Bull. Aust. Math. Soc. **85** (2012), 395–414
doi:10.1017/S000497271100298X

TWO NONTRIVIAL WEAK SOLUTIONS FOR THE DIRICHLET PROBLEM ON THE SIERPIŃSKI GASKET

DENISA STANCU-DUMITRU

(Received 6 June 2011)

Abstract

We study a Dirichlet problem involving the weak Laplacian on the Sierpiński gasket, and we prove the existence of at least two distinct nontrivial weak solutions using Ekeland's Variational Principle and standard tools in critical point theory combined with corresponding variational techniques.

2010 *Mathematics subject classification*: primary 28A80; secondary 35B38, 35J05, 31C25.

Keywords and phrases: Sierpiński gasket, weak Laplacian, Dirichlet form, weak solution, Ekeland's variational principle.

1. Introduction

The word 'fractal' is derived from the Latin *fractus*, meaning broken, and is due to Mandelbrot in 1975. A fractal often has the following properties: it has a simple recursive definition, it has a fine structure at arbitrary small scales, it is self-similar, and it has a Hausdorff dimension which is greater than its topological dimension. A simple example of a fractal is the Sierpiński gasket (triangle). It was introduced in 1915 in an influential paper [12] by the Polish mathematician Waclaw Sierpiński and it plays an important role in the theory of curves. It is one of the basic examples of post critically finite fractals (see [9]). The complement of it is a union of triangles.

The basic differential operator in the theory of fractals is the Laplacian. Therefore, when speaking of differential equations on fractals or fractal differential equations, one means equations involving the Laplacian.

The pioneering works in analysis on fractal sets are the probabilistic approaches of Kusuoka [11] and Barlow and Perkins [1]. They have constructed and investigated Brownian motion on the Sierpiński gasket. In their standpoint, the Laplace operator

This work was partially supported by the strategic grant POSDRU/88/1.5/S/49516, Project ID 49516 (2009), co-financed by the European Social Fund–Investing in People, within the Sectorial Operational Programme Human Resources Development 2007–2013. The author was also partially supported by the grant CNCSIS-UEFISCSU PN-II-ID-PCE-2011-3-0075 Analysis, Control and Numerical Approximations of Partial Differential Equations.

has been formulated as the infinitesimal generator of the diffusion process. On the other hand, a direct and natural construction of a Laplacian on the Sierpiński gasket as a limit of difference quotients was given by Kigami [7], who later extended the method to the class of post critically finite fractals; for details see [8, 9]. Kigami gives a general construction of self-similar energies and Laplacians on a family of self-similar fractals that includes the familiar Sierpiński gasket.

Many problems on fractal domains lead to nonlinear models, such as reaction–diffusion equations and problems on elastic fractal media or fluid flow through fractal regions.

Let V be the Sierpiński gasket in $\mathbb{R}^{N-1}$ $(N \geq 2)$ and V_0 be the boundary of the Sierpiński gasket V. In this paper we are concerned with the problem

$$\begin{cases} -\Delta u(x) = f(x)|u(x)|^{p-2}u(x) + (1 - g(x))|u(x)|^{q-2}u(x), & \text{for } x \in V \setminus V_0, \\ u(x) = 0, & \text{for } x \in V_0, \end{cases} \tag{1.1}$$

where Δ is the Laplacian on V, $1 < p < 2 < q$ are real numbers, $f, g \in C(V)$ satisfy $f^+ = \max\{f, 0\} \neq 0$ and $0 \leq g(x) < 1$ for all $x \in V$.

2. Preliminary results

We start by recalling the definition of the Sierpiński gasket in $\mathbb{R}^{N-1}$ for $N \geq 2$. Let $q_1, q_2, \ldots, q_N \in \mathbb{R}^{N-1}$ satisfy $|q_i - q_j| = 1$ for $i \neq j$. For every $i \in \{1, \ldots, N\}$, define the map $S_i : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by

$$S_i(x) = \tfrac{1}{2}(x - q_i) + q_i.$$

It is clear that S_i is a similarity with ratio $\frac{1}{2}$. Let $\mathcal{S} := \{S_1, \ldots, S_N\}$ and define the map $F : \mathcal{P}(\mathbb{R}^{N-1}) \to \mathcal{P}(\mathbb{R}^{N-1})$ by

$$F(A) = \bigcup_{i=1}^{N} S_i(A) \text{ for every subset } A \text{ of } \mathbb{R}^{N-1}.$$

Theorem 9.1 in [5] shows that there exists a unique nonempty compact subset V of $\mathbb{R}^{N-1}$ such that $F(V) = V$, that is, the set V is a fixed point of the map F. The set V is called the *Sierpiński gasket* in $\mathbb{R}^{N-1}$.

The Sierpiński gasket V can be constructed inductively. Put $V_0 = \{q_1, \ldots, q_N\}$, $V_m = F(V_{m-1})$ for $m \geq 1$ and $V_\star = \bigcup_{m\geq 0} V_m$. The points in $V_m \setminus V_0$ are called junction points. We note $q_i = S_i(q_i)$ for every $i \in \{1, \ldots, N\}$ and thus we have $V_0 \subseteq V_1$, hence $V_\star = F(V_\star)$. Since the maps S_i are homeomorphisms for $i \in \{1, \ldots, N\}$, we deduce that $\overline{V_\star}$ is a fixed point of the map F. On the other hand, taking H to be the convex hull of the set V_0 we notice that $S_i(H) \subseteq H$ for $i \in \{1, \ldots, N\}$. Thus, $V_m \subseteq H$ for every natural number $m \geq 0$ and it follows that $\overline{V_\star} \subseteq H$. Thus, we conclude that $\overline{V_\star}$ is a nonempty and compact set, hence $V = \overline{V_\star}$.

By [5, Theorem 9.3], the Hausdorff dimension d of V satisfies the equality $\sum_{i=1}^{N}(\frac{1}{2})^d = 1$. Hence $d = \ln N / \ln 2$ and $0 < \mathcal{H}^d(V) < +\infty$, where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^{N-1}$. Let μ be the normalised restriction of $\mathcal{H}^d$ to the subsets of V, so $\mu(V) = 1$.

The measure μ has the following property: $\mu(B) > 0$, *for every nonempty open subset B of V*. In other words, the support of μ coincides with V.

We recall from [6] the following notation:

$$C(V) = \{u : V \to \mathbb{R} \mid u \text{ is continuous}\}$$

and

$$C_0(V) = \{u \in C(V) \mid u|_{V_0} = 0\}.$$

The spaces $C(V)$ and $C_0(V)$ are endowed with the sup-norm denoted by $\|\cdot\|_{\sup}$.

For each function $v : V \to \mathbb{R}$ and each nonnegative integer m let

$$W_m(v) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y \in V_m,\\ |x-y|=2^{-m}}} (v(x) - v(y))^2.$$

Since for each nonnegative integer m we have $W_m(v) \le W_{m+1}(v)$, so we can define

$$W(v) = \lim_{m\to\infty} W_m(v),$$

(possibly $W(v) = +\infty$).

We also recall the following lemma that plays an important role in our analysis below.

Lemma 2.1 (The Sobolev type inequality). *For all $v \in C(V)$,*

$$\sup_{\substack{x,y \in V_\star\\ x \neq y}} \frac{|v(x) - v(y)|}{|x-y|^\alpha} \le (2N+3)\sqrt{W(v)}, \tag{2.1}$$

where $\alpha = (1/(2\log 2))\log((N+2)/N)$.

