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THE WEAK VOPĚNKA PRINCIPLE FOR DEFINABLE CLASSES OF STRUCTURES

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Abstract. We give a level-by-level analysis of the Weak Vopěnka Principle for definable classes of relational structures (WVP), in accordance with the complexity of their definition, and we determine the large-cardinal strength of each level. Thus, in particular, we show that WVP for Σ_2-definable classes is equivalent to the existence of a strong cardinal. The main theorem (Theorem 5.11) shows, more generally, that WVP for Σ_n-definable classes is equivalent to the existence of a Σ_n-strong cardinal (Definition 5.1). Hence, WVP is equivalent to the existence of a Σ_n-strong cardinal for all $n < \omega$.

§1. Introduction. The Vopěnka Principle (VP), which asserts that there is no rigid proper class of graphs, is a well-known strong large-cardinal principle[1] (see [9]). Properly formulated as a first-order assertion, VP is a schema, that is, an infinite collection of statements, one for each first-order formula defining a proper class of graphs and asserting that the class defined by the formula is not rigid, i.e., there is some non-identity morphism. An equivalent formulation of VP as a first-order schema is given by restricting VP to proper classes of graphs that are definable with a certain degree of complexity, according to the Levy hierarchy of formulas Σ_n, $n < \omega$ (see [8]). Thus, writing $\mathbf{\Sigma_n}$-VP for the first-order assertion that every Σ_n-definable (with parameters) proper class of graphs is rigid, we have that VP is equivalent to the schema consisting of $\mathbf{\Sigma_n}$-VP for every n. As shown in [4, 5], $\mathbf{\Sigma_1}$-VP is provable in ZFC, while $\mathbf{\Sigma_2}$-VP is equivalent to the existence of a proper class of supercompact cardinals, $\mathbf{\Sigma_3}$-VP is equivalent to the existence of a proper class of extendible cardinals, and $\mathbf{\Sigma_n}$-VP is equivalent to the existence of a proper class of $C^{(n-2)}$-extendible cardinals, for $n \geq 3$. The level-by-level analysis of $\mathbf{\Sigma_n}$-VP, $n < \omega$, and the determination of their large-cardinal strength yielded new results in category theory, homology theory, homotopy theory, and universal

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[1] As related in [2], *"The story of Vopěnka's principle* [...] *is that of a practical joke which misfired: In the* 1960*'s P. Vopěnka was repelled by the multitude of large cardinals which emerged in set theory. When he constructed, in collaboration with Z. Hedrlín and A. Pultr, a rigid graph on every set* [...], *he came to the conclusion that, with some more effort, a large rigid class of graphs must surely be also constructible. He then decided to tease set-theorists: he introduced a new principle* (*known today as Vopěnka's principle*), *and proved some consequences concerning large cardinals. He hoped that some set-theorists would continue this line of research* (*which they did*) *until somebody showed that the principle was nonsense. However the latter never materialized—after a number of unsuccessful attempts at constructing a large rigid class of graphs, Vopěnka's principle received its name from Vopěnka's disciples."*

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algebra (see [5]). For example, the existence of cohomological localizations in the homotopy category of simplicial sets (Bousfield Conjecture) follows from $\mathbf{\Sigma_2}$-VP.

The role of VP in category theory has a rich history. The first equivalences of VP with various category-theoretic statements were announced by Fisher in [6]. Further equivalences were proved over the next two decades by Adámek, Rosický, Trnková, and others. Their work showed that under VP *"the structure of locally presentable categories becomes much more transparent"* [2, p. 241]. For example, the statement that a category is locally presentable if and only if it is complete and bounded is equivalent to VP. And so is the statement that every orthogonality class in a locally presentable category is a small-orthogonality class [2, 6.9 and 6.14], [10]. Of the many category-theoretic statements now known to be equivalent to VP, the following one (see [2, 6.D]) turned out to be of particular interest:

(1) Every full subcategory of a locally presentable category $\mathcal{K}$ closed under colimits is coreflective in $\mathcal{K}$.

What made (1) particularly interesting is that its dual statement

(2) Every full subcategory of a locally presentable category $\mathcal{K}$ closed under limits is reflective in $\mathcal{K}$,

while being a consequence of (1), could not be proved equivalent to it. Since VP—hence also (1)—was known to be equivalent to

$$\mathbf{Ord} \text{ cannot be fully embedded into } \mathbf{Gra}$$

(see [2, 6.3]), while statement (2) was proved equivalent to

$$\mathbf{Ord}^{op} \text{ cannot be fully embedded into } \mathbf{Gra}$$

(see [2, 6.22 and 6.23]), the latter assertion was then called the *Weak Vopěnka Principle* (WVP). The term *Weak* was aptly given, for it is readily shown that VP implies WVP [3] (Proposition 2.1). The question then remained if WVP implied VP. Using a result of Isbell [7], which showed that $\mathbf{Ord}^{op}$ is bounded iff there is no proper class of measurable cardinals, Adámek and Rosický [2] proved that WVP implies the existence of a proper class of measurable cardinals. This was seen as a first step in showing that WVP was indeed a strong large-cardinal principle, perhaps even equivalent to VP. Much work was devoted to trying to obtain stronger large cardinals from it, e.g., strongly compact or supercompact cardinals, but to no avail. A further natural principle, between VP and WVP, called the *Semi-Weak Vopěnka Principle* (SWVP), was introduced in [1] and the further question of the equivalence between the three principles, WVP, SWVP, and VP, remained open. The problem was finally solved in 2019 by the second author of the present paper. In [13] he showed that WVP and SWVP are equivalent, and they are also equivalent to the large-cardinal principle "OR is Woodin," whose consistency strength is known to be well below the existence of a supercompact cardinal, thereby showing that WVP cannot imply VP (if consistent with ZFC).

In the present paper we carry out a level-by-level analysis of WVP and SWVP similar to the analysis of VP done in [4, 5]. Thus, for every $n \geq 2$ we prove the equivalence of both $\mathbf{\Sigma_n}$-WVP and $\mathbf{\Sigma_n}$-SWVP (see Definition 2.2) with the existence of certain large cardinals. In particular, we show that $\mathbf{\Sigma_2}$-WVP and $\mathbf{\Sigma_2}$-SWVP are equivalent to the existence of a proper class of strong cardinals. The main theorems

(Theorems 5.11 and 5.13) show, more generally, that $\mathbf{\Sigma_n}$-WVP and $\mathbf{\Sigma_n}$-SWVP are equivalent to the existence of a proper class of Σ_n-strong cardinals (Definition 5.1). It follows that WVP and SWVP are equivalent to the schema asserting the existence of a Σ_n-strong cardinal for every $n < \omega$. Our arguments yield also a new proof of the second author's result from [13] that WVP implies "OR is Woodin" (Corollary 5.15). The main difference between the two proofs is that while in the present paper we derive the extenders witnessing "OR is Woodin" from homomorphisms on products of relational structures with universe of the form V_α, the proof in [13] uses homomorphisms of so-called $\mathcal{P}$-structures. We think, however, that it should be possible to do a similar level-by-level analysis as done here by using $\mathcal{P}$-structures instead. A number of consequences in category theory should follow from our results. For instance, the statement that every $\mathbf{\Sigma_2}$-definable full subcategory of a locally presentable category $\mathcal{K}$ closed in $\mathcal{K}$ under limits is reflective in $\mathcal{K}$, should be equivalent to the existence of a proper class of strong cardinals. See [2, Chapter 6] for more examples.

§2. Preliminaries. Recall that a *graph* is a structure $G = \langle G, E_G \rangle$, where G is a non-empty set and E_G is a binary relation on G. If $G = \langle G, E_G \rangle$ and $H = \langle H, E_H \rangle$ are graphs, a map $h : G \to H$ is a *homomorphism* if it preserves the binary relation, meaning that for all $x, y \in G$, if $x E_G y$, then $h(x) E_H h(y)$.

A class $\mathcal{G}$ of graphs is called *rigid* if there are no non-trivial homomorphisms between graphs in $\mathcal{G}$, i.e., the only homomorphisms are the identity morphisms $G \to G$, for $G \in \mathcal{G}$.

The original formulation of the *Vopěnka Principle* (VP) (P. Vopěnka, ca. 1960) asserts that there is no rigid proper class of graphs. As shown in [2, 6.A], VP is equivalent to the statement that the category **Ord** of ordinals cannot be fully embedded into the category **Gra** of graphs. That is, there is no sequence $\langle G_\alpha : \alpha \in \mathrm{OR} \rangle$ of graphs such that for every $\alpha \leq \beta$ there exists exactly one homomorphism $G_\alpha \to G_\beta$, and no homomorphism $G_\beta \to G_\alpha$ whenever $\alpha < \beta$.

The *Weak Vopěnka Principle* (WVP) (first introduced in [3]) is the statement *dual* to VP, namely that the opposite category of ordinals, $\mathbf{Ord}^{op}$, cannot be fully embedded into **Gra**. That is, there is no sequence $\langle G_\alpha : \alpha \in \mathrm{OR} \rangle$ of graphs such that for every $\alpha \leq \beta$ there exists exactly one homomorphism $G_\beta \to G_\alpha$, and no homomorphism $G_\alpha \to G_\beta$ whenever $\alpha < \beta$.

The *Semi-Weak Vopěnka Principle* (SWVP) [1] asserts that there is no sequence $\langle G_\alpha : \alpha \in \mathrm{OR} \rangle$ of graphs such that for every $\alpha \leq \beta$ there exists some (not necessarily unique) homomorphism $G_\beta \to G_\alpha$, and no homomorphism $G_\alpha \to G_\beta$ whenever $\alpha < \beta$.

Clearly, SWVP implies WVP. The second author showed in [12] that SWVP is in fact equivalent to WVP. As shown in [3], VP implies WVP, and the same argument also shows that VP implies SWVP. In fact, the argument shows the following:

PROPOSITION 2.1. VP *implies that for every sequence* $\langle G_\alpha : \alpha \in \mathrm{OR} \rangle$ *of graphs there exist* $\alpha < \beta$ *with a homomorphism* $G_\alpha \to G_\beta$.

PROOF. Suppose $\langle G_\alpha : \alpha \in \mathrm{OR} \rangle$ is a sequence of graphs. Without loss of generality, if $\alpha < \beta$, then G_α and G_β are not isomorphic. Since there are only set-many (as opposed to proper-class-many) non-isomorphic graphs of any given

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cardinality, there exists a proper class $C \subseteq \mathrm{OR}$ such that $|G_\alpha| < |G_\beta|$ whenever $\alpha < \beta$ are in C. For each $\alpha \in C$, add to $G_\alpha = \langle G_\alpha, E_\alpha \rangle$ a rigid binary relation S_α on G_α [11], as well as the non-identity relation $\neq$, and consider the structure $A_\alpha = \langle G_\alpha, E_\alpha, S_\alpha, \neq \rangle$. Since the cardinalities are strictly increasing, by the $\neq$ relation there cannot be any homomorphism $A_\beta \to A_\alpha$ with $\alpha < \beta$. Also, because of the rigid relation S_α, the identity is the only homomorphism $A_\alpha \to A_\alpha$. Since the category whose objects are the A_α, $\alpha \in C$ and the morphisms are the homomorphisms can be fully embedded into **Gra** (see [2, 2.65]), by VP the class $\{A_\alpha : \alpha \in C\}$ is not rigid, and so there must exist $\alpha < \beta$ with a homomorphism $A_\alpha \to A_\beta$, hence also a homomorphism $G_\alpha \to G_\beta$. ⊣

The definitions of VP, WVP, and SWVP given above quantify over arbitrary classes, so they are not first-order. Thus, a proper study of these principles must be carried out in some adequate class theory, such as NBG. In particular, the proof of the last proposition can only be formally given in such class theory. We shall however be interested in the forthcoming in the first-order versions of VP, WVP, and SWVP, which require us to restrict our attention to definable classes.

