we only need to prove the lemma under the assumption that u is smooth and strictly (ω, m)-subharmonic, that is, $\omega_u \in \Gamma_m(\omega)$ (achieved by considering the sequence $(1-1/j)u_j$, $j \geqslant 1$).

Moreover, since $\varepsilon\omega_u^{m-1} \wedge \omega^{n-m+1} \leqslant \omega_{(1-\varepsilon)v} \wedge \omega_u^{m-1} \wedge \omega^{n-m}$, it suffices to prove that

$$\int_{U(\varepsilon,t)} \omega_{(1-\varepsilon)v} \wedge \omega_u^{m-1} \wedge \omega^{n-m} \leqslant \left(1 + \frac{Ct}{\varepsilon^m}\right) \int_{U(\varepsilon,t)} \omega_u^m \wedge \omega^{n-m}. \tag{2.13}$$

Since $\omega_u^{m-1} \wedge \omega^{n-m} > 0$, applying [Mic82, Equation (4.8)], we can write

$$\gamma^{n-1} := \omega_u^{m-1} \wedge \omega^{n-m} \tag{2.14}$$

for some Hermitian metric γ. By the definition of an (ω, m)-subharmonic function,

$$\omega_v \wedge \gamma^{n-1} \geqslant 0.$$

Solving the linear elliptic equation, we can write $\omega \wedge \gamma^{n-1} = dd^c w \wedge \gamma^{n-1}$ for some smooth γ-subharmonic function w in Ω. Therefore, if we set $\tilde v := v + w$, then $\tilde v$ is a γ-subharmonic function. With this property in hand, we can use the proof of [BT76, Proposition 3.1] and the quasi-continuity of $\tilde v$ (or equivalently, that of v), from Proposition 2.14 to get that

$$\int_{U(\varepsilon,t)} dd^c(1-\varepsilon)v \wedge \gamma^{n-1} \leqslant \int_{U(\varepsilon,t)} dd^c u \wedge \gamma^{n-1} + \int_{U(\varepsilon,t)} [(1-\varepsilon)v + S_\varepsilon + t - u] dd^c\gamma^{n-1}.$$

This implies that

$$\int_{U(\varepsilon,t)} \omega_{(1-\varepsilon)v} \wedge \gamma^{n-1} \leqslant \int_{U(\varepsilon,t)} \omega_u \wedge \gamma^{n-1} + t \int_{U(\varepsilon,t)} \|dd^c\gamma^{n-1}\|, \tag{2.15}$$

where $\|dd^c\gamma^{n-1}\|$ is the total variation of $dd^c\gamma^{n-1}$. Furthermore, we can use Lemma 2.3 to bound $\|dd^c\gamma^{n-1}\|$ from above by

$$R := C(\omega_u^{m-1} \wedge \omega^{n-m+1} + \omega_u^{m-2} \wedge \omega^{n-m+2} + \omega_u^{m-3} \wedge \omega^{n-m+3}),$$

where C depends only on X, ω, n, m. Therefore, the inequality (2.13) will follow if we have

$$\int_{U(\varepsilon,t)} R \leqslant \frac{C}{\varepsilon^m} \int_{U(\varepsilon,t)} \omega_u^m \wedge \omega^{n-m}$$

for every $0 < t < \min\{\varepsilon^3/16B_\omega, t_0\}$. Writing $a_k := \int_{U(\varepsilon,t)} \omega_u^k \wedge \omega^{n-k}$, for $0 \leqslant k \leqslant m$, we need to show that

$$a_k \leqslant \frac{Ca_m}{\varepsilon^m}.$$

As in [KN15a, Theorem 2.3] we shall verify that for $0 < t < \delta := \min\{\varepsilon^3/16B_\omega, t_0\}$,

$$\varepsilon a_k \leqslant a_{k+1} + \delta B_\omega(a_k + a_{k-1} + a_{k-2}),$$

where we understand that $a_k \equiv 0$ if $k < 0$. Indeed, since u is smooth and strictly (ω, m)-subharmonic, inequality (2.15) applied for $\gamma_k^{n-1} := \omega_u^k \wedge \omega^{n-k-1} > 0$, $0 \leqslant k \leqslant m-2$ (see (2.14)), gives that

$$\int_{U(\varepsilon,t)} \omega_{(1-\varepsilon)v} \wedge \gamma_k^{n-1} \leqslant \int_{U(\varepsilon,t)} \omega_u \wedge \gamma_k^{n-1} + t\int_{U(\varepsilon,t)} \|dd^c\gamma_k^{n-1}\|.$$

By (2.11), (2.12) and Lemma 2.3 we have

$$\int_{U(\varepsilon,t)} \|dd^c\gamma_k^{n-1}\| \leqslant B_\omega(a_k + a_{k-1} + a_{k-2}).$$

Moreover, since v is a bounded (ω, m)-subharmonic function, we also have

$$\varepsilon\int_{U(\varepsilon,t)} \omega \wedge \gamma_k^{n-1} \leqslant \int_{U(\varepsilon,t)} \omega_{(1-\varepsilon)v} \wedge \gamma_k^{n-1}.$$

Combining the last three inequalities, we get that for $0 < t < \delta$,

$$\varepsilon a_k \leqslant a_{k+1} + \delta B_\omega(a_k + a_{k-1} + a_{k-2}).$$

Thus the proof of the lemma follows. □

Proof of Theorem 2.16. Suppose that $\{u < v\}$ is not empty. Then for $\varepsilon > 0$ small enough, we have $\{u < (1-\varepsilon)v + \inf_\Omega[w - (1-\varepsilon)v] + t\} \subset \{u < v\}$ for any $0 < t \leqslant t_0$, where $t_0 > 0$ depends on u, v, ε. Applying Lemma 2.17, we have for $0 < t \leqslant \min\{\varepsilon^{m+3}/16B_\omega, t_0\}$,

$$\varepsilon\int_{U(\varepsilon,t)} \omega_u^{m-1} \wedge \omega^{n-m+1} \leqslant C\int_{U(\varepsilon,t)} \omega_u^m \wedge \omega^{n-m} = 0,$$

where C is independent of t. Therefore, $\omega_u^{m-1} \wedge \omega^{n-m+1} = 0$ in $U(\varepsilon, t)$ for $0 < t \leqslant t_1$, where $t_1 := \min\{\varepsilon^{m+3}/16B_\omega, t_0\}$. Thus we can iterate this argument to get that $\omega_u^{m-2} \wedge \omega^{n-m+2} = \cdots = \omega^n = 0$ in $U(\varepsilon, t_1)$. This is impossible and the proof of the theorem follows. □

Remark 2.18. The statement of Theorem 2.16 holds true if we replace $\bar{\Omega}$ by a compact Hermitian manifold, with the same proof modulo obvious modifications.