Next, we define

$$H_0^1(V) = \{u \mid u \in C_0(V) \text{ and } W(u) < +\infty\}.$$

The space $H_0^1(V)$ appears as a dense linear subspace of $L^2(V,\mu)$ endowed with the usual norm $\|\cdot\|_2$. The space $H_0^1(V)$ is endowed with the norm

$$\|u\| = \sqrt{W(u)}.$$

Actually, there is an inner product defining this norm. For $u, v \in H_0^1(V)$ and each nonnegative integer m we set

$$W_m(u,v) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y \in V_m,\\ |x-y|=2^{-m}}} (u(x) - u(y))(v(x) - v(y)).$$

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It is easy to see using Cauchy's inequality that the following limit

$$W(u, v) = \lim_{m\to\infty} W_m(u, v),$$

exists and is finite if $u, v \in H_0^1(V)$. The space $H_0^1(V)$ endowed with the inner product $W(\cdot, \cdot)$ is a real Hilbert space. Following [4] we know that $W(\cdot, \cdot)$ is a densely defined closed, nonnegative and symmetric bilinear form. Further, $W(\cdot, \cdot)$ is a Dirichlet form on $L^2(V, \mu)$.

By (2.1), for all $v \in H_0^1(V)$, we have

$$|v(x) - v(y)| \leq (2N + 3)|x - y|^\alpha \|v\|,$$

and, taking $y = q_1$, we readily get

$$|v(x)| \leq (2N + 3)\|v\|. \tag{2.2}$$

Furthermore, from Lemma 2.1 and the Ascoli–Arzéla Theorem, the embedding

$$(H_0^1(V), \|\cdot\|) \hookrightarrow (C_0(V), \|\cdot\|_{\sup}) \tag{2.3}$$

is compact.

Now we define the Laplacian on the Sierpiński gasket V. Let $H^{-1}(V)$ be the closure of $L^2(V)$ with respect to the pre-norm

$$\|w\|_{-1} = \sup_{\substack{g\in H_0^1(V),\\ \|g\|=1}} |\langle w, g\rangle|,$$

where $\langle w, g\rangle = \int_V wg\, d\mu$, for $w \in L^2(V)$ and $g \in H_0^1(V)$. Then $H^{-1}(V)$ is a Hilbert space. Now we have three Hilbert spaces $H_0^1(V)$, $L^2(V, \mu)$ and $H^{-1}(V)$ and the embeddings

$$H_0^1(V) \subset L^2(V, \mu) \subset H^{-1}(V).$$

The relation

$$-W(u, v) = \langle \Delta u, v\rangle, \quad \text{for all } v \in H_0^1(V),$$

uniquely defines a function $\Delta u \in H^{-1}(V)$ for all $u \in H_0^1(V)$ and we call Δ the weak Laplacian on V; see [10]. This operator is linear, self-adjoint and defined on a linear subset of $H_0^1(V)$ which is dense in $L^2(V, \mu)$ and also in $(H_0^1(V), \|\cdot\|)$. A complete presentation of this operator in the general setting can be found in the books of Kigami [9] and Strichartz [13].

3. Main results

A function $u \in H_0^1(V)$ is called a *weak solution* for Problem (1.1) if it satisfies the equality

$$W(u, v) - \int_V f(x)|u|^{p-2}uv\, d\mu - \int_V (1 - g(x))|u|^{q-2}uv\, d\mu = 0$$

for all $v \in H_0^1(V)$.

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Theorem 3.1. *Problem (1.1) has at least two distinct nontrivial weak solutions, for $f \in C(V)$ with $f^+ \neq 0$ and $\|f^+\|_{\sup}$ small enough.*

We consider the following problem

$$\begin{cases} -\Delta u(x) = \lambda |u(x)|^{p-2}u(x) + |u(x)|^{q-2}u(x), & \text{for } x \in V \setminus V_0, \\ u(x) = 0, & \text{for } x \in V_0, \end{cases} \tag{3.1}$$

where p, q, λ are real numbers such that $1 < p < 2 < q$ and $\lambda > 0$.

Corollary 3.2. *There exists $\overline{\lambda} > 0$ such that for every $\lambda \in (0, \overline{\lambda})$, Problem (3.1) has at least two nontrivial weak solutions.*

4. Proofs

We consider the energy functional corresponding to Problem (1.1) defined as $I : H_0^1(V) \to \mathbb{R}$,

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_V f(x)|u|^p \, d\mu - \frac{1}{q}\int_V (1 - g(x))|u|^q \, d\mu$$

for all $u \in H_0^1(V)$.

Proposition 2.19 in [6] shows that the functional $I \in C^1(H_0^1(V), \mathbb{R})$ and the Fréchet derivative is given by

$$\langle I'(u), v\rangle = W(u, v) - \int_V f(x)|u|^{p-2}uv \, d\mu - \int_V (1 - g(x))|u|^{q-2}uv \, d\mu$$

for all $u, v \in H_0^1(V)$.

Thus, the weak solutions of Problem (1.1) are exactly the critical points of I. We shall prove that the functional I possesses nontrivial critical points.

Since the energy functional I is not bounded on $H_0^1(V)$, it is useful to consider the functional on the so called Nehari manifold, defined by

$$\mathcal{N} = \{u \in H_0^1(V) \setminus \{0\} \mid \langle I'(u), u\rangle = 0\}.$$

Thus $u \in \mathcal{N}$ if and only if

$$\|u\|^2 - \int_V f(x)|u|^p \, d\mu - \int_V (1 - g(x))|u|^q \, d\mu = 0. \tag{4.1}$$

Furthermore, we have the following results.

Lemma 4.1. *The energy functional I is coercive and bounded below on $\mathcal{N}$.*

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Proof. For every $u \in \mathcal{N}$,

$$
\begin{aligned}
I(u) &= \frac{q-2}{2q}\|u\|^2 - \Big(\frac{1}{p} - \frac{1}{q}\Big) \int_V f(x)|u|^p \, d\mu \\
&\geq \frac{q-2}{2q}\|u\|^2 - \Big(\frac{1}{p} - \frac{1}{q}\Big) \int_V f^+(x)|u|^p \, d\mu \\
&\geq \frac{q-2}{2q}\|u\|^2 - \Big(\frac{1}{p} - \frac{1}{q}\Big) \|f^+\|_{\sup}\|u\|_{\sup}^p \\
&\geq \frac{q-2}{2q}\|u\|^2 - \Big(\frac{1}{p} - \frac{1}{q}\Big) \|f^+\|_{\sup}(2N+3)^p\|u\|^p,
\end{aligned}
$$

where the latter inequality follows by (2.2). Thus, I is coercive and bounded below on $\mathcal{N}$. The proof of Lemma 4.1 is complete. □

For every $u \in H_0^1(V)$ we define the function $h_u : (0, \infty) \to \mathbb{R}$ by $h_u(t) = I(tu)$ for all $t > 0$, that is,

$$
h_u(t) = \frac{t^2}{2}\|u\|^2 - \frac{t^p}{p} \int_V f(x)|u|^p \, d\mu - \frac{t^q}{q} \int_V (1 - g(x))|u|^q \, d\mu, \quad \text{for all } t > 0.
$$

We have

$$
h_u'(t) = t\|u\|^2 - t^{p-1} \int_V f(x)|u|^p \, d\mu - t^{q-1} \int_V (1 - g(x))|u|^q \, d\mu, \quad \text{for all } t > 0
$$

and

$$
\begin{aligned}
h_u''(t) = \|u\|^2 - (p-1)t^{p-2} \int_V f(x)|u|^p \, d\mu - (q-1)t^{q-2} \\
\times \int_V (1 - g(x))|u|^q \, d\mu, \quad \text{for all } t > 0.
\end{aligned}
$$

It is easy to see that

$$
th_u'(t) = \|tu\|^2 - \int_V f(x)|tu|^p \, d\mu - \int_V (1 - g(x))|tu|^q \, d\mu, \quad \text{for all } t > 0
$$

and so, for $u \in H_0^1(V) \setminus \{0\}$ and $t > 0$,

$$
h_u'(t) = 0 \quad \text{if and only if } tu \in \mathcal{N}, \tag{4.2}
$$

that is, the positive critical points of h_u correspond to points on the Nehari manifold. In particular, $h_u'(1) = 0$ if and only if $u \in \mathcal{N}$. Thus, it is natural to divide $\mathcal{N}$ into three subsets corresponding to the local minima, the local maxima and the points of inflection. Accordingly, we define

$$
\begin{aligned}
\mathcal{N}^+ &= \{u \in \mathcal{N} \mid h_u''(1) > 0\}, \\
\mathcal{N}^0 &= \{u \in \mathcal{N} \mid h_u''(1) = 0\}, \\
\mathcal{N}^- &= \{u \in \mathcal{N} \mid h_u''(1) < 0\}.
\end{aligned}
$$

