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# A HECKE ACTION ON $G_1T$-MODULES

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*Abstract* We construct an action of the affine Hecke category on the principal block $\mathrm{Rep}_0(G_1T)$ of $G_1T$-modules where $G$ is a connected reductive group over an algebraically closed field of characteristic $p>0$, $T$ a maximal torus of $G$ and $G_1$ the Frobenius kernel of $G$. To define it, we define a new category with a Hecke action which is equivalent to the combinatorial category defined by Andersen-Jantzen-Soergel.

## 1. Introduction

Let $G$ be a connected reductive algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic $p>0$. One of the most important goals in representation theory is to describe the characters of irreducible representations. In the case of rational representations of $G$, Lusztig gave a conjecture which gives the characters of irreducible representations of $G$ in terms of Kazhdan-Lusztig polynomials of the affine Weyl group for $p>h$, where $h$ is the Coxeter number. Thanks to the works of Kazhdan-Lusztig [KL93, KL94a, KL94b], Kashiwara-Tanisaki [KT95, KT96] and Andersen-Jantzen-Soergel [AJS94], this is proved for $p$ large enough. An explicit bound on $p$ is known by Fiebig [Fie12].

However, as Williamson [Wil17] showed, Lustzig's conjecture fails for many $p$. Therefore, we need a new approach for such $p$. Riche-Williamson [RW18] gave such an approach, and now we explain it. Assume that $p>h$. Let $\mathrm{Rep}_0(G)$ be the principal block of the category of rational representations of $G$. For each affine simple reflection $s$, we have the wall-crossing functor $\theta_s\colon \mathrm{Rep}_0(G)\to\mathrm{Rep}_0(G)$. The Grothendieck group of $\mathrm{Rep}_0(G)$ is isomorphic to the anti-spherical quotient of the group algebra of the affine Weyl group. Here, the action of the affine Weyl group on a representation is given by $[M](s+1)=[\theta_s(M)]$ for $M\in\mathrm{Rep}_0(G)$ and a simple affine reflection $s$. Riche-Williamson [RW18] conjectured the existence of a categorification of this anti-spherical quotient. More precisely, they conjectured that there is an action of $\mathcal{D}$ on $\mathrm{Rep}_0(G)$ where $\mathcal{D}$ is

*2020 Mathematics subject classification*: 20G05, 22E47 the diagrammatic Hecke category defined by Elias-Williamson [EW16]. Assuming this conjecture, they proved that the anti-spherical quotient of $\mathcal{D}$ is a graded version of the category of tilting modules in $\mathrm{Rep}_0(G)$. In particular, one can describe the character of indecomposable tilting modules in terms of $p$-Kazhdan-Lusztig polynomials. Recently this description was proved by Achar-Makisumi-Riche-Williamson [AMRW19] when $p > h$, and for any $p$ by Riche-Williamson [RW22]. We note that if $p \geq 2h-2$, then characters for irreducible modules are described by characters of tilting modules [And98]. We also remark that Sobaje [Sob20] gave for all $p$ an algorithm to calculate the characters of irreducible modules by the characters of indecomposable tilting modules.

Achar-Makisumi-Riche-Williamson also proved a big part of the conjecture, but not a full statement. In the case of $G = \mathrm{GL}_n$, the original conjecture is proved by Riche-Williamson [RW18]. Recently, the conjecture is proved by Bezrukavnikov-Riche [BR22] for $p > h$.

In this paper, we consider the $G_1T$-version of this conjecture, where $T \subset G$ is a maximal torus and $G_1$ is the Frobenius kernel of $G$. Namely, we define an action of the category $\mathcal{D}$ on the principal block of $G_1T$-modules.

Next, we state our main theorem. We remark that we have an object $B_s \in \mathcal{D}$ for any affine simple reflection $s$ (see the next subsection for the details). Assume that $p > h$. Let $\mathrm{Rep}_0(G_1T)$ be the principal block of the category of $G_1T$-modules.

**Theorem 1.1** (Theorem 3.31). *The category $\mathcal{D}$ acts on $\mathrm{Rep}_0(G_1T)$, where $B_s \in \mathcal{D}$ acts as the wall-crossing functor for any affine simple reflection $s$.*

Kaneda (private communication) proved this theorem for $\mathrm{GL}_n$ using the arguments of Riche-Williamson [RW18].

Let $X^\vee$ be the cocharacter group of $T$ and set $X^\vee_{\mathbb{K}} = X^\vee \otimes_{\mathbb{Z}} \mathbb{K}$. Let $S = \mathrm{Sym}(X^\vee_{\mathbb{K}})$ be the symmetric algebra of $X^\vee_{\mathbb{K}}$. This is a graded algebra via $\deg(X^\vee_{\mathbb{K}}) = 2$. Andersen-Jantzen-Soergel defined a combinatorial category $\mathcal{K}_{\mathrm{AJS}}$. This category is an $S$-linear category with a grading. We define a category $\mathbb{K} \otimes_S \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS}}$ with the same objects as $\mathcal{K}_{\mathrm{AJS}}$; however, the space of morphisms is defined as $\mathrm{Hom}_{\mathbb{K}\otimes_S \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS}}}(M,N) = \mathbb{K} \otimes_S \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{K}_{\mathrm{AJS}}}(M,N(i))$, where $N(i)$ denotes the grading shift (the upperscript f means forgetting the gradings). Let $\mathrm{Proj}(\mathrm{Rep}_0(G_1T))$ be the category of projective objects in $\mathrm{Rep}_0(G_1T)$. Andersen-Jantzen-Soergel constructed a functor $\mathcal{V}\colon \mathrm{Proj}(\mathrm{Rep}_0(G_1T)) \to \mathbb{K} \otimes_S \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS}}$ and proved that it is fully faithful. They also determined the essential image of $\mathcal{V}$, and using this functor, they proved Lusztig's conjecture for large $p$.

In order to obtain an action of $\mathcal{D}$ on $\mathrm{Rep}_0(G_1T)$, it is sufficient to define an action on $\mathrm{Proj}(\mathrm{Rep}_0(G_1T))$ (see 3.7). Therefore, by the results of Andersen-Jantzen-Soergel, it is sufficient to construct the action of $\mathcal{D}$ on the essential image of $\mathcal{V}$. The main obstructions to do it are the following.

(1) Elias-Williamson defined $\mathcal{D}$ via generators and relations. Since the relations are very complicated, it is hard to check that the action is well-defined.

(2) The category $\mathcal{K}_{\mathrm{AJS},P}$ contains only "local" information. Hence, it is difficult to construct the action directly.

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### 1.1. The category $\mathcal{S}$Bimod

We use the category $\mathcal{S}$Bimod [Abe21] instead of the category $\mathcal{D}$. The category $\mathcal{S}$Bimod is equivalent to the category $\mathcal{D}$. We recall the definition of $\mathcal{S}$Bimod. Let $W_{\mathrm{aff}}$ be the affine Weyl group attached to $G$ and $\mathrm{Frac}(S)$ the field of fractions of $S$. An object $M$ in $\mathcal{S}$Bimod is a graded $S$-bimodule and submodules $M_x^{\mathrm{Frac}(S)} \subset M \otimes_S \mathrm{Frac}(S)$ ($x \in W_{\mathrm{aff}}$) with the property $M \otimes_S \mathrm{Frac}(S) = \bigoplus_{x \in W_{\mathrm{aff}}} M_x^{\mathrm{Frac}(S)}$ and $mf = \overline{x}(f)m$ for $f \in S$ and $m \in M_x^{\mathrm{Frac}(S)}$. Here, $\overline{x}$ is the image of $x$ in the finite Weyl group. For $M, N \in \mathcal{S}$Bimod, we have the tensor product $M \otimes N = M \otimes_S N$ with the decomposition $(M \otimes N) \otimes_S \mathrm{Frac}(S) = \bigoplus_{x \in W_{\mathrm{aff}}} (M \otimes N)_x^{\mathrm{Frac}(S)}$, where $(M \otimes N)_x^{\mathrm{Frac}(S)} = \bigoplus_{yz = x} M_y^{\mathrm{Frac}(S)} \otimes_{\mathrm{Frac}(S)} N_z^{\mathrm{Frac}(S)}$. A homomorphism $M \to N$ is a degree zero $S$-bimodule homomorphism which sends $M_x^{\mathrm{Frac}(S)}$ to $N_x^{\mathrm{Frac}(S)}$ for any $x \in W_{\mathrm{aff}}$.

Let $X$ be the character group of $T$. An alcove is a connected component of $X \otimes_{\mathbb{Z}} \mathbb{R} \setminus \bigcup_t H_t$, where $t$ runs through the affine reflections in $W_{\mathrm{aff}}$, and $H_t$ is the fixed hyperplane of $t$. We fix an alcove $A_0$ and let $S_{\mathrm{aff}}$ be the reflections with respect to the walls of $A_0$. Then, $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is a Coxeter system. For each $s \in S_{\mathrm{aff}}$, put $S^s = \{f \in S \mid s(f) = f\}$. Then, the $S$-bimodule $S \otimes_{S^s} S(1)$ has, when tensored by $\mathrm{Frac}(S)$, a unique decomposition as described above such that $(S \otimes_{S^s} S(1))_w^{\mathrm{Frac}(S)} \neq 0$ only when $w = e, s$. Let $B_s$ be this object. Now $\mathcal{S}$Bimod consists of the objects $M$ which are direct summands of direct sums of objects of the form $B_{s_1} \otimes \cdots \otimes B_{s_l}(n)$ where $s_1, \ldots, s_l \in S_{\mathrm{aff}}$ and $n \in \mathbb{Z}$. It is proved in [Abe21] that the category $\mathcal{S}$Bimod is equivalent to the diagrammatic Hecke category defined by Elias-Williamson. As shown in [EW16, Abe21], this gives a categorification of the Hecke algebra of the affine Weyl group; namely, the split Grothendieck group of $\mathcal{S}$Bimod is isomorphic to the Hecke algebra.

### 1.2. Another combinatorial category

We also give another realization of the category of Andersen-Jantzen-Soergel $\mathcal{K}_{\mathrm{AJS}}$ [AJS94]. As in [Lus80], we use the combinatorics of alcoves to define the category. Let $\mathcal{A}$ be the set of alcoves. We fix a positive system $\Delta^+$ of the root system $\Delta$ of $G$. Then, this defines an order on $\mathcal{A}$[Lus80]. Recall that we have fixed $A_0 \in \mathcal{A}$. The action of $W_{\mathrm{aff}}$ on $X \otimes_{\mathbb{Z}} \mathbb{R}$ induces the action of $W_{\mathrm{aff}}$ on $\mathcal{A}$ such that the map $w \mapsto w(A_0)$ gives a bijection $W_{\mathrm{aff}} \to \mathcal{A}$.

Set $S^\emptyset = S[(\alpha^\vee)^{-1} \mid \alpha \in \Delta]$. We define the category $\widetilde{\mathcal{K}}'$ as follows. An object of $\widetilde{\mathcal{K}}'$ is a graded $S$-bimodule $M$ with a decomposition $S^\emptyset \otimes_S M = \bigoplus_{A \in \mathcal{A}} M_A^\emptyset$, such that $mf = \overline{x}(f)m$ for $m \in M_A^\emptyset$, $f \in S^\emptyset$, $x \in W_{\mathrm{aff}}$ such that $A = x(A_0)$ and $\overline{x}$ is the image of $x$ in the finite Weyl group. A morphism $f\colon M \to N$ is a degree zero $S$-bimodule homomorphism, such that $f(M_A^\emptyset) \subset \bigoplus_{A' \geq A} N_{A'}^\emptyset$. We will also define some subcategories of $\widetilde{\mathcal{K}}'$. Particularly, the category denoted by $\widetilde{\mathcal{K}}_P$ plays an important role in our construction. Since it is technical, we do not say anything about its definitions in the Introduction, but instead refer to Definition 2.16. We only note that, for each $A \in \mathcal{A}$, the module $M_{\{A\}} = (M \cap \bigoplus_{A' \geq A} M_{A'}^\emptyset)/(M \cap \bigoplus_{A' > A} M_{A'}^\emptyset)$ is graded free for $M \in \widetilde{\mathcal{K}}_P$.

We define an action $\mathcal{S}$Bimod on $\widetilde{\mathcal{K}}'$ as follows. Let $B \in \mathcal{S}$Bimod and note that we have a submodule $B_x^\emptyset \subset B \otimes_S S^\emptyset$, such that $B_x^\emptyset \otimes_{S^\emptyset} \mathrm{Frac}(S) = B_x^{\mathrm{Frac}(S)}$. Let $M \in \widetilde{\mathcal{K}}'$.

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Then, we define $M*B$ by $M*B = M\otimes_S B$ as a graded $S$-bimodule and $(M*B)^{\emptyset}_{w(A_0)} = \bigoplus_{x\in W_{\mathrm{aff}}} M^{\emptyset}_{wx^{-1}(A_0)} \otimes_{S^{\emptyset}} B^{\emptyset}_x$ for $w\in W_{\mathrm{aff}}$. We can prove that the above action of $\mathcal{S}$Bimod on $\widetilde{\mathcal{K}}'$ induces a well-defined action also on $\widetilde{\mathcal{K}}_P$ (Proposition 2.24). Therefore, the split Grothendieck group $[\widetilde{\mathcal{K}}_P]$ of $\widetilde{\mathcal{K}}_P$ has a structure of $[\mathcal{S}$Bimod$]$-module defined by $[M][B] = [M*B]$. Hence, $[\widetilde{\mathcal{K}}_P]$ is a module of the Hecke algebra. This category satisfies the following.