We end this subsection by proving a volume–capacity inequality which corresponds to the one in [DK14]. This inequality was the key ingredient in studying local integrability of m-subharmonic functions.

Definition 2.19 (Capacity). For any Borel set $E \subset \Omega$,

$$\mathrm{cap}_{m,\omega}(E) := \sup\bigg\{\int_E (\omega + dd^c v)^m \wedge \omega^{n-m} : v \in \mathcal{A}_m(\omega) \cap C(\Omega),\ 0 \leqslant v \leqslant 1\bigg\}.$$

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Lemma 2.20 (Local volume–capacity inequality). *Let $1 < \tau < n/(n-m)$. There exists a constant $C = C(\tau)$ such that for any Borel set $E \subset \Omega$,*

$$V_\omega(E) \leqslant C[\mathrm{cap}_{m,\omega}(E)]^\tau,$$

where $V_\omega(E) := \int_E \omega^n$.

The exponent here is optimal because if we take $\omega = dd^c|z|^2$, then the explicit formula for $\mathrm{cap}_m(B(0,r))$ in $\Omega = B(0,1)$, with $0 < r < 1$, provides an example.

Proof. From [DK14, Proposition 2.1] we know that $V_\omega(E) \leqslant C[\mathrm{cap}_m(E)]^\tau$ with

$$\mathrm{cap}_m(E) = \sup\left\{\int_E (dd^c w)^m \wedge \omega^{n-m} : w \in \mathcal{A}_m \cap C(\Omega), 0 \leqslant w \leqslant 1\right\},$$

which is the capacity related to $m-\omega$-subharmonic functions in Ω and the class $\mathcal{A}_m$ consists of all $m-\omega$-subharmonic functions which are locally approximable by a decreasing sequence of smooth $m-\omega$-subharmonic functions in Ω.

Note that the argument in [DK14] remains valid for non-Kähler ω since the mixed form type inequality used there still holds by stability estimates for the Monge–Ampère equation. Therefore, the proof will follow if we can show that $\mathrm{cap}_m(E)$ is less than $\mathrm{cap}_{m,\omega}(E)$. Since ω is globally defined there exists a constant $C > 0$ such that

$$\frac{1}{C} dd^c\rho \leqslant \omega \leqslant C dd^c \rho,$$

where $\rho = |z|^2 - A \leqslant 0$. We can choose C such that $|\rho/C| \leqslant 1/2$. Take $0 \leqslant w \leqslant 1/2$ a continuous $m-\omega$-subharmonic in $\mathcal{A}_m$. Then it is easy to see that

$$\int_E (dd^c w)^m \wedge \omega^{n-m} \leqslant \int_E \left(\omega + dd^c\left(w - \frac{\rho}{C}\right)\right)^m \wedge \omega^{n-m} \leqslant \mathrm{cap}_{m,\omega}(E).$$

Hence, $\mathrm{cap}_m(E) \leqslant 2^n \mathrm{cap}_{m,\omega}(E)$. □

3. Hessian equations on compact Hermitian manifolds

In this section we study Hessian equations on a compact n-dimensional Hermitian manifold (X,ω). To do this we need first to transfer the local results from the previous section to the manifold setting. Then we apply them to prove results on the existence and stability of solutions of Hessian equations. Finally, we prove that every (ω,m)-subharmonic function can be approximated by a decreasing sequence of smooth (ω,m)-subharmonic function on X. This allows us to replace assumptions on $\mathcal{A}_m(\omega)$ by just $\mathrm{SH}_m(\omega)$ in statements. In what follows our notation is as in [KN15a, KN15b], we write $L^1(\omega^n)$ for $L^1(X,\omega^n)$, $\|\cdot\|_p := \|\cdot\|_{L^p(X,\omega^n)}$ and $\|\cdot\|_\infty := \sup_X |\cdot|$.

3.1 Pluripotential estimates for (ω, m)-subharmonic functions

Fix an integer $1 \leqslant m < n$. By means of partition of unity we carry over the local construction from § 2 onto the compact Hermitian manifold X.

Definition 3.1. An upper semicontinuous function $u : X \to [-\infty, +\infty[$ is called (ω,m)-subharmonic in X if $u \in L^1(\omega^n)$ and $u \in \mathrm{SH}_m(U,\omega)$ for each coordinate patch $U \subset\subset X$.

We denote by $\mathrm{SH}_m(X,\omega)$ or $\mathrm{SH}_m(\omega)$ the set of all (ω,m)-subharmonic functions in X. Similarly, we say that $u \in \mathcal{A}_m(\omega)$ if $u \in \mathrm{SH}_m(\omega)$ and there exists a decreasing sequence of smooth (ω,m)-subharmonic functions on X which converges to u (globally). So, if $u \in \mathcal{A}_m(\omega)$, then for any coordinate patch $U \subset\subset X$ we have $u \in \mathcal{A}_m(U,\omega)$. Thus the properties of $\mathcal{A}_m(U,\omega)$ (Proposition 2.8, Hessian measures, the Bedford–Taylor convergence theorem, etc.) are also valid for $\mathcal{A}_m(\omega)$.

Below we state several results which are analogues of those from [DK12a]. We omit the proofs which are similar and require only local properties.

PROPOSITION 3.2 (CLN inequalities). *Let $\varphi_1,\dots,\varphi_m \in \mathcal{A}_m(\omega)\cap C(X)$ and $0 \leqslant \varphi_1,\dots,\varphi_m \leqslant 1$. Then there exists a uniform constant $C>0$ such that*

$$\int_X \omega_{\varphi_1}\wedge\cdots\wedge\omega_{\varphi_m}\wedge\omega^{n-m} \leqslant C.$$

The following lemma is classical (see, for example, Hörmander's book [Hör94]).

LEMMA 3.3. *Let $\varphi \in \mathrm{SH}_m(\omega)$ with $\sup_X \varphi = 0$. There exists a uniform constant $C = C(X,\omega) > 0$ such that*

$$\int_X |\varphi|\omega^n \leqslant C.$$

Consequently, the family $\{\varphi \in \mathrm{SH}_m(\omega) : \sup_X \varphi = 0\}$ is compact in $\mathrm{SH}_m(\omega)$ with respect to $L^1(\omega^n)$-topology, that is, for any sequence $\varphi_j \in \mathrm{SH}_m(\omega)$ with $\sup_X \varphi_j = 0$, $j \geqslant 1$, there exists a subsequence $\{\varphi_{j_k}\}$ such that φ_{j_k} converges to $\varphi \in \mathrm{SH}_m(\omega)$ as $j_k \to +\infty$ in $L^1(\omega^n)$.