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LEMMA 4.2. *Suppose that $\overline{u}$ is a local minimiser for I on $\mathcal{N}$ and $\overline{u} \notin \mathcal{N}^0$. Then $I'(\overline{u}) = 0$ in $H^{-1}(V)$.*

PROOF. If $\overline{u}$ is a local minimiser for I on $\mathcal{N}$, then $\overline{u}$ is a solution of the optimisation problem

$$\text{minimise } I(u) \text{ subject to } \gamma(u) = 0, \quad u \in H_0^1(V),$$

where

$$\gamma(u) = \|u\|^2 - \int_V f(x)|u|^p \, d\mu - \int_V (1 - g(x))|u|^q \, d\mu.$$

Since $\overline{u} \in \mathcal{N}$,

$$\|\overline{u}\|^2 = \int_V f(x)|\overline{u}|^p \, d\mu + \int_V (1 - g(x))|\overline{u}|^q \, d\mu;$$

thus

$$\begin{aligned}\langle \gamma'(\overline{u}), \overline{u} \rangle &= 2\|\overline{u}\|^2 - p \int_V f(x)|\overline{u}|^p \, d\mu - q \int_V (1 - g(x))|\overline{u}|^q \, d\mu \\ &= \|\overline{u}\|^2 - (p - 1) \int_V f(x)|\overline{u}|^p \, d\mu - (q - 1) \int_V (1 - g(x))|\overline{u}|^q \, d\mu,\end{aligned}$$

so

$$\langle \gamma'(\overline{u}), \overline{u} \rangle = h''_{\overline{u}}(1). \tag{4.3}$$

We notice that $\gamma'(\overline{u}) \neq 0$. Otherwise, if $\gamma'(\overline{u}) = 0$, we have $\langle \gamma'(\overline{u}), \overline{u} \rangle = 0$ and taking into account (4.3) we deduce that it is a contradiction with $\overline{u} \notin \mathcal{N}^0$.

Hence, by the theory of Lagrange multipliers, there exists $a \in \mathbb{R}$ such that

$$I'(\overline{u}) = a\gamma'(\overline{u}). \tag{4.4}$$

Accordingly,

$$\langle I'(\overline{u}), \overline{u} \rangle = a\langle \gamma'(\overline{u}), \overline{u} \rangle. \tag{4.5}$$

Since $\overline{u} \in \mathcal{N}$ it follows that $\langle I'(\overline{u}), \overline{u} \rangle = 0$, and since $\langle \gamma'(\overline{u}), \overline{u} \rangle \neq 0$ by (4.5) we deduce that $a = 0$; therefore by (4.4) we conclude that $I'(\overline{u}) = 0$ in $H^{-1}(V)$. Thus, Lemma 4.2 is proved. □

LEMMA 4.3. (i) *For any $u \in \mathcal{N}^+$ we have $\int_V f(x)|u|^p \, d\mu > 0$.*

(ii) *For any $u \in \mathcal{N}^0$ we have $\int_V f(x)|u|^p \, d\mu > 0$ and $\int_V (1 - g(x))|u|^q \, d\mu > 0$.*

(iii) *For any $u \in \mathcal{N}^-$ we have $\int_V (1 - g(x))|u|^q \, d\mu > 0$.*

PROOF. We get

$$h''_u(1) = \|u\|^2 - (p - 1) \int_V f(x)|u|^p \, d\mu - (q - 1) \int_V (1 - g(x))|u|^q \, d\mu$$

for every $u \in H_0^1(V)$. If $u \in \mathcal{N}$, by condition (4.1), we deduce that

$$h''_u(1) = (2 - q)\|u\|^2 - (p - q) \int_V f(x)|u|^p \, d\mu \tag{4.6}$$

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and

$$h_u''(1) = (2-p)\|u\|^2 - (q-p)\int_V (1-g(x))|u|^q \, d\mu. \tag{4.7}$$

If $u \in \mathcal{N}^+$, by (4.6),

$$\int_V f(x)|u|^p \, d\mu > \frac{q-2}{q-p}\|u\|^2 \tag{4.8}$$

so (i) holds.

If $u \in \mathcal{N}^0$, by (4.6),

$$\int_V f(x)|u|^p \, d\mu = \frac{q-2}{q-p}\|u\|^2 \tag{4.9}$$

and, by (4.7),

$$\int_V (1-g(x))|u|^q \, d\mu = \frac{2-p}{q-p}\|u\|^2. \tag{4.10}$$

Therefore (ii) is true.

If $u \in \mathcal{N}^-$, by (4.7),

$$\int_V (1-g(x))|u|^q \, d\mu > \frac{2-p}{q-p}\|u\|^2 \tag{4.11}$$

so we find (iii). The proof of Lemma 4.3 is complete. □

Let

$$M := \frac{q-2}{q-p}\Big(\frac{2-p}{q-p}\Big)^{(2-p)/(q-2)}(2N+3)^{-2(q-p)/(q-2)} > 0.$$

LEMMA 4.4. *The set $\mathcal{N}^0$ is empty for all $f \in C(V)$ with $f^+ \neq 0$ and $\|f^+\|_{\sup} < M$.*

PROOF. We assume by contradiction that there exists $f \in C(V)$ with $f^+ \neq 0$ and $\|f^+\|_{\sup} < M$ such that $\mathcal{N}^0$ is not empty. For $u \in \mathcal{N}^0$, using (4.9),

$$\|u\|^2 = \frac{q-p}{q-2}\int_V f(x)|u|^p \, d\mu \le \frac{q-p}{q-2}\int_V f^+(x)|u|^p \, d\mu \le \frac{q-p}{q-2}\|f^+\|_{\sup}(2N+3)^p\|u\|^p$$

or

$$\|u\|^2 \le \Big(\frac{q-p}{q-2}(2N+3)^p\|f^+\|_{\sup}\Big)^{2/(2-p)}.$$

Similarly, by (4.10),

$$\|u\|^2 = \frac{q-p}{2-p}\int_V (1-g(x))|u|^q \, d\mu \le \frac{q-p}{2-p}(2N+3)^q\|u\|^q$$

or

$$\|u\|^2 \ge \Big(\frac{2-p}{q-p}(2N+3)^{-q}\Big)^{2/(q-2)}.$$

Thus,

$$\|f^+\|_{\sup} \ge \frac{q-2}{q-p}\Big(\frac{2-p}{q-p}\Big)^{(2-p)/(q-2)}(2N+3)^{-2(q-p)/(q-2)} = M,$$

which is a contradiction. This completes the proof of Lemma 4.4. □

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We consider, for any $u \in H_0^1(V)$, the function $\phi_u : (0, \infty) \to \mathbb{R}$ defined by

$$\phi_u(t) = t^{2-p}\|u\|^2 - t^{q-p}\int_V (1 - g(x))|u|^q \, d\mu, \quad \text{for all } t > 0.$$

Clearly,

$$tu \in \mathcal{N} \quad \text{if and only if } \phi_u(t) = \int_V f(x)|u|^p \, d\mu. \tag{4.12}$$

Moreover,

$$\phi_u'(t) = (2 - p)t^{1-p}\|u\|^2 - (q - p)t^{q-p-1}\int_V (1 - g(x))|u|^q \, d\mu, \tag{4.13}$$

and so it is easy to see that if $tu \in \mathcal{N}$ then

$$t^{p-1}\phi_u'(t) = h_u''(t). \tag{4.14}$$

Thus, $tu \in \mathcal{N}^+$, respectively $tu \in \mathcal{N}^-$, if and only if $\phi_u'(t) > 0$, respectively $\phi_u'(t) < 0$.