**Theorem 1.2** (Theorem 2.35, 2.40). *We have the following.*

(1) *For each $A\in\mathcal{A}$, we have an indecomposable module $Q(A)\in\widetilde{\mathcal{K}}_P$, such that $Q(A)_{\{A\}}\simeq S$ and $Q(A)_{\{A'\}}\neq 0$ implies $A'\geq A$.*

(2) *Any object in $\widetilde{\mathcal{K}}_P$ is isomorphic to a direct sum of $Q(A)(n)$ for $A\in\mathcal{A}$ and $n\in\mathbb{Z}$.*

(3) *The split Grothendieck group $[\widetilde{\mathcal{K}}_P]$ is isomorphic to a certain submodule $\mathcal{P}^0$ of the periodic Hecke module (the submodule was introduced in [Lus80]).*

### 1.3. A relation with a work of Fiebig-Lanini

Fiebig-Lanini [FL15] had a similar work (earlier than this work) and defined a certain category. Logically, results in this paper do not depend on their work. However, in the proofs in this paper, we borrow many ideas from their work. Moreover, in subsection 2.9, we prove that our category $\widetilde{\mathcal{K}}_P$ is equivalent to the category of Fiebig-Lanini. The author thinks it is possible to establish the theory on top of the theory of Fiebig-Lanini, but the existence of a Hecke action does not easily follow from their theory.

### 1.4. Relations with representation theory

The category $\widetilde{\mathcal{K}}_P$ is not the category we really need. We modify this category as follows. Objects in $\mathcal{K}_P$ are the same as those in $\widetilde{\mathcal{K}}_P$, and the space of homomorphisms is defined by

$$\mathrm{Hom}_{\mathcal{K}_P}(M,N) = \mathrm{Hom}_{\widetilde{\mathcal{K}}_P}(M,N)/\{\varphi\colon M\to N \mid \varphi(M^{\emptyset}_A)\subset \bigoplus_{A'>A} N^{\emptyset}_{A'}\}.$$

We prove that the action of $B\in\mathcal{S}$Bimod on $\mathcal{K}_P$ is well-defined.

**Theorem 1.3** (Proposition 3.3, Theorem 3.9). *We have the following.*

(1) *The object $Q(A)$ is also indecomposable as an object of $\mathcal{K}_P$.*

(2) *We have $[\mathcal{K}_P]\simeq[\widetilde{\mathcal{K}}_P]$. Hence $[\mathcal{K}_P]$ is also isomorphic to $\mathcal{P}^0$.*

We also define a functor $\mathcal{F}\colon\mathcal{K}_P\to\mathcal{K}_{\mathrm{AJS}}$. Recall that we have a wall-crossing functor $\vartheta_s\colon\mathcal{K}_{\mathrm{AJS}}\to\mathcal{K}_{\mathrm{AJS}}$ for each $s\in S_{\mathrm{aff}}$, see [Fie11, 5.3].

**Theorem 1.4** (Proposition 3.14, 3.26). *Let $M\in\mathcal{K}_P$. We have the following.*

(1) *We have $\mathcal{F}(M*B_s)\simeq\vartheta_s(\mathcal{F}(M))$ for each $s\in S_{\mathrm{aff}}$.*

(2) *The functor $\mathcal{F}$ is fully faithful.*

Let $\mathcal{K}_{\mathrm{AJS},P}$ be the essential image of $\mathcal{F}$. We define $\mathbb{K}\otimes_S\mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}$ and $\mathbb{K}\otimes_S\mathcal{K}^{\mathrm{f}}_P$ in the same way as $\mathbb{K}\otimes_S\mathcal{K}^{\mathrm{f}}_{\mathrm{AJS}}$. One of the main results in [AJS94] says that $\mathbb{K}\otimes_S\mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}\simeq$

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Proj($\mathrm{Rep}_0(G_1T)$) (see 3.7). Since the action of $\mathcal{S}$Bimod on $\mathcal{K}_P \simeq \mathcal{K}_{\mathrm{AJS},P}$ gives an action on $\mathbb{K}\otimes_S \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}$, we now get the action of $\mathcal{S}$Bimod on Proj($\mathrm{Rep}_0(G_1T)$). We can extend this action to $\mathrm{Rep}_0(G_1T)$ (see 3.7).

Let $A_0$ be the alcove containing $\rho/p$ where $\rho$ is the half sum of positive roots. We have an equivalence $\mathbb{K}\otimes_S \mathcal{K}^{\mathrm{f}}_P \simeq \mathbb{K}\otimes_S \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P} \simeq \mathrm{Proj}(\mathrm{Rep}_0(G_1T))$ and $Q(A)$ corresponds to $P(\lambda_A)$, where $\lambda_{w(A_0)} = pw(\rho/p)-\rho$ for $w\in W_{\mathrm{aff}}$ and $P(\lambda_A)$ is the projective cover of the irreducible representation with highest weight $\lambda_A$. Let $Z(\mu)\in \mathrm{Rep}(G_1T)$ be the baby Verma module with highest weight $\mu$ and $(P(\lambda):Z(\mu))$ the multiplicity of $Z(\mu)$ in a Verma flag of $P(\lambda)$. By the constructions, we have the following.

**Theorem 1.5** (Corollary 3.36). *The multiplicity $(P(\lambda_A):Z(\lambda_{A'}))$ is equal to the rank of $Q(A)_{\{A'\}}$.*

In 3.9, we discuss Lusztig's conjecture on irreducible characters of rational representations. We give a proof of the conjecture based on the theory developed in this paper.

## 2. Our combinatorial category

We shall use a different notation than the Introduction. In particular, we do not fix the alcove $A_0$. So, we distinguish two actions (from the right and left) of $W_{\mathrm{aff}}$ on $\mathcal{A}$ as in [Lus80]. We will also work in a more general situation than in the Introduction. Forget every notation and the assumptions from the Introduction. Notation used in the main body of this paper will be explained.

### 2.1. Notation

Let $(X,\Delta,X^\vee,\Delta^\vee)$ be a root datum. Let $\mathcal{A}$ the set of alcoves, namely the set of connected components of $X_\mathbb{R}\setminus\bigcup_{\alpha\in\Delta,n\in\mathbb{Z}}\{\lambda\in X_\mathbb{R}\mid\langle\lambda,\alpha^\vee\rangle=n\}$ where $X_\mathbb{R}=X\otimes_\mathbb{Z}\mathbb{R}$. Let $W_{\mathrm{f}}$ be the finite Weyl group and $W'_{\mathrm{aff}}=W_{\mathrm{f}}\ltimes\mathbb{Z}\Delta$ the affine Weyl group with the natural surjective homomorphism $W'_{\mathrm{aff}}\to W_{\mathrm{f}}$. For each $\alpha\in\Delta$ and $n\in\mathbb{Z}$, let $s_{\alpha,n}\colon X\to X$ be the reflection with respect to $\{\lambda\in X_\mathbb{R}\mid\langle\lambda,\alpha^\vee\rangle=n\}$. As in [Lus80], let $S_{\mathrm{aff}}$ be the set of $W'_{\mathrm{aff}}$-orbits on the set of faces. Then, for each $s\in S_{\mathrm{aff}}$ and $A\in\mathcal{A}$, we denote $As$ as the alcove $\neq A$, which has a common face of type $s$ with $A$. The subgroup of $\mathrm{Aut}(\mathcal{A})$ (permutations of elements in $\mathcal{A}$) generated by $S_{\mathrm{aff}}$ is denoted by $W_{\mathrm{aff}}$. Then, $(W_{\mathrm{aff}},S_{\mathrm{aff}})$ is a Coxeter system isomorphic to the affine Weyl group. The Bruhat order on $W_{\mathrm{aff}}$ is denoted by $\geq$. We shall consider the right action of $W_{\mathrm{aff}}$ on $\mathcal{A}$.

We give related notation and also some facts. If we fix an alcove $A_0$, then $W'_{\mathrm{aff}}\simeq\mathcal{A}$ via $w\mapsto wA_0$ and $W'_{\mathrm{aff}}$ acts on $\mathcal{A}$ by $(w(A_0))x=wx(A_0)$. This gives an isomorphism $W'_{\mathrm{aff}}\simeq W_{\mathrm{aff}}$. The facts stated below are obvious from this description.

Let $\Lambda$ be the set of maps $\lambda\colon\mathcal{A}\to X$ such that $\lambda(xA)=\overline{x}\lambda(A)$ for any $x\in W'_{\mathrm{aff}}$ and $A\in\mathcal{A}$, where $\overline{x}\in W_{\mathrm{f}}$ is the image of $x$. We write $\lambda_A=\lambda(A)$ for $\lambda\in\Lambda$ and $A\in\mathcal{A}$. For each $A\in\mathcal{A}$, $\lambda\mapsto\lambda_A$ gives an isomorphism $\Lambda\xrightarrow{\sim}X$, and the inverse of this isomorphism is denoted by $\nu\mapsto\nu^A$. The group $W_{\mathrm{aff}}$ acts on $\Lambda$ by $(x(\lambda))(A)=\lambda(Ax)$.

Let $\Lambda_{\mathrm{aff}}$ be the set of $\lambda\in\Lambda$ such that $\lambda_A\in\mathbb{Z}\Delta$ for any, or equivalently, some $A\in\mathcal{A}$. For $\lambda\in\Lambda_{\mathrm{aff}}$ and $A\in\mathcal{A}$, we define $A\lambda=A+\lambda_A$. Then, for $\lambda_1,\lambda_2\in\Lambda_{\mathrm{aff}}$, $(A\lambda_1)\lambda_2=(A+(\lambda_1)_A)\lambda_2=A+(\lambda_1)_A+(\lambda_2)_{A+(\lambda_1)_A}$. Since elements in $\Lambda$ are constant on $\mathbb{Z}\Delta$-orbits,

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we have $(\lambda_2)_{A+(\lambda_1)_A} = (\lambda_2)_A$. Hence, $(A\lambda_1)\lambda_2 = A + (\lambda_1+\lambda_2)_A$; namely, $(A,\lambda)\mapsto A\lambda$ gives an action of $\Lambda_{\mathrm{aff}}$ on $\mathcal{A}$. Therefore, we get $\Lambda_{\mathrm{aff}} \hookrightarrow \mathrm{Aut}(\mathcal{A})$ and the image is contained in $W_{\mathrm{aff}}$. Thus, we may regard $\Lambda_{\mathrm{aff}}$ as a subgroup of $W_{\mathrm{aff}}$.

Let $\lambda\in\Lambda$ and $A,A'\in\mathcal{A}$ and assume that $A,A'$ are in the same $\Lambda_{\mathrm{aff}}$-orbit. Namely, there exists $\mu\in\Lambda_{\mathrm{aff}}$ such that $A = A'\mu = A' + \mu_{A'}$. Since elements in $\Lambda$ are constant on $\mathbb{Z}\Delta$-orbits, we get $\lambda_{A'} = \lambda_A$. Namely, the isomorphism $\lambda\mapsto\lambda_A$ only depends on $\Lambda_{\mathrm{aff}}$-orbit in $\mathcal{A}$. Hence, we also write the isomorphism by $\lambda\mapsto\lambda_\Omega$ where $\Omega\in\mathcal{A}/\Lambda_{\mathrm{aff}}$. The inverse is denoted by $\lambda\mapsto\lambda^\Omega$. The $\Lambda_{\mathrm{aff}}$-orbit through $A$ is equal to $\{A+\lambda \mid \lambda\in\mathbb{Z}\Delta\}$. Let $A+\mathbb{Z}\Delta$ be this set.

The following lemma is obvious from the definitions.

**Lemma 2.1.** *Let $\lambda\in\Lambda$, $\nu\in X$, $x\in W_{\mathrm{aff}}$, $y\in W'_{\mathrm{aff}}$ and $A\in\mathcal{A}$.*

(1) $x(\lambda)_A = \lambda_{Ax}$.
(2) $y(\lambda_A) = \lambda_{yA}$.
(3) $\nu^A = x(\nu^{Ax})$.
(4) $\nu^A = y(\nu)^{yA}$.

Fix a positive system $\Delta^+\subset\Delta$. Let $\alpha\in\Delta^+$ and $n\in\mathbb{Z}$. We say $A\le s_{\alpha,n}(A)$ if, for all $a\in A$, we have $\langle a,\alpha^\vee\rangle < n$. The generic Bruhat order $\le$ on $\mathcal{A}$ is the partial order generated by the relations $A\le s_{\alpha,n}(A)$. The following lemma is obvious from the definition.

**Lemma 2.2.** *Let $A\in\mathcal{A}$, $w\in W'_{\mathrm{aff}}$ and $a$ is in the closure of $A$. If $A\le w(A)$, then $w(a)-a\in\mathbb{R}_{\ge 0}\Delta^+$.*

**Lemma 2.3.** *Let $A,A'\in\mathcal{A}$ such that $A+\nu = A'$ for $\nu\in\mathbb{Z}\Delta$. Then, $A\le A'$ if and only if $\nu\in\mathbb{Z}_{\ge0}\Delta^+$.*

**Proof.** We assume $\nu\in\mathbb{Z}_{\ge0}\Delta^+$ and prove that $A\le A'$. We may assume $\nu=\alpha\in\Delta^+$. Take $n\in\mathbb{Z}$ such that $n-1<\langle a,\alpha^\vee\rangle<n$ for any $a\in A$. For $a\in A$, we have $\langle s_{\alpha,n}(a),\alpha^\vee\rangle = \langle a-(\langle a,\alpha^\vee\rangle - n)\alpha,\alpha^\vee\rangle = 2n-\langle a,\alpha^\vee\rangle$. Hence, $n<\langle s_{\alpha,n}(a),\alpha^\vee\rangle<n+1$. Therefore, $A\le s_{\alpha,n}(A)\le s_{\alpha,n+1}s_{\alpha,n}(A) = A+\alpha$.

However, assume that $A\le A'$. Take $a\in A$. Then by Lemma 2.2, we have $(a+\nu)-a\in\mathbb{R}_{\ge0}\Delta^+$. Hence, $\nu\in\mathbb{R}_{\ge0}\Delta^+$. Since $\nu\in\mathbb{Z}\Delta$, we get $\nu\in\mathbb{Z}_{\ge0}\Delta^+$. □

A subset $I\subset\mathcal{A}$ is called open (resp. closed) if $A\in I$, $A'\le A$ (resp. $A'\ge A$), which implies $A'\in I$. This defines a topology on $\mathcal{A}$. The following lemma is an immediate result of the previous lemma, and it plays an important role throughout this paper.