2.1. The VP, WVP, and SWVP for definable classes. Each of VP, WVP, and SWVP can be formulated in the first-order language of set theory as a definition schema, namely as an infinite list of definitions, one for every natural number n, as follows:

DEFINITION 2.2. Let n be a natural number, and let P be a set or a proper class.

The $\Sigma_n(P)$-*Vopěnka Principle* ($\Sigma_n(P)$-VP *for short*) asserts that there is no Σ_n-definable, with parameters in P, sequence $\langle G_\alpha : \alpha \in \mathrm{OR} \rangle$ of graphs such that for every $\alpha \leq \beta$ there exists exactly one homomorphism $G_\alpha \to G_\beta$, and no homomorphism $G_\beta \to G_\alpha$ whenever $\alpha < \beta$.

The $\Sigma_n(P)$-*Weak Vopěnka Principle* ($\Sigma_n(P)$-WVP *for short*) asserts that there is no Σ_n-definable, with parameters in P, sequence $\langle G_\alpha : \alpha \in \mathrm{OR} \rangle$ of graphs such that for every $\alpha \leq \beta$ there exists exactly one homomorphism $G_\beta \to G_\alpha$, and no homomorphism $G_\alpha \to G_\beta$ whenever $\alpha < \beta$.

The boldface versions $\boldsymbol{\Sigma}_n$-VP *and* $\boldsymbol{\Sigma}_n$-WVP are defined as $\Sigma_n(V)$-VP and $\Sigma_n(V)$-WVP respectively, i.e., any set is allowed as a parameter in the definitions.

$\Pi_n(P)$-VP and $\Pi_n(P)$-WVP, as well as $\boldsymbol{\Pi}_\mathbf{n}$-VP and $\boldsymbol{\Pi}_\mathbf{n}$-WVP, and the lightface (i.e., without parameters) versions Σ_n-VP, Σ_n-WVP and Π_n-VP, Π_n-WVP, are defined similarly.

The *Vopěnka Principle* (VP) is the schema asserting that the $\boldsymbol{\Sigma}_\mathbf{n}$-VP holds for every $n < \omega$. And the *Weak Vopěnka Principle* (WVP) is the schema asserting that the $\boldsymbol{\Sigma}_\mathbf{n}$-WVP holds for every $n < \omega$.

If instead of requiring that for $\alpha \leq \beta$ there is exactly one homomorphism $G_\beta \to G_\alpha$ we only require that there is at least one, then we obtain the *Semi-Weak Vopěnka Principle* (SWVP), formulated as the first-order schema consisting of $\boldsymbol{\Sigma}_\mathbf{n}$-SWVP for all $n < \omega$.

It is well-known that the category of structures in any fixed (many-sorted, infinitary) relational language can be fully embedded into **Gra** (see [2, 2.65]). Thus, if in the original definitions of VP, WVP, and SWVP one replaces "graphs" by "structures in a fixed (many-sorted, infinitary) relational language," one obtains

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equivalent notions. The same is true for the first-order formulations of these principles, but some extra care is needed to ensure there is no increase in the complexity of the definitions. In particular, in the case of infinite language signatures, an extra parameter for the language signature τ, as well as a parameter for a rigid binary relation on a binary signature associated with τ, may be needed in the definition. Namely, suppose Γ is one of the definability classes Σ_n, Π_n, with $n \geq 1$, P is a set or a proper class, and $\mathcal{C}$ is a Γ-definable, with parameters in P, class of (possibly many-sorted) relational structures in a language type τ, i.e., $\tau = \langle R_\alpha : \alpha < \lambda \rangle$, where each R_α is an n_α-ary relation symbol, n_α being some ordinal, possibly infinite. As in [2, 2.65], there is a Δ_1-definable (i.e., both Σ_1-definable and Π_1-definable), using τ as a parameter, one-sorted binary type τ' (meaning that all the relations are binary), and also a Γ-definable, with parameters in P plus τ as an additional parameter, full embedding of $\mathcal{C}$ into the category $\mathbf{Rel}\,\tau'$ of τ'-structures and homomorphisms. Furthermore, there is a Δ_1-definable, using τ and a rigid binary relation r on τ' as parameters, full embedding of $\mathbf{Rel}\,\tau'$ into $\mathbf{Gra}$. Hence, there is a Γ-definable (with parameters in P, plus τ and r as additional parameters) full embedding of $\mathcal{C}$ into $\mathbf{Gra}$. Therefore, in the definitions of Γ-VP, Γ-WVP, and Γ-SWVP we may replace "graphs" by "structures in a fixed (many-sorted, infinitary) relational language" and obtain equivalent principles, provided we allow for the additional parameters (τ and r) involved. Let us, however, stress the fact that in the case of finite τ, or even if τ is countable infinite and definable without parameters (e.g., recursive), then no additional parameters are involved, and therefore the versions of Γ-VP, Γ-WVP, and Γ-SWVP for graphs and for relational structures are equivalent.

2.2. Strong cardinals. Recall that a cardinal κ is λ-*strong*, where λ is a cardinal greater than κ, if there exists an elementary embedding $j : V \to M$, with M transitive, with critical point κ, and with V_λ contained in M. A cardinal κ is *strong* if it is λ-strong for every cardinal $\lambda > \kappa$.

If κ is a strong cardinal, then for every cardinal $\lambda > \kappa$ there exists an elementary embedding $j : V \to M$, with M transitive, critical point κ, V_λ contained in M, and $j(\kappa) > \lambda$. Moreover, if κ is strong, then $V_\kappa \preceq_{\Sigma_2} V$. (See [9].)

It is well-known that the notion of strong cardinal can be formulated in terms of extenders (see [9, Section 26]). Namely,

Definition 2.3. Given a cardinal κ, and $\beta > \kappa$, a (κ, β)-*extender* is a collection $\mathcal{E} := \{E_a : a \in [\beta]^{<\omega}\}$ such that:

(1) Each E_a is a κ-complete ultrafilter over $[\kappa]^{|a|}$, and E_a is not κ^+-complete for some a.
(2) For each $\xi < \kappa$, there is some a with $\{s \in [\kappa]^{|a|} : \xi \in s\} \in E_a$.
(3) *Coherence:* If $a \subseteq b$ are in $[\beta]^{<\omega}$, with $b = \{\alpha_1, \dots, \alpha_n\}$ and $a = \{\alpha_{i_1}, \dots, \alpha_{i_n}\}$, and $\pi_{ba} : [\kappa]^{|b|} \to [\kappa]^{|a|}$ is the map given by $\pi_{ba}(\{\xi_1, \dots, \xi_n\}) = \{\xi_{i_1}, \dots, \xi_{i_n}\}$, then
$$X \in E_a \quad \text{if and only if} \quad \{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\} \in E_b\,.$$
(4) *Normality*: Whenever $a \in [\beta]^{<\omega}$ and $f : [\kappa]^{|a|} \to V$ are such that $\{s \in [\kappa]^{|a|} : f(s) \in \max(s)\} \in E_a$, there is $b \in [\beta]^{<\omega}$ with $a \subseteq b$ such that
$$\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\} \in E_b\,.$$

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(5) *Well-foundedness*: Whenever $a_m \in [\beta]^{<\omega}$ and $X_m \in E_{a_m}$ for $m \in \omega$, there is a function $d : \bigcup_m a_m \to \kappa$ such that $d``a_m \in X_m$ for every m.

PROPOSITION 2.4. *A cardinal κ is λ-strong if and only if there exists a $(\kappa, |V_\lambda|^+)$-extender $\mathcal{E}$ such that $V_\lambda \subseteq \overline{M}_{\mathcal{E}}$ and $\lambda < j_{\mathcal{E}}(\kappa)$ (where $\overline{M}_{\mathcal{E}}$ is the transitive collapse of the direct limit ultrapower $M_{\mathcal{E}}$ of V by $\mathcal{E}$, and $j_{\mathcal{E}} : V \to \overline{M}_{\mathcal{E}}$ is the corresponding elementary embedding).*

PROOF. See [9, Exercise 26.7]. ⊣

§3. The Product Reflection Principle. For any set S of relational structures $\mathcal{A} = \langle A, \dots \rangle$ of the same type, the set-theoretic product $\prod S$ is the structure whose universe is the set of all functions f with domain S such that $f(\mathcal{A}) \in A$ for every $\mathcal{A} \in S$, and whose relations are defined pointwise.

DEFINITION 3.1 (The Product Reflection Principle (PRP)). For Γ a definability class (i.e., one of Σ_n, Π_n, some $n > 0$), and P a set or proper class, $\Gamma(P)$-PRP asserts that for every class $\mathcal{C}$ of graphs that is Γ-definable with parameters in P, the following holds:

PRP: There is a non-empty subset S of $\mathcal{C}$ such that for every G in $\mathcal{C}$ there is a homomorphism $\prod S \to G$.

If $P = \varnothing$, then we simply write Γ-PRP. If $P = V$, then we write Γ in boldface, e.g., $\mathbf{\Sigma_n}$-PRP.

Note that in the case where $\mathcal{C}$ is a set, PRP is trivial because we may take $S = \mathcal{C}$ and use the coordinate projection homomorphisms from $\prod S$, so the nontrivial case is the one where $\mathcal{C}$ is a proper class (and S is still required to be a set.)

In the definition of $\Gamma(P)$-PRP we may replace "graphs" by "structures in a fixed (many-sorted, infinitary) relational language" and obtain equivalent principles, provided we allow for some additional parameters (see our remarks after Definition 2.2). Thus, the boldface principle $\mathbf{\Sigma_n}$-PRP for classes of graphs is equivalent to its version for classes of relational structures.

We shall denote by $C^{(n)}$ the Π_n-definable closed and unbounded class of ordinals κ that are Σ_n-*correct in* V, i.e., $V_\kappa \preceq_{\Sigma_n} V$. (See [4].)

PROPOSITION 3.2. $\mathbf{\Sigma_1}$-PRP *holds. In fact, for every $\kappa \in C^{(1)}$ and every Σ_1-definable with parameters in V_κ proper class $\mathcal{C}$ of structures in a fixed relational language $\tau \in V_\kappa$, the set $S := \mathcal{C} \cap V_\kappa$ witnesses $\Sigma_1(V_\kappa)$*-PRP.

PROOF. Let $\kappa \in C^{(1)}$, and let $\mathcal{C}$ be a Σ_1-definable, with a set of parameters $P \in V_\kappa$, proper class of structures in a relational language $\tau \in V_\kappa$. Note that since $\kappa \in C^{(1)}$, $V_\kappa = H_\kappa$, and hence $|\mathrm{TC}(\{\tau\} \cup P)| < \kappa$. Let $\varphi(x)$ be a Σ_1 formula, with parameters in P, defining $\mathcal{C}$. We claim that $S := \mathcal{C} \cap V_\kappa$ satisfies PRP. Given $\mathcal{A} \in \mathcal{C}$, let $\lambda \in C^{(1)}$ be greater than κ and such that $\mathcal{A} \in V_\lambda$. Let $N \preceq V_\lambda$ be of cardinality less than κ and such that $\mathcal{A} \in N$ and $\mathrm{TC}(\{\tau\} \cup P) \subseteq N$. Let $\pi : M \to N$ be the inverse transitive collapse isomorphism, and let $\mathcal{B} \in M$ be such that $\pi(\mathcal{B}) = \mathcal{A}$. Notice that π fixes τ and the parameters of $\varphi(x)$. Since M is transitive and of cardinality less than κ, $\mathcal{B} \in H_\kappa = V_\kappa$. Also, since $V_\lambda \models \varphi(\mathcal{A})$, we have $N \models \varphi(\mathcal{A})$, and therefore $M \models \varphi(\mathcal{B})$. Hence, since M is transitive and φ is upwards absolute for transitive

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sets, $\mathcal{B} \in \mathcal{C}$. Thus, $\mathcal{B} \in S$. Then the composition of π with the projection $\prod S \to \mathcal{B}$ yields the desired homomorphism. $\dashv$

PROPOSITION 3.3. *If κ is a strong cardinal, then $\Sigma_2(V_\kappa)$-PRP holds.*

PROOF. Let κ be a strong cardinal and let $\mathcal{C}$ be a Σ_2-definable, with parameters in V_κ, proper class of structures in a fixed relational language $\tau \in V_\kappa$. Let $\varphi(x)$ be a Σ_2 formula defining it. We will show that $S := \mathcal{C} \cap V_\kappa$ witnesses PRP.