We assume that $u \in H_0^1(V) \setminus \{0\}$. By (4.13), ϕ_u has a unique critical point at $t = t_0$, where

$$t_0 = \left(\frac{(2 - p)\|u\|^2}{(q - p)\int_V (1 - g(x))|u|^q \, d\mu}\right)^{1/(q-2)} > 0$$

and ϕ_u is strictly increasing on $(0, t_0)$ and strictly decreasing on (t_0, ∞). Moreover, $\lim_{t\to 0} \phi_u(t) = 0$ and $\lim_{t\to\infty} \phi_u(t) = -\infty$. Furthermore,

$$\begin{aligned}\phi_u(t_0) &= \frac{q-2}{q-p}\left(\frac{2-p}{q-p}\right)^{(2-p)/(q-2)}\|u\|^p\left(\frac{\|u\|^q}{\int_V (1 - g(x))|u|^q \, d\mu}\right)^{(2-p)/(q-2)} \\ &\geq \|f^+\|_{\sup}^{-1}\frac{q-2}{q-p}\left(\frac{2-p}{q-p}\right)^{(2-p)/(q-2)}(2N + 3)^{-2(q-p)/(q-2)}\int_V f(x)|u|^p \, d\mu \\ &> \int_V f(x)|u|^p \, d\mu,\end{aligned}$$

since

$$\|f^+\|_{\sup} < M = \frac{q-2}{q-p}\left(\frac{2-p}{q-p}\right)^{(2-p)/(q-2)}(2N + 3)^{-2(q-p)/(q-2)}.$$

Lemma 4.5. *For each $f \in C(V)$ with $f^+ \neq 0$, $\|f^+\| < M$ and each $u \in H_0^1(V) \setminus \{0\}$, we have the following results.*

(i) *If $\int_V f(x)|u|^p \, d\mu \leq 0$, then there exists a unique $t_1 = t_1(u) > t_0$ such that $t_1 u \in \mathcal{N}^-$ and h_u is increasing on $(0, t_1)$ and decreasing on (t_1, ∞). Moreover,*

$$I(t_1 u) = \sup_{t\geq 0} I(tu). \tag{4.15}$$

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(ii) *If* $\int_V f(x)|u|^p\,d\mu > 0$, *then there exist unique* $0 < t_2 = t_2(u) < t_0 < t_1$ *such that* $t_2 u \in \mathcal{N}^+$, $t_1 u \in \mathcal{N}^-$ *and* h_u *is decreasing on* $(0, t_2)$, *increasing on* (t_2, t_1) *and decreasing on* (t_1, ∞). *Moreover,*

$$I(t_2 u) = \inf_{0\le t\le t_0} I(tu), \qquad I(t_1 u) = \sup_{t\ge t_0} I(tu). \tag{4.16}$$

(iii) $t_1(u)$ *is a continuous function for* $u \in H_0^1(V) \setminus \{0\}$.
(iv) $\mathcal{N}^- = \{u \in H_0^1(V) \setminus \{0\} : (1/\|u\|)t_1(u/\|u\|) = 1\}$.

Proof. We fix an arbitrary $u \in H_0^1(V) \setminus \{0\}$ and we have $\int_V (1 - g(x))|u|^q\,d\mu > 0$.

(i) We assume that $\int_V f(x)|u|^p\,d\mu \le 0$. Since ϕ_u is strictly increasing on $(0, t_0)$ and $\lim_{t\to 0}\phi_u(t) = 0$ it follows that $\phi_u(t) > 0 \ge \int_V f(x)|u|^p\,d\mu$ on $(0, t_0)$, so the equation with respect to t,

$$\phi_u(t) = \int_V f(x)|u|^p\,d\mu, \tag{4.17}$$

has no solution on $(0, t_0)$.

Since $\phi_u(t_0) > \int_V f(x)|u|^p\,d\mu$, ϕ_u is strictly decreasing on (t_0, ∞) and $\lim_{t\to\infty}\phi_u(t) = -\infty$, (4.17) with respect to t has a unique solution t_1 on (t_0, ∞) and $\phi_u'(t_1) < 0$, which depends on u. Taking into account the above facts we get t_1 as the unique solution on $(0, \infty)$ of (4.17).

Thus, by (4.12), we have $t_1 u \in \mathcal{N}$ which is equivalent, by (4.2), to $h_u'(t_1) = 0$. Therefore, h_u has a unique critical point at $t = t_1$ on $(0, \infty)$. Using (4.14) and $\phi_u'(t_1) < 0$ it follows that $h_u''(t_1) < 0$, so $t_1 u \in \mathcal{N}^-$ and t_1 is a maximum point of h_u on $(0, \infty)$. Given that, we deduce that h_u is increasing on $(0, t_1)$ and decreasing on (t_1, ∞), and

$$h_u(t_1) = \sup_{t\ge 0} h_u(t).$$

This means that (4.15) holds.

(ii) We assume that $\int_V f(x)|u|^p\,d\mu > 0$. Since $\phi_u(t_0) > \int_V f(x)|u|^p\,d\mu$, ϕ_u is strictly increasing on $(0, t_0)$ and $\lim_{t\to 0}\phi_u(t) = 0$, (4.17) with respect to t has a unique solution t_2 on $(0, t_0)$. Also, since $\phi_u(t_0) > \int_V f(x)|u|^p\,d\mu$, ϕ_u is strictly decreasing on (t_0, ∞) and $\lim_{t\to\infty}\phi_u(t) = -\infty$, Equation (4.17) with respect to t has a unique solution t_1 on (t_0, ∞). Thus, the equation with respect to t,

$$\phi_u(t) = \int_V f(x)|u|^p\,d\mu,$$

has exactly two solutions t_1, t_2 depending on u with $0 < t_2 < t_0 < t_1$ such that $\phi_u'(t_2) > 0$ and $\phi_u'(t_1) < 0$. Therefore, there exist two multiples of u such that $t_2 u \in \mathcal{N}^+$ and $t_1 u \in \mathcal{N}^-$. Hence, h_u has two critical points at $t = t_2$ and $t = t_1$ such that $h_u''(t_2) > 0$ and $h_u''(t_1) < 0$. It follows that t_2 is a minimum point of h_u and t_1 is a maximum point of h_u. This yields that h_u is decreasing on $(0, t_2)$, increasing on (t_2, t_1), decreasing on (t_1, ∞) and $h_u(t_2) = \inf_{0\le t\le t_0} h_u(t)$, $h_u(t_1) = \sup_{t\ge t_0} h_u(t)$. The above facts imply that (4.16) holds.

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(iii) By the uniqueness of $t_1(u)$ and the extremal property of $t_1(u)$, we deduce that $t_1(u)$ is a continuous function for $u \in H_0^1(V) \setminus \{0\}$.

(iv) For $u \in \mathcal{N}^-$, we consider $w = u/\|u\|$. We claim that $w \in H_0^1(V) \setminus \{0\}$. Taking $h : \mathbb{R} \to \mathbb{R}$, $h(t) = t/\|u\|$ (that is a Lipschitz function with constant $L \geq 0$ and $h(0) = 0$) it follows by [2, Lemma 3.1] that $w = h \circ u = u/\|u\| \in H_0^1(V)$. Also $w \neq 0$ since $u \neq 0$. Our claim is proved.