**Lemma 2.4.** *For each $\Omega\in\mathcal{A}/\Lambda_{\mathrm{aff}}$ and $x\in W_{\mathrm{aff}}$, the map $x\colon\Omega\to\Omega x$ preserves the order.*

For $A,A'\in\mathcal{A}$, set $[A,A'] = \{A''\in\mathcal{A}\mid A\le A''\le A'\}$. For $\alpha\in\Delta^+$ and $A\in\mathcal{A}$, take $n\in\mathbb{Z}$ such that $n-1<\langle a,\alpha^\vee\rangle<n$ for all $a\in A$ and define $\alpha\uparrow A = s_{\alpha,n}(A)$. By the definition, $A\le\alpha\uparrow A$. We define $\alpha\downarrow A$ as the unique element such that $\alpha\uparrow(\alpha\downarrow A) = A$.

In this paper, graded module (resp. ring) means $\mathbb{Z}$-graded module (resp. ring). Let $M=\bigoplus_i M^i$ be a graded module. For $k\in\mathbb{Z}$, we define $M(k)$ by $M(k)^i = M^{i+k}$. For a graded ring $S$, a graded $S$-module $M$ is called graded free if it is isomorphic to $\bigoplus_i S(n_i)$

where $n_1,\dots,n_r \in \mathbb{Z}$ (in this paper, graded free means graded free of finite rank). We set $\mathrm{grk}(M) = \sum_i v^{n_i} \in \mathbb{Z}[v,v^{-1}]$, where $v$ is an indeterminate.

### 2.2. The categories

Fix a Noetherian integral domain $\mathbb{K}$ (in the Introduction, it was an algebraically closed field. Since our arguments work with a Noetherian integral domain, we assume $\mathbb{K}$ is a Noetherian integral domain. Later we will add more assumptions). We define $\Lambda^\vee$ using $X^\vee$ exactly in the same way as we defined $\Lambda$ using $X$. As $\Lambda$, $W_{\mathrm{aff}}$ acts on $\Lambda^\vee$. We put $\Lambda^\vee_{\mathbb{K}} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{K}$, $X^\vee_{\mathbb{K}} = X^\vee \otimes_{\mathbb{Z}} \mathbb{K}$ and $R = \mathrm{Sym}(\Lambda^\vee_{\mathbb{K}})$. The algebra $R$ is equipped with a grading such that $\deg(\Lambda^\vee_{\mathbb{K}}) = 2$. As for the case of $\Lambda$ and $X$, for $f \in \Lambda^\vee$, we put $f_A = f(A)$. Then, $f \mapsto f_A$ gives an isomorphism $\Lambda^\vee \to X^\vee$ and this induces an isomorphism $R \to S$ for which we also write $f \mapsto f_A$. The inverse of this map is denoted by $g \mapsto g^A$.

**Assumption 2.5.** In the rest of this section, we assume the following.

(1) We have $2 \in \mathbb{K}^\times$, and any $\alpha^\vee \neq \beta^\vee \in (\Delta^\vee)^+$ are linearly independent in $X^\vee_{\mathbb{K}/\mathfrak{m}}$ for any maximal ideal $\mathfrak{m} \subset \mathbb{K}$. This is the GKM-property of the moment graph attached to the finite Weyl group [Fie11, 9.1].

(2) The torsion primes of the root system $(X^\vee,\Delta^\vee,X,\Delta)$ [JMW14, Definition 2.43] are invertible in $\mathbb{K}$.

**Lemma 2.6.** *The representation $X^\vee_{\mathbb{K}}$ of $W_{\mathrm{f}}$ is faithful.*

**Proof.** If $w \in W_{\mathrm{f}}$ fixes any element in $X^\vee_{\mathbb{K}}$, it fixes any image of $\alpha \in \Delta$. By the assumption, $\Delta^\vee \to X^\vee_{\mathbb{K}}$ is injective. Therefore, $w$ fixes any coroot. Hence, $w$ is identity. □

The image of $\alpha^\vee \in \Delta^\vee$ in $X^\vee_{\mathbb{K}}$ is denoted by the same letter. We also put $S = \mathrm{Sym}(X^\vee_{\mathbb{K}})$. We give a grading to $S$ via $\deg(X^\vee_{\mathbb{K}}) = 2$. Set $S^\emptyset = S[(\alpha^\vee)^{-1} \mid \alpha \in \Delta]$. For an $S$-module $M$, set $M^\emptyset = S^\emptyset \otimes_S M$. If $M$ is an $S$-algebra, then $M^\emptyset$ is an $S^\emptyset$-algebra. Let $S_0$ be a flat commutative graded $S$-algebra. If $M$ is an $S_0$-module, then $M^\emptyset \simeq S_0^\emptyset \otimes_{S_0} M$ is an $S_0^\emptyset$-module. We consider the category $\widetilde{\mathcal{K}}'(S_0)$ consisting of $M = (M,\{M_A^\emptyset\}_{A\in\mathcal{A}})$ such that

- $M$ is a graded $(S_0,R)$-bimodule which is finitely generated torsion-free as a left $S_0$-module.
- $M_A^\emptyset$ is an $(S_0^\emptyset,R)$-bimodule such that $mf = f_A m$ for any $m \in M_A^\emptyset$ and $f \in R$.
- $M^\emptyset = \bigoplus_{A\in\mathcal{A}} M_A^\emptyset$.

A morphism $M \to N$ in $\widetilde{\mathcal{K}}'(S_0)$ is an $(S_0,R)$-bimodule $\varphi$ homomorphism of degree zero such that

$$\varphi(M_A^\emptyset) \subset \bigoplus_{A'\geq A} N_{A'}^\emptyset$$

for any $A \in \mathcal{A}$. We put $\mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}'(S_0)}(M,N) = \bigoplus_i \mathrm{Hom}_{\widetilde{\mathcal{K}}'(S_0)}(M,N(i))$. This is a graded $(S_0,R)$-bimodule. For $M \in \widetilde{\mathcal{K}}'(S_0)$, we put $\mathrm{supp}_{\mathcal{A}}(M) = \{A \in \mathcal{A} \mid M_A^\emptyset \neq 0\}$.

**Remark 2.7.** Let $\Omega \in \mathcal{A}/\Lambda_{\mathrm{aff}}$. For any $m \in \bigoplus_{A\in\Omega} M_A^\emptyset$ and $f \in R$, we have $mf = f_\Omega m$. The action of $W'_{\mathrm{aff}}$ on $\mathcal{A}/\Lambda_{\mathrm{aff}}$ factors through $W'_{\mathrm{aff}} \to W_{\mathrm{f}}$, and $W_{\mathrm{f}}$ acts on $\mathcal{A}/\Lambda_{\mathrm{aff}}$

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simply transitively. We have $M^\emptyset = \bigoplus_{w\in W_{\mathrm{f}}}(\bigoplus_{A\in w(\Omega)} M_A^\emptyset)$ and for $m \in \bigoplus_{A\in w(\Omega)} M_A^\emptyset$, $mf = w(f_\Omega)m$. Therefore, the decomposition of $M^\emptyset$ into $\bigoplus_{A\in w(\Omega)} M_A^\emptyset$ is determined by the $(S_0,R)$-bimodule structure. Hence, any $(S_0,R)$-bimodule homomorphism $M \to N$ sends $\bigoplus_{A\in\Omega} M_A^\emptyset$ to $\bigoplus_{A\in\Omega} N_A^\emptyset$. We will often use this fact.

**Remark 2.8.** Here, we do not assume that a morphism $M \to N$ in $\widetilde{\mathcal{K}}'(S_0)$ sends $M_A^\emptyset$ to $N_A^\emptyset$.

For each closed subset $I \subset \mathcal{A}$, we define $M_I = M \cap \bigoplus_{A\in I} M_A^\emptyset$. Set

$$(M_I)_A^\emptyset = \begin{cases} M_A^\emptyset & (A \in I), \\ 0 & (A \notin I). \end{cases}$$

By the following lemma, $M_I \in \widetilde{\mathcal{K}}'(S_0)$ and therefore, $M \mapsto M_I$ is an endofunctor of $\widetilde{\mathcal{K}}'(S_0)$. We have a natural monomorphism $M_I \to M$ in $\widetilde{\mathcal{K}}'(S_0)$.

**Lemma 2.9.** *The module $M_I$ is an $(R_0,S)$-submodule of $M$, and we have*

$$(M_I)^\emptyset = \bigoplus_{A\in I} M_A^\emptyset.$$

*We also have $M_{I_1\cap I_2} = M_{I_1} \cap M_{I_2}$ for any closed subsets $I_1, I_2 \subset \mathcal{A}$.*

**Proof.** The first part is obvious, and for the second part, the left-hand side is contained in the right-hand side. Take $m$ from the right-hand side and let $f \in S$ such that $fm \in M$. Then, we have $fm \in M_I$, and $m$ is in the left-hand side. The last assertion is obvious. □

If $S_0'$ is a commutative flat graded $S_0$-algebra, then for $M \in \widetilde{\mathcal{K}}'(S_0)$, the $(S_0',R)$-bimodule $S_0' \otimes_{S_0} M$ has a decomposition $(S_0' \otimes_{S_0} M)^\emptyset \simeq \bigoplus_{A\in\mathcal{A}}((S_0')^\emptyset \otimes_{S_0^\emptyset} M_A^\emptyset)$, and this decomposition gives a structure of an object in $\widetilde{\mathcal{K}}'(S_0')$. It is easy to see that $M \mapsto S_0' \otimes_{S_0} M$ is a functor $\widetilde{\mathcal{K}}'(S_0) \to \widetilde{\mathcal{K}}'(S_0')$.

For each $\alpha \in \Delta$, set $W'_{\alpha,\mathrm{aff}} = \{1, s_\alpha\} \ltimes \mathbb{Z}\alpha \subset W'_{\mathrm{aff}}$. We also put $S^\alpha = S[(\beta^\vee)^{-1} \mid \beta \in \Delta \setminus \{\pm\alpha\}]$ and $M^\alpha = S^\alpha \otimes_S M$ for any left $S$-module $M$. Again, if $M$ is an $S$-algebra then $M^\alpha$ is an $S^\alpha$-algebra. If $M \in \widetilde{\mathcal{K}}'(S_0)$, then $M^\alpha \in \widetilde{\mathcal{K}}'(S_0^\alpha)$ as mentioned above. Note that, from our assumption, $\bigcap_{\alpha\in\Delta^+} S^\alpha = S$ [AJS94, 9.1 Lemma]. We say $M \in \widetilde{\mathcal{K}}(S_0)$ if $M \in \widetilde{\mathcal{K}}'(S_0)$ and satisfies the following two conditions which are taken from [FL15]. These are important properties in our arguments.

(S) $M_{I_1\cup I_2} = M_{I_1} + M_{I_2}$ for any two closed subsets $I_1, I_2$.

(LE) For any $\alpha \in \Delta^+$, there exist $M^{(\Omega)} \in \widetilde{\mathcal{K}}'(S_0^\alpha)$ for all $\Omega \in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}$ with an injective morphism $M^{(\Omega)} \hookrightarrow M^\alpha$ in $\widetilde{\mathcal{K}}'(S_0^\alpha)$ such that $\mathrm{supp}_{\mathcal{A}} M^{(\Omega)} \subset \Omega$ and the induced morphism $\bigoplus_{\Omega\in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}} M^{(\Omega)} \to M^\alpha$ is an isomorphism in $\widetilde{\mathcal{K}}'(S_0^\alpha)$.

(S) stands for "sheaf" and (LE) stands for "local extension condition" [FL15, Definition 5.4].

Let $M \in \widetilde{\mathcal{K}}'(S_0)$. If $M^\alpha = \bigoplus_{\Omega\in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}}(\bigoplus_{A\in\Omega} M_A^\emptyset \cap M^\alpha)$ for any $\alpha \in \Delta$, $M$ satisfies (LE). The converse is not true, in general. For example, assume $\#\Delta > 1$. Fix $\alpha \in \Delta^+$.

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Take $\beta\in\Delta^+\setminus\{\alpha\}$ and $A\in\mathcal{A}$. Define $N\in\widetilde{\mathcal{K}}'(S)$ by $N=\{(f,g)\mid f,g\in S, f\equiv g(\mathrm{mod}\ \alpha)\}$, $N_A^{\emptyset}=S^{\emptyset}\oplus 0$, $N_{A-\beta}^{\emptyset}=0\oplus S^{\emptyset}$ and $N_{A'}^{\emptyset}=0$ for any $A'\in\mathcal{A}\setminus\{A,A-\beta\}$ (these determine the right $R$-action on $N$ uniquely). Then, it is easy to see that $N^{\alpha}\neq\bigoplus_{\Omega\in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}}(\bigoplus_{A\in\Omega}N_A^{\emptyset}\cap N^{\alpha})$. Define $N^{(\Omega)}\in\widetilde{\mathcal{K}}'(S)$ for $\Omega\in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}$ as follows: $N^{(W'_{\alpha,\mathrm{aff}}A)}=S^{\alpha}$, $(N^{(W'_{\alpha,\mathrm{aff}}A)})_A^{\emptyset}=S^{\emptyset}$, $(N^{(W'_{\alpha,\mathrm{aff}}A)})_{A'}^{\emptyset}=0$ for $A'\in\mathcal{A}\setminus\{A\}$, $N^{(W'_{\alpha,\mathrm{aff}}(A-\beta))}=S^{\alpha}$, $(N^{(W'_{\alpha,\mathrm{aff}}(A-\beta))})_{A-\beta}^{\emptyset}=S^{\emptyset}$, $(N^{(W'_{\alpha,\mathrm{aff}}(A-\beta))})_{A'}^{\emptyset}=S^{\emptyset}$ for $A'\in\mathcal{A}\setminus\{A-\beta\}$ and $N^{(\Omega)}=0$ for $\Omega\neq W'_{\alpha,\mathrm{aff}}A, W'_{\alpha,\mathrm{aff}}(A-\beta)$. Then, we have $\mathrm{supp}_{\mathcal{A}}N^{(\Omega)}\subset\Omega$. We define $N^{(W'_{\alpha,\mathrm{aff}}A)}\to N^{\alpha}$ (resp. $N^{(W'_{\alpha,\mathrm{aff}}(A-\beta))}\to N^{\alpha}$) by $f\mapsto(\alpha f,0)$ (resp. $f\mapsto(f,f)$). Then, $\bigoplus_{\Omega\in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}}N^{(\Omega)}\to N^{\alpha}$ is an isomorphism. We can also easily verify that $N^{\gamma}=\bigoplus_{\Omega\in W'_{\gamma,\mathrm{aff}}\backslash\mathcal{A}}(\bigoplus_{A\in\Omega}N_A^{\emptyset}\cap N^{\gamma})$ for any $\gamma\in\Delta^+\setminus\{\alpha\}$. Hence, $N$ satisfies (LE).