Proof. The first part is from [TW13, § 2, p. 8], where the proof used only the fact that φ is a smooth $(\omega,1)$-subharmnic function, that is,

$$n dd^c\varphi\wedge\omega^{n-1}/\omega^n \geqslant -n,$$

coupled with the existence of the Green function for the Gauduchon metric in the conformal class of ω. Since every $(\omega,1)$-subharmonic function is approximated by a decreasing sequence of smooth $(\omega,1)$-subharmonic functions, we get the statement for general (ω,m)- subharmonic functions. The second part follows from Proposition 2.8 and requires only properties of $(\omega,1)$-subharmonic functions. □

The estimates of the decay of volume of sublevel sets follow directly from Lemma 3.3. We use the notation

$$V_\omega(E) := \int_E \omega^n.$$

COROLLARY 3.4. *Let $\varphi \in \mathrm{SH}_m(\omega)$ with $\sup_X \varphi = 0$. Then, for any $t>0$,*

$$V_\omega(\{\varphi < -t\}) \leqslant C/t,$$

where $C>0$ is a uniform constant.

Following [BT82] and [Koł03] we define the capacity related to the Hessian equations.

Definition 3.5 (Capacity). For a Borel set $E \subset X$,

$$\mathrm{cap}_{m,\omega}(E) := \sup\left\{ \int_E \omega_\rho^m \wedge \omega^{n-m} : \rho \in \mathcal{A}_m(\omega) \cap C(X),\ 0 \leqslant \rho \leqslant 1 \right\}.$$

Then, as in the local case, we have the estimate with the sharp exponent.

Proposition 3.6. *Fix $1 < \tau < n/(n-m)$. There exists a uniform constant $C = C(\tau, X, \omega) > 0$ such that for any Borel set $E \subset X$,*

$$V_\omega(E) \leqslant C[\mathrm{cap}_{m,\omega}(E)]^\tau.$$

Proof. The basic idea is from [DK14]. Surprisingly, it is enough to use the estimates for the Monge–Ampère equation to obtain a sharp bound related to capacity defined in terms of more general Hessian equations. One could infer the statement from the local counterpart, but due to the difficulties with approximation by smooth (ω, m)-subharmonic functions that approach would be more technical than a direct proof (like [Lu13] in the Kähler case). This requires the estimates in the Hermitian setting [KN15a].

Without loss of generality we assume that $V_\omega(E) > 0$. Denote by $\mathbf{1}_E$ the characteristic function of E. By [KN15a, Theorem 0.1] we can find a continuous ω-plurisubharmonic function u on X with $\sup_X u = 0$ and a constant $b > 0$ solving

$$\omega_u^n = b\,\mathbf{1}_E \omega^n.$$

Set $p = m\tau/n(\tau-1) > 1$. We will need the lower bound for L^p-norm of $b\,\mathbf{1}_E$.

Fact. There exists a uniform constant $c_0 > 0$ depending on X, ω, p such that

$$\|b\,\mathbf{1}_E\|_p \geqslant c_0.$$

Indeed, suppose that this were not true; then there would be a sequence of Borel sets $\{E_j\}_{j=1}^\infty$ such that

$$1 \geqslant \|b_j\,\mathbf{1}_{E_j}\|_p \searrow 0 \quad \text{as } j \to +\infty.$$

By [KN15a, KN15b] we know that for $0 < t \leqslant t_{\min}$ ($t_{\min} > 0$ depending only on X, ω)

$$t^n \hbar(t) \leqslant C\|b_j\,\mathbf{1}_{E_j}\|_1 \leqslant C\|b_j\,\mathbf{1}_{E_j}\|_p \searrow 0,$$

where the function $\hbar(t)$ is the inverse function of $\kappa(t)$ defined in [KN15a, Theorem 5.3]. This leads to a contradiction for a fixed $t = t_{\min}$.

Thus, by *a priori* estimates for Monge–Ampère equations [KN15a, Corollary 5.6] we have

$$\|u\|_\infty \leqslant C\|b\,\mathbf{1}_E\|_p^{1/n} = Cb^{1/n}[V_\omega(E)]^{1/pn}. \tag{3.1}$$

We observe that by the proof of [Ngu16, Proposition 1.5] for $-1 \leqslant w \leqslant 0$,

$$\int_X \omega_w^n \geqslant \int_X \omega^n - C\|w\|_\infty,$$

where $C = C(X, \omega)$. Hence, there exists $0 < \delta = \delta(X, \omega) < 1$ such that if $\|u\|_\infty \leqslant \delta$ then $\int_X \omega_u^n \geqslant V_\omega(X)/2$, that is,$b \geqslant V_\omega(X)/2V_\omega(E)$. We now consider two cases.

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Case 1. If $\|u\|_\infty > \delta$, then, by (3.1)

$$\|u\|_\infty + 1 \leqslant (C + C/\delta)\, b^{1/n} [V_\omega(E)]^{1/pn}. \tag{3.2}$$

The mixed form type inequality [Ngu16, Lemma 1.9] gives $\omega_u^m \wedge \omega^{n-m} \geqslant b^{m/n} \mathbf{1}_E$. Hence, by definition of capacity we have

$$\begin{aligned}
\mathrm{cap}_{m,\omega}(E) &\geqslant \frac{1}{(1+\|u\|_\infty)^m} \int_E (\omega + dd^c u)^m \wedge \omega^{n-m} \\
&\geqslant \frac{1}{(1+\|u\|_\infty)^m} \int_E b^{m/n} \mathbf{1}_E \omega^n \\
&\geqslant \frac{b^{m/n} V_\omega(E)}{C_1 b^{m/n} [V_\omega(E)]^{m/pn}} \\
&= \frac{[V_\omega(E)]^{1-m/pn}}{C_1},
\end{aligned}$$

where we used (3.2) for the last inequality and $C_1 = (C + C/\delta)^m$. Therefore, we have

$$V_\omega(E) \leqslant C[\mathrm{cap}_{m,\omega}(E)]^{1+m/(pn-m)}.$$

Plugging the value of $p = m\tau/n(\tau - 1)$ gives the desired inequality.