Given any $\mathcal{A} \in \mathcal{C}$, let $\lambda \in C^{(2)}$ be greater than or equal to κ and with $\mathcal{A} \in V_\lambda$.

Let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$, and $j(\kappa) > \lambda$.

By elementarity, the restriction of j to $\prod S$ yields a homomorphism

$$h : \prod S \to \prod(\{X : M \models \varphi(X)\} \cap V^M_{j(\kappa)}).$$

Since $\mathcal{A} \in V_\lambda$, and $\lambda \in C^{(2)}$, we have that $V_\lambda \models \varphi(\mathcal{A})$. Since $\lambda \in C^{(1)}$ is Π_1-expressible and therefore downwards absolute for transitive classes, and since $V_\lambda \subseteq M$, it follows that $V_\lambda \preceq_{\Sigma_1} M$ and therefore $M \models \varphi(\mathcal{A})$. Moreover $\mathcal{A} \in V_\lambda \subseteq V^M_{j(\kappa)}$. Thus, letting

$$g : \prod(\{X : M \models \varphi(X)\} \cap V^M_{j(\kappa)}) \to \mathcal{A}$$

be the projection map, we have that

$$g \circ h : \prod S \to \mathcal{A}$$

is a homomorphism, as wanted. $\dashv$

COROLLARY 3.4. *If there exists a proper class of strong cardinals, then $\mathbf{\Sigma}_2$-PRP holds.*

We shall next show that SWVP is equivalent to the assertion that PRP holds for all definable proper classes of structures. Let Γ_n be either Σ_n or Π_n.

PROPOSITION 3.5. *$\Gamma_n(P)$-PRP implies $\Gamma_n(P)$-SWVP, for every $n > 0$ and every class P.*

PROOF. Assume $\mathcal{G} = \langle G_\alpha : \alpha \in \mathrm{OR}\rangle$ is a sequence of graphs that is definable by a Γ_n formula φ from a parameter $p \in P$, such that whenever $\alpha \leq \beta$ there is a homomorphism $G_\beta \to G_\alpha$. We shall use $\Gamma_n(P)$-PRP to produce a homomorphism $G_\beta \to G_\gamma$ for some $\gamma > \beta$, thereby witnessing the desired instance of SWVP.

In the case $\Gamma_n = \Sigma_n$ we define the class of graphs $\mathcal{C} = \{G_\alpha : \alpha \in \mathrm{OR}\}$, which is Σ_n-definable from the parameter $p \in P$ because a graph G is in $\mathcal{C}$ if and only if $\exists \alpha \in \mathrm{OR}\ \varphi(\alpha, G, p)$ and the formula φ is Σ_n. Therefore by $\Sigma_n(P)$-PRP, there is a subset $\mathcal{S}$ of the class $\mathcal{C}$ such that for every $G \in \mathcal{C}$ there is a homomorphism $\prod \mathcal{S} \to G$. This gives a subset I of the class OR such that for every $\gamma \in \mathrm{OR}$ there is a homomorphism $\prod_{\alpha \in I} G_\alpha \to G_\gamma$. In particular, letting $\beta = \sup I$ and $\gamma > \beta$ (for example, $\gamma = \beta + 1$), there is a homomorphism $\prod_{\alpha \in I} G_\alpha \to G_\gamma$. Composing this with a homomorphism $G_\beta \to \prod_{\alpha \in I} G_\alpha$ (which exists because for each $\alpha \in I$ there is a homomorphism $G_\beta \to G_\alpha$) we obtain a homomorphism $G_\beta \to G_\gamma$, as desired.

In the case $\Gamma_n = \Pi_n$, we will need the following preliminary observation. Namely, we may assume without loss of generality that for every κ the class $\{\alpha \in \mathrm{OR} :$

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$|G_\alpha| \leq \kappa\}$ is bounded by some ordinal $f(\kappa)$, as otherwise it would contain two distinct α and α' such that G_α and $G_{\alpha'}$ are isomorphic, trivially witnessing the desired instance of SWVP. It follows that the class $\{\alpha \in \mathrm{OR} : |G_\alpha| \geq \alpha\}$ contains a club class (namely the class of all closure points of this bounding function f) and in particular it is a proper class. We therefore define the class of graphs

$$\mathcal{C} = \{G_\alpha : \alpha \in \mathrm{OR} \text{ and } |G_\alpha| \geq \alpha\},$$

which is Π_n-definable from the parameter $p \in P$ because a graph G is in $\mathcal{C}$ if and only if

$$\forall \beta \in \mathrm{OR}\ \big(|\beta| = |G| \implies \exists \alpha \leq \beta\ \varphi(\alpha, G, p)\big)$$

and the formula φ is Π_n. (The main point is that the existential quantification over α is bounded.) Therefore, by $\Pi_n(P)$-PRP, there is a subset $\mathcal{S}$ of the class $\mathcal{C}$ such that for every graph $G \in \mathcal{C}$ there is a homomorphism $\prod \mathcal{S} \to G$. This gives a subset I of the class $\{\alpha \in \mathrm{OR} : |G_\alpha| \geq \alpha\}$ such that for every ordinal γ in that class, there is a homomorphism $\prod_{\alpha \in I} G_\alpha \to G_\gamma$. Letting $\beta = \sup I$, because the class $\{\alpha \in \mathrm{OR} : |G_\alpha| \geq \alpha\}$ is a proper class, it contains some ordinal $\gamma > \beta$. We may then proceed as in the Σ_n case to obtain a homomorphism $G_\beta \to G_\gamma$, as desired. ⊣

The converse also holds, and in fact more is true. Namely,

Proposition 3.6. $\Pi_n(P)$-WVP *implies* $\Sigma_{n+1}(P)$-PRP, *for every* $n \in \omega$, *and every* P.

Proof. Let $\mathcal{C}$ be a Σ_{n+1}-definable, with parameter $p \in P$, class of graphs that is a counterexample to PRP. We may represent this class as an increasing union: $\mathcal{C} = \bigcup_{\beta \in C^{(n)}} \mathcal{C}^{V_\beta}$ where $\mathcal{C}^{V_\beta}$ is the relativization of $\mathcal{C}$ to V_β. Note that $\mathcal{C}^{V_\beta}$ is a subset of $\mathcal{C} \cap V_\beta$ for any $\beta \in C^{(n)}$, and (unless $\mathcal{C}$ happens to be Π_n) it may be a proper subset, since the least witness to a Σ_{n+1} property of a graph may have much larger rank than the graph itself.

We recursively define a function $f : \mathrm{Ord} \to \mathrm{Ord}$ by letting $f(0)$ be the least $\beta \in C^{(n)}$ such that $\mathcal{C}^{V_\beta} \neq \varnothing$, letting $f(\alpha+1)$ be the least $\beta \in C^{(n)}$ such that there is no homomorphism $\prod \mathcal{C}^{V_{f(\alpha)}} \to \prod \mathcal{C}^{V_\beta}$, and letting $f(\lambda) = \sup_{\alpha<\lambda} f(\alpha)$ if λ is a limit ordinal. To see that f is a total function, note that the ordinal β in the definition of $f(\alpha+1)$ exists: otherwise for any graph $G \in \mathcal{C}$ we could take $\beta \in C^{(n)}$ sufficiently large that $G \in \mathcal{C}^{V_\beta}$ and then compose a homomorphism $\prod \mathcal{C}^{V_{f(\alpha)}} \to \prod \mathcal{C}^{V_\beta}$ with a projection homomorphism to obtain a homomorphism $\prod \mathcal{C}^{V_{f(\alpha)}} \to G$, thereby witnessing PRP for $\mathcal{C}$.

For every ordinal α we define the product graph

$$H_\alpha = \prod \mathcal{C}^{V_{f(\alpha)}}.$$

Note that the sequence $\langle H_\alpha : \alpha \in \mathrm{Ord}\rangle$ is a counterexample to SWVP (which we do not claim to be Π_n or even Σ_{n+1}): for every pair of ordinals $\alpha \leq \alpha'$ there is a homomorphism from $H_{\alpha'}$ to H_α given by restriction, and for every ordinal α the definition of f implies there is no homomorphism from H_α to $H_{\alpha+1}$ (nor to $H_{\alpha'}$ for any larger α', or else we could compose with a restriction homomorphism to get a homomorphism to $H_{\alpha+1}$.) We will use this sequence to build a counterexample to WVP, which moreover will be Π_n.

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Let Λ be the class of all limit ordinals that are fixed points of f. Equivalently because f is continuous and increasing, Λ is the class of all limit ordinals that are closed under f. Note that Λ is a closed unbounded class and $\Lambda \subseteq C^{(n)}$ (so in particular the elements of λ are limit cardinals, although we won't use this fact directly.) For every pair of ordinals $\lambda \leq \lambda'$ in Λ we define the function $h_{\lambda'\lambda} : V_{\lambda'+1} \to V_{\lambda+1}$ by

$$h_{\lambda'\lambda}(x) = x \cap V_\lambda.$$

We will define some structure (constants and relations) on the sets $V_{\lambda+1}$ ($\lambda \in \Lambda$) that is preserved by these functions $h_{\lambda'\lambda}$ and not by any other functions. To ensure that the structure is not preserved by any other functions, it will encode our counterexample $\langle H_\alpha : \alpha \in \text{Ord} \rangle$ to SWVP (among other things.) The coding will use a "stratified" version H^*_α of the product graph H_α, which we define as a double product of length $1 + \alpha$, having $\prod \mathcal{C}^{V_{f(0)}}$ as initial factor, namely:

$$H^*_\alpha = \prod \mathcal{C}^{V_{f(0)}} \times \prod (\mathcal{C}^{V_{f(1)}} \setminus \mathcal{C}^{V_{f(0)}}) \times \prod (\mathcal{C}^{V_{f(2)}} \setminus \mathcal{C}^{V_{f(1)}}) \times \cdots .$$

Note that the graph H^*_α is isomorphic to H_α (the initial factor was added to ensure this in the case $\alpha = 0$), and it has the advantage that for every pair of ordinals $\lambda \leq \lambda'$ in Λ, the function $h_{\lambda'\lambda}$ is a homomorphism from $H^*_{\lambda'}$ to H^*_λ because it just restricts the outermost product from $\prod_{\alpha_0 < \lambda'}$ to $\prod_{\alpha_0 < \lambda}$, whereas it might not be a homomorphism from $H_{\lambda'}$ to H_λ because $\mathcal{C}^{V_{\lambda'}} \cap V_\lambda$ might not be equal to $\mathcal{C}^{V_\lambda}$.

Our structures will be defined as in [12] except using the graph H^*_α in place of H_α. Namely, we let Σ be the signature with a constant symbol c and ternary relation symbols R, S, and T, and for every ordinal $\lambda \in \Lambda$ we define a corresponding Σ-structure

$$\mathcal{M}_\lambda = \langle V_{\lambda+1}, c^{\mathcal{M}_\lambda}, R^{\mathcal{M}_\lambda}, S^{\mathcal{M}_\lambda}, T^{\mathcal{M}_\lambda} \rangle,$$

where $c^{\mathcal{M}_\lambda} = \lambda$ and the interpretations of R, S, and T are defined as follows:

$$R^{\mathcal{M}_\lambda}(\alpha, x, y) \iff (\alpha = \text{rank}(x) \text{ and } x \in y) \text{ or } \alpha = \lambda,$$
$$S^{\mathcal{M}_\lambda}(\alpha, x, y) \iff (\alpha = \text{rank}(x) \text{ and } x \notin y) \text{ or } \alpha = \lambda,$$
$$T^{\mathcal{M}_\lambda}(\alpha, x, y) \iff x \text{ is adjacent to } y \text{ in } H^*_\alpha.$$

(In the definition of $T^{\mathcal{M}_\lambda}$ we take x and y to be vertices of H^*_α.)