We have the following. $M$ satisfies (LE) if and only if for any $\alpha\in\Delta$, there exists $N\in\widetilde{\mathcal{K}}'(S_0^{\alpha})$ which is isomorphic to $M^{\alpha}$ and satisfies $N=\bigoplus_{\Omega\in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}}(\bigoplus_{A\in\Omega}N_A^{\emptyset}\cap N)$.

**Lemma 2.10.** *Let $M\in\widetilde{\mathcal{K}}'(S_0)$, $\alpha\in\Delta$ and $A\in\mathcal{A}$. Assume that* $\mathrm{supp}_{\mathcal{A}}(M)\subset W'_{\alpha,\mathrm{aff}}A$. *Then, $M$ satisfies (S). In particular, if $M$ satisfies (LE), then $M^{\alpha}$ satisfies (S).*

**Proof.** Set $\Omega=W'_{\alpha,\mathrm{aff}}A$ and let $I_1,I_2\subset\mathcal{A}$ be closed subsets. We have $\Omega=\{A,\alpha\uparrow A,\alpha\uparrow(\alpha\uparrow A),\dots\}\cup\{\alpha\downarrow A,\alpha\downarrow(\alpha\downarrow A),\dots\}$ and $\Omega$ is a totally ordered subset of $\mathcal{A}$. Since $\Omega$ is totally ordered, $I_1\cap\Omega\subset I_2\cap\Omega$ or $I_2\cap\Omega\subset I_1\cap\Omega$. We may assume $I_1\cap\Omega\subset I_2\cap\Omega$. We can take closed subsets $I_1'$ and $I_2'$ such that $I_1'\subset I_2'$, $I_1'\cap\Omega=I_1\cap\Omega$ and $I_2'\cap\Omega=I_2\cap\Omega$. Then, we have $M_{I_1'}=M_{I_1}$, $M_{I_2'}=M_{I_2}$ and $M_{I_1'\cup I_2'}=M_{I_1\cup I_2}$. Hence, we may assume $I_1=I_1'$ and $I_2=I_2'$. In this case, (S) obviously holds. □

Let $K\subset\mathcal{A}$ be a locally closed subset; namely, $K$ is the intersection of a closed subset $I$ with an open subset $J$. It is easy to see that, if $M\in\widetilde{\mathcal{K}}'(S_0)$ satisfies (S), then $M_I/M_{I\setminus J}\simeq M_{I'}/M_{I'\setminus J'}$ naturally for closed subsets $I,I'$ and open subsets $J,J'$ such that $K=I\cap J=I'\cap J'$. We define $M_K=M_I/M_{I\setminus J}$ for $M\in\widetilde{\mathcal{K}}(S_0)$. By Lemma 2.9, we have

$$\bigoplus_{A\in K}M_A^{\emptyset}\xrightarrow{\sim}M_K^{\emptyset}.$$

By putting $(M_K)_A^{\emptyset}$ equal to the image of $M_A^{\emptyset}$ in $M_K^{\emptyset}$ by this isomorphism, we have an object $M_K$ of $\widetilde{\mathcal{K}}'(S_0)$. The following lemma is obvious.

**Lemma 2.11.** *We have* $\mathrm{supp}_{\mathcal{A}}(M_K)=\mathrm{supp}_{\mathcal{A}}(M)\cap K$ *for any locally closed subset $K\subset\mathcal{A}$.*

**Lemma 2.12.** *Let $K_1,K_2\subset\mathcal{A}$ be locally closed subsets. If $M\in\widetilde{\mathcal{K}}(S_0)$, then $(M_{K_1})_{K_2}\simeq M_{K_1\cap K_2}$*

**Proof.** The proof is divided into 4 steps.

(1) Assume that both $K_1,K_2$ are closed. Then, the lemma follows from the definitions.

(2) Assume that $K_1$ is open and $K_2$ is closed. Set $I_1=\mathcal{A}\setminus K_1$. Then, we have

$$(M_{K_1})_{K_2}=M/M_{I_1}\cap\bigoplus_{A\in K_2}(M/M_{I_1})_A^{\emptyset}.$$

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Note that $M_{K_2}/(M_{K_2} \cap M_{I_1}) = M_{K_2}/M_{K_2 \cap I_1} = M_{K_1 \cap K_2}$. There is a canonical embedding from $M_{K_2}/(M_{K_2} \cap M_{I_1})$ to $(M_{K_1})_{K_2}$. Let $m \in M$ such that $m + M_{I_1} \in \bigoplus_{A \in K_2}(M/M_{I_1})_A^{\emptyset}$. Then, $M_A^{\emptyset}$-component $m_A$ of $m$ is 0 for $A \notin I_1 \cup K_2$. Hence, $m \in M_{I_1 \cup K_2} = M_{I_1} + M_{K_2}$. Therefore, the canonical embedding is surjective. We get the lemma.

(3) Assume that $K_2$ is closed. Take a closed subset $I_1$ and an open subset $J_1$ such that $K_1 = I_1 \cap J_1$. Then, by (2), $(M_{J_1})_{I_1} \simeq M_{K_1}$. Hence, $(M_{K_1})_{K_2} \simeq ((M_{J_1})_{I_1})_{K_2} = (M_{J_1})_{I_1 \cap K_2}$ by (1). This is isomorphic to $M_{J_1 \cap I_1 \cap K_2} = M_{K_1 \cap K_2}$ by (2).

(4) Now we prove the lemma in general. Let $I_i$ be a closed subset and $J_i$ be an open subset such that $K_i = I_i \cap J_i$ and put $J_i^c = \mathcal{A} \setminus J_i$ for $i = 1,2$. Then,

$$(M_{K_1})_{K_2} = (M_{K_1})_{I_2}/(M_{K_1})_{I_2 \cap J_2^c} \simeq M_{K_1 \cap I_2}/M_{K_1 \cap I_2 \cap J_2^c}$$

by (3). We have $M_{K_1 \cap I_2} = M_{I_1 \cap I_2}/M_{I_1 \cap I_2 \cap J_1^c}$ and $M_{K_1 \cap I_2 \cap J_2^c} = M_{I_1 \cap I_2 \cap J_2^c}/M_{I_1 \cap I_2 \cap J_2^c \cap J_1^c}$. Hence,

$$(M_{K_1})_{K_2} \simeq M_{I_1 \cap I_2}/(M_{I_1 \cap I_2 \cap J_1^c} + M_{I_1 \cap I_2 \cap J_2^c}).$$

Since $M_{I_1 \cap I_2 \cap J_1^c} + M_{I_1 \cap I_2 \cap J_2^c} = M_{(I_1 \cap I_2 \cap J_1^c) \cup (I_2 \cap I_2 \cap J_2^c)} = M_{(I_1 \cap I_2) \setminus (J_1 \cap J_2)}$, we get the lemma. □

**Lemma 2.13.** *If $M \in \widetilde{\mathcal{K}}(S_0)$, then $M_K \in \widetilde{\mathcal{K}}(S_0)$.*

**Proof.** Take a closed subset $I$ and an open subset $J$ such that $K = I \cap J$.

We prove $M_K$ satisfies (S). Let $I_1, I_2$ be closed subsets. Since $(M_K)_{I_i} = M_{K \cap I_i}$ is a quotient of $M_{I \cap I_i}$, it is sufficient to prove that $M_{I \cap I_1} \oplus M_{I \cap I_2} \to (M_K)_{I_1 \cup I_2}$ is surjective. The module $(M_K)_{I_1 \cup I_2} = M_{K \cap (I_1 \cup I_2)}$ is a quotient of $M_{I \cap (I_1 \cup I_2)}$, and since $M_{I \cap (I_1 \cup I_2)} = M_{I \cap I_1} + M_{I \cap I_2}$, the map is surjective.

We prove $M_K$ satisfies (LE). We may assume $M = \bigoplus_{\Omega \in W'_{\alpha,\mathrm{aff}} \setminus \mathcal{A}}(\bigoplus_{A \in \Omega} M_A^{\emptyset} \cap M^{\alpha})$. Let $m \in M_I^{\alpha}$. Then, for each $\Omega \in W'_{\alpha,\mathrm{aff}} \setminus \mathcal{A}$, we have $m_\Omega \in M^{\alpha} \cap \bigoplus_{A \in \Omega} M_A^{\emptyset}$ such that $m = \sum m_\Omega$. Then, for each $A \in \mathcal{A}$, we have $m_A = (m_\Omega)_A$, where $\Omega$ is the unique $W'_{\alpha,\mathrm{aff}}$-orbit containing $A$. Therefore, since $m \in M_I^{\alpha}$, we have $m_\Omega \in M_I^{\alpha}$. Hence, $m_\Omega \in M_I^{\alpha} \cap \bigoplus_{A \in \Omega}(M_I)_A^{\emptyset}$. Namely, $M_I$ satisfies (LE). Since $M_K$ is a quotient of $M_I$, it also satisfies (LE). □

## 2.3. Standard filtration

Note that $\{A\} = \{A' \in \mathcal{A} \mid A' \geq A\} \cap \{A' \in \mathcal{A} \mid A' \leq A\}$ is locally closed. Let $S_0$ be a flat commutative graded $S$-algebra. We say that an object $M$ of $\widetilde{\mathcal{K}}(S_0)$ admits a standard filtration if $M_{\{A\}}$ is a graded free $S_0$-module for any $A \in \mathcal{A}$. Let $\widetilde{\mathcal{K}}_\Delta(S_0)$ be the full subcategory of $\widetilde{\mathcal{K}}(S_0)$ consisting of an object $M$ which admits a standard filtration and for which $\mathrm{supp}_{\mathcal{A}}(M)$ is finite. By Lemma 2.12, if $M \in \widetilde{\mathcal{K}}_\Delta(S_0)$, then $M_K \in \widetilde{\mathcal{K}}_\Delta(S_0)$ for any locally closed subset $K \subset \mathcal{A}$.

**Lemma 2.14.** *Let $M_1, \ldots, M_l \in \widetilde{\mathcal{K}}(S_0)$ and assume that $\mathrm{supp}_{\mathcal{A}}(M_1), \ldots, \mathrm{supp}_{\mathcal{A}}(M_l)$ are all finite. Let $I \subset \mathcal{A}$ be a closed subset and $A \in I$ such that $I \setminus \{A\}$ is closed. Then, there exist closed subsets $I_0 \subset I_1 \subset \cdots \subset I_r$ and $k \in \{1, \ldots, r\}$ such that $\#(I_j \setminus I_{j-1}) = 1$ for any*

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*$j=1,\dots,r$, $I_k\cap(\bigcup_i \mathrm{supp}_{\mathcal{A}}(M_i)) = I\cap(\bigcup_i \mathrm{supp}_{\mathcal{A}}(M_i))$, $I_{k-1}=I_k\setminus\{A\}$, $(M_i)_{I_0}=0$ and $(M_i)_{I_r}=M$ for any $i=1,\dots,l$. In particular, we have $(M_i)_I\simeq(M_i)_{I_k}$ for all $i=1,\dots,l$.*

**Proof.** There exist $A_0^-,A_0^+$ such that $\mathrm{supp}_{\mathcal{A}}(M_i)\subset[A_0^-,A_0^+]$ for any $i=1,\dots,l$ by [Lus80, Proposition 3.7]. Put $I_0=\{A'\in\mathcal{A}\mid A'\not\leq A_0^+\}\cap I$. We enumerate the elements in $(I\setminus\{A\})\cap[A_0^-,A_0^+]$ (resp. $[A_0^-,A_0^+]\setminus I$) as $\{A_1,\dots,A_{k-1}\}$ (resp. $\{A_{k+1},\dots,A_r\}$) such that $A_i\geq A_j$ implies $i\leq j$. Put $A_k=A$. Then, it is easy to see that $I_i=I_0\cup\{A_1,\dots,A_i\}$ is closed and satisfies the conditions of the lemma. □

**Lemma 2.15.** *Let $M\in\widetilde{\mathcal{K}}_\Delta(S_0)$, and let $K$ be a locally closed subset. Then $M_K$ is graded free as a left $S_0$-module.*

**Proof.** Since $M_K\in\widetilde{\mathcal{K}}_\Delta(S_0)$, we may assume $K=\mathcal{A}$. Take closed subsets $I_0\subset I_1\subset\cdots\subset I_r$ such that $I_{i+1}\setminus I_i=\{A_i\}$, $M_{I_0}=0$ and $M_{I_r}=M$. Then, $M_{I_{i+1}}/M_{I_i}=M_{\{A_i\}}$ is a graded free $S_0$-module. Hence, $M_{I_r}/M_{I_0}=M$ is also graded free. □

Finally, we define the category $\widetilde{\mathcal{K}}_P(S_0)$, which plays an important role later. The definitions are taken from [FL15, Lemma 4.11].