By (i) and (ii) there exists a unique $t_1(w) > 0$ such that $t_1(w)w \in \mathcal{N}^-$ or $(u/\|u\|)t_1(u/\|u\|) \in \mathcal{N}^-$. Since $u \in \mathcal{N}^-$ it follows that $(1/\|u\|)t_1(u/\|u\|) = 1$ and this implies that

$$\mathcal{N}^- \subset \Big\{u \in H_0^1(V) \setminus \{0\} : \frac{1}{\|u\|} t_1\Big(\frac{u}{\|u\|}\Big) = 1\Big\}.$$

Conversely, we assume that $u \in H_0^1(V) \setminus \{0\}$ is such that $(1/\|u\|)t_1(u/\|u\|) = 1$. Then

$$\frac{u}{\|u\|} t_1\Big(\frac{u}{\|u\|}\Big) \in \mathcal{N}^-,$$

so $u \in \mathcal{N}^-$.

We conclude that

$$\mathcal{N}^- = \Big\{u \in H_0^1(V) \setminus \{0\} : \frac{1}{\|u\|} t_1\Big(\frac{u}{\|u\|}\Big) = 1\Big\}.$$

Lemma 4.5 is proved. □

For all $f \in C(V)$ with $f^+ \neq 0$ and $\|f^+\|_{\sup} < M$, by Lemma 4.5 we deduce that the sets $\mathcal{N}^+$ and $\mathcal{N}^-$ are nonempty, and combining this result with Lemma 4.4 we conclude that

$$\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-.$$

We define

$$I_1 = \inf_{u \in \mathcal{N}^+} I(u) \quad \text{and} \quad I_2 = \inf_{u \in \mathcal{N}^-} I(u)$$

and we try to see if they are attained.

LEMMA 4.6. *The following assertions hold.*

(i) $I_1 < 0$ *for all* $f \in C(V)$ *with* $f^+ \neq 0$ *and* $\|f^+\|_{\sup} < M$.

(ii) *If* $\|f^+\|_{\sup} < \overline{M} := (p/2)M$, *then* $I_2 > C > 0$ *for some constant* C. *In particular,* $I_1 = \inf_{u \in \mathcal{N}} I(u)$ *for all* $f \in C(V)$ *with* $f^+ \neq 0$ *and* $\|f^+\|_{\sup} < \overline{M}$.

PROOF. (i) We consider $u \in \mathcal{N}^+$ and taking into account (4.8) we obtain

$$\|u\|^2 < \frac{q-p}{q-2} \int_V f(x)|u|^p \, d\mu.$$

Thus, using the above inequality and Lemma 4.3(i),

$$I(u) = \frac{q-2}{2q}\|u\|^2 - \Big(\frac{1}{p} - \frac{1}{q}\Big) \int_V f(x)|u|^p \, d\mu < -\frac{(q-p)(2-p)}{2pq} \int_V f(x)|u|^p \, d\mu < 0$$

so $I_1 < 0$.

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(ii) Let $u \in \mathcal{N}^-$. By (4.11), we get

$$\|u\|^2 < \frac{q-p}{2-p} \int_V (1-g(x))|u|^q \, d\mu \le \frac{q-p}{2-p}(2N+3)^q \|u\|^q$$

and this yields that

$$\|u\|^2 > \Big(\frac{2-p}{q-p}(2N+3)^{-q}\Big)^{2/(q-2)}.$$

Therefore

$$\begin{aligned} I(u) &= \frac{q-2}{2q}\|u\|^2 - \Big(\frac{1}{p}-\frac{1}{q}\Big)\int_V f(x)|u|^p \, d\mu \\ &\ge \|u\|^p \Big(\frac{q-2}{2q}\|u\|^{2-p} - \Big(\frac{1}{p}-\frac{1}{q}\Big)\|f^+\|_{\sup}(2N+3)^p\Big) \\ &> \Big(\frac{2-p}{q-p}(2N+3)^{-q}\Big)^{p/(q-2)} \\ &\quad \times \Big(\frac{q-2}{2q}\Big(\frac{2-p}{q-p}(2N+3)^{-q}\Big)^{(2-p)/(q-2)} - \Big(\frac{1}{p}-\frac{1}{q}\Big)\|f^+\|_{\sup}(2N+3)^p\Big). \end{aligned}$$

If $\|f^+\|_{\sup} < (p/2)M = \overline{M}$, then $I_2 > C > 0$ for some constant C. This completes the proof of Lemma 4.6. □

LEMMA 4.7. *We assume that $f \in C(V)$ with $f^+ \ne 0$ and $\|f^+\|_{\sup} < M$.*

(i) *For each $u \in \mathcal{N}$, there exist $\tau > 0$ and a differentiable function $\omega : B(0,\tau) \subset H_0^1(V) \to [0,+\infty)$ such that $\omega(0) = 1$, $\omega(w)(u-w) \in \mathcal{N}$ for all $w \in B(0,\tau)$ and*

$$\langle \omega'(0), v\rangle = \frac{2W(u,v) - p\int_V f(x)|u|^{p-2}uv \, d\mu - q\int_V (1-g(x))|u|^{q-2}uv \, d\mu}{(2-p)\|u\|^2 - (q-p)\int_V (1-g(x))|u|^q \, d\mu} \tag{4.18}$$

for all $v \in H_0^1(V)$.

(ii) *For each $u \in \mathcal{N}^-$, there exist $\tau > 0$ and a differentiable function $\omega : B(0,\tau) \subset H_0^1(V) \to [0,+\infty)$ such that $\omega(0) = 1$, $\omega(w)(u-w) \in \mathcal{N}^-$ for all $w \in B(0,\tau)$ and*

$$\langle \omega'(0), v\rangle = \frac{2W(u,v) - p\int_V f(x)|u|^{p-2}uv \, d\mu - q\int_V (1-g(x))|u|^{q-2}uv \, d\mu}{(2-p)\|u\|^2 - (q-p)\int_V (1-g(x))|u|^q \, d\mu} \tag{4.19}$$

for all $v \in H_0^1(V)$.

PROOF. (i) For $u \in \mathcal{N}$, we define a function that depends on u, $\psi_u : [0,+\infty) \times H_0^1(V) \to \mathbb{R}$ by

$$\begin{aligned} \psi_u(t,w) &= \langle I'(t(u-w)), t(u-w)\rangle \\ &= t^2\|u-w\|^2 - t^p \int_V f(x)|u-w|^p \, d\mu - t^q \int_V (1-g(x))|u-w|^q \, d\mu. \end{aligned}$$

Then $\psi_u(1,0) = \langle I'(u), u\rangle = 0$ since $u \in \mathcal{N}$.

Using the fact that $u \in \mathcal{N}$ and $\mathcal{N}^0$ is empty, we have

$$\frac{\partial}{\partial t}\psi_u(1,0) = 2\|u\|^2 - p\int_V f(x)|u|^p\,d\mu - q\int_V (1-g(x))|u|^q\,d\mu$$
$$= (2-p)\|u\|^2 - (q-p)\int_V (1-g(x))|u|^q\,d\mu = h_u''(1) \neq 0.$$

According to the implicit function theorem, there exist $\tau > 0$ and a differentiable function $\omega : B(0,\tau) \subset H_0^1(V) \to [0,+\infty)$ such that $\omega(0) = 1$ and

$$\psi_u(\omega(w), w) = 0, \qquad \text{for all } w \in B(0,\tau),$$

which is equivalent to

$$\langle I'(\omega(w)(u-w)), \omega(w)(u-w)\rangle = 0, \qquad \text{for all } w \in B(0,\tau);$$

that means $\omega(w)(u-w) \in \mathcal{N}$ for all $w \in B(0,\tau)$. We also have

$$\langle \omega'(0), v\rangle = \frac{2W(u,v) - p\int_V f(x)|u|^{p-2}uv\,d\mu - q\int_V (1-g(x))|u|^{q-2}uv\,d\mu}{(2-p)\|u\|^2 - (q-p)\int_V (1-g(x))|u|^q\,d\mu}$$

for all $v \in H_0^1(V)$.