Essentially the same argument as in [12] shows that the only homomorphisms among the structures $\mathcal{M}_\lambda$ for $\lambda \in \Lambda$ are the homomorphisms $h_{\lambda'\lambda}$ for $\lambda \leq \lambda'$. Here we will just remind the reader of the main idea of that argument, which is that for any ordinals λ and λ' in Λ:

(1) If $\lambda < \lambda'$, then preservation of the T relation and the constant c ensures that any "forward" homomorphism from $\mathcal{M}_\lambda$ to $\mathcal{M}_{\lambda'}$ would produce a homomorphism from H^*_λ to $H^*_{\lambda'}$, or equivalently from H_λ to $H_{\lambda'}$, contradicting the fact that $\langle H_\alpha : \alpha \in \text{Ord} \rangle$ is a counterexample to SWVP.

(2) If $\lambda \leq \lambda'$, then preservation of the R and S relations (which encode the membership relation $\in$) ensures that any "reverse" homomorphism from $\mathcal{M}_{\lambda'}$ to $\mathcal{M}_\lambda$ that is not equal to $h_{\lambda'\lambda}$ would have a critical point that is mapped forward, which would then yield a contradiction using preservation of the T relation by an argument similar to (1).

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Enumerating Λ in increasing order as $\langle \lambda_\xi : \xi \in \mathrm{Ord} \rangle$, or in other words letting λ_ξ be the ξth fixed point of f, the sequence of structures

$$\langle \mathcal{M}_{\lambda_\xi} : \xi \in \mathrm{Ord} \rangle$$

is therefore a counterexample to WVP. It remains to check that this sequence of structures is Π_n. For any ordinal ξ and any structure $\mathcal{M}$, the condition $\mathcal{M} = \mathcal{M}_{\lambda_\xi}$ is equivalent to the conjunction of the following conditions, each of which can be expressed by a Π_n formula (or simpler):

(1) $\mathcal{M}$ is a Σ-structure whose underlying set M contains a largest ordinal λ. This condition can be expressed by a bounded formula.
(2) M is a rank initial segment of V (which must therefore be equal to $V_{\lambda+1}$.) This condition is Π_1.
(3) λ is in the class $C^{(n)}$. This condition is Π_n.
(4) λ is equal to λ_ξ, the ξth closure point of f. Given that $M = V_{\lambda+1}$ where $\lambda \in C^{(n)}$, this condition can be expressed by a bounded formula over M. To see this, first note that the definition of the class $C^{(n)}$ is Π_n and therefore absolute to V_λ. Second, note that the definition of f (as a relation and in particular as a partial function) using $C^{(n)}$ is also absolute to V_λ. For the absoluteness of f, note that the complexity of $\mathcal{C}$ doesn't make f complex, because f is defined using relativizations of $\mathcal{C}$ that can be computed locally; also note that the construction of product graphs is absolute to V_λ, as is the existence of a homomorphism between any two given graphs.

   Third, note that because f is absolute to V_λ, it follows that V_λ can see whether or not λ is a closure point of f. (This holds if and only if the relativization of f to V_λ is a total function on λ.) Finally, note that M can see more specifically whether or not λ is the ξth closure point of f: in the case that $\xi \in M$, this again follows from the absoluteness of f to V_λ, but actually the case that $\xi \in M$ is the only case we need to consider, because $M = V_{\lambda+1}$ and for λ to be the ξth closure point of f would require $\xi \leq \lambda$ and therefore $\xi \in M$.
(5) The constant and the three relations of the Σ-structure $\mathcal{M}$ are defined correctly, meaning $c^{\mathcal{M}} = c^{\mathcal{M}_\lambda}$, $R^{\mathcal{M}} = R^{\mathcal{M}_\lambda}$, $S^{\mathcal{M}} = S^{\mathcal{M}_\lambda}$, and $T^{\mathcal{M}} = T^{\mathcal{M}_\lambda}$, defined as above. Given that the underlying set M of this structure is equal to $V_{\lambda+1}$ where λ is a closure point of f (and is therefore in $C^{(n)}$) these four conditions can be expressed by bounded formulas over the structure $\mathcal{M}$. We show this only for the condition $T^{\mathcal{M}} = T^{\mathcal{M}_\lambda}$, since the other three are relatively straightforward.

   First, note that the definition of the sequence of graphs $\langle H^*_\alpha : \alpha < \lambda \rangle$ is absolute to V_λ: again the complexity of the class $\mathcal{C}$ doesn't matter because f and H^*_α are defined using local relativizations of it. Second, note that the vertex set and edge relation of the graph H^*_λ are definable by bounded formulas over $V_{\lambda+1}$ for a similar reason. These two observations show that the correct relation $T^{\mathcal{M}_\lambda}$ is definable by a bounded formula over the set $M = V_{\lambda+1}$, so the structure $\mathcal{M}$ can see whether or not $T^{\mathcal{M}_\lambda}$ agrees with its own relation $T^{\mathcal{M}}$. ⊣

The following is now an immediate consequence of Propositions 3.5 and 3.6.

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THEOREM 3.7. *For every* $n > 0$ *and every set or proper class* P, *the following are equivalent*:

(1) $\Sigma_{n+1}(P)$-PRP.
(2) $\Pi_n(P)$-PRP.
(3) $\Sigma_{n+1}(P)$-SWVP.
(4) $\Pi_n(P)$-SWVP.
(5) $\Sigma_{n+1}(P)$-WVP.
(6) $\Pi_n(P)$-WVP.

PROOF. The implications (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), and (5) $\Rightarrow$ (6) are trivial. The implications (3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6) are clear. Proposition 3.5 yields (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4). Finally, Proposition 3.6 yields (6) $\Rightarrow$ (1). ⊣

In the next two sections we shall prove, for each $n > 0$, exact equivalences of (1)–(6) above with large cardinals.

§4. The main theorem for strong cardinals.

THEOREM 4.1. *The following are equivalent*:

(1) *There exists a strong cardinal.*
(2) Σ_2-PRP.
(3) Π_1-PRP.
(4) Σ_2-SWVP.
(5) Π_1-SWVP.
(6) Σ_2-WVP.
(7) Π_1-WVP.

PROOF. (1)$\Rightarrow$(2) is given by Proposition 3.3. The equivalence of (2)–(7) is given by Theorem 3.7. So, it will be sufficient to prove (3)$\Rightarrow$(1).

(3)$\Rightarrow$(1): Let $\mathcal{A}$ be the class of all structures

$$\mathcal{A}_\alpha := \langle V_{\alpha+1}, \in, \alpha, \{R^\alpha_\varphi\}_{\varphi\in\Pi_1}\rangle,$$

where the constant α is the α-th element of $C^{(1)}$ and $\{R^\alpha_\varphi\}_{\varphi\in\Pi_1}$ is the Π_1 relational diagram for $V_{\alpha+1}$, i.e., if $\varphi(x_1, \dots, x_n)$ is a Π_1 formula in the language of $\langle V_{\alpha+1}, \in, \alpha\rangle$, then

$$R^\alpha_\varphi = \{\langle x_1, \dots, x_n\rangle : \langle V_{\alpha+1}, \in, \alpha\rangle \models \text{“}\varphi(x_1, \dots, x_n)\text{”}\}.$$

We claim that $\mathcal{A}$ is Π_1-definable without parameters. For $X \in \mathcal{A}$ if and only if $X = \langle X_0, X_1, X_2, X_3\rangle$, where:

(1) X_2 belongs to $C^{(1)}$,
(2) $X_0 = V_{X_2+1}$,
(3) $X_1 = \in \restriction X_0$,
(4) X_3 is the Π_1 relational diagram of $\langle X_0, X_1, X_2\rangle$, and
(5) $\langle X_0, X_1, X_2\rangle \models$ “X_2 is the X_2-th element of $C^{(1)}$.”

Note that $\mathcal{A}$ is a proper class. In fact, the class C of ordinals α such that $\mathcal{A}_\alpha \in \mathcal{A}$ is a closed and unbounded proper class. By Π_1-PRP there exists a subset S of C such that for every $\beta \in C$ there is a homomorphism $j_\beta : \prod_{\alpha\in S} \mathcal{A}_\alpha \to \mathcal{A}_\beta$. By enlarging S, if necessary, we may assume that sup(S) $\in$ S. Let us denote $\prod_{\alpha\in S}\mathcal{A}_\alpha$ by M. Notice that

$$M = \langle \prod_{\alpha\in S} V_{\alpha+1}, \overline{\in}, \langle\alpha\rangle_{\alpha\in S}, \{\overline{R}^{\alpha}_{\varphi}\}_{\varphi\in\Pi_1}\rangle,$$

where $\overline{\in}$ is the pointwise membership relation, and $\overline{R}^{\alpha}_{\varphi}$ is the pointwise R^{α}_{φ} relation. Let $\kappa :=$ sup(S).

Now assume, aiming for a contradiction, that no cardinal $\leq\kappa$ is strong, and fix some $\beta\in C$ greater than κ, of uncountable cofinality, such that no cardinal $\leq\kappa$ is β-strong. Let $j = j_\beta$. ⊣

Claim 4.2. *j preserves the Boolean operations $\cap$, $\cup$, $-$, and also the $\subseteq$ relation.*

Proof of claim. For every $X, Y, Z \in M$,

$$M \models \text{“}X = Y\cap Z\text{”} \quad\text{iff}\quad V_{\alpha+1}\models \text{“}X(\alpha) = Y(\alpha)\cap Z(\alpha)\text{,” all } \alpha\in S\,.$$

So, letting $\varphi(x,y,z)$ be the bounded formula expressing $x = y\cap z$, we have that $\langle X(\alpha), Y(\alpha), Z(\alpha)\rangle \in R^{\alpha}_{\varphi}$, for all $\alpha\in S$. Hence $\langle X, Y, Z\rangle \in \overline{R}_{\varphi}$, and since j is a homomorphism $\langle j(X), j(Y), j(Z)\rangle \in R^{\beta}_{\varphi}$, which yields $\mathcal{A}_\beta \models$ “$j(X) = j(Y)\cap j(Z)$.”

Similarly for the operations $\cup, -$, and for the relation $\subseteq$. ⊣

Now define $k : V_{\kappa+1}\to V_{\beta+1}$ by

$$k(X) = j(\langle X\cap V_\alpha\rangle_{\alpha\in S})\,.$$

Claim 4.3. *k also preserves the Boolean operations, as well as the $\subseteq$ relation.*

Proof of claim. Suppose $V_{\kappa+1}\models$ “$X = Y\cap Z$.” Then $X\cap V_\alpha = (Y\cap V_\alpha)\cap(Z\cap V_\alpha)$, for every $\alpha\in S$. Hence,

$$M \models \text{“}\langle X\cap V_\alpha\rangle_{\alpha\in S} = \langle Y\cap V_\alpha\rangle_{\alpha\in S}\cap\langle Z\cap V_\alpha\rangle_{\alpha\in S}.\text{”}$$

Since j preserves the $\cap$ operation,

$$\mathcal{A}_\beta \models \text{“}k(X) = k(Y)\cap k(Z).\text{”}$$

Hence,

$$V_{\beta+1} \models \text{“}k(X) = k(Y)\cap k(Z).\text{”}$$

Similarly for the operations $\cup, -$, and the relation $\subseteq$. ⊣

Claim 4.4. *k maps ordinals to ordinals, and is the identity on $\omega+1$.*

Proof of claim. Let $\varphi(x)$ be the bounded formula expressing that x is an ordinal. Let $\gamma\leq\kappa$. Then $\gamma\cap V_\alpha$ is an ordinal, for all $\alpha<\kappa$, and so

$$M \models \text{“}\langle\gamma\cap V_\alpha\rangle_{\alpha\in S}\,\overline{\in}\,\overline{R}_{\varphi}.\text{”}$$

Since j is a homomorphism,

$$\mathcal{A}_\beta \models \text{“}j(\langle\gamma\cap V_\alpha\rangle_{\alpha\in S})\in R^{\beta}_{\varphi},\text{”}$$

which yields that $k(\gamma) = j(\langle\gamma\cap V_\alpha\rangle_{\alpha\in S})$ is an ordinal in $\mathcal{A}_\beta$, hence also in $V_{\beta+1}$.