**Definition 2.16.** We say that a sequence $M_1\to M_2\to M_3$ in $\widetilde{\mathcal{K}}_\Delta(S_0)$ satisfies (ES) if the composition $M_1\to M_2\to M_3$ is zero and

$$0\to(M_1)_{\{A\}}\to(M_2)_{\{A\}}\to(M_3)_{\{A\}}\to 0$$

is exact for any $A\in\mathcal{A}$.

We define the category $\widetilde{\mathcal{K}}_P(S_0)\subset\widetilde{\mathcal{K}}_\Delta(S_0)$ as follows: $M\in\widetilde{\mathcal{K}}_P(S_0)$ if and only if for any sequence $M_1\to M_2\to M_3$ in $\widetilde{\mathcal{K}}_\Delta(S_0)$ which satisfies (ES), the induced sequence

$$0\to\mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}_\Delta(S_0)}(M,M_1)\to\mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}_\Delta(S_0)}(M,M_2)\to\mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}_\Delta(S_0)}(M,M_3)\to 0$$

is exact.

**Lemma 2.17.** *Assume that $M_1,M_2,M_3\in\widetilde{\mathcal{K}}(S_0)$ satisfy $\#\mathrm{supp}_{\mathcal{A}}(M_i)<\infty$ ($i=1,2,3$), and the sequence $M_1\to M_2\to M_3$ satisfies (ES). Then, $0\to(M_1)_K\to(M_2)_K\to(M_3)_K\to 0$ is exact for any locally closed subset $K$.*

**Proof.** Replacing $M_i$ with $(M_i)_K$ for $i=1,2,3$, we may assume $K=\mathcal{A}$. We can take closed subsets $I_0\subset I_1\subset\cdots\subset I_r$ such that $(M_i)_{I_0}=0$, $(M_i)_{I_r}=M_i$ and $\#(I_{j+1}\setminus I_j)=1$ for $i=1,2,3$ and $j=0,\dots,r$, as in Lemma 2.14. Then, the exactness of $0\to(M_1)_{I_j}\to(M_2)_{I_j}\to(M_3)_{I_j}\to 0$ follows by induction on $j$ and a standard diagram argument. □

**Lemma 2.18.** *Let $M\in\widetilde{\mathcal{K}}_\Delta(S_0)$, and $I_1\supset I_2$ are closed subsets of $\mathcal{A}$. Then, $M_{I_2}\to M_{I_1}\to M_{I_1}/M_{I_2}$ satisfies (ES).*

**Proof.** Note that $M_{I_1}/M_{I_2}=M_{I_1\setminus I_2}$. The lemma follows from Lemma 2.12. □

We put $\widetilde{\mathcal{K}}'=\widetilde{\mathcal{K}}'(S)$, $\widetilde{\mathcal{K}}=\widetilde{\mathcal{K}}(S)$, $\widetilde{\mathcal{K}}_\Delta=\widetilde{\mathcal{K}}_\Delta(S)$ and $\widetilde{\mathcal{K}}_P=\widetilde{\mathcal{K}}_P(S)$. We also put $(\widetilde{\mathcal{K}}')^*=\widetilde{\mathcal{K}}'(S^*)$, $\widetilde{\mathcal{K}}^*=\widetilde{\mathcal{K}}(S^*)$, $\widetilde{\mathcal{K}}^*_\Delta=\widetilde{\mathcal{K}}_\Delta(S^*)$ and $\widetilde{\mathcal{K}}^*_P=\widetilde{\mathcal{K}}_P(S^*)$ for $*\in\Delta\cup\{\emptyset\}$.

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### 2.4. Hecke action

For $\lambda \in \Lambda_{\mathbb{K}}$ and $f \in \Lambda_{\mathbb{K}}^{\vee}$, we put $\langle \lambda, f\rangle = f_A(\lambda_A)$ for $A \in \mathcal{A}$. It is easy to see that this does not depend on $A$ and gives an isomorphism $\Lambda_{\mathbb{K}}^{\vee} \simeq \mathrm{Hom}_{\mathbb{K}}(\Lambda_{\mathbb{K}},\mathbb{K})$. Let $s \in S_{\mathrm{aff}}$, and we define $\alpha_s \in \Lambda_{\mathbb{K}}$ and $\alpha_s^{\vee} \in \Lambda_{\mathbb{K}}^{\vee}$ as follows: let $A \in \mathcal{A}$ and $\alpha \in \Delta^+$ such that $s_{\alpha,n} = As$ for some $n \in \mathbb{Z}$. Then, we put $\alpha_s = \alpha^A$ and $\alpha_s^{\vee} = (\alpha^{\vee})^A$. These depend on a choice of $A$ and $\alpha$. For each $s \in S_{\mathrm{aff}}$, we fix such $A$ and $\alpha$ and define $\alpha_s, \alpha_s^{\vee}$.

**Lemma 2.19.** *The pair $(\alpha_s,\alpha_s^{\vee})$ does not depend on $A,\alpha$ up to sign.*

**Proof.** Let $A' \in \mathcal{A}$ and take $\beta \in \Delta^+$ and $m \in \mathbb{Z}$ such that $A's = s_{\beta,m}A'$. Take $x \in W'_{\mathrm{aff}}$ such that $A' = xA$. Then, $A's = xAs = xs_{\alpha,n}A$. Since the action of $W'_{\mathrm{aff}}$ on $X_{\mathbb{R}}$ preserves the set $\{\{\lambda \in X_{\mathbb{R}} \mid \langle \lambda,\alpha^{\vee}\rangle = n\} \mid \alpha \in \Delta, n \in \mathbb{Z}\}$, there exists $(\gamma,k) \in \Delta \times \mathbb{Z}$ such that $xs_{\alpha,n} = s_{\gamma,k}x$. Moreover, $\gamma \in \{\pm \overline{x}(\alpha)\}$, where $\overline{x} \in W_{\mathrm{f}}$ is the image of $x$ under $W'_{\mathrm{aff}} \to W_{\mathrm{f}}$. We may assume $\gamma = \overline{x}(\alpha)$. We have $A's = s_{\gamma,k}xA = s_{\gamma,k}A'$. Hence, $s_{\gamma,k} = s_{\beta,m}$ and therefore, $\beta = \varepsilon\gamma = \varepsilon\overline{x}(\alpha)$ for $\varepsilon = 1$ or $\varepsilon = -1$. We have $\beta^{A'} = \varepsilon \overline{x}(\alpha)^{xA} = \varepsilon\alpha^A$ and $(\beta^{\vee})^{A'} = \varepsilon(\alpha^{\vee})^A$. □

We have that $(\Lambda_{\mathbb{K}}, \{\alpha_s\}_{s\in S_{\mathrm{aff}}}, \{\alpha_s^{\vee}\}_{s \in S_{\mathrm{aff}}})$ is a realization which satisfies Demazure surjectivity [EW16, Definition 3.1]. Let $\mathcal{S}$Bimod be the category introduced in [Abe21]. We remark that [Abe21, Assumption 3.2] is satisfied in this case by [Abe20a, Theorem 1.2, Proposition 3.7]. Set $R^{\emptyset} = R[((\alpha^{\vee})^A)^{-1} \mid \alpha \in \Delta]$ for $A \in \mathcal{A}$. It is easy to see that this does not depend on $A$. We put $B^{\emptyset} = R^{\emptyset} \otimes_R B$ for $B \in \mathcal{S}$Bimod.

Recall that we have an object $B_s \in \mathcal{S}$Bimod. Set $R^s = \{f \in R \mid s(f) = f\}$. As an $R$-bimodule, $B_s = R \otimes_{R^s} R(1) \simeq \{(f,g) \in R^2 \mid f \equiv g (\mathrm{mod}\ \alpha_s)\}$ and we have the decomposition of $B_s^{\emptyset} = \bigoplus_{w \in W}(B_s)_w^{\emptyset}$, where

$$\begin{aligned}(B_s)_e^{\emptyset} &= R^{\emptyset}(\delta_s \otimes 1 - 1 \otimes s(\delta_s)),\\ (B_s)_s^{\emptyset} &= R^{\emptyset}(\delta_s \otimes 1 - 1 \otimes \delta_s),\\ (B_s)_w^{\emptyset} &= 0 \quad (w \neq e,s).\end{aligned}$$

Here, $\delta_s \in \Lambda_{\mathbb{K}}^{\vee}$ is chosen such that $\langle \alpha_s, \delta_s\rangle = 1$. The decomposition does not depend on our choice of $\delta_s$.

**Lemma 2.20.** *Let $B \in \mathcal{S}$Bimod. Then, there exists a decomposition $B^{\emptyset} = \bigoplus_{x \in W_{\mathrm{aff}}} B_x^{\emptyset}$ such that $\mathrm{Frac}(R) \otimes_{R^{\emptyset}} B_x^{\emptyset} \simeq B_x^{\mathrm{Frac}(R)}$. Here, $B_x^{\mathrm{Frac}(R)}$ is the $\mathrm{Frac}(R)$-bimodule as in the definition of $\mathcal{S}$Bimod.*

**Proof.** Assume that $B_1 \in \mathcal{S}$Bimod is a direct summand of $B \in \mathcal{S}$Bimod, and let $e \in \mathrm{End}_{\mathcal{S}\mathrm{Bimod}}(B)$ be the idempotent such that $B_1 = e(B)$. If $B$ satisfies the lemma, then by putting $(B_1)_x^{\emptyset} = e(B_x^{\emptyset})$, we see that $B_1$ also satisfies the lemma. Therefore, we may assume $B = B_{s_1} \otimes \cdots \otimes B_{s_l}$ for $s_i \in S_{\mathrm{aff}}$. Note that for $B = B_s$, the lemma holds as we have seen in the above. Hence, it is sufficient to prove that if $B_1, B_2$ satisfy the lemma, then $B = B_1 \otimes B_2$ also satisfies the lemma.

For $x \in W_{\mathrm{aff}}$ and $b \in (B_1)_x^{\emptyset}$, we have $bf = x(f)b$ for $f \in R$. Since $\{(\alpha^{\vee})^A \mid \alpha \in \Delta\}$ is stable under the action of $x$, the formula says that $(B_1)_x^{\emptyset}$ is also a right $R^{\emptyset}$-module. Therefore, $B_1^{\emptyset}$ is also a right $R^{\emptyset}$-module. Hence, $R^{\emptyset} \otimes_R B_1 \otimes_R B_2 \simeq B_1^{\emptyset} \otimes_R B_2 \simeq B_1^{\emptyset} \otimes_{R^{\emptyset}} R^{\emptyset} \otimes_R B_2 \simeq$

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$B_1^\emptyset \otimes_{R^\emptyset} B_2^\emptyset$. We put $B_x^\emptyset = \bigoplus_{yz=x} (B_1)_y^\emptyset \otimes_{R^\emptyset} (B_2)_z^\emptyset$. Then, we get $B^\emptyset = \bigoplus_{x\in W_{\mathrm{aff}}} B_x^\emptyset$ and we have $\mathrm{Frac}(R) \otimes_{R^\emptyset} B_x^\emptyset \simeq B_x^{\mathrm{Frac}(R)}$. □

Let $S_0$ be a flat commutative graded $S$-algebra. For $M \in \widetilde{\mathcal{K}}'(S_0)$ and $B \in \mathcal{S}\mathrm{Bimod}$, we define $M * B \in \widetilde{\mathcal{K}}'(S_0)$ by

- As an $(S_0,R)$-bimodule, $M * B = M \otimes_R B$.
- We put $(M*B)_A^\emptyset = \bigoplus_{x\in W_{\mathrm{aff}}} M_{Ax^{-1}}^\emptyset \otimes_{R^\emptyset} B_x^\emptyset$.

Let $f\colon M \to N$ be a morphism in $\widetilde{\mathcal{K}}'(S_0)$. We have $f(M_{Ax^{-1}}^\emptyset) \subset \bigoplus_{A'\in Ax^{-1}+\mathbb{Z}\Delta,\, A'\geq Ax^{-1}} N_{A'}^\emptyset$. By Lemma 2.3, for $A' \in Ax^{-1} + \mathbb{Z}\Delta$, $A' \geq Ax^{-1}$ if and only if $A'x \geq A$. Therefore, $\bigoplus_{A'\in Ax^{-1}+\mathbb{Z}\Delta,\, A'\geq Ax^{-1}} N_{A'}^\emptyset = \bigoplus_{A'\in A+\mathbb{Z}\Delta,\, A'\geq A} N_{A'x^{-1}}^\emptyset$ by replacing $A'x$ with $A'$. Hence,

$$(f\otimes \mathrm{id})(M_{Ax^{-1}}^\emptyset \otimes B_x^\emptyset) \subset \bigoplus_{A'\in A+\mathbb{Z}\Delta,\, A'\geq A} N_{A'x^{-1}}^\emptyset \otimes B_x^\emptyset \subset \bigoplus_{A'\geq A} (N*B)_{A'}^\emptyset.$$

Therefore, $(f\otimes \mathrm{id})$ gives a morphism in $\widetilde{\mathcal{K}}'(S_0)$. Similarly, if $f\colon B_1 \to B_2$ is a morphism in $\mathcal{S}\mathrm{Bimod}$, then $\mathrm{id}\otimes f\colon M*B_1 \to M*B_2$ is a morphism in $\mathcal{K}'(S_0)$.

For each $B \in \mathcal{S}\mathrm{Bimod}$, $B_x^\emptyset$ is free as a left $R^\emptyset$-module. We put $\mathrm{supp}_{W_{\mathrm{aff}}}(B) = \{x \in W_{\mathrm{aff}} \mid B_x^\emptyset \neq 0\}$. The following lemma follows.

**Lemma 2.21.** *We have* $\mathrm{supp}_{\mathcal{A}}(M*B) = \{Ax \mid A \in \mathrm{supp}_{\mathcal{A}}(M), x \in \mathrm{supp}_{W_{\mathrm{aff}}}(B)\}$.