Case 2. If $\|u\|_\infty \leqslant \delta < 1$, then $b \geqslant V_\omega(X)/2V_\omega(E)$. Again, by definition we have

$$\begin{aligned}
\mathrm{cap}_{m,\omega}(E) &\geqslant \int_E \omega_u^m \wedge \omega^{n-m} \\
&\geqslant \int_E b^{m/n} \mathbf{1}_E \omega^n \\
&\geqslant \left(\frac{V_\omega(X)}{2V_\omega(E)}\right)^{m/n} \cdot V_\omega(E).
\end{aligned}$$

This implies that $V_\omega(E) \leqslant C[\mathrm{cap}_{m,\omega}(E)]^{n/(n-m)}$. Thus we complete the proof. □

Let us recall that, by the definition, the constant $B > 0$ satisfies, on X,

$$-B\omega^2 \leqslant 2n\, dd^c\omega \leqslant B\omega^2, \quad -B\omega^3 \leqslant 4n^2\, d\omega \wedge d^c\omega \leqslant B\omega^3. \tag{3.3}$$

For general Hermitian metric ω the Hessian measures do not preserve the volume of manifolds, so the classical comparison principle [BT82, Koł05] is no longer true (see [DK12a]). However, a weaker form will be enough for several applications as it is proven in [KN15a, KN15b]. We state below the analogue for Hessian operators.

Theorem 3.7 (Weak comparison principle). *Let $\varphi, \psi \in \mathcal{A}_m(\omega) \cap C(X)$. Fix $0 < \varepsilon < 1$ and use the notation $S(\varepsilon) := \inf_X [\varphi - (1-\varepsilon)\psi]$ and $U(\varepsilon, s) := \{\varphi < (1-\varepsilon)\psi + S(\varepsilon) + s\}$ for $s > 0$. Then, for $0 < s < \varepsilon^3/16B$,*

$$\int_{U(\varepsilon,s)} \omega_{(1-\varepsilon)\psi}^m \wedge \omega^{n-m} \leqslant \left(1 + \frac{Cs}{\varepsilon^m}\right) \int_{U(\varepsilon,s)} \omega_\varphi^m \wedge \omega^{n-m},$$

where $C > 0$ is a uniform constant depending only on n, m, ω.

Proof. This follows from the argument in [KN15a, Theorem 0.2] with the aid of Corollary 2.4. □

Thanks to the weak comparison principle we can estimate the rate of the decay of capacity of sublevel sets not far from the minimum point.

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Lemma 3.8. *Fix $0 < \varepsilon < \frac{3}{4}$ and $\varepsilon_B := \frac{1}{3}\min\{\varepsilon^m, \varepsilon^3/16B\}$. Consider $\varphi, \psi \in \mathcal{A}_m(\omega) \cap C(X)$ with $\varphi \leqslant 0$ and $-1 \leqslant \psi \leqslant 0$. With $U(\varepsilon, s)$ defined as in the previous theorem, for any $0 < s, t < \varepsilon_B$, we have*

$$t^m \mathrm{cap}_{m,\omega}(U(\varepsilon, s)) \leqslant C \int_{U(\varepsilon, s+t)} \omega_\varphi^m \wedge \omega^{n-m}, \tag{3.4}$$

where $C > 0$ depends only on X, ω.

Proof. See the arguments in [KN15a, Lemma 5.4, Remark 5.5] by using the above weak comparison principle (Theorem 3.7). □

The preparations above were needed for the proof of *a priori* estimates for solutions to Hessian equations with the right-hand side in L^p, $p > n/m$. We follow the method from [Koł98, Koł03] with small variations.

Lemma 3.9. *We retain the assumptions and notation of Lemma 3.8. Assume, furthermore, that*

$$\omega_\varphi^m \wedge \omega^{n-m} = f\omega^n$$

for $f \in L^p(\omega^n)$, $p > n/m$. Fix $0 < \alpha < (p - n/m)/p(n-m)$. Then there exists a constant $C_\alpha = C(\alpha, \omega)$ such that for any $0 < s, t < \varepsilon_B$,

$$t[V_\omega(U(\varepsilon, s))]^{1/m\tau} \leqslant C_\alpha \|f\|_p^{1/m} [V_\omega(U(\varepsilon, s+t))]^{(1+m\alpha)/m\tau},$$

where $\tau = (1 + m\alpha)p/(p-1) < n/(n-m)$.

Proof. It is elementary that

$$0 < \alpha < \frac{p - n/m}{p(n-m)} \Leftrightarrow \frac{p}{p-1} < \tau = \frac{(1+m\alpha)p}{p-1} < \frac{n}{n-m}. \tag{3.5}$$

By the volume–capacity inequality (Proposition 3.6) and Lemma 3.8 we have

$$t^m [V_\omega(U(\varepsilon, s))]^{1/\tau} \leqslant C_\alpha\, t^m\, \mathrm{cap}_{m,\omega}(U(\varepsilon, s)) \leqslant C_\alpha \cdot C \int_{U(\varepsilon, s+t)} f\omega^n.$$

The Hölder inequality implies that

$$t^m [V_\omega(U(\varepsilon, s))]^{1/\tau} \leqslant C_\alpha \|f\|_p [V_\omega(U(\varepsilon, s+t))]^{p-1/p}.$$

Taking the mth root of both sides and plugging in the value of τ, we get the desired inequality. □

Thanks to this lemma we get a uniform estimate for the solution of Hessian equations with L^p, $p > n/m$, control of the right-hand side.

Theorem 3.10. *Fix $0 < \varepsilon < \frac{3}{4}$ and $\varepsilon_B := \frac{1}{3}\min\{\varepsilon^m, \varepsilon^3/16B\}$. Let $\varphi, \psi \in \mathcal{A}_m(\omega) \cap C(X)$ satisfy $-1 \leqslant \psi \leqslant 0$ and $\varphi \leqslant 0$. Assume that*

$$\omega_\varphi^m \wedge \omega^{n-m} = f\omega^n$$

with $f \in L^p(\omega^n)$, $p > n/m$. Put

$$U(\varepsilon, s) = \Big\{\varphi < (1-\varepsilon)\psi + \inf_X [\varphi - (1-\varepsilon)\psi] + s\Big\},$$

and fix $0 < \alpha < (p - n/m)/p(n-m)$. Then there exists a constant $C_\alpha = C(\alpha, \omega)$ such that for $0 < s < \varepsilon_B$,

$$s \leqslant 4C_\alpha \|f\|_p^{1/m} [V_\omega(U(\varepsilon, s))]^{\alpha/\tau},$$

where $\tau = (1 + m\alpha)p/(p-1)$.

Proof. First, for $0 < \alpha < (p - n/m)/p(n - m)$, we define

$$a(s) := [V_\omega(U(\varepsilon, s))]^{1/m\tau}, \quad C := C_\alpha \|f\|_p^{1/m}.$$

It follows from Lemma 3.9 that for any $0 < s, t < \varepsilon_B$,

$$ta(s) \leqslant C[a(s + t)]^{1+m\alpha}. \tag{3.6}$$

The function $a(x)$ satisfies

$$\lim_{x \to s^-} a(x) = a(s) \quad \text{and} \quad \lim_{x \to s^+} a(x) =: a(s^+) \geqslant a(s). \tag{3.7}$$

To finish the proof, we shall show that for any $0 < s < \varepsilon_B$,

$$s \leqslant \frac{2^{1+m\alpha}}{2^{m\alpha} - 1} \cdot C[a(s)]^{m\alpha}.$$

The argument is similar to the proof of [KN15a, Theorem 5.3], but simpler here, so we include the proof for the sake of completeness.