For every ordinal $\gamma \leq \omega$, we have that $\gamma \cap V_\alpha = \gamma$, for all $\alpha \in S$. Moreover, γ is definable by some bounded formula φ_γ. Hence,

$$M \models \text{“}\langle \gamma \cap V_\alpha \rangle_{\alpha \in S} \,\overline{\in}\, \overline{R}_{\varphi_\gamma}\text{”}$$

and therefore

$$\mathcal{A}_\beta \models \text{“}j(\langle \gamma \cap V_\alpha \rangle_{\alpha \in S}) \in R^\beta_{\varphi_\gamma},\text{”}$$

which yields $k(\gamma) = \gamma$. ⊣

Note that $k(\kappa) = j(\langle \alpha \rangle_{\alpha \in S}) = \beta$.

For each $a \in [\beta]^{<\omega}$, define E_a by

$$X \in E_a \quad \text{iff} \quad X \subseteq [\kappa]^{|a|} \text{ and } a \in k(X).$$

Since $k(\kappa) = \beta$ and $k(|a|) = |a|$, we also have $k([\kappa]^{|a|}) = [\beta]^{|a|}$, and hence $[\kappa]^{|a|} \in E_a$. Moreover, since k preserves Boolean operations and the $\subseteq$ relation, E_a is an ultrafilter over $[\kappa]^{|a|}$.

Claim 4.5. E_a is ω_1-complete.

Proof of claim. Given $\{X_n : n < \omega\} \subseteq E_a$, let $Y = \{\langle n, x \rangle : x \in X_n\}$. So, $Y \subseteq V_\kappa$. We can express that $X = \bigcap_{n<\omega} X_n$ by a bounded sentence φ in the parameters X, Y, and ω. Moreover, since α is a limit ordinal, for every $\alpha \in S$, the sentence $\varphi(X \cap V_\alpha, Y \cap V_\alpha, \omega)$ expresses that $X \cap V_\alpha = \bigcap_{n<\omega} X_n \cap V_\alpha$. So,

$$M \models \text{“}\langle X \cap V_\alpha, Y \cap V_\alpha, \omega \rangle_{\alpha \in S} \,\overline{\in}\, \overline{R}_\varphi.\text{”}$$

Since j is a homomorphism,

$$\mathcal{A}_\beta \models \text{“}\langle j(X \cap V_\alpha), j(Y \cap V_\alpha), j(\omega) \rangle_{\alpha \in S} \in R^\beta_\varphi\text{”}$$

and so $\langle k(X), k(Y), k(\omega) \rangle$ satisfies φ. Since $k(\omega) = \omega$, we thus have $k(X) = \bigcap_{n<\omega} k(X_n)$. Hence, $a \in k(X)$, and so $X \in E_a$. ⊣

Let $\mathcal{E} := \{E_a : a \in [\beta]^{<\omega}\}$.

Claim 4.6. $\mathcal{E}$ is normal. That is, whenever $a \in [\beta]^{<\omega}$ and f is a function with domain $[\kappa]^{|a|}$ such that $\{s \in [\kappa]^{|a|} : f(s) \in \max(s)\} \in E_a$, there is $b \supseteq a$ such that $\{s \in [\kappa]^{|b|} : f(\pi^\kappa_{ba}(s)) \in s\} \in E_b$, where $\pi^\kappa_{ba} : [\kappa]^{|b|} \to [\kappa]^{|a|}$ is the standard projection function.

Proof. Fix a and f, and suppose the set

$$X := \{s \in [\kappa]^{|a|} : f(s) \in \max(s)\}$$

belongs to E_a. For every $\alpha \in S$,

$$V_{\alpha+1} \models \text{“}X \cap V_\alpha = \{s \in [\alpha]^{|a|} : (f \cap V_\alpha)(s) \in \max(s)\}.\text{”}$$

Thus, letting $\varphi[X \cap V_\alpha, \alpha, |a|, f \cap V_\alpha]$ be the sentence

$$\forall x (x \in (X \cap V_\alpha) \leftrightarrow x \in [\alpha]^{|a|} \wedge (f \cap V_\alpha)(x) \in \max(x)),$$

we have that

$$V_{\alpha+1} \models \varphi[X \cap V_\alpha, \alpha, |a|, f \cap V_\alpha].$$

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Since the formula $\varphi(x, y, z)$ is equivalent to a bounded formula, we have $\langle X \cap V_\alpha, \alpha, |a|, f \cap V_\alpha \rangle \in R^\alpha_\varphi$, for every $\alpha \in S$. Hence,

$$M \models \text{“}\langle \langle X \cap V_\alpha \rangle_{\alpha \in S}, \langle \alpha \rangle_{\alpha \in S}, \langle |a| \rangle_{\alpha \in S}, \langle f \cap V_\alpha \rangle_{\alpha \in S} \rangle \mathrel{\overline{\in}} \overline{R}^\alpha_\varphi\text{”}$$

and therefore

$$V_{\beta+1} \models \text{“}\langle j(\langle X \cap V_\alpha \rangle_{\alpha \in S}), j(\langle \alpha \rangle_{\alpha \in S}), j(\langle |a| \rangle_{\alpha \in S}), j(\langle f \cap V_\alpha \rangle_{\alpha \in S}) \rangle \in R^\beta_\varphi.\text{”}$$

Since $j(\langle \alpha \rangle_{\alpha \in S}) = k(\kappa) = \beta$ and $k(|a|) = |a|$,

$$V_{\beta+1} \models \text{“}\langle k(X), \beta, |a|, k(f) \rangle \in R^\beta_\varphi,\text{”}$$

which yields

$$k(X) = \{s \in [\beta]^{|a|} : k(f)(s) \in \max(s)\}.$$

Also, since $X \in E_a$, we have that $k(f)(a) \in \max(a)$.

Let $\delta = k(f)(a)$, and let $b = a \cup \{\delta\}$. Thus,

$$b \in \{s \in [\beta]^{|b|} : k(f)(\pi^\beta_{ba}(s)) \in s\},$$

where $\pi^\beta_{ba} : [\beta]^{|b|} \to [\beta]^{|a|}$ is the standard projection function. So, since $\{s \in [\beta]^{|b|} : k(f)(\pi^\beta_{ba}(s)) \in s\} = k(\{s \in [\kappa]^{|b|} : f(\pi^\kappa_{ba}(s)) \in s\})$, we have

$$\{s \in [\kappa]^{|b|} : f(\pi^\kappa_{ba}(s)) \in s\} \in E_b,$$

which shows that $\mathcal{E}$ is normal. ⊣

For each $a \in [\beta]^{<\omega}$, the ultrapower $\mathrm{Ult}(V, E_a)$ of V by the ω_1-complete ultrafilter E_a is well-founded. So, let

$$j_a : V \to M_a \cong \mathrm{Ult}(V, E_a),$$

with M_a transitive, be the corresponding ultrapower embedding. As usual, we denote the elements of M_a by their corresponding elements in $\mathrm{Ult}(V, E_a)$.

Claim 4.7. $\mathcal{E}$ is coherent. That is, for every $a \subseteq b$ in $[\beta]^{<\omega}$,

$$X \in E_a \text{ if and only if } \{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\} \in E_b.$$

Proof. Let $a \subseteq b$ in $[\beta]^{<\omega}$, and suppose $X \in E_a$. Thus, $X \subseteq [\kappa]^{|a|}$ and $a \in k(X)$. We need to see that $b \in k(\{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\})$. Now notice that, since k is the identity on natural numbers, and $k(\kappa) = \beta$,

$$k(\{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\}) = \{s \in [\beta]^{|b|} : \pi_{ba}(s) \in k(X)\}.$$

Hence, since $\pi_{ba}(b) = a$, and $a \in k(X)$, we have that $b \in \{s \in [\beta]^{|b|} : \pi_{ba}(s) \in k(X)\}$, as wanted.

Conversely, if $\{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\} \in E_b$, we have that $b \in k(\{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\}) = \{s \in [\beta]^{|b|} : \pi_{ba}(s) \in k(X)\}$. Hence, $\pi_{ba}(b) = a \in k(X)$, and therefore $X \in E_a$. ⊣

For each $a \subseteq b$ in $[\beta]^{<\omega}$, let $i_{ab} : M_a \to M_b$ be given by

$$i_{ab}([f]_{E_a}) = [f \circ \pi_{ba}]_{E_b}$$

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for all $f : [\kappa]^{|a|} \to V$. By coherence, the maps i_{ab} are well-defined and commute with the ultrapower embeddings j_a (see [9, Section 26]).

Let $M_{\mathcal{E}}$ be the direct limit of

$$\langle\langle M_a : a \in [\beta]^{<\omega}\rangle, \langle i_{ab} : a \subseteq b\rangle\rangle.$$

For notational simplicity, whenever we write $[a, [f]]$ what we mean is that $[f] = [f]_{E_a}$, which belongs to M_a. Thus, when we say, e.g., that $[a, [f]] \in [b, [g]]$ in $M_{\mathcal{E}}$, what we mean is that $[f] = [f]_{E_a} \in M_a$, $[g] = [g]_{E_b} \in M_b$, and $[\langle a, [f]_{E_a}\rangle]_{\mathcal{E}} \in_{\mathcal{E}} [\langle b, [g]_{E_b}\rangle]_{\mathcal{E}}$.

Let $j_{\mathcal{E}} : V \to M_{\mathcal{E}}$ be the corresponding limit elementary embedding, i.e.,

$$j_{\mathcal{E}}(x) = [a, [c^a_x]_{E_a}]$$

for some (any) $a \in [\beta]^{<\omega}$, and where $c^a_x : [\kappa]^{|a|} \to \{x\}$.

Let $k_a : M_a \to M_{\mathcal{E}}$ be given by

$$k_a([f]_{E_a}) = [a, [f]_{E_a}].$$

It is easily checked that $j_{\mathcal{E}} = k_a \circ j_a$ and $k_b \circ i_{ab} = k_a$, for all $a \subseteq b$, $a, b \in [\beta]^{<\omega}$. Thus, letting $\mathrm{Id}_{|a|} : [\kappa]^{|a|} \to [\kappa]^{|a|}$ be the identity function, we have

$$M_{\mathcal{E}} = \{j_{\mathcal{E}}(f)(k_a([\mathrm{id}_{|a|}]_{E_a})) : a \in [\beta]^{<\omega} \text{ and } f : [\kappa]^{|a|} \to V\}.$$

Let $M^*_{\mathcal{E}} := \{[a, [f]] \in M_{\mathcal{E}} : f \cap V_\alpha : [\alpha]^{|a|} \to V_\alpha, \text{ all } \alpha \in S\}$. Suppose $[a, [f]]$, $[b, [g]] \in M^*_{\mathcal{E}}$. Then the following can be easily verified:

(1) $[a, [f]] \in_{\mathcal{E}} [b, [g]]$ iff $k(f)(a) \in k(g)(b)$.
(2) $[a, [f]] =_{\mathcal{E}} [b, [g]]$ iff $k(f)(a) = k(g)(b)$.

Claim 4.8. $M^*_{\mathcal{E}}$ is well-founded and downward closed under $\in_{\mathcal{E}}$.