Consider $M \otimes_R B_s = M \otimes_{R^s} R(1) = M(1)\otimes 1 \oplus M(1)\otimes \delta_s$. In $(M\otimes_R B_s)^\emptyset = M^\emptyset(1)\otimes 1 \oplus M^\emptyset(1)\otimes \delta_s$, we have

$$\begin{aligned}(M*B_s)_A^\emptyset &= \{m\delta_s \otimes 1 - m\otimes s(\delta_s) \mid m \in M_A^\emptyset\} \oplus \{m\delta_s\otimes 1 - m\otimes \delta_s \mid m \in M_{As}^\emptyset\} \\ &\simeq M_A^\emptyset \oplus M_{As}^\emptyset.\end{aligned} \tag{2.1}$$

The isomorphism is given by $m\otimes f \mapsto (mf, ms(f))$. Note that the last isomorphism is an isomorphism as left $S_0^\emptyset$-modules. As right $R$-modules, if $m \in (M*B_s)_A^\emptyset$ corresponds to $(m_1,m_2) \in M_A^\emptyset \oplus M_{As}^\emptyset$, then $mf$ corresponds to $(m_1 f, m_2 s(f))$.

**Proposition 2.22.** *Let $M,N \in \widetilde{\mathcal{K}}'(S_0)$. We have* $\mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}'(S_0)}(M, N*B_s) \simeq \mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}'(S_0)}(M*B_s, N)$.

**Proof.** Take $\delta \in \Lambda_{\mathbb{K}}^\vee$ such that $\langle \alpha_s, \delta\rangle = 1$. As $(S_0,R)$-bimodules, we have $N*B_s = N\otimes_{R^s} R(1)$ and $M*B_s = M\otimes_{R^s} R(1)$. For $\varphi\colon M\otimes_{R^s} R(1) \to N$, define $\psi\colon M \to N\otimes_{R^s} R(1)$ by $\psi(m) = \varphi(m\delta\otimes 1)\otimes 1 - \varphi(m\otimes 1)\otimes s(\delta)$. We know that if $\varphi$ is an $(S_0,R)$-bimodule homomorphism, $\psi$ is also an $(S_0,R)$-bimodule homomorphism and it induces a bijection between the spaces of $(S_0,R)$-bimodule homomorphisms (ee, for example, [Lib08, Lemma 3.3]). We prove that $\varphi$ is a morphism in $\widetilde{\mathcal{K}}'(S_0)$ if and only if $\psi$ is a morphism in $\widetilde{\mathcal{K}}'(S_0)$.

Set $a(m) = m\delta\otimes 1 - m\otimes s(\delta)$ and $b(m) = ms(\delta)\otimes 1 - m\otimes s(\delta)$ for $m \in M^\emptyset$. We also define $a'(n), b'(n) \in N^\emptyset \otimes_{R^s} R$ for $n \in N^\emptyset$ in the same way. Then, we have $(M*B_s)_A^\emptyset = a(M_A^\emptyset) + b(M_{As}^\emptyset)$ and the same for $N$ by (2.1) for $A \in \mathcal{A}$.

Let $A \in \mathcal{A}$ and $m \in M_A^\emptyset$. By the definition, $\psi(m) = \varphi(a(m))\otimes 1 + b'(\varphi(m\otimes 1))$. Since $a(m) \in (M*B_s)_A^\emptyset$, $\varphi(a(m))\otimes 1 = (\alpha_s)_A^{-1}\varphi(a(m))\alpha_s \otimes 1 = (\alpha_s)_A^{-1} a'(\varphi(a(m))) - (\alpha_s)_A^{-1} b'(\varphi(a(m)))$. However, we have $m\otimes 1 = (\alpha_s)_A^{-1} m\alpha_s \otimes 1 = (\alpha_s)_A^{-1} a(m) -$

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$(\alpha_s)_A^{-1}b(m)$. Since $\varphi$ and $b'$ are left $S_0$-equivariant, we get $\psi(m) = (\alpha_s)_A^{-1}a'(\varphi(a(m))) - (\alpha_s)_A^{-1}b'(\varphi(b(m)))$.

Assume that $\varphi$ is a morphism in $\widetilde{\mathcal{K}}'(S_0)$. Then, for any $m \in M_A^\emptyset$, $\varphi(a(m)) \in \bigoplus_{A' \ge A} N_{A'}^\emptyset$. Hence, $a'(\varphi(a(m))) \in \bigoplus_{A' \ge A}(N * B_s)_{A'}^\emptyset$. Since $b(m) \in (M * B_s)_{As}^\emptyset$, we have $\varphi(b(m)) \in \bigoplus_{A' \ge As, A' \in As + \mathbb{Z}\Delta} N_{A'}^\emptyset$. Therefore, $b'(\varphi(b(m))) \in \bigoplus_{A' \ge As, A' \in As+\mathbb{Z}\Delta}(N * B_s)_{A's}^\emptyset$. If $A' \in As + \mathbb{Z}\Delta$ satisfies $A' \ge As$, since $s\colon As + \mathbb{Z}\Delta \to A + \mathbb{Z}\Delta$ preserves the order, we get $A's \ge A$. Hence, $b'(\varphi(b(m))) \in \bigoplus_{A' \ge A}(N * B_s)_{A'}^\emptyset$. Therefore, $\psi$ is a morphism in $\widetilde{\mathcal{K}}'(S_0)$.

However, assume that $\psi$ is a morphism in $\widetilde{\mathcal{K}}'(S_0)$. Consider the map $\Phi\colon N \otimes_{R^s} R \to N$ defined by $n \otimes f \mapsto nf$. Then, $\Phi(a'(n)) = n\alpha_s$ and $\Phi(b'(n)) = 0$. Therefore, $\Phi((N * B_s)_A^\emptyset) = \Phi(a'(N_A^\emptyset) + b'(N_{As}^\emptyset)) \subset N_A^\emptyset$. Let $m \in M_A^\emptyset$. Then applying $\Phi$ to $\psi(m) = (\alpha_s)_A^{-1}a'(\varphi(a(m))) - (\alpha_s)_A^{-1}b'(\varphi(b(m)))$, we get $(\alpha_s)_A^{-1}\varphi(a(m))\alpha_s \in \bigoplus_{A' \in A+\mathbb{Z}\Delta, A' \ge A} M_{A'}^\emptyset$. Hence, $\varphi(a(M_A^\emptyset)) \subset \bigoplus_{A' \ge A} N_{A'}^\emptyset$. Similarly, using $N \otimes_{R^s} R \to N$ defined by $n \otimes f \mapsto ns(f)$, we get $\varphi(b(M_{As}^\emptyset)) \subset \bigoplus_{A' \ge A} N_{A'}^\emptyset$. Since $(M * B_s)_A^\emptyset = a(M_A^\emptyset) + b(M_{As}^\emptyset)$, $\varphi$ is a morphism in $\widetilde{\mathcal{K}}'(S_0)$. $\square$

**Lemma 2.23.** *Let $M \in \widetilde{\mathcal{K}}'(S_0)$.*

(1) *For $\alpha \in \Delta$, $s \in S_{\mathrm{aff}}$ and $\Omega \in W'_{\alpha,\mathrm{aff}} \backslash \mathcal{A}$, set $M^{(\Omega)} = M^\alpha \cap \bigoplus_{A \in \Omega} M_A^\emptyset$. Then, we have the following.*

  (a) *If $\Omega s = \Omega$, then $(M * B_s)^{(\Omega)} \simeq M^{(\Omega)} * B_s$.*

  (b) *If $\Omega s \neq \Omega$, then the right action of $\alpha_s$ on $M^{(\Omega)}$ is invertible and we have*
$$(M * B_s)^{(\Omega)} \simeq M^{(\Omega)} \otimes (\delta_s \otimes 1 - 1 \otimes s(\delta_s)) \oplus M^{(\Omega s)} \otimes (\delta_s \otimes 1 - 1 \otimes \delta_s)$$
  *where $\langle \alpha_s, \delta_s \rangle = 1$.*

(2) *If $M \in \widetilde{\mathcal{K}}'(S_0)$ satisfies (LE), then $M * B$ also satisfies (LE) for any $B \in \mathcal{S}\mathrm{Bimod}$.*

**Proof.** We have
$$\begin{aligned}(M * B_s)^{(\Omega)} &= M^\alpha * B_s \cap \bigoplus_{A \in \Omega}(M * B_s)_A^\emptyset \\ &= M^\alpha * B_s \cap \left(\bigoplus_{A \in \Omega} M_A^\emptyset \otimes (B_s)_e^\emptyset \oplus \bigoplus_{A \in \Omega} M_{As}^\emptyset \otimes (B_s)_s^\emptyset\right).\end{aligned}$$

If $\Omega s = \Omega$, then in the second direct sum, we can replace $As$ with $A$. Therefore,
$$\begin{aligned}(M * B_s)^{(\Omega)} &= M^\alpha * B_s \cap \left(\bigoplus_{A \in \Omega} M_A^\emptyset \otimes (B_s)_e^\emptyset \oplus \bigoplus_{A \in \Omega} M_A^\emptyset \otimes (B_s)_s^\emptyset\right) \\ &= M^\alpha * B_s \cap \bigoplus_{A \in \Omega} M_A^\emptyset \otimes (B_s)^\emptyset \\ &= (M^\alpha \cap \bigoplus_{A \in \Omega} M_A^\emptyset) \otimes B_s \\ &= M^{(\Omega)} * B_s.\end{aligned}$$

Assume that $\Omega s \neq \Omega$ and take $A \in \Omega$. Set $\beta^\vee = (\alpha_s^\vee)_A$. Then the assumption $\Omega s \neq \Omega$ tells us that $\beta^\vee \neq \pm\alpha^\vee$. Hence, $\beta^\vee$ is invertible in $S^\alpha$. The element $s_\alpha(\beta^\vee)$ is also invertible.

Let $\delta \in X_{\mathbb{K}}^{\vee}$ such that $\langle \alpha,\delta\rangle = 1$. For $m \in M^{(\Omega)}$, there exists $m_1 \in \bigoplus_{A' \in A+\mathbb{Z}\alpha} M_{A'}^{\emptyset}$ and $m_2 \in \bigoplus_{A' \in s_{(\alpha,0)}A+\mathbb{Z}\alpha} M_{A'}^{\emptyset}$ such that $m = m_1 + m_2$. For each $f \in R$, $m_1 f = f_A m_1$ and $m_2 f = s_\alpha(f_A) m_2$. By calculations using this, we have

$$\left(\frac{1}{\beta^\vee} m + \frac{\langle \alpha,\beta^\vee\rangle}{\beta s_\alpha(\beta^\vee)}(\delta m - m\delta^A)\right)\alpha_s^\vee = m.$$

Hence, the right action of $\alpha_s^\vee$ is invertible.

Therefore, we have $(M * B_s)^{(\Omega)} = (M * B_s[\alpha_s^{-1}])^{(\Omega)}$ where $B_s[(\alpha_s^\vee)^{-1}] = B_s \otimes_R R[(\alpha_s^\vee)^{-1}]$. Since $B_s[(\alpha_s^\vee)^{-1}] = R[(\alpha_s^\vee)^{-1}](\delta_s \otimes 1 - 1 \otimes s(\delta_s)) \oplus R[(\alpha_s^\vee)^{-1}](\delta_s \otimes 1 - 1 \otimes \delta_s)$ with $R[(\alpha_s^\vee)^{-1}](\delta_s \otimes 1 - 1 \otimes s(\delta_s)) \subset (B_s)_e^{\emptyset}$ and $R[(\alpha_s^\vee)^{-1}](\delta_s \otimes 1 - 1 \otimes \delta_s) \subset (B_s)_s^{\emptyset}$, the definition of $(M * B_s)^{(\Omega)}$ implies (b).

(2) Fix $\alpha \in \Delta$. By replacing $M^\alpha$ with an object which is isomorphic to $M^\alpha$, we may assume $M^\alpha = \bigoplus_{\Omega \in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}}(\bigoplus_{A\in\Omega} M_A^{\emptyset} \cap M^\alpha)$. Let $\{\Omega_i\}$ be a complete set of representatives for $\{\Omega \in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A} \mid \Omega s \neq \Omega\}/\{e,s\}$. Then, we have

$$\begin{aligned}\bigoplus_{\Omega \in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}} (M^\alpha * B_s)^{(\Omega)} &= \bigoplus_{\Omega s = \Omega} (M * B_s)^{(\Omega)} \oplus \bigoplus_i ((M * B_s)^{(\Omega_i)} \oplus (M * B_s)^{(\Omega_i s)}) \\ &= \bigoplus_{\Omega s = \Omega} M^{(\Omega)} * B_s \oplus \bigoplus_i ((M * B_s)^{(\Omega_i)} \oplus (M * B_s)^{(\Omega_i s)}).\end{aligned}$$

From the argument of the proof of (1)(b), we have $M^{(\Omega_i)} \otimes (\delta_s \otimes 1 - 1 \otimes s(\delta_s)) \oplus M^{(\Omega_i)} \otimes (\delta_s \otimes 1 - 1 \otimes \delta_s) = M^{(\Omega_i)} \otimes B_s[\alpha_s^{-1}] = M^{(\Omega_i)} \otimes B_s$. Therefore, by (1)(b), $((M * B_s)^{(\Omega_i)} \oplus (M * B_s)^{(\Omega_i s)}) = M^{(\Omega_i)} \otimes B_s \oplus M^{(\Omega_i s)} \otimes B_s$. Hence,

$$\begin{aligned}\bigoplus_{\Omega \in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}} (M * B_s)^{(\Omega)} &= \bigoplus_{\Omega s = \Omega} M^{(\Omega)} * B_s \oplus \bigoplus_i (M^{(\Omega_i)} * B_s \oplus M^{(\Omega_i s)} * B_s) \\ &= \bigoplus_{\Omega \in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}} M^{(\Omega)} * B_s \\ &= M^\alpha * B_s.\end{aligned}$$

Hence, $M * B_s$ satisfies (LE). □

## 2.5. An example

We give an example of our category. Let $(X = \mathbb{Z}, \Delta = \{\alpha = 2\}, X^\vee = \mathbb{Z}, \Delta^\vee = \{\alpha^\vee = 1\})$ be the root system of type $A_1$. The Weyl group $W_{\mathrm{f}}$ is $\{e, s_\alpha\}$. Let $s_1 \in S_{\mathrm{aff}}$ (resp. $s_0 \in S_{\mathrm{aff}}$) be the element corresponding to $W'_{\mathrm{aff}}\{0\}$ (resp. $W'_{\mathrm{aff}}\{1\}$). Then, $S_{\mathrm{aff}} = \{s_0, s_1\}$. The set of alcoves is given by $\mathcal{A} = \{A_n = \{r \in \mathbb{R} = X \otimes_{\mathbb{Z}} \mathbb{R} \mid n < r < n+1\} \mid n \in \mathbb{Z}\}$. We have $A_n s_1 = A_{n-1}$ if $n$ is even and $A_n s_1 = A_{n+1}$ if $n$ is odd. The algebra $S = \mathrm{Sym}(X_{\mathbb{K}}^\vee)$ is isomorphic to the polynomial ring $\mathbb{K}[\alpha^\vee]$.