Fix $s_0 := s \in (0, \varepsilon_B)$. Let us define by induction the sequence $s_i, i \geqslant 1$ as follows:

$$s_i := \sup\{0 \leqslant x \leqslant s_{i-1} : a(s_{i-1}) \geqslant 2a(x)\}. \tag{3.8}$$

Since $a(0) = 0$ and $a(x) > 0$ for $x > 0$, it follows from the first equality in (3.7) that

$$s_0 > s_1 > \cdots > s_i \searrow 0 \quad \text{as } i \to +\infty.$$

(If $a(0^+) > 0$, then $s_N = s_{N+1} = \cdots = 0$ for some $1 \leqslant N < +\infty$.) By (3.7) and definition (3.8) we get

$$2a(s_i) \leqslant a(s_{i-1}) \leqslant 2a(s_i^+).$$

Hence, by (3.6),

$$s_{i-1} - s_i = \lim_{x \to s_i^+} (s_{i-1} - x) \leqslant C[a(s_{i-1})]^{1+m\alpha}/a(s_i^+).$$

It follows that

$$\begin{aligned} s_{i-1} - s_i \leqslant 2C[a(s_{i-1})]^{m\alpha} &\leqslant 2C(1/2^{m\alpha})[a(s_{i-2})]^{m\alpha} \\ &\leqslant \cdots \\ &\leqslant 2C(1/2^{m\alpha})^{i-1}[a(s_0)]^{m\alpha}. \end{aligned}$$

Thus,

$$\begin{aligned} s = \sum_{i=1}^{\infty} (s_{i-1} - s_i) &\leqslant 2^{1+m\alpha} C \sum_{i=1}^{\infty} (1/2^{m\alpha})^i [a(s_0)]^{m\alpha} \\ &= \frac{2^{1+m\alpha} C}{2^{m\alpha} - 1} [a(s)]^{m\alpha}. \end{aligned}$$

This completes the proof. □

From the statement of Theorem 3.10, we can derive the uniform estimate by taking $\varepsilon = 1/2$ and $\psi = 0$ and combining it with the estimate of the decay of volume of sublevel sets (Corollary 3.4). Thus we get that if $\omega_\varphi^m \wedge \omega^{n-m} = f\omega^n$ with $0 \leqslant f \in L^p(\omega^n)$, $p > n/m$, and φ is normalized by $\sup_X \varphi = -1$, then for any $0 < s < \varepsilon_B$,

$$s \leqslant \frac{C_\alpha \|f\|_p^{1/m}}{|-\inf_X \varphi - s|^{(p-1)\alpha/p(1+m\alpha)}},$$

where $0 < \alpha < (p - n/m)/p(n-m)$ is fixed. This leads to

$$\|\varphi\|_\infty \leqslant C\|f\|_p^{1/m \cdot p(1+m\alpha)/(p-1)\alpha}, \tag{3.9}$$

where $C = C(\alpha, p, \omega, X)$. Note that here we have used the fact that there exists a uniform lower bound for $\|f\|_p$ similar to the one in [KN15a, KN15b], though the present case is simpler. Indeed, it follows from Theorem 3.10 that for $s = \varepsilon_B/2$,

$$\|f\|_p^{1/m} \geqslant \frac{\varepsilon_B}{8C_\alpha [V_\omega(X)]^{\alpha/\tau}}.$$

This gives an explicit bound.

3.2 Existence of weak solutions and stability

The existence of weak solutions to the Monge–Ampère equations on compact Hermitian manifolds has recently been obtained in [KN15a] where the technique is quite different from that of [Koł05]. We will adapt those techniques to the Hessian equation.

Let us start with a quantitative version of [KN15a, Corollary 5.10] (see also [DK14, Theorem 3.1] for a similar result in the Kähler case).

Theorem 3.11. *Let $u, v \in \mathcal{A}_m(\omega) \cap C(X)$ be such that $\sup_X u = 0$ and $v \leqslant 0$. Suppose that $\omega_u^m \wedge \omega^{n-m} = f\omega^n$, where $f \in L^p(\omega^n), p > n/m$. Fix $0 < \alpha < (p - n/m)/p(n-m)$. Then*

$$\sup_X (v - u) \leqslant C\|(v-u)_+\|_1^{1/ap^*},$$

where the constant $a = 1/p^* + m(m+2) + (m+2)/\alpha$, and C depends only on $\alpha, p, \omega, \|f\|_p$ and $\|v\|_\infty$.

Proof. By the uniform estimate (3.9), $\|u\|_\infty$ is controlled by $\|f\|_p$. After a rescaling we may assume that $\|u\|_\infty, \|v\|_\infty \leqslant 1$. We wish to estimate $-S := \sup_X(v-u) > 0$ in terms of $\|(v-u)_+\|_1$ as in the Kähler case [Koł03]. Suppose that

$$\|(v-u)_+\|_1 \leqslant \varepsilon^{ap^*} \tag{3.10}$$

for $0 < \varepsilon \ll 3/4$ and $a > 0$ (to be determined later). Let

$$\hbar(s) := (s/4C_\alpha \|f\|_p^{1/m})^{1/\alpha}$$

be the inverse function of $4C_\alpha \|f\|_p^{1/m} s^\alpha$ in Theorem 3.10. Consider sublevel sets $U(\varepsilon, t) = \{u < (1-\varepsilon)v + S_\varepsilon + t\}$, where $S_\varepsilon = \inf_X[u - (1-\varepsilon)v]$. It is clear that

$$S - \varepsilon \leqslant S_\varepsilon \leqslant S.$$

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Therefore, $U(\varepsilon, 2t) \subset \{u < v + S + \varepsilon + 2t\}$. Then $(v-u)_+ \geqslant |S| - \varepsilon - 2t > 0$ for $0 < t < \varepsilon_B$ and $0 < \varepsilon < |S|/2$ on the latter set (if $|S| \leqslant 2\varepsilon$ then we are done).