Proof. Well-foundedness follows from items (1) and (2) above, as any infinite $\in_{\mathcal{E}}$-descending sequence in $M^*_{\mathcal{E}}$ would yield an infinite $\in$-descending sequence in $V_{\beta+1}$.

Now suppose $[a, [f]] \in_{\mathcal{E}} [b, [g]]$, with $[b, [g]] \in M^*_{\mathcal{E}}$. Then, for some $c \supseteq a, b$, and some $X \in E_c$,

$$(f \circ \pi_{ca})(s) \in (g \circ \pi_{cb})(s)$$

for every $s \in X$. Let $Y = \{\pi_{ca}(s) : s \in X\} \in E_a$. Define $h : [\kappa]^{|a|} \to V$ by: $h(s) = f(s)$ for all $s \in Y$, and $h(s) = 0$, otherwise. Then $[h]_{E_a} = [f]_{E_a}$, and $[a, [f]] = [a, [h]] \in M^*_{\mathcal{E}}$. ⊣

By the last claim, $M^*_{\mathcal{E}}$ is well-founded and extensional. So, let M^* be the transitive collapse of $M^*_{\mathcal{E}}$.

Claim 4.9. $V_\beta \subseteq M^*$.

Proof of claim. Since κ and α, for $\alpha \in S$, belong to $C^{(1)}$, we have that $|V_\kappa| = \kappa$ and $|V_\alpha| = \alpha$, all $\alpha \in S$. Let $f \in V$ be a bijection between $[\kappa]^1$ and V_κ such that $f \restriction [\alpha]^1$ is a bijection between $[\alpha]^1$ and V_α, all $\alpha \in S$. Let $\varphi(x, y, z)$ be a Π_1 formula expressing that $x = [u]^1$, with u an ordinal, $y = V_u$, and $z : x \to V_u$ is a bijection.

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Thus,

$$M \models \text{“}\langle [\alpha]^1, V_\alpha, f \cap V_\alpha\rangle_{\alpha\in S} \mathrel{\overline{\in}} \overline{R}_\varphi.\text{”}$$

Hence,

$$\mathcal{A}_\beta \models \text{“}\langle k([\kappa]^1), k(V_\kappa), k(f)\rangle \in R^\beta_\varphi\text{”}$$

and so $k(f) : [\beta]^1 \to V_\beta$ is a bijection. Therefore, for every $x \in V_\beta$ there exists $\gamma < \beta$ such that $k(f)(\{\gamma\}) = x$.

Thus, letting $D := \{[\{\gamma\}, [f]] : \gamma < \beta\}$, we have just shown that the map $i : \langle D, \in_{\mathcal{E}} \restriction D\rangle \to \langle V_\beta, \in\rangle$ given by

$$i([\{\gamma\}, [f]]) = k(f)(\{\gamma\})$$

is onto. Moreover, if $[\{\gamma\}, [f]] \in_{\mathcal{E}} [\{\delta\}, [f]]$, then for some $X \in E_{\{\gamma,\delta\}}$, we have

$$(f \circ \pi_{\{\gamma,\delta\}\{\gamma\}})(s) \in (f \circ \pi_{\{\gamma,\delta\}\{\delta\}})(s)$$

for every $s \in X$. Letting φ be the bounded formula expressing this, since V_α is closed under f for every α, we have

$$M \models \text{“}\langle X \cap V_\alpha, (f \circ \pi_{\{\gamma,\delta\}\{\gamma\}}) \cap V_\alpha, (f \circ \pi_{\{\gamma,\delta\}\{\delta\}}) \cap V_\alpha\rangle_{\alpha\in S} \mathrel{\overline{\in}} \overline{R}_\varphi.\text{”}$$

Hence, in $\mathcal{A}_\beta$, for every $s \in k(X)$,

$$(k(f) \circ \pi_{\{\gamma,\delta\}\{\gamma\}})(s) \in (k(f) \circ \pi_{\{\gamma,\delta\}\{\delta\}})(s).$$

In particular, since $\{\gamma,\delta\} \in k(X)$,

$$k(f)(\{\gamma\}) \in k(f)(\{\delta\}).$$

A similar argument shows that i is one-to-one. Hence, i is an isomorphism, and so i is just the transitive collapsing map. Since $D \subseteq M^*_{\mathcal{E}}$, to conclude that $V_\beta \subseteq M^*$ it will be sufficient to show that the transitive collapse of D is the same as the restriction to D of the transitive collapse of $M^*_{\mathcal{E}}$. For this, it suffices to see that every $\in_{\mathcal{E}}$-element of an element of D is $=_{\mathcal{E}}$-equal to an element of D. So, suppose $[\{\gamma\}, [f]] \in D$ and $[a, [g]] \in_{\mathcal{E}} [\{\gamma\}, [f]]$, with $[a, [g]] \in M^*_{\mathcal{E}}$. Then $k(g)(a) \in k(f)(\gamma)$, by (1) above (just before Claim 4.8). Now $k(f) : [\beta]^1 \to V_\beta$ is surjective and V_β is transitive, so there is some $\delta < \beta$ such that $k(f)(\{\delta\}) = k(g)(a)$. Hence, by (2) above, $[\{\delta\}, [f]] =_{\mathcal{E}} [a, [g]]$. ⊣

Claim 4.10. $M_{\mathcal{E}}$ is closed under ω-sequences, and hence it is well-founded.

Proof of claim. Let $\langle j_{\mathcal{E}}(f_n)(k_{a_n}([\mathrm{id}_{|a_n|}]_{E_{a_n}}))\rangle_{n<\omega}$ be a sequence of elements of $M_{\mathcal{E}}$. On the one hand, the sequence $\langle j_{\mathcal{E}}(f_n)\rangle_{n<\omega} = j_{\mathcal{E}}(\langle f_n\rangle_{n<\omega})$ belongs to $M_{\mathcal{E}}$. On the other hand, $k_{a_n}([\mathrm{Id}_{|a_n|}]_{E_{a_n}}) = [a_n, [\mathrm{Id}_{|a_n|}]_{E_{a_n}}]$ belongs to $M^*_{\mathcal{E}}$ for all $n < \omega$. Since $\mathcal{E}$ is normal (Claim 4.6), as in [9, Lemma 26.2(a)] we can show that the transitive collapse of $[a_n, [\mathrm{Id}_{|a_n|}]_{E_{a_n}}]$ is precisely a_n. The sequence $\langle a_n\rangle_{n<\omega}$ belongs to V_β, because β has uncountable cofinality. Hence, since $V_\beta \subseteq M^*$, the preimage of $\langle a_n\rangle_{n<\omega} \in V_\beta$ under the transitive collapsing map of $M^*_{\mathcal{E}}$ to M^* is precisely the sequence $\langle k_{a_n}([\mathrm{Id}_{|a_n|}]_{E_{a_n}})\rangle_{n<\omega}$ and belongs to $M_{\mathcal{E}}$. It now follows that the sequence $\langle j_{\mathcal{E}}(f_n)(k_{a_n}([\mathrm{id}_{|a_n|}]_{E_{a_n}}))\rangle_{n<\omega}$ is also in $M_{\mathcal{E}}$. ⊣

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Let $\pi : M_{\mathcal{E}} \to N$ be the transitive collapsing isomorphism, and let $j_N : V \to N$ be the corresponding elementary embedding, i.e., $j_N = \pi \circ j_{\mathcal{E}}$.

CLAIM 4.11. $j_N(\kappa) \geq \beta$.

PROOF OF CLAIM. Let $\alpha < \beta$. Let Id_1 be the identity function on $[\kappa]^1$, and let $c_{[\kappa]^1} : [\kappa]^1 \to \{[\kappa]^1\}$ and $c_\kappa : [\kappa]^1 \to \{\kappa\}$. In $M_{\{\alpha\}}$, we have

$$[\mathrm{Id}_1]_{E_{\{\alpha\}}} \in [c_{[\kappa]^1}]_{E_{\{\alpha\}}} = [[c_\kappa]_{E_{\{\alpha\}}}]^1 = [j_{\{\alpha\}}(\kappa)]^1,$$

and hence in $M_{\mathcal{E}}$,

$$k_{\{\alpha\}}([\mathrm{Id}_1]_{E_{\{\alpha\}}}) \in k_{\{\alpha\}}([j_{\{\alpha\}}(\kappa)]^1) = [j_{\mathcal{E}}(\kappa)]^1,$$

and therefore, since $\pi(k_{\{\alpha\}}([\mathrm{Id}_1]_{E_{\{\alpha\}}})) = \{\alpha\}$, in N we have

$$\{\alpha\} \in [j_N(\kappa)]^1,$$

that is, $\alpha < j_N(\kappa)$. ⊣

Since $\beta > \kappa$, the last claim implies that the critical point of j_N is less than or equal to κ. Since $V_\beta \subseteq N$ by Claim 4.9, j_N witnesses that its critical point is a β-strong cardinal, in contradiction to our choice of β. This completes the proof of Theorem 4.1. ⊣

The boldface version of Theorem 4.1, i.e., with parameters, also holds by essentially the same arguments. Namely,

THEOREM 4.12. *The following are equivalent*:

(1) *There exists a proper class of strong cardinals.*
(2) $\mathbf{\Sigma_2}$-PRP.
(3) $\mathbf{\Pi_1}$-PRP.
(4) $\mathbf{\Sigma_2}$-SWVP.
(5) $\mathbf{\Pi_1}$-SWVP.
(6) $\mathbf{\Sigma_2}$-WVP.
(7) $\mathbf{\Pi_1}$-WVP.

For the proof of (3) implies (1), in order to show that there exists a strong cardinal greater than or equal to a fixed ordinal γ we need to consider the class of structures

$$\mathcal{A}_\alpha := \langle V_{\alpha+1}, \in, \alpha, \{R^\alpha_\varphi\}_{\varphi \in \Pi_1}, \langle \delta \rangle_{\delta < \gamma} \rangle,$$

where the structure

$$\langle V_{\alpha+1}, \in, \alpha, \{R^\alpha_\varphi\}_{\varphi \in \Pi_1} \rangle$$

is as in the proof of Theorem 4.1, and we have a constant δ for every $\delta < \gamma$.

§5. The general case. We shall now consider the general case of definable proper classes of structures with any degree of definable complexity. For this we shall need the following new kind of large cardinals.

If $j : V \to M$ is an elementary embedding, with M transitive and critical point κ, and A is a class definable by a formula φ (possibly with parameters in V_κ),

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we define

$$j(A) := \{X \in M : M \models \varphi(X)\}.$$

Note that

$$j(A) = \bigcup\{j(A \cap V_\alpha) : \alpha \in \mathrm{OR}\}$$

as $j(A \cap V_\alpha) = \{X \in M : M \models \varphi(X)\} \cap V^M_{j(\alpha)}$. Also note that if A is a class of structures of the same type $\tau \in V_\kappa$, then by elementarity $j(A)$ is a subclass of M of structures of type τ.

5.1. Γ_n-strong cardinals. In the sequel, let Γ_n stand for one of the definability classes Σ_n, Π_n.

DEFINITION 5.1. For $n \geq 1$, a cardinal κ is λ-Γ_n-*strong* if for every Γ_n-definable (without parameters) class A there is an elementary embedding $j : V \to M$ with M transitive, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $A \cap V_\lambda \subseteq j(A)$.

κ is Γ_n-*strong* if it is λ-Γ_n-strong for every ordinal λ.

Note that in the definition above, $A \cap V_\lambda$ is only required to be contained in $j(A) \cap V_\lambda$ and not equal to it. The reason is that in the Σ_2 case, if A is the class of non-strong cardinals (which is Σ_2) and κ is the least strong cardinal, then $\kappa \notin A$, but $\kappa \in j(A)$. See however the equivalence given in Proposition 5.9.

As with the case of strong cardinals, standard arguments show (cf. [9, Exercise 26.7(b)]) that κ is Γ_n-strong if and only if for every Γ_n-definable (without parameters) class A and every ordinal λ there is an elementary embedding $j : V \to M$ with M transitive, $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$, and $A \cap V_\lambda \subseteq j(A)$.