(ii) Similarly, there exist $\tau > 0$ and a differentiable function $\omega : B(0,\tau) \subset H_0^1(V) \to [0,+\infty)$ such that $\omega(0) = 1$, $\omega(w)(u-w) \in \mathcal{N}$ for all $w \in B(0,\tau)$ and

$$\langle \omega'(0), v\rangle = \frac{2W(u,v) - p\int_V f(x)|u|^{p-2}uv\,d\mu - q\int_V (1-g(x))|u|^{q-2}uv\,d\mu}{(2-p)\|u\|^2 - (q-p)\int_V (1-g(x))|u|^q\,d\mu}$$

for all $v \in H_0^1(V)$. Since $u \in \mathcal{N}^-$ we have

$$h_u''(1) = (2-p)\|u\|^2 - (q-p)\int_V (1-g(x))|u|^q\,d\mu < 0$$

and thus, by the continuity of the function ω, we obtain

$$h''_{\omega(w)(u-w)}(1) = (2-p)\|\omega(w)(u-w)\|^2 - (q-p)\int_V (1-g(x))|\omega(w)(u-w)|^q\,d\mu < 0,$$

for τ small enough, and this implies that $\omega(w)(u-w) \in \mathcal{N}^-$ for all $w \in B(0,\tau)$. □

Remark 4.8. Since $\omega(0) = 1$ it follows that $\omega(w) \neq 0$ for all $w \in B(0,\tau)$ with τ sufficiently small.

Lemma 4.9. *We assume that $f \in C(V)$ with $f^+ \neq 0$ and $\|f^+\|_{\sup} < \overline{M}$.*

(i) *There exists a minimising sequence $(u_n) \subset \mathcal{N}$ such that*

$$I(u_n) \to I_1 \quad \textit{and} \quad I'(u_n) \to 0 \quad \textit{in } H^{-1}(V).$$

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(ii) *There exists a minimising sequence* $(v_n) \subset \mathcal{N}^-$ *such that*

$$I(v_n) \to I_2 \quad \textit{and} \quad I'(v_n) \to 0 \quad \textit{in } H^{-1}(V).$$

PROOF. Since I is bounded from below on $\mathcal{N}$ (by Lemma 4.1) and by Ekeland's variational principle (see [3]), there exists a minimising sequence $(u_n) \subset \mathcal{N}$ such that

$$I(u_n) < \inf_{\mathcal{N}} I + \frac{1}{n} = I_1 + \frac{1}{n}, \tag{4.20}$$

$$I(u_n) \le I(w) + \frac{1}{n}\|w - u_n\|, \quad \text{for each } w \in \mathcal{N}. \tag{4.21}$$

Since I is coercive on $\mathcal{N}$, (u_n) is bounded in $H_0^1(V)$.

We will show that $I'(u_n) \to 0$ in $H^{-1}(V)$. Applying Lemma 4.7(i), for u_n, we obtain for every $\tau_n > 0$ functions $\omega_n : B(0, \tau_n) \subset H_0^1(V) \to [0, +\infty)$ satisfying $\omega_n(w)(u_n - w) \in \mathcal{N}$. For a fixed n, we choose $0 < \delta < \tau_n$ and for $u \in H_0^1(V) \setminus \{0\}$ arbitrarily fixed we set

$$w_\delta = \delta \frac{u}{\|u\|} \in B(0, \delta) \subset H_0^1(V)$$

and $w_\delta \ne 0$. We denote

$$\Omega_\delta := \omega_n(w_\delta)(u_n - w_\delta) \in \mathcal{N}.$$

By (4.21),

$$I(\Omega_\delta) - I(u_n) \ge -\frac{1}{n}\|\Omega_\delta - u_n\|,$$

and by the mean value theorem we obtain

$$\langle I'(u_n), \Omega_\delta - u_n\rangle + o(\|\Omega_\delta - u_n\|) \ge -\frac{1}{n}\|\Omega_\delta - u_n\|,$$

which is equivalent to

$$\langle I'(u_n), -w_\delta\rangle + (\omega_n(w_\delta) - 1)\langle I'(u_n), u_n - w_\delta\rangle \ge -\frac{1}{n}\|\Omega_\delta - u_n\| + o(\|\Omega_\delta - u_n\|).$$

Taking into account that $\Omega_\delta \in \mathcal{N}$ and $\omega_n(w_\delta) \ne 0$ for δ small enough (by Remark 4.8), the above inequality implies

$$\begin{aligned}
-\delta\Big\langle I'(u_n), \frac{u}{\|u\|}\Big\rangle &+ (\omega_n(w_\delta) - 1)\langle I'(u_n) - I'(\Omega_\delta), u_n - w_\delta\rangle \\
&\ge -\frac{1}{n}\|\Omega_\delta - u_n\| + o(\|\Omega_\delta - u_n\|).
\end{aligned}$$

Thus,

$$\Big\langle I'(u_n), \frac{u}{\|u\|}\Big\rangle \le \frac{\|\Omega_\delta - u_n\|}{n\delta} + \frac{o(\|\Omega_\delta - u_n\|)}{\delta} + \frac{\omega_n(w_\delta) - 1}{\delta}\langle I'(u_n) - I'(\Omega_\delta), u_n - w_\delta\rangle. \tag{4.22}$$

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Since $\Omega_\delta - u_n = (\omega(w_\delta) - 1)u_n - \omega_n(w_\delta)w_\delta$, we have $\|\Omega_\delta - u_n\| \le |\omega(w_\delta) - 1|\|u_n\| + \delta|\omega_n(w_\delta)|$ and, taking into account the facts that

$$\lim_{\delta\to 0} \frac{|\omega_n(w_\delta) - 1|}{\delta} \le \|\omega_n'(0)\|$$

and (u_n) is bounded, passing to the limit in (4.22) as $\delta \to 0$ we deduce that there exists a constant $C_1 = C_1(\delta) > 0$ such that

$$\left\langle I'(u_n), \frac{u}{\|u\|}\right\rangle \le \frac{C_1}{n}(1 + \|\omega_n'(0)\|).$$

Our aim is to show that $\|\omega_n'(0)\|$ is uniformly bounded in n. By (4.18) and using the fact that the sequence (u_n) is bounded, we have

$$\langle \omega_n'(0), v\rangle \le \frac{C_2\|v\|}{(2-p)\|u_n\|^2 - (q-p)\int_V (1-g(x))|u_n|^q \, d\mu} \quad \text{for some } C_2 > 0.$$

Thus, we must show that $(2-p)\|u_n\|^2 - (q-p)\int_V (1-g(x))|u_n|^q \, d\mu > C_3$ for a constant $C_3 > 0$ and n sufficiently large. Arguing by contradiction we assume that there exists a subsequence of (u_n), still denoted by (u_n), such that

$$(2-p)\|u_n\|^2 - (q-p)\int_V (1-g(x))|u_n|^q \, d\mu = o(1). \tag{4.23}$$

Since $u_n \in \mathcal{N}$, by (4.23), we have

$$\int_V f(x)|u_n|^p \, d\mu = \|u_n\|^2 - \int_V (1-g(x))|u_n|^q \, d\mu = \frac{q-2}{2-p}\int_V (1-g(x))|u_n|^q \, d\mu + o(1) \tag{4.24}$$

and

$$\|u_n\| \le \left(\frac{q-p}{q-2}\|f^+\|_{\sup}(2N+3)^p\right)^{1/(2-p)} + o(1). \tag{4.25}$$

Let $K : \mathcal{N} \to \mathbb{R}$ be defined by

$$K(u) = \frac{q-2}{2-p}\left(\frac{2-p}{q-p}\right)^{(q-1)/(q-2)}\left(\frac{\|u\|^{2(q-1)}}{\int_V (1-g(x))|u|^q \, d\mu}\right)^{1/(q-2)} - \int_V f(x)|u|^p \, d\mu.$$

We have $K(u_n) = o(1)$. Indeed, from (4.23) and (4.24) we obtain

$$\begin{aligned} K(u_n) &= \frac{q-2}{2-p}\left(\frac{2-p}{q-p}\right)^{(q-1)/(q-2)}\left(\frac{q-p}{2-p}\right)^{(q-1)/(q-2)}\int_V (1-g(x))|u_n|^q \, d\mu \\ &\quad - \frac{q-2}{2-p}\int_V (1-g(x))|u_n|^q \, d\mu + o(1) \\ &= o(1). \end{aligned}$$