By Lemma 3.8 and the Hölder inequality, we have

$$\begin{aligned}\operatorname{cap}_{m,\omega}(U(\varepsilon,t)) \leqslant \frac{C}{t^m}\int_{U(\varepsilon,2t)} f\omega^n &\leqslant \frac{C}{t^m}\int_X \frac{(v-u)_+^{1/p^*}}{(|S|-\varepsilon-2t)^{1/p^*}} f\omega^m \\ &\leqslant \frac{C\|f\|_p}{t^m(|S|-\varepsilon-2t)^{1/p^*}}\|(v-u)_+\|_1^{1/p^*}.\end{aligned}$$

Moreover, by Theorem 3.10,

$$\hbar(t) \leqslant [V_\omega(U(\varepsilon,t))]^{1/\tau} \leqslant C \operatorname{cap}_{m,\omega}(U(\varepsilon,t)),$$

where $\tau = (1+m\alpha)p^*$ and C also depends on α. Combining these inequalities, we obtain

$$(|S|-\varepsilon-2t)^{1/p^*} \leqslant \frac{C\|f\|_p}{t^m\hbar(t)}\|(v-u)_+\|_1^{1/p^*}.$$

Therefore, using (3.10),

$$\begin{aligned}|S| &\leqslant \varepsilon + 2t + \left(\frac{C\|f\|_p}{t^m\hbar(t)}\right)^{p^*}\|(v-u)_+\|_1 \\ &\leqslant 3\varepsilon + \left(\frac{C\|f\|_p\varepsilon^a}{t^m\hbar(t)}\right)^{p^*}.\end{aligned}$$

Recall that $\varepsilon_B = (1/3)\min\{\varepsilon^m, \varepsilon^3/16B\}$. So, taking $t = \varepsilon_B/2 \geqslant \varepsilon^{m+2}$, we have

$$\hbar(t) = \left(\frac{t}{4C_\alpha\|f\|_p^{1/m}}\right)^{1/\alpha} \geqslant \frac{C\varepsilon^{(m+2)/\alpha}}{\|f\|_p^{1/m\alpha}}.$$

If we choose $a = 1/p^* + m(m+2) + (m+2)/\alpha$, then

$$(\varepsilon^a/\varepsilon^{m(m+2)+(m+2)/\alpha})^{p^*} = \varepsilon.$$

Hence $|S| \leqslant C\varepsilon$ with $C = C(\alpha, p, \omega, \|f\|_p)$. Thus,

$$\sup_X (v-u) \leqslant C\|(v-u)_+\|_1^{1/ap^*}.$$

This is the stability estimate we wished to show. □

Applying the above theorem twice, we get the symmetric (with respect to u and v) form of this result.

Corollary 3.12. *Fix $\alpha > 0$ and $a > 0$ as in Theorem 3.11. Suppose that $u, v \in \mathcal{A}_m(\omega) \cap C(X)$, normalized $\sup_X u = \sup_X v = 0$, satisfy*

$$\omega_u^m \wedge \omega^{n-m} = f\omega^n, \quad \omega_v^m \wedge \omega^{n-m} = g\omega^n,$$

where $0 \leqslant f, g \in L^p(\omega^n)$, $p > n/m$. Then

$$\|u-v\|_\infty \leqslant C\|u-v\|_1^{1/ap^*},$$

where $C = C(\alpha, p, \|f\|_p, \|g\|_p, X, \omega) > 0$.

On compact non-Kähler manifolds we can only expect to solve the Hessian equation up to a multiplicative constant on the right-hand side. We need to know that those constants stay bounded as long as the given functions on the right-hand side are bounded in L^p.

Lemma 3.13. *Suppose that* $u \in \mathrm{SH}_m(\omega) \cap C^\infty(X)$ *satisfies*

$$\omega_u^m \wedge \omega^{n-m} = cf\omega^n,$$

where $f \in L^p(\omega^n)$, $p > n/m$, *and* $\int_X f\omega^n > 0$. *Then*

$$c_{\min} \leqslant c \leqslant 1/c_{\min}$$

for a uniform constant $c_{\min} = C(\|f\|_p, \|f^{1/m}\|_1, X, \omega) > 0$.

Proof. This is a consequence of mixed form type inequality and the *a priori* estimate in Theorem 3.10. The proof is similar to that for the Monge–Ampère equation [KN15a, Lemma 5.9]. □

Thanks to the work of Székelyhidi [Szé15] and Zhang [Zha15], the Hessian equation has a smooth solution when the right-hand side is smooth and positive. Using the approximation procedure as in [KN15a] and the stability (Corollary 3.12), we get the following existence result. Note that the solution is obtained as a uniform limit of a sequence of smooth functions, therefore it automatically belongs to $\mathcal{A}_m(\omega)$.

Theorem 3.14 (Existence). *Let* $0 \leqslant f \in L^p(\omega^n)$, $p > n/m$, *satisfy* $\int_X f\omega^n > 0$. *There exist* $u \in \mathcal{A}_m(\omega) \cap C(X)$ *and a constant* $c > 0$ *satisfying*

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = cf\omega^n.$$

Remark 3.15. As in [Ngu16], it follows from the weak comparison principle (Theorem 3.7) that the constant $c > 0$ is uniquely defined by f.

By adapting the method in [KN15b] we get the following stability statement for the Hessian equation on compact Hermitian manifolds.

Proposition 3.16. *Suppose that* $u, v \in \mathrm{SH}_m(\omega) \cap C^\infty(X)$, $\sup_X u = \sup_X v = 0$, *satisfy*

$$\omega_u^m \wedge \omega^{n-m} = f\omega^n, \quad \omega_v^m \wedge \omega^{n-m} = g\omega^n,$$

where $f, g \in L^p(\omega^n)$, $p > n/m$. *Assume that*

$$f \geqslant c_0 > 0$$

for some constant c_0. *Fix* $0 < a < 1/(m+1)$. *Then*

$$\|u - v\|_\infty \leqslant C\|f - g\|_p^a$$

where the constant C *depends on* $c_0, a, p, \|f\|_p, \|g\|_p, \omega, X$.

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Proof. The proof follows the one in [KN15b, Theorem 3.1], with the difference that we need here the smoothness assumption on u, v in order to use the mixed form type inequality [Går59]. This inequality is likely to be true in the general setting (see [Koł05, Ngu16]), but at the moment we do not have it. In § 2 we have provided estimates for elementary symmetric functions which are needed to make the arguments in [KN15b] go through. We only point out where those arguments should be modified.