PROPOSITION 5.2. *Every strong cardinal is Σ_2-strong.*

PROOF. Let κ be a strong cardinal and let A be a class that is Σ_2-definable (even allowing for parameters in V_κ). Let $\lambda \in C^{(2)}$ be greater than κ. Let $j : V \to M$ be elementary, with M transitive, $\mathrm{crit}(j) = \kappa$, and $V_\lambda \subseteq M$. Let φ be a Σ_2 formula defining A. If $a \in A \cap V_\lambda$, then $V_\lambda \models \varphi(a)$. Hence, since $V_\lambda \preceq_{\Sigma_1} M$, $M \models \varphi(a)$, and so $a \in j(A) = \{x : M \models \varphi(x)\}$. ⊣

PROPOSITION 5.3. *If $\lambda \in C^{(n+1)}$, then a cardinal κ is λ-Π_n-strong if and only if it is λ-Σ_{n+1}-strong.*

PROOF. Assume κ is λ-Π_n-strong, with $\lambda \in C^{(n+1)}$, and let A be a Σ_{n+1}-definable class. Let $\varphi(x) \equiv \exists y\, \psi(x, y)$ be a Σ_{n+1} formula, with $\psi(x, y)$ being Π_n, that defines A. Now define B as the class of all structures of the form $\langle V_\alpha, \in, a\rangle$, where $\alpha \in C^{(n)}$, $a \in V_\alpha$, and $V_\alpha \models \varphi(a)$. Then B is Π_n-definable. By our assumption, let $j : V \to M$ be an elementary embedding with M transitive, $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$, and $B \cap V_\lambda \subseteq j(B)$. We just need to show that $A \cap V_\lambda \subseteq j(A)$. So, suppose $a \in A \cap V_\lambda$. Since $\lambda \in C^{(n+1)}$, we have that $V_\lambda \models \varphi(a)$. Let $b \in V_\lambda$ be a witness, so that $V_\lambda \models \psi(a, b)$. For some $\alpha < \lambda$ in $C^{(n)}$ we have that $a, b \in V_\alpha$. Hence, $V_\alpha \models \varphi(a)$. So $\langle V_\alpha, \in, a\rangle \in B \cap V_\lambda$, and therefore $\langle V_\alpha, \in, a\rangle \in j(B)$. Thus, $M \models$ "$\alpha \in C^{(n)}$, $a \in V_\alpha$, and $V_\alpha \models \varphi(a)$." Hence, $M \models \varphi(a)$, i.e., $a \in j(A)$. ⊣

COROLLARY 5.4. *A cardinal κ is Π_n-strong if and only if it is Σ_{n+1}-strong.*

Proposition 5.5. *Suppose that* $n \geq 2$ *and* $\lambda \in C^{(n)}$. *Then the following are equivalent for a cardinal* $\kappa < \lambda$:

(1) κ *is* λ-Σ_n-*strong.*
(2) *There is an elementary embedding* $j : V \to M$ *with* M *transitive,* $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$, $j(\kappa) > \lambda$, *and* $M \models$ "$\lambda \in C^{(n-1)}$."

Proof. (1)⇒(2): Suppose κ is λ-Σ_n-strong. Let $A = C^{(n-1)}$. Since A is Π_{n-1}-definable, hence also Σ_n-definable, by (1) there is an elementary embedding $j : V \to M$ with M transitive, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $A \cap V_\lambda \subseteq j(A)$. Since $\lambda \in C^{(n)}$, $C^{(n-1)} \cap \lambda$ is a club subset of λ. For every $\alpha < \lambda$ in $C^{(n-1)}$, $\alpha \in j(A)$, and hence $M \models$ "$\alpha \in C^{(n-1)}$" and so $M \models$ "λ is a limit point of $C^{(n-1)}$," which yields $M \models$ "$\lambda \in C^{(n-1)}$."

(2)⇒(1): Let A be a class definable by a Σ_n formula φ, and let $j : V \to M$ be an elementary embedding with M transitive, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $M \models$ "$\lambda \in C^{(n-1)}$." Let $a \in A \cap V_\lambda$. Since $\lambda \in C^{(n)}$, $V_\lambda \models \varphi(a)$. And since $V_\lambda \subseteq M$ and $M \models$ "$\lambda \in C^{(n-1)}$," $M \models \varphi(a)$, i.e., $a \in j(A)$. ⊣

Proposition 5.6. *If* κ *is* Π_n-*strong, then* $\kappa \in C^{(n+1)}$.

Proof. By induction on n. So let κ be Π_n-strong and assume, inductively, that $\kappa \in C^{(n)}$. Let $\exists x \varphi(x)$ be a formula, with $\varphi(x)$ being a Π_n formula which may contain parameters in V_κ, and suppose that $\exists x \varphi(x)$ holds in V. Pick a witness b and let $\lambda \in C^{(n)}$ be such that $b \in V_\lambda$. Thus, $V_\lambda \models \exists x \varphi(x)$. By Corollary 5.4 and Proposition 5.5, let $j : V \to M$ be an elementary embedding, with M transitive, $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$, $j(\kappa) > \lambda$, and $M \models$ "$\lambda \in C^{(n)}$." Then by elementarity of j, in V there exists some $\lambda' \in C^{(n)}$ less than κ such that $V_{\lambda'} \models$ "$\exists x \varphi(x)$," and since $\kappa \in C^{(n)}$, $V_\kappa \models$ "$\exists x \varphi(x)$," as wanted. ⊣

The last proposition suggests the following definition and the ensuing characterization of Σ_n-strong cardinals in terms of extenders.

Definition 5.7. Given $n \geq 1$ and given cardinals $\kappa < \lambda$, a Σ_n-*strong* (κ, λ)-*extender* is a $(\kappa, |V_\lambda|^+)$-extender $\mathcal{E}$ (see Definition 2.3) such that $V_\lambda \subseteq \overline{M}_{\mathcal{E}}$, $\lambda < j(\kappa)$, and $\overline{M}_{\mathcal{E}} \models$ "$\lambda \in C^{(n-1)}$," where $\overline{M}_{\mathcal{E}}$ is the transitive collapse of the direct limit ultrapower $M_{\mathcal{E}}$ of V by $\mathcal{E}$, and $j : V \to \overline{M}_{\mathcal{E}}$ is the corresponding elementary embedding.

Proposition 5.8. *If* $n \geq 2$ *and* $\lambda \in C^{(n)}$, *then a cardinal* $\kappa < \lambda$ *is* λ-Σ_n-*strong if and only if there exists a* Σ_n-*strong* (κ, λ)-*extender.*

Proof. If $\mathcal{E}$ is a Σ_n-strong (κ, λ)-extender, then the extender embedding $j : V \to \overline{M}_{\mathcal{E}}$ witnesses that κ is λ-Σ_n-strong by Proposition 5.5.

Conversely, suppose $j : V \to M$ is an elementary embedding, with M transitive, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $M \models$ "$\lambda \in C^{(n-1)}$." Note that since $\lambda \in C^{(1)}$, $|V_\lambda| = \lambda$. Let $\mathcal{E}$ be the (κ, λ^+)-extender derived from j. Namely, for every $a \in [\lambda^+]^{<\omega}$ let E_a be defined by

$$X \in E_a \text{ if and only if } X \subseteq [\kappa]^{|a|} \text{ and } a \in j(X).$$

One can easily check that $\mathcal{E}$ satisfies conditions (1)–(5) of Definition 2.3 (see [9, Exercise 26.7]). So we only need to check that $\overline{M}_{\mathcal{E}} \models$ "$\lambda \in C^{(n-1)}$."

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Let $j_{\mathcal{E}} : V \to M_{\mathcal{E}}$ and $k_{\mathcal{E}} : \overline{M}_{\mathcal{E}} \to M$ be the standard maps given by: $j_{\mathcal{E}}(x) = [a, [c^a_x]]$ (any a), where $c^a_x : [\kappa]^{|a|} \to \{x\}$; and $k_{\mathcal{E}}(\pi([a, [f]])) = j(f)(a)$, for $f : [\kappa]^{|a|} \to V$, where $\pi : M_{\mathcal{E}} \to \overline{M}_{\mathcal{E}}$ is the transitive collapse isomorphism. The maps $j_{\mathcal{E}}$ and $k_{\mathcal{E}}$ are elementary and $j = k_{\mathcal{E}} \circ \pi \circ j_{\mathcal{E}}$. Moreover, $k_{\mathcal{E}} \restriction V_\lambda$ is the identity.

Since $M \models$ "$\lambda \in C^{(n-1)}$," for each $\mu < \lambda$ in $C^{(n-1)}$, we have that $M \models$ "$\mu \in C^{(n-1)}$." So, since $k_{\mathcal{E}}$ is elementary and is the identity on V_λ, we have that $M_{\mathcal{E}} \models$ "$\mu \in C^{(n-1)}$." Hence, $M_{\mathcal{E}} \models$ "λ is a limit point of $C^{(n-1)}$," which yields $M_{\mathcal{E}} \models$ "$\lambda \in C^{(n-1)}$." ⊣

Similar characterizations may also be given for Π_n-strong cardinals. Namely, if $n \geq 2$ and $\lambda \in C^{(n)}$, then a cardinal $\kappa < \lambda$ is λ-Π_n-strong if and only if there exists a Π_n-strong (κ, λ)-extender. Notice that (3) of the following proposition characterizes Π_n-strong cardinals as witnessing "OR is Woodin" restricted to Π_n-definable classes (see Definition 5.14). In particular, it says $A \cap V_\lambda = j(A) \cap V_\lambda$, not just $A \cap V_\lambda \subseteq j(A) \cap V_\lambda$.

Proposition 5.9. *Suppose that $n \geq 1$ and λ is a limit point of $C^{(n)}$. Then the following are equivalent for a cardinal $\kappa < \lambda$:*

(1) *κ is λ-Π_n-strong.*
(2) *There is an elementary embedding $j : V \to M$ with M transitive,* $\text{crit}(j) = \kappa$, *$j(\kappa) => \lambda$, $V_\lambda \subseteq M$, and $M \models$ "$\lambda \in C^{(n)}$."*
(3) *For every Π_n-definable class A there is an elementary embedding $j : V \to M$ with M transitive,* $\text{crit}(j) = \kappa$, *$j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $A \cap V_\lambda = j(A) \cap V_\lambda$.*

Proof. (1)⇒(2): Suppose κ is λ-Π_n-strong. Let $A = C^{(n)}$. Since A is Π_n-definable, by (1) there is an elementary embedding $j : V \to M$ with M transitive, $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $A \cap V_\lambda \subseteq j(A)$. Thus, for every $\alpha < \lambda$ in A, $\alpha \in j(A)$, and hence $M \models$ "$\alpha \in C^{(n)}$" and so $M \models$ "λ is a limit point of $C^{(n)}$," which yields $M \models$ "$\lambda \in C^{(n)}$."

(2)⇒(3): Let A be a class definable by a Π_n formula $\varphi(x)$, and let $j : V \to M$ be an elementary embedding with M transitive, $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $M \models$ "$\lambda \in C^{(n)}$." Let $a \in A \cap V_\lambda$. Since $\lambda \in C^{(n)}$, $V_\lambda \models \varphi(a)$. And since $V_\lambda \subseteq M$ and $M \models$ "$\lambda \in C^{(n)}$," $M \models \varphi(a)$, i.e., $a \in j(A)$. Conversely, suppose $a \in j(A) \cap V_\lambda$, i.e., $M \models$ "$\varphi(a)$." Since $M \models$ "$\lambda \in C^{(n)}$," $V_\lambda \models \varphi(a)$. And since $\lambda \in C^{(n)}$, $a \in A$.

(3)⇒(1) is immediate. ⊣

Let us remark that the implication (2)⇒(3) above also holds for classes A that are Π_n-definable with parameters in V_κ. Thus a corollary of Proposition 5.9 is that a cardinal κ is Π_n-strong if and only if it is λ-Π_n-strong for every Π_n-definable, with parameters in V_κ, class A.