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We show that (u_n) does not converge to 0. Assuming the contrary, the facts that (u_n) converges to 0 and $I \in C^1(H_0^1(V), \mathbb{R})$ imply that $I(u_n)$ converges to $I(0) = 0$, which is a contradiction with (4.20) and $I_1 < 0$. On the other hand, by (4.25) we deduce that

$$\begin{aligned}
K(u_n) &\geq \|u_n\|^p\Big(\frac{q-2}{2-p}\Big(\frac{2-p}{q-p}\Big)^{(q-1)/(q-2)}\Big(\frac{\|u_n\|^{2(q-1)}}{\int_V(1-g(x))|u|^q\,d\mu}\Big)^{1/(q-2)}\|u_n\|^{-p} \\
&\qquad - \|f^+\|_{\sup}(2N+3)^p\Big) \\
&\geq \|u_n\|^p\Big(\frac{q-2}{2-p}\Big(\frac{2-p}{q-p}\Big)^{(q-1)/(q-2)}\Big(\frac{\|u_n\|^{2(q-1)}}{(2N+3)^q\|u_n\|^q}\Big)^{1/(q-2)}\|u_n\|^{-p} \\
&\qquad - \|f^+\|_{\sup}(2N+3)^p\Big) \\
&= \|u_n\|^p\Big(\frac{q-2}{2-p}\Big(\frac{2-p}{q-p}\Big)^{(q-1)/(q-2)}(2N+3)^{-q/(q-2)}\|u_n\|^{1-p} - \|f^+\|_{\sup}(2N+3)^p\Big) \\
&\geq \|u_n\|^p\Big(\frac{q-2}{2-p}\Big(\frac{2-p}{q-p}\Big)^{(q-1)/(q-2)}(2N+3)^{-q/(q-2)} \\
&\quad \times \Big(\frac{q-p}{q-2}\|f^+\|_{\sup}(2N+3)^p\Big)^{(1-p)/(2-p)} - \|f^+\|_{\sup}(2N+3)^p\Big) \\
&> C_4,
\end{aligned}$$

where C_4 is a positive constant, since $\|f^+\|_{\sup} < \overline{M} < M$ and (u_n) does not converge to 0. This inequality contradicts the fact that $K(u_n) = o(1)$. Thus, our supposition is false and consequently we have proved that

$$\Big\langle I'(u_n), \frac{u}{\|u\|}\Big\rangle \leq \frac{C_1}{n}$$

for every n and every $u \in H_0^1(V) \setminus \{0\}$, and this implies that $I'(u_n) \to 0$ in $H^{-1}(V)$.
(ii) The proof is similar to (i), but in this case we use Lemma 4.7(ii) instead of Lemma 4.7(i). □

Lemma 4.10. *For each $f \in C(V)$ with $f^+ \neq 0$ and $\|f^+\|_{\sup} < \overline{M}$, there exists $\widetilde{u} \in \mathcal{N}^+$ that is a minimiser of I. Moreover, $\widetilde{u}$ is a nontrivial weak solution of Problem (1.1) and satisfies*

$$I(\widetilde{u}) \to 0 \quad \text{as } \|f^+\|_{\sup} \to 0.$$

Proof. By Lemma 4.9(i), there exists a sequence $(u_n) \subset \mathcal{N}$ such that

$$\lim_{n\to\infty} I(u_n) = I_1 < 0 \tag{4.26}$$

and

$$I'(u_n) \to 0 \quad \text{in } H^{-1}(V). \tag{4.27}$$

Since I is coercive on $\mathcal{N}$, (u_n) is bounded in $H_0^1(V)$. As $H_0^1(V)$ is a Hilbert space, there exist a subsequence of (u_n), still denoted by (u_n), and $\widetilde{u} \in H_0^1(V)$ such that (u_n)

weakly converges to $\widetilde{u}$ in $H_0^1(V)$, and by the compact embedding (2.3) we infer that (u_n) strongly converges to $\widetilde{u}$ in $(C_0(V), \|\cdot\|_{\sup})$.

Taking into account that (u_n) strongly converges to $\widetilde{u}$ in $(C_0(V), \|\cdot\|_{\sup})$ it follows that

$$\lim_{n\to\infty} \int_V f(x)|u_n|^p \, d\mu = \int_V f(x)|\widetilde{u}|^p \, d\mu$$

and

$$\lim_{n\to\infty} \int_V (1 - g(x))|u_n|^q \, d\mu = \int_V (1 - g(x))|\widetilde{u}|^q \, d\mu.$$

Using (4.26),

$$I(\widetilde{u}) \le \liminf_{n\to\infty} I(u_n) = I_1 < 0 = I(0)$$

and we deduce that $\widetilde{u} \neq 0$. Moreover, by (4.27),

$$0 = \lim_{n\to\infty} \langle I'(u_n), w\rangle = \langle I'(\widetilde{u}), w\rangle, \quad \text{for all } w \in H_0^1(V).$$

Hence $\widetilde{u} \in \mathcal{N}$.

Now we prove that (u_n) strongly converges to $\widetilde{u}$ in $H_0^1(V)$. Assuming by contradiction that the sequence (u_n) does not strongly converge to $\widetilde{u}$ in $H_0^1(V)$ we infer that

$$\|\widetilde{u}\|^2 < \liminf_{n\to\infty} \|u_n\|^2.$$

Since $u_n \in \mathcal{N}$ for every natural n, we have

$$\begin{aligned}\|\widetilde{u}\|^2 &- \int_V f(x)|\widetilde{u}|^p \, d\mu - \int_V (1 - g(x))|\widetilde{u}|^q \, d\mu \\ &< \liminf_{n\to\infty}\Big(\|u_n\|^2 - \int_V f(x)|u_n|^p \, d\mu - \int_V (1 - g(x))|u_n|^q \, d\mu\Big) = 0,\end{aligned}$$

which contradicts the fact that $\widetilde{u} \in \mathcal{N}$. Thus, (u_n) strongly converges to $\widetilde{u}$ in $H_0^1(V)$. We have $\|\widetilde{u}\|^2 = \liminf_{n\to\infty} \|u_n\|^2$. Therefore,

$$I(\widetilde{u}) = \lim_{n\to\infty} I(u_n) = I_1 < 0,$$

which means $\widetilde{u}$ is a minimiser of I. Since $\widetilde{u} \in \mathcal{N}$ and $I(\widetilde{u}) < 0$,

$$\int_V f(x)|\widetilde{u}|^p \, d\mu > \frac{p(q-2)}{2(q-p)}\|\widetilde{u}\|^2 > 0.$$

Now we show that $\widetilde{u} \in \mathcal{N}^+$. Otherwise, if $\widetilde{u} \in \mathcal{N}^-$, we have $I_1 < 0 < I_2 \le I(\widetilde{u}) = I_1$, which is a contradiction. Thus, $\widetilde{u} \in \mathcal{N}^+$.

By Lemma 4.2, we infer that $\widetilde{u}$ is a nontrivial weak solution of Problem (1.1). By (4.8), we obtain

$$\|\widetilde{u}\|^2 < \frac{q-p}{q-2} \int_V f(x)|\widetilde{u}|^p \, d\mu \le \frac{q-p}{q-2}\|f^+\|_{\sup}(2N+3)^p\|\widetilde{u}\|^p$$

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or

$$\|\widetilde{u}\|^{2-p} < \frac{q-p}{q-2}\|f^+\|_{\sup}(2N+3)^p,$$

and this yields that $\|\widetilde{u}\| \to 0$ as $\|f^+\|_{\sup} \to 0$ so $I(\widetilde{u}) \to 0$ as $\|f^+\|_{\sup} \to 0$. Hence, Lemma 4.10 is proved. □

Lemma 4.11. *For each $f \in C(V)$ with $f^+ \neq 0$ and $\|f^+\|_{\sup} < \overline{M}$, there exists $\widehat{u}$ that is a minimiser of I on $\mathcal{N}^-$. Moreover, $\widehat{u}$ is a nontrivial weak solution of Problem* (1.1).