Note that now both f and u are smooth. Use the notation

$$\varphi := u - v \quad \text{and} \quad T = \sum_{k=0}^{m-1} \omega_u^k \wedge \omega_v^{m-1-k} \wedge \omega^{n-m}.$$

By Corollary 2.4 we still have, for a continuous function $w \geqslant 0$ on X and a Borel set $E \subset X$, that

$$\left| \int_E w dd^c T \right| \leqslant C \|w\|_{L^\infty(E)} (1 + \|u\|_\infty)^m (1 + \|v\|_\infty)^m.$$

So the inequality [KN15b, Equation (3.16)] is valid. Next, the inequality corresponding to the one in the proof of [KN15b, Lemma 3.6] has the form

$$\frac{\omega_u \wedge \omega^{n-1}}{\omega^n} \cdot \frac{\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega_u^{m-1} \wedge \omega^{n-m}}{\omega^n} \geqslant \frac{\omega_u^m \wedge \omega^{n-m}}{\omega^n} \cdot \frac{\theta\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^{n-1}}{\omega^n},$$

where $\omega_u \in \Gamma_m$. This is exactly the content of Lemma 2.5 applied for $\gamma = \omega_u$ and φ. There is an extra constant $\theta > 0$ here, but it causes no harm as it only depends on n, m. □

3.3 Approximation property for (ω, m)-subharmonic functions

We will show the approximation property for (ω, m)-subharmonic functions on X for every $1 < m < n$. The case $m = 1$ is classical. The case $m = n$, that is to say, for quasi-plurisubharmonic functions, is a result due to Demailly (see [BK07] for a simple proof). When ω is Kähler the approximation property for (ω, m)-subharmonic functions has recently been proven by Lu and Nguyen [LN15]. They use the viscosity solutions and ideas from [Ber13, EGZ15]. By a similar approach, but without reference to viscosity solutions, we generalize the approximation theorem in [LN15] to the case of general Hermitian metric ω.

The following theorem is essentially contained in the work of Székelyhidi [Szé15].

Theorem 3.17. *Let H be a smooth function on X. Then, there exists a unique $u \in \mathrm{SH}_m(\omega) \cap C^\infty(X)$ solving the Hessian equation*

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = e^{u+H} \omega^n.$$

Proof. The uniform estimate follows from the maximum principle. We claim that there exists a constant $C = C(H, \omega)$ such that

$$\|u\|_\infty \leqslant C.$$

Indeed, suppose that u attains a maximum at $x \in X$. Then $dd^c u(x) \leqslant 0$. Hence, at x,

$$e^{u(x)+H(x)} = (\omega + dd^c u)^m \wedge \omega^{n-m}/\omega^n \leqslant \omega^n/\omega^n = 1.$$

This implies that $e^{\sup_X u} \leqslant e^{-\inf_X H}$. Similarly, $e^{\inf_X u} \geqslant e^{-\sup_X H}$.

Lemma 3.18 (Hou–Ma–Wu Laplacian estimate). *We have*

$$\sup_X |\partial\overline{\partial}u| \leqslant C\Big(1 + \sup_X |\nabla u|^2\Big),$$

where the constant C depends on $\|u\|_\infty, \omega, H$.

Proof. We follow the proof in [Szé15] which generalized the Hou–Ma–Wu result [HMW10] to Hermitian manifolds. We only need to adjust our notation to that in [Szé15]. Write

$$\omega = \sqrt{-1}\sum \omega_{j\bar{k}}\, dz_j \wedge d\bar{z}_k.$$

Let $(\omega^{j\bar{k}})$ be the inverse matrix of $(\omega_{j\bar{k}})$ and consider

$$A^{ij} = \omega^{j\bar{p}}(\omega_{i\bar{p}} + u_{i\bar{p}}) =: \omega^{j\bar{p}} g_{i\bar{p}}.$$

Then the equation is equivalent to

$$F(A) = u + H,$$

where

$$F(A) = \log S_m(\lambda([A^{ij}])),$$

with S_m denoting the elementary symmetric polynomial of degree m. Without loss of generality we may assume that z_0 is the origin 0 and the coordinates z are chosen as in [Szé15, § 4].

From now on we use the notation and the computations in [Szé15, § 4] with $\alpha \equiv \chi \equiv \omega$. Since $\|u\|_\infty \leqslant C$, where C is a uniform constant and ω is a positive form, it follows that $\underline{u} \equiv 0$ is the subsolution in the sense used in [Szé15]. When the right-hand side is independent of u the proof is given in [Szé15]. A small modification is required for the present case. As the equation is now

$$F(A) = u + H,$$

the computations will change accordingly at each step. We need to use differentiation at 0 to get

$$u_p + H_p = F^{kk} g_{k\bar{k}p},$$
$$u_{1\bar{1}} + H_{1\bar{1}} = F^{pq,rs} g_{p\bar{q}1} g_{r\bar{s}\bar{1}} + F^{kk} g_{k\bar{k}1\bar{1}}.$$

Since $\mathcal{F} = \sum F^{kk} > \tau$ and $u_{1\bar{1}}$ is controlled by $\lambda_1 > 1$, the second equation above is enough to get inequality (81) in [Szé15]:

$$F^{kk}\tilde{\lambda}_{1,k\bar{k}} \geqslant -F^{pq,rs} g_{p\bar{q}1} g_{r\bar{s}\bar{1}} - 2F^{kk}\mathrm{Re}(g_{k\bar{1}1}\overline{T^1_{k1}}) - C_0\lambda_1\mathcal{F}.$$

Again, if we replace h_p there by $u_p + H_p$, then inequality (95) in [Szé15] holds true:

$$F^{kk} u_{pk\bar{k}} u_{\bar{p}} \geqslant -C_0 K\mathcal{F} - \epsilon_1 F^{kk}\lambda_k^2 - C_{\epsilon_1}\mathcal{F}K.$$

The rest of the proof is unchanged. So we get the lemma. □

Thus, we have proven the Hou–Ma–Wu type second-order estimate which enables us to use the blow-up argument, due to Dinew and Kołodziej [DK12b], to get the gradient estimate (see also its variations due to Tosatti and Weinkove [TW13] and to Székelyhidi [Szé15]). Consequently, we also get *a priori* estimates for $|\partial\bar{\partial}u|$. Then, $C^{2,\alpha}$ estimates follow from the Evans–Krylov theorem (see, for example, [TWWY15]). By bootstrapping arguments we get C^∞ estimates for the equation.

Finally, the existence follows by the standard continuity method through the family

$$\log(\omega_{u_t}^m \wedge \omega^{n-m}/\omega^n) = u_t + tH$$

for $t \in [0, 1]$. The uniqueness is a simple consequence of the maximum principle. □

We also need the existence and uniqueness of weak solutions of the Hessian type equation. We refer to [Ngu16] for more details about weak solutions to this equation in the case $m = n$.