Define $Q_{A_n} \in \widetilde{\mathcal{K}}' = \widetilde{\mathcal{K}}'(S)$ as follows. As an $(S,R)$-bimodule, we define $Q_{A_n} = \{(f,g) \in S^2 \mid f \equiv g (\mathrm{mod}\ \alpha^\vee)\}$. Here, $S$ acts naturally and $r \in R$ acts by $(f,g)r = (r_{A_n} f, r_{A_{n+1}} g)$.

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We put $(Q^{\emptyset}_{A_n})_{A_n} = S^{\emptyset} \oplus 0$, $(Q^{\emptyset}_{A_n})_{A_{n+1}} = 0 \oplus S^{\emptyset}$ and $(Q^{\emptyset}_{A_n})_{A_m} = 0$ for $m \neq n, n+1$ (we denote this object $Q_{A_n,\alpha}$ later in 3.5).

We have $\operatorname{supp}_{\mathcal{A}}(Q_{A_n}) = \{A_n, A_{n+1}\}$. We prove $Q_{A_0} * B_{s_1} \simeq Q_{A_{-1}} \oplus Q_{A_1}$. We have $\operatorname{supp}_{\mathcal{A}}(Q_{A_0} * B_{s_1}) = \{A_0, A_1, A_0 s_1, A_1 s_1\} = \{A_{-1}, A_0, A_1, A_2\}$.

Below, by an isomorphism $f \mapsto f_{A_0}$, we identify $R \simeq S = \mathbb{K}[\alpha^\vee]$. Hence, $Q_{A_n} = \{(a,b) \in \mathbb{K}[\alpha^\vee]^2 \mid a \equiv b (\mathrm{mod}\ \alpha^\vee)\}$. Put $s = s_\alpha$ which acts on $\mathbb{K}[\alpha^\vee]$. The right actions of $R \simeq \mathbb{K}[\alpha^\vee]$ on $Q_{A_0}, Q_{A_1}, Q_{A_{-1}}$ are given as follows: for $(a,b) \in Q_{A_0}$, we have $(a,b)f = (af, bs(f))$ and for $(c,d) \in Q_{A_1}, Q_{A_{-1}}$, we have $(c,d)f = (cs(f), df)$.

We have $B_{s_1} \simeq \{(f,g) \in \mathbb{K}[\alpha^\vee] \mid f \equiv g (\mathrm{mod}\ \alpha^\vee)\}$, $(B^{\emptyset}_{s_1})_e = \mathbb{K}[\alpha^\vee]^{\emptyset} \oplus 0$ and $(B^{\emptyset}_{s_1})_{s_1} = 0 \oplus \mathbb{K}[\alpha^\vee]^{\emptyset}$ where $\mathbb{K}[\alpha^\vee]^{\emptyset} = \mathbb{K}[(\alpha^\vee)^{\pm 1}]$. We have

$$\begin{aligned}
(Q_{A_0} * B_{s_1})^{\emptyset}_{A_{-1}} &= (\mathbb{K}[\alpha^\vee]^{\emptyset} \oplus 0) \otimes (0 \oplus \mathbb{K}[\alpha^\vee]^{\emptyset}),\\
(Q_{A_0} * B_{s_1})^{\emptyset}_{A_0} &= (\mathbb{K}[\alpha^\vee]^{\emptyset} \oplus 0) \otimes (\mathbb{K}[\alpha^\vee]^{\emptyset} \oplus 0),\\
(Q_{A_0} * B_{s_1})^{\emptyset}_{A_1} &= (0 \oplus \mathbb{K}[\alpha^\vee]^{\emptyset}) \otimes (\mathbb{K}[\alpha^\vee]^{\emptyset} \oplus 0),\\
(Q_{A_0} * B_{s_1})^{\emptyset}_{A_2} &= (0 \oplus \mathbb{K}[\alpha^\vee]^{\emptyset}) \otimes (0 \oplus \mathbb{K}[\alpha^\vee]^{\emptyset}).
\end{aligned}$$

These correspond to $A_{-1} = A_0 s_1$, $A_0 = A_0 e$, $A_1 = A_1 e$ and $A_2 = A_1 s_1$, respectively.

We define $p_1 \colon Q_{A_0} * B_s \to Q_{A_{-1}}$ by $p_1((a,b) \otimes (f,g)) = (ag, af)$ and $p_2 \colon Q_{A_0} * B_s \to Q_{A_1}$ by $p_2((a,b) \otimes (f,g)) = ((bs(f) - ag)/\alpha^\vee, (bs(g) - af)/\alpha^\vee)$. In the definition of $p_2$, we note that $bs(f) \equiv ag, bs(g) \equiv af (\mathrm{mod}\ \alpha^\vee)$ since $a \equiv b, s(f) \equiv f, s(g) \equiv g, f \equiv g (\mathrm{mod}\ \alpha^\vee)$. These are $\mathbb{K}[\alpha^\vee]$-bimodule homomorphisms, and from the above description, $p_1$ is a morphism in $\widetilde{\mathcal{K}}'$. We have $p_2((1,0) \otimes (0,1)) = (-1/\alpha^\vee, 0)$. Hence, $p_2((Q_{A_0} * B_{s_1})^{\emptyset}_{A_{-1}}) \subset (Q_{A_1})^{\emptyset}_{A_1}$. We also have $p_2((Q_{A_0} * B_{s_1})^{\emptyset}_{A_1}) \subset (Q_{A_1})^{\emptyset}_{A_1}$, $p_2((Q_{A_0} * B_{s_1})^{\emptyset}_{A_0}), p_2((Q_{A_0} * B_{s_1})^{\emptyset}_{A_2}) \subset (Q_{A_1})^{\emptyset}_{A_2}$. Therefore, $p_2$ is also a morphism in $\widetilde{\mathcal{K}}'$.

We define $i_1 \colon Q_{A_{-1}} \to Q_{A_0} * B_{s_1}$ by $i_1(a,b) = (b,a) \otimes (1,1) + ((a-b)/\alpha^\vee, (a-b)/\alpha^\vee) \otimes (0, \alpha^\vee)$. In $(Q_{A_0} * B_{s_1})^{\emptyset}$, $i_1$ is given by $i_1(a,b) = (b,a) \otimes (1,0) + (a,b) \otimes (0,1)$. It is easy to see that $i_1$ is a left $\mathbb{K}[\alpha^\vee]$-module homomorphism. For $f \in \mathbb{K}[\alpha^\vee]$, we have $i_1(a,b)f = (b,a) \otimes (f,0) + (a,b) \otimes (0, s(f)) = (b,a)f \otimes (1,0) + (a,b)s(f) \otimes (0,1) = (bf, as(f)) \otimes (1,0) + (as(f), bf) \otimes (0,1) = i_1(as(f), bf) = i_1((a,b)f)$. Therefore, $i_1$ is a $\mathbb{K}[\alpha^\vee]$-bimodule homomorphism. We can also check that $i_1$ is a morphism in $\widetilde{\mathcal{K}}'$. We also define $i_2 \colon Q_{A_1} \to Q_{A_0} * B_{s_1}$ by $i_2(a,b) = (0, \alpha^\vee) \otimes (s(a), s(b))$. Then, it is straightforward to check that $i_2$ is a morphism in $\widetilde{\mathcal{K}}'$. Finally, straightforward calculations imply $p_1 \circ i_1 = \mathrm{id}$, $p_2 \circ i_2 = \mathrm{id}$, $i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}$. Hence, $Q_{A_0} * B_{s_1} \simeq Q_{A_{-1}} \oplus Q_{A_1}$.

Note that the decomposition $Q_{A_0} * B_{s_1} = \operatorname{Im} i_1 \oplus \operatorname{Im} i_2$ is not compatible with respect to the decomposition over $\mathbb{K}[\alpha^\vee]^{\emptyset}$ since $i_1$ is not compatible with the decomposition.

### 2.6. Hecke actions preserve $\widetilde{\mathcal{K}}_\Delta$

We assume that $\mathbb{K}$ is local. Then, since any direct summand of a graded free $S$-module is also graded free, a direct summand of an object in $\widetilde{\mathcal{K}}_\Delta$ is also in $\widetilde{\mathcal{K}}_\Delta$. The aim of this subsection is to prove the following proposition.

**Proposition 2.24.** *We have* $\widetilde{\mathcal{K}}_\Delta * \mathcal{S}\mathrm{Bimod} \subset \widetilde{\mathcal{K}}_\Delta$.

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We fix $M \in \widetilde{\mathcal{K}}_\Delta$ and $s \in S_{\mathrm{aff}}$ in this subsection and prove $M * B_s \in \widetilde{\mathcal{K}}_\Delta$. The most difficult part is to prove that $M * B_s$ satisfies (S). First we remark that, since $M * B_s$ satisfies (LE) by Lemma 2.23, $(M * B_s)^\alpha$ satisfies (S) by Lemma 2.10.

**Lemma 2.25.** *If $I$ is a closed $s$-invariant subset of $\mathcal{A}$, then $(M * B_s)_I \simeq M_I * B_s$.*

**Proof.** We have $(M * B_s)_I^\emptyset = \bigoplus_{A \in I} M_A^\emptyset \otimes (B_s)_e^\emptyset \oplus \bigoplus_{A \in I} M_{As}^\emptyset \otimes (B_s)_s^\emptyset$. Since $I$ is $s$-invariant, $\bigoplus_{A \in I} M_{As}^\emptyset \otimes (B_s)_s^\emptyset = \bigoplus_{A \in I} M_A^\emptyset \otimes (B_s)_s^\emptyset$. Hence, $(M * B_s)_I^\emptyset = \bigoplus_{A \in I} M_A^\emptyset \otimes ((B_s)_e^\emptyset \oplus (B_s)_s^\emptyset) = \bigoplus_{A \in I} M_A^\emptyset \otimes B_s^\emptyset = M_I^\emptyset \otimes B_s^\emptyset$. □

**Lemma 2.26.** *Let $A \in \mathcal{A}$ such that $As < A$ and $I$ (resp. $J$) be an $s$-invariant closed (resp. open) subset such that $I \cap J = \{A, As\}$. Set $N = M * B_s$. Then, we have*

$$N_{I \setminus \{As\}} / N_{I \setminus \{A, As\}} \simeq M_{\{A, As\}}(-1), \quad N_I / N_{I \setminus \{As\}} \simeq M_{\{A, As\}}(1).$$

*as left $S$-modules.*

**Proof.** First we note that $I \setminus \{A, As\} = I \setminus J$ and $I \setminus \{As\} = (I \setminus J) \cup \{A' \in \mathcal{A} \mid A' \geq A\}$ are closed. We have an exact sequence

$$0 \to N_{I \setminus \{As\}} / N_{I \setminus \{A, As\}} \to N_I / N_{I \setminus \{A, As\}} \to N_I / N_{I \setminus \{As\}} \to 0. \tag{2.2}$$

We have $(N_I / N_{I \setminus \{A, As\}})^\emptyset = N_A^\emptyset \oplus N_{As}^\emptyset$ and we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (N_{I \setminus \{As\}} / N_{I \setminus \{A, As\}})^\emptyset & \longrightarrow & (N_I / N_{I \setminus \{A, As\}})^\emptyset & \longrightarrow & (N_I / N_{I \setminus \{As\}})^\emptyset & \longrightarrow & 0 \\ & & \big\| \wr & & \big\| \wr & & \big\| \wr & & \\ 0 & \longrightarrow & N_A^\emptyset & \longrightarrow & N_A^\emptyset \oplus N_{As}^\emptyset & \longrightarrow & N_{As}^\emptyset & \longrightarrow & 0. \end{array}$$

Therefore, $N_{I \setminus \{As\}} / N_{I \setminus \{A, As\}} = (N_I / N_{I \setminus \{A, As\}}) \cap (N_A^\emptyset \oplus 0)$.

Set $L = N_I / N_{I \setminus \{A, As\}}$. By Lemma 2.25, $L \simeq M_{\{A, As\}} \otimes_{R^s} R(1)$. We have $L^\emptyset = L_A^\emptyset \oplus L_{As}^\emptyset$. We determine $L \cap (L_A^\emptyset \oplus 0)$.