It easily follows from the last proposition that being a Π_n-strong cardinal is a Π_{n+1} property. Moreover, if κ is Π_n-strong, then $\kappa \in C^{(n+1)}$. Hence, if κ is Π_{n+1}-strong, then there are many Π_n-strong cardinals below κ, which shows that the Π_n-strong cardinals, $n > 0$, form a hierarchy of strictly increasing strength.

Similarly as in Proposition 3.3 we can prove the following.

Proposition 5.10. *If κ is a Σ_n-strong cardinal, where $n \geq 2$, then* $\Sigma_n(V_\kappa)$-PRP *holds.*

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PROOF. Let $n \geq 2$. Let κ be Σ_n-strong and let $\mathcal{C}$ be a definable, by a Σ_n formula with parameters in V_κ, proper class of structures in a fixed countable relational language. We will show that the set $S := \mathcal{C} \cap V_\kappa$ witnesses PRP.

Given any $\mathcal{A} \in \mathcal{C}$, let $\lambda \geq \kappa$ be an ordinal with $\mathcal{A} \in V_\lambda$.

Let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$, $j(\kappa) > \lambda$, and $\mathcal{C} \cap V_\lambda \subseteq j(\mathcal{C})$.

By elementarity, the restriction of j to $\prod S$ yields a homomorphism

$$h : \prod S \to \prod(j(\mathcal{C}) \cap V^M_{j(\kappa)}).$$

Since $\mathcal{A} \in \mathcal{C} \cap V_\lambda$, we have that $\mathcal{A} \in j(\mathcal{C})$. Moreover $\mathcal{A} \in V_\lambda \subseteq V^M_{j(\kappa)}$. Thus, letting

$$g : \prod(j(\mathcal{C}) \cap V^M_{j(\kappa)}) \to \mathcal{A}$$

be the projection map, we have that

$$g \circ h : \prod S \to \mathcal{A}$$

is a homomorphism, as wanted. ⊣

5.2. The main theorem for Γ_n-strong cardinals. Using similar arguments as in Theorem 4.1, we can now prove the main theorem of this section.

THEOREM 5.11. *The following are equivalent for $n \geq 2$:*

(1) *There exists a Σ_n-strong cardinal.*
(2) *There exists a Π_{n-1}-strong cardinal.*
(3) Σ_n-PRP.
(4) Π_{n-1}-PRP.
(5) Σ_n-SWVP.
(6) Π_{n-1}-SWVP.
(7) Σ_n-WVP.
(8) Π_{n-1}-WVP.

PROOF. (1)⇒(3) is given by Proposition 5.10. The equivalence of (3)–(8) is given by Theorem 3.7. So, we only need to prove (4)⇒(2).

The proof is analogous to the proof of Theorem 4.1, so we shall only indicate the relevant differences. Theorem 4.1 proves the case $n = 2$ (see Proposition 5.2). Thus, we shall assume in the sequel that $n > 2$.

Let $\mathcal{A}$ be the class of all structures

$$\mathcal{A}_\alpha := \langle V_{\alpha+1}, \in, \alpha, C^{(n-1)} \cap \alpha, \{R^\alpha_\varphi\}_{\varphi\in\Pi_1}\rangle,$$

where the constant α is the α-th element of $C^{(n-1)}$, and $\{R^\alpha_\varphi\}_{\varphi\in\Pi_1}$ is the Π_1 relational diagram for $V_{\alpha+1}$, i.e., if $\varphi(x_1, \dots, x_n)$ is a Π_1 formula in the language of $\langle V_{\alpha+1}, \in, \alpha, C^{(n-1)} \cap \alpha\rangle$, then

$$R^\alpha_\varphi = \{\langle x_1, \dots, x_n\rangle : \langle V_{\alpha+1}, \in, \alpha, C^{(n-1)} \cap \alpha\rangle \models \text{“}\varphi(x_1, \dots, x_n)\text{”}\}.$$

Then $\mathcal{A}$ is Π_{n-1}-definable without parameters. For $X \in \mathcal{A}$ if and only if $X = \langle X_0, X_1, X_2, X_3, X_4\rangle$, where:

(1) X_2 belongs to $C^{(n-1)}$.

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(2) $X_0 = V_{X_2+1}$.
(3) $X_1 = \in \restriction X_0$.
(4) X_0 satisfies that $X_3 = C^{(n-1)} \cap X_2$.
(5) X_4 is the Π_1 relational diagram of $\langle X_0, X_1, X_2, X_3\rangle$.
(6) $\langle X_0, X_1, X_2, X_3\rangle \models$ "X_2 is the X_2-th element of $C^{(n-1)}$."

Note that the class C of ordinals α such that $\mathcal{A}_\alpha \in \mathcal{A}$ is a closed and unbounded proper class. By Π_{n-1}-PRP there exists a subset S of C such that for every $\beta \in C$ there is a homomorphism $j_\beta : \prod_{\alpha\in S} \mathcal{A}_\alpha \to \mathcal{A}_\beta$. By enlarging S, if necessary, we may assume that $\kappa := \sup(S) \in S$.

Now fix some $\beta \in C$ greater than κ, of uncountable cofinality, and assume, towards a contradiction, that no cardinal $\leq \kappa$ is β-Π_{n-1}-strong. Let $j = j_\beta$.

From this point, the proof proceeds as in Theorem 4.1. Namely, we define $k : V_{\kappa+1} \to V_{\beta+1}$ by

$$k(X) = j(\langle X \cap V_\alpha\rangle_{\alpha\in S})$$

and note that $k(\kappa) = \beta$.

For each $a \in [\beta]^{<\omega}$, define E_a by

$$X \in E_a \quad \text{iff} \quad X \subseteq [\kappa]^{|a|} \text{ and } a \in k(X)\,.$$

As in Theorem 4.1, E_a is an ω_1-complete ultrafilter over $[\kappa]^{|a|}$. Moreover, $\mathcal{E} := \{E_a : a \in [\beta]^{<\omega}\}$ is normal and coherent.

For each $a \in [\beta]^{<\omega}$, the ultrapower $\mathrm{Ult}(V, E_a)$ is well-founded by ω_1-completeness. So, let

$$j_a : V \to M_a \cong \mathrm{Ult}(V, E_a),$$

with M_a transitive, be the corresponding ultrapower embedding, and let $M_\mathcal{E}$ be the direct limit of

$$\langle\langle M_a : a \in [\beta]^{<\omega}\rangle, \langle i_{ab} : a \subseteq b\rangle\rangle,$$

where the $i_{ab} : M_a \to M_b$ are the usual commuting maps. The corresponding limit embedding $j_\mathcal{E} : V \to M_\mathcal{E}$ is elementary. As in Theorem 4.1, $M_\mathcal{E}$ is closed under ω-sequences, and hence it is well-founded. Moreover, letting $\pi : M_\mathcal{E} \to N$ be the transitive collapsing isomorphism, and $j_N : V \to N$ the corresponding elementary embedding, i.e., $j_N = \pi \circ j_\mathcal{E}$, we have that $V_\beta \subseteq N$ and $j_N(\kappa) \geq \beta$. Since $\beta > \kappa$, this implies that the critical point of j_N is less than or equal to κ. The only additional argument needed, with respect to the proof of Theorem 4.1, is the following:

CLAIM 5.12. $N \models$ "$\beta \in C^{(n-1)}$."

PROOF. Since β is a limit point of $C^{(n-1)}$, it suffices to show that if $\gamma < \beta$ belongs to $C^{(n-1)}$, then $N \models$ "$\gamma \in C^{(n-1)}$." So, fix some $\gamma < \beta$ in $C^{(n-1)}$.

Let $f : [\kappa]^1 \to \kappa$ be such that $f(\{x\}) = x$. It is well known that $k_{\{\gamma\}}([f]_{E_{\{\gamma\}}}) = \gamma$, where $k_{\{\gamma\}} : M_{\{\gamma\}} \to N$ is the standard map given by $k_{\{\gamma\}}([f]_{E_{\{\gamma\}}}) = \pi([\{\gamma\}, [f]_{E_{\{\gamma\}}}])$ (see [9, Lemma 26.2(a)]).

Let $X := \{\{x\} \in [\kappa]^1 : x \in C^{(n-1)}\}$. Note that, since being a singleton is a property expressible by a bounded formula, and $C^{(n-1)} \cap \alpha$ is a predicate in the language of every structure $\mathcal{A}_\alpha$, the homomorphism k maps X to the set

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$\{\{x\} \in [\beta]^1 : x \in C^{(n-1)}\}$. Thus, $\{\gamma\} \in k(X)$, and therefore $X \in E_{\{\gamma\}}$. Hence, $M_{\{\gamma\}} \models$ "$[f] \in C^{(n-1)}$," and therefore $M_{\mathcal{E}} \models$ "$[\{\gamma\}, [f]] \in C^{(n-1)}$," which yields $N \models$ "$\gamma \in C^{(n-1)}$," as wanted. ⊣

Thus, by Proposition 5.9, j_N witnesses that the critical point of j_N is less than or equal to κ and is β-Π_{n-1}-strong, in contradiction to our choice of β. ⊣

In a similar way we may obtain the following parameterized version of Theorem 5.11. For the proof of (4) implies (2), we need to consider the class of structures

$$\mathcal{A}_\alpha := \langle V_{\alpha+1}, \in, \alpha, C^{(n-1)} \cap \alpha, \{R^\alpha_\varphi\}_{\varphi \in \Pi_1}, \langle \delta \rangle_{\delta < \gamma} \rangle,$$

where the structure

$$\langle V_{\alpha+1}, \in, \alpha, C^{(n-1)} \cap \alpha, \{R^\alpha_\varphi\}_{\varphi \in \Pi_1} \rangle$$

is as in the proof of Theorem 5.11, and we have a constant δ for every $\delta < \gamma$.

Theorem 5.13. *The following are equivalent for $n \geq 2$:*

(1) *There exists a proper class of Σ_n-strong cardinals.*
(2) *There exists a proper class of Π_{n-1}-strong cardinals.*
(3) $\mathbf{\Sigma_n}$-PRP.
(4) $\mathbf{\Pi_{n-1}}$-PRP.
(5) $\mathbf{\Sigma_n}$-SWVP.
(6) $\mathbf{\Pi_{n-1}}$-SWVP.
(7) $\mathbf{\Sigma_n}$-WVP.
(8) $\mathbf{\Pi_{n-1}}$-WVP.

Recall that a cardinal κ is *Woodin* if for every $A \subseteq V_\kappa$ there is $\alpha < \kappa$ such that α is $<\kappa$-A-strong, i.e., for every $\gamma < \kappa$ there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \alpha$, $\gamma < j(\alpha)$, $V_\gamma \subseteq M$, and $A \cap V_\gamma = j(A) \cap V_\gamma$. (See [9, Theorem 26.14].)

Definition 5.14. OR *is Woodin* if for every definable (with set parameters) class A there exists some α which is A-strong, i.e., for every γ there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \alpha$, $\gamma < j(\alpha)$, $V_\gamma \subseteq M$, and $A \cap V_\gamma = j(A) \cap V_\gamma$.

The statement "OR is Woodin" is first-order expressible as a schema, namely as "There exists α which is A-strong," for each definable, with parameters, class A. Or equivalently, by Proposition 5.9 and the remark that follows it, as the schema "There exists α which is Π_n-strong," for every n. Let us note that, by Theorem 5.13, "OR is Woodin" is also equivalent to the schema asserting "There exist a proper class of α which are Π_n-strong," for every n. Thus, Theorem 5.13 yields the following corollary, first proved by the second author in [13] for arbitrary classes (not necessarily definable), which gives the exact large-cardinal strength of WVP and SWVP.

Corollary 5.15. *The following are equivalent*:

(1) OR *is Woodin*.
(2) SWVP.
(3) WVP.

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