THEOREM 3.19. *Let $0 \leqslant f \in L^p(\omega^n)$, $p > n/m$, be such that $\int_X f\omega^n > 0$. Assume that $\{f_j\}_{j\geqslant 1}$ are smooth and positive functions on X converging in $L^p(\omega^n)$ to f as $j \to +\infty$. Assume that $u_j \in \mathrm{SH}_m(\omega) \cap C^\infty(X)$ solves*

$$\omega_{u_j}^m \wedge \omega^{n-m} = e^{u_j} f_j \omega^n. \tag{3.11}$$

Then u_j converges uniformly to $u \in \mathcal{A}_m(\omega) \cap C(X)$ as $j \to +\infty$, which is the unique solution in $\mathcal{A}_m(\omega) \cap C(X)$ of

$$\omega_u^m \wedge \omega^{n-m} = e^u f \omega^n. \tag{3.12}$$

Proof. Set $M_j := \sup_X u_j$. Using the argument in [Ngu16, Claim 2.6] we get that the M_j are uniformly bounded. Set $\tilde{u}_j := u_j - M_j$. Equation (3.11) reads

$$\omega_{\tilde{u}_j}^m \wedge \omega^{n-m} = e^{\tilde{u}_j + M_j} f_j \omega^n.$$

Then $\{\tilde{u}_j\}_{j\geqslant 1}$ is relatively compact in $L^1(\omega^n)$ (Lemma 3.3). Passing to a subsequence, still writing $\tilde{u}_j$, we obtain a Cauchy sequence in $L^1(\omega^n)$. By Corollary 3.12 it follows that $\{\tilde{u}_j\}_{j\geqslant}$ is a Cauchy sequence in $C(X)$. Therefore, it converges uniformly to a solution $\tilde{u} \in \mathcal{A}_m(\omega)$ of $\omega_{\tilde{u}}^m \wedge \omega^{n-m} = e^{\tilde{u}+M} f\omega$, where $M = \lim_j M$. Rewriting $u = \tilde{u} + M$, we get that u_j converges uniformly to u which satisfies $\omega_u^m \wedge \omega^{n-m} = e^u f \omega^n$.

By the weak comparison principle (Theorem 3.7) the equation (3.12) has at most one solution in $\mathcal{A}_m(\omega) \cap C(X)$ (see, for example, [Ngu16, Lemma 2.3]). Thanks to this, we conclude that the sequence u_j converges uniformly to the unique solution u because every convergent subsequence in $L^1(\omega^n)$ does. □

We are ready to prove the main result of this subsection.

LEMMA 3.20 (Approximation property). *For any $u \in \mathrm{SH}_m(X, \omega)$ there exists a decreasing sequence of smooth (ω, m)-subharmonic functions on X converging to u pointwise. In particular, $\mathrm{SH}_m(X, \omega) \equiv \mathcal{A}_m(X, \omega)$.*

Proof. The general scheme is borrowed from Berman [Ber13] and Eyssidieux *et al.* [EGZ15] (used also in [LN15]). However, to make the argument work we have to employ results which allow us to extend the proof from the Kähler context to the Hermitian context.

Take u an (ω, m)-subharmonic function. As $\max\{u, -j\} \in \mathrm{SH}_m(\omega)$ for any $j \geqslant 1$, without loss of generality we may assume that u is bounded. Suppose that $u \leqslant h \in C^\infty(X)$, where the function h may not belong to $\mathrm{SH}_m(\omega)$. Consider the largest (ω, m)-subharmonic function $\tilde{h}$ which is smaller than or equal to h. The function $\tilde{h}$ can be obtained by taking the upper semicontinuous regularization of

$$\sup\{v \in \mathrm{SH}_m(\omega) \cap L^\infty(X) : v \leqslant h\}.$$

Then it is clear that $\tilde{h}$ is a (ω, m)-subharmonic and $u \leqslant \tilde{h} \leqslant h$. We will show that $\tilde{h}$ can be approximated by a decreasing sequence of smooth (ω, m)-subharmonic functions, $\tilde{h} \in \mathcal{A}_m(\omega)$. Once this is done, we also obtain $u \in \mathcal{A}_m(\omega)$ by letting $h \searrow u$ and choosing an appropriate sequence of approximants of $\tilde{h} \searrow u$.

Since $h \in C^\infty(X)$, we can write $\omega_h^m \wedge \omega^{n-m} = F\omega^n$, with F being a smooth function on X. We take the non-negative part $F_* = \max\{F, 0\}$, and then a smooth approximation of it to obtain a non-negative and smooth function $\tilde{F} \geqslant F_*$. Using the existence of a smooth (ω, m)-solution to the complex Hessian type equation (Theorem 3.17), we get for $0 < \varepsilon \leqslant 1$,

$$\omega_{\tilde{w}_\varepsilon}^m \wedge \omega^{n-m} = e^{(1/\varepsilon)(\tilde{w}_\varepsilon - h)}[\tilde{F} + \varepsilon]\omega^n,$$

where $\tilde{w}_\varepsilon \in \mathrm{SH}_m(\omega) \cap C^\infty(X)$.

It is easy to see, by the maximum principle, that $\tilde{w}_\varepsilon \leqslant h$ and $\tilde{w}_\varepsilon$ is decreasing in ε. That means $\tilde{w}_\varepsilon \nearrow$ as $\varepsilon \searrow 0$ and is bounded from above by h. Taking limits on both sides as $\tilde{F} \to F_*$ uniformly, by Theorem 3.19 we get (for any fixed ε) that $\tilde{w}_\varepsilon \to w_\varepsilon \in \mathcal{A}_m(\omega) \cap C(X)$ uniformly and w_ε is also increasing as $\varepsilon \searrow 0$. Moreover, at the limit we have

$$\omega_{w_\varepsilon}^m \wedge \omega^{n-m} = e^{(1/\varepsilon)(w_\varepsilon - h)}(F_* + \varepsilon)\omega^n.$$

Since $w_\varepsilon \leqslant h$, the right-hand side is uniformly bounded in $L^\infty(X)$. The monotone sequence of continuous (ω, m)-subharmonic functions $\{w_\varepsilon\}_{\varepsilon > 0}$ is bounded by h, therefore it is Cauchy in $L^1(X)$. Letting $\varepsilon \searrow 0$, it follows from Corollary 3.12 that $w_\varepsilon \nearrow w \in \mathcal{A}_m(\omega) \cap C(X)$ uniformly and w satisfies

$$\omega_w^m \wedge \omega^{n-m} \leqslant \mathbf{1}_{\{w=h\}} F_* \, \omega^n.$$

We now claim that $w = \tilde{h}$. Indeed, as $w_\varepsilon \leqslant \tilde{h}$, it follows that $w \leqslant \tilde{h}$. It remains to show that $w \geqslant \tilde{h}$ on $\{w < h\}$. Take $v \in \mathrm{SH}_m(\omega) \cap L^\infty(X)$ and $v \leqslant h$. First, we observe that $\omega_w^m \wedge \omega^{n-m} = 0$ on $\{w < v\} \subset \{w < h\}$. If $\{w < v\}$ were non-empty then by the maximality of w on this set would we would get a contradiction (see Theorem 2.16, Remark 2.18). ☐

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