By (2.1), we have $L_A^\emptyset \simeq M_A^\emptyset \oplus M_{As}^\emptyset$ and $L_{As}^\emptyset \simeq M_{As}^\emptyset \oplus M_A^\emptyset$. In general, we write $m_{A'}$ for the image of $m \in M$ in $M_{A'}^\emptyset$, where $A' \in \mathcal{A}$. The image of $m_1 \otimes 1 + m_2 \otimes \delta \in L = M_{\{A, As\}} \otimes_{R^s} R(1)$ in each direct summand is

$$m_{1,A} + m_{2,A}\delta \in M_A^\emptyset \subset L_A^\emptyset,$$
$$m_{1,As} + m_{2,As}s(\delta) \in M_{As}^\emptyset \subset L_A^\emptyset,$$
$$m_{1,As} + m_{2,As}\delta \in M_{As}^\emptyset \subset L_{As}^\emptyset,$$
$$m_{1,A} + m_{2,A}s(\delta) \in M_A^\emptyset \subset L_{As}^\emptyset.$$

Therefore, $m_1 \otimes 1 + m_2 \otimes \delta \in L_A^\emptyset$ if and only if $m_{1,As} + m_{2,As}\delta = 0$, $m_{1,A} + m_{2,A}s(\delta) = 0$. Note that $m_{2,As}\delta = (s(\delta))_A m_{2,As}$ and $m_{2,A}s(\delta) = (s(\delta))_A m_{2,A}$. Therefore, $(m_1 + (s(\delta))_A m_2)_{A'} = 0$ for $A' = A, As$. Hence, $m_1 + (s(\delta))_A m_2 = 0$. Therefore, we have

$$L \cap (L_A^\emptyset \oplus 0) = \{m_2 \otimes \delta - (s(\delta))_A m_2 \otimes 1 \mid m_2 \in M_{\{A, As\}}\}(1)$$

which is isomorphic to $M_{\{A, As\}}(-1)$.

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The map $L \simeq M_{\{A,As\}} \otimes_{R^s} R(1) \ni m \otimes f \mapsto (s(f))_A m \in M_{\{A,As\}}(1)$ is surjective and, by the above argument, the kernel is $L \cap (L_A^{\emptyset} \oplus 0) \simeq N_{I \setminus \{As\}}/N_{I \setminus \{A,As\}}$. Therefore, by the exact sequence (2.2), we have $N_I/N_{I \setminus \{As\}} \simeq M_{\{A,As\}}(1)$. □

**Lemma 2.27.** *Let $A \in \mathcal{A}$ such that $As < A$, $I$ is a closed subset and $J$ is an open subset. Then, we have the following.*

(1) *If $I \cap J = \{As\}$, then $(M * B_s)_I/(M * B_s)_{I \setminus J} \simeq M_{\{A,As\}}(1)$ as left $S$-modules.*
(2) *If $I \cap J = \{A\}$, then $(M * B_s)_I/(M * B_s)_{I \setminus J} \simeq M_{\{A,As\}}(-1)$ as left $S$-modules.*

**Proof.** Set $N = M * B_s \in \widetilde{\mathcal{K}}'$.

(1) Put $I_1 = \{A' \in \mathcal{A} \mid A' \geq As\}$. This is $s$-invariant. Since $I$ is closed and contains $As$, we have $I_1 \subset I$. Hence, $N_{I_1}/N_{I_1 \setminus \{As\}} \hookrightarrow N_I/N_{I \setminus \{As\}}$. By Lemma 2.26, we have $N_{I_1}/N_{I_1 \setminus \{As\}} \simeq M_{\{A,As\}}(-1)$. Hence, we have $M_{\{A,As\}}(-1) \hookrightarrow N_I/N_{I \setminus \{As\}}$.

Let $\nu \in X_{\mathbb{K}}^{\vee}$ and write $S_{(\nu)}$ for the localization at the prime ideal $(\nu)$. Set $N_{(\nu)} = S_{(\nu)} \otimes_S N$. The algebra $S_{(\nu)}$ is an $S^{\alpha}$-algebra for a certain $\alpha \in \Delta$. Therefore, $N_{(\nu)}$ satisfies (S). Hence, the above embedding $(M_{(\nu)})_{\{A,As\}}(-1) \hookrightarrow (N_{(\nu)})_I/(N_{(\nu)})_{I \setminus \{As\}}$ is an isomorphism. Since $M$ admits a standard filtration, $M_{\{A,As\}}$ is graded free as an $S$-module. Therefore, $M_{\{A,As\}}(-1) = \bigcap_{\nu \in X_{\mathbb{K}}}(S_{(\nu)} \otimes_S M_{\{A,As\}}(-1)) = \bigcap_{\nu \in X_{\mathbb{K}}}((N_{(\nu)})_I/(N_{(\nu)})_{I \setminus \{As\}}) \supset N_I/N_{I \setminus \{As\}}$. We get the lemma.

(2) First, we prove that there exists an embedding $(M * B_s)_I/(M * B_s)_{I \setminus J} \hookrightarrow M_{\{A,As\}}(-1)$. We may assume $J = \{A' \in \mathcal{A} \mid A' \leq A\}$ since $I \setminus J$ is not changed. Then, $J$ is $s$-invariant. Put $I_1 = I \cup Is$. Then $I_1$ is an $s$-invariant closed subset and $I_1 \cap J = (I \cap J) \cup (Is \cap J) = (I \cap J) \cup (I \cap J)s = \{A, As\}$. We have $I_1 \setminus \{As\} \supset I$. Hence, we have an embedding $N_I/N_{I \setminus J} \hookrightarrow N_{I_1 \setminus \{As\}}/N_{I_1 \setminus \{A,As\}} \simeq M_{\{A,As\}(-1)}$. We prove that this embedding is surjective.

First, we assume that $\mathbb{K}$ is a field. Take a sequence of closed subsets $I_0 \subset \cdots \subset I_r$ such that $\#(I_{i+1} \setminus I_i) = 1$, $N_{I_0} = 0$, $N_{I_r} = N$, and there exists $k = 1, \ldots, r$ such that $I_{k-1} \cap \operatorname{supp}_{\mathcal{A}}(N) = I \cap \operatorname{supp}_{\mathcal{A}}(N)$ and $I_k = I_{k-1} \cup \{A\}$ (Lemma 2.14). Let $A_i \in \mathcal{A}$ such that $I_i = I_{i-1} \cup \{A_i\}$. Since $N_{I_i}$ is a filtration of $N$, for each $l$, the $l$-th graded piece $N^l$ satisfies $\dim_{\mathbb{K}} N^l = \sum_i (N_{I_i}/N_{I_{i-1}})^l$. By the existence of an embedding we have proved, $\dim_{\mathbb{K}}(N_{I_i}/N_{I_{i-1}})^l \leq \dim_{\mathbb{K}}(M_{\{A_i,A_is\}})^{l+\varepsilon(A_i)}$, where $\varepsilon(A_i) = 1$ if $A_is > A_i$ and $\varepsilon(A_i) = -1$ otherwise. We have

$$
\begin{aligned}
&\dim_{\mathbb{K}}(M_{\{A_i,A_is\}})^{l+\varepsilon(A_i)} \\
&= \sum_i (\dim_{\mathbb{K}}(M_{\{A_i\}})^{l+\varepsilon(A_i)} + \dim_{\mathbb{K}}(M_{\{A_is\}})^{l+\varepsilon(A_i)}) \\
&= \sum_{A_is > A_i} \dim_{\mathbb{K}}(M_{\{A_i\}}^{l+1}) + \sum_{A_is > A_i} \dim_{\mathbb{K}}(M_{\{A_is\}}^{l+1}) \\
&\quad + \sum_{A_is < A_i} \dim_{\mathbb{K}}(M_{\{A_i\}}^{l-1}) + \sum_{A_is < A_i} \dim_{\mathbb{K}}(M_{\{A_is\}}^{l-1}).
\end{aligned}
$$

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By replacing $A_i$ with $A_is$ in the second and fourth sum, we have

$$\begin{aligned}
&\sum_i(\dim_{\mathbb{K}}(M_{\{A_i\}})^{l+\varepsilon(A_i)}+\dim_{\mathbb{K}}(M_{\{A_is\}})^{l+\varepsilon(A_i)})\\
&=\sum_{A_is>A_i}\dim_{\mathbb{K}}(M^{l+1}_{\{A_i\}})+\sum_{A_is<A_i}\dim_{\mathbb{K}}(M^{l+1}_{\{A_i\}}))\\
&\quad+\sum_{A_is<A_i}\dim_{\mathbb{K}}(M^{l-1}_{\{A_i\}})+\sum_{A_is>A_i}\dim_{\mathbb{K}}(M^{l-1}_{\{A_i\}})\\
&=\sum_i(\dim_{\mathbb{K}}M^{l+1}_{\{A_i\}}+\dim_{\mathbb{K}}M^{l-1}_{\{A_i\}}).
\end{aligned}$$

Since $\{M_{\{A_i\}}\}$ are subquotients of a filtration $\{M_{I_i}\}$ on $M$, we have $\sum_i\dim_{\mathbb{K}}(M_{\{A_i\}})^{l'}=\dim_{\mathbb{K}}M^{l'}$. Hence, $\sum_i(\dim_{\mathbb{K}}M^{l+1}_{\{A_i\}}+\dim_{\mathbb{K}}M^{l-1}_{\{A_i\}})=\dim_{\mathbb{K}}M^{l+1}+\dim_{\mathbb{K}}M^{l-1}$.

However, since $N=M*B_s=M\otimes_{R^s}R(1)=M(1)\otimes 1\oplus M(1)\otimes\delta_s$ where $\delta_s$ satisfies $\langle\delta_s,\alpha_s^\vee\rangle=1$, we have $\dim_{\mathbb{K}}N^l=\dim_{\mathbb{K}}M^{l+1}+\dim_{\mathbb{K}}M^{l-1}$. Therefore, we get

$$\dim_{\mathbb{K}}N^l=\sum_i\dim_{\mathbb{K}}(N_{I_i}/N_{I_{i-1}})^l\le\sum_i\dim_{\mathbb{K}}(M_{\{A_i,A_is\}})^{l+\varepsilon(A_i)}=\dim_{\mathbb{K}}N^l.$$

Hence, the embedding has to be a bijection.

Now, let $\mathbb{K}$ be a general Noetherian integral domain. Assume that we can prove that $(N_{I_i}/N_{I_{i-1}})\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\simeq(M_{\{A_i,A_is\}}(\varepsilon(A_i)))\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$ for each maximal ideal $\mathfrak{m}$ in $\mathbb{K}$. Since $M^l_{\{A_i,A_is\}}$ is finitely generated as a $\mathbb{K}$-module, by Nakayama's lemma, $(N_{I_i}/N_{I_{i-1}})^l_{\mathfrak{m}}\to(M_{\{A_i,A_is\}})^{l+\varepsilon(A_i)}_{\mathfrak{m}}$ is surjective, where $(\bullet)_{\mathfrak{m}}$ means the localization at $\mathfrak{m}$. Since this is true for any maximal ideal $\mathfrak{m}$, the map $(N_{I_i}/N_{I_{i-1}})^l\to M^l_{\{A_i,A_is\}}$ is surjective for any $l\in\mathbb{Z}$. Hence, it is an isomorphism. Therefore, it is sufficient to prove $(N_{I_i}/N_{I_{i-1}})\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\simeq(M_{\{A_i,A_is\}}(\varepsilon(A_i)))\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$. In the rest of the proof, we omit the grading.

To prove this, we need some properties on the base change to $\mathbb{K}/\mathfrak{m}$. Let $L\in\widetilde{\mathcal{K}}'$. Then, $L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$ is an $(S/\mathfrak{m}S,R/\mathfrak{m}R)$-bimodule and we have $S^\emptyset\otimes_S L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\simeq\bigoplus_{A\in\mathcal{A}}L^\emptyset_A\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$. Therefore, it defines an object in $\widetilde{\mathcal{K}}'_{\mathbb{K}/\mathfrak{m}}$. Here, the suffix $\mathbb{K}/\mathfrak{m}$ means that, in the definition of $\widetilde{\mathcal{K}}'$, we replace $\mathbb{K}$ with $\mathbb{K}/\mathfrak{m}$. We also have $B\otimes_{\mathbb{K}}\mathbb{K}/\mathfrak{m}\in\mathcal{S}\mathrm{Bimod}_{\mathbb{K}/\mathfrak{m}}$ (the meaning of the suffix $\mathbb{K}/\mathfrak{m}$ is the same as above) and we have $(M*B)\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\simeq(M\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))*(B\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))$. Let $K\subset\mathcal{A}$ be a closed subset. Then, we have a map $L_K\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\to L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$. Since $\operatorname{supp}_{\mathcal{A}}(L_K\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))\subset K$, the image of this homomorphism is in $(L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_K$. Hence, we get a map $L_K\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\to(L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_K$. We claim:

(a) The map is surjective.

(b) If $L/L_K$ is graded free, then this map is an isomorphism.

We prove (a) first. By the exact sequence $0\to L_K\to L\to L/L_K\to 0$, we have an exact sequence $L_K\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\to L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\to(L/L_K)\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\to 0$. Since $\operatorname{supp}_{\mathcal{A}}((L/L_K)\otimes_{\mathbb{K}}$

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$(\mathbb{K}/\mathfrak{m})) \subset \mathcal{A}\setminus K$, the map $L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \to (L/L_K)\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$ factors through $L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \to (L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))/(L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_K$. Hence, $(L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_K \subset \operatorname{Ker}(L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \to (L/L_K)\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})) = \operatorname{Im}(L_K\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \to L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))$. Therefore, we get (a). If $L/L_K$ is graded free, then $L/L_K$ is free as a $\mathbb{K}$-module. Hence, $L_K\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \to L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$ is injective. Therefore, we have (b).

In particular, if $L$ satisfies (S), then $L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$ also satisfies (S). Indeed, let $K_1, K_2$ be closed subsets. Then, we have a commutative diagram

$$\begin{array}{ccc} L_{K_1}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \oplus L_{K_2}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) & \twoheadrightarrow & (L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_{K_1} \oplus (L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_{K_2} \\ \downarrow\!\!\downarrow & & \downarrow \\ L_{K_1\cup K_2}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) & \twoheadrightarrow & (L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_{K_1\cup K_2}. \end{array}$$

Here, the horizontal maps are surjective by (a) in the above, and the left vertical map is surjective since $L$ satisfies (S). Hence, the right vertical map