Proof. By Lemma 4.9(ii), there exists a sequence $(v_n) \subset \mathcal{N}^-$ such that

$$\lim_{n\to\infty} I(v_n) = \inf_{u\in\mathcal{N}^-} I(u) = I_2 > 0 \tag{4.28}$$

and

$$I'(v_n) \to 0 \quad \text{in } H^{-1}(V). \tag{4.29}$$

Since I is coercive, (v_n) is bounded in $H_0^1(V)$. As $H_0^1(V)$ is a Hilbert space, there exist a subsequence of (v_n), again denoted by (v_n), and $\widehat{u} \in H_0^1(V)$ such that (v_n) weakly converges to $\widehat{u}$ in $H_0^1(V)$, and by the compact embedding (2.3) we deduce that (v_n) strongly converges to $\widehat{u}$ in $(C_0(V), \|\cdot\|_{\sup})$.

Taking into account that (v_n) strongly converges to $\widehat{u}$ in $(C_0(V), \|\cdot\|_{\sup})$ it follows that

$$\lim_{n\to\infty} \int_V f(x)|v_n|^p\, d\mu = \int_V f(x)|\widehat{u}|^p\, d\mu \tag{4.30}$$

and

$$\lim_{n\to\infty} \int_V (1-g(x))|v_n|^q\, d\mu = \int_V (1-g(x))|\widehat{u}|^q\, d\mu. \tag{4.31}$$

We claim that $\widehat{u} \neq 0$. We first notice that

$$I(u_n) - \frac{1}{2}\langle I'(v_n), v_n\rangle = \left(\frac{1}{2} - \frac{1}{p}\right)\int_V f(x)|v_n|^p\, d\mu + \left(\frac{1}{2} - \frac{1}{q}\right)\int_V (1-g(x))|v_n|^q\, d\mu.$$

Using (4.28), (4.29), (4.30) and (4.31),

$$I_2 = \left(\frac{1}{2} - \frac{1}{p}\right)\int_V f(x)|\widehat{u}|^p\, d\mu + \left(\frac{1}{2} - \frac{1}{q}\right)\int_V (1-g(x))|\widehat{u}|^q\, d\mu.$$

If $\widehat{u} = 0$, it follows that $I_2 = 0$, which is a contradiction since $I_2 > 0$. Our claim is proved.

Moreover by (4.29), we have

$$0 = \lim_{n\to\infty}\langle I'(v_n), w\rangle = \langle I'(\widehat{u}), w\rangle, \quad \text{for all } w \in H_0^1(V).$$

Hence $\widehat{u} \in \mathcal{N}$. Since $v_n \in \mathcal{N}^-$, it follows that

$$\begin{aligned} h''_{\widehat{u}}(1) &= (2-p)\|\widehat{u}\|^2 - (q-p)\int_V (1-g(x))|\widehat{u}|^q \, d\mu \\ &\leq \liminf_{n\to\infty}\Big((2-p)\|v_n\|^2 - (q-p)\int_V (1-g(x))|v_n|^q \, d\mu\Big) \\ &= \liminf_{n\to\infty} h''_{v_n}(1) < 0; \end{aligned}$$

thus $\widehat{u} \in \mathcal{N}^-$.

Now we prove that (v_n) strongly converges to $\widehat{u}$ in $H_0^1(V)$. Otherwise,

$$\|\widehat{u}\|^2 < \liminf_{n\to\infty} \|v_n\|^2.$$

Since $v_n \in \mathcal{N}$ for every natural n, we have

$$\begin{aligned} &\|\widehat{u}\|^2 - \int_V f(x)|\widehat{u}|^p \, d\mu - \int_V (1-g(x))|\widehat{u}|^q \, d\mu \\ &\quad < \liminf_{n\to\infty}\Big(\|v_n\|^2 - \int_V f(x)|v_n|^p \, d\mu - \int_V (1-g(x))|v_n|^q \, d\mu\Big) = 0, \end{aligned}$$

which contradicts the fact that $\widehat{u} \in \mathcal{N}$. Therefore, (v_n) strongly converges to $\widehat{u}$ in $H_0^1(V)$. This implies that

$$I(\widehat{u}) = \lim_{n\to\infty} I(v_n) = I_2 > 0,$$

which means $\widehat{u}$ is a minimiser of I on $\mathcal{N}^-$. By Lemma 4.2, we deduce that $\widehat{u}$ is a nontrivial weak solution of Problem (1.1). This concludes the proof of Lemma 4.11. □

Proof of theorem 3.1 By Lemmas 4.10 and 4.11 we conclude that there exist $\widetilde{u} \in \mathcal{N}^+$ and $\widehat{u} \in \mathcal{N}^-$ such that

$$I(\widetilde{u}) = I_1 < 0 < I_2 = I(\widehat{u}).$$

Thus $\widetilde{u}$ and $\widehat{u}$ are distinct. Therefore $\widetilde{u} \in \mathcal{N}^+$ and $\widehat{u} \in \mathcal{N}^-$ are two distinct nontrivial weak solutions of Problem (1.1). This completes the proof of Theorem 3.1. □

Proof of corollary 3.2 If we take $f(x) = \lambda > 0$ and $g(x) = 0$ for every $x \in V$, Problem (1.1) becomes Problem (3.1). Thus, by Theorem 3.1, we deduce that the conclusion of Corollary 3.2 is valid. □

References

[1] M. T. Barlow and E. A. Perkins, 'Brownian motion on the Sierpinski gasket', *Probab. Theory Related Fields* **79** (1988), 543–624.

[2] B. Breckner, V. Rădulescu and C. Varga, 'Infinitely many solutions for the Dirichlet problem on the Sierpinski gasket', *Anal. Appl.* **9** (2011), 235–248.

[3] I. Ekeland, 'On the variational principle', *J. Math. Anal. Appl.* **17** (1974), 324–353.

[4] K. Falconer, 'Semilinear PDEs on self-similar fractals', *Comm. Math. Phys.* **206** (1999), 235–245.

https://doi.org/10.1017/S000497271100298X Published online by Cambridge University Press

[5] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 2nd edn (John Wiley and Sons, Chichester, 2003).

[6] K. Falconer and J. Hu, 'Nonlinear elliptical equations on the Sierpinski gasket', *J. Math. Anal. Appl.* **240** (1999), 552–573.

[7] J. Kigami, 'A harmonic calculus on the Sierpiński spaces', *Japan J. Appl. Math.* **6** (1989), 259–290.

[8] J. Kigami, 'Harmonic calculus on p.c.f. self-similar sets', *Trans. Amer. Math. Soc.* **335** (1993), 721–755.

[9] J. Kigami, *Analysis on Fractals* (Cambridge University Press, Cambridge, 2001).

[10] S. M. Kozlov, 'Harmonization and homogenization on fractals', *Comm. Math. Phys.* **153** (1993), 339–357.

[11] S. Kusuoka, 'A Diffusion Process on a Fractal', *Probabilistic Methods in Mathematical Physics (Katata/Kyoto, 1985)* (Academic Press, Boston, MA, 1987), pp. 251–274.

[12] W. Sierpiński, 'Sur une courbe dont tout point est un point de ramification', *Comptes Rendus (Paris)* **160** (1915), 302–305.

[13] R. S. Strichartz, *Differential Equations on Factals: A Tutorial* (Princeton University Press, Princeton, NJ, 2006).

DENISA STANCU-DUMITRU, Department of Mathematics, University of Craiova, 200585 Craiova, Romania
e-mail: denisa.stancu@yahoo.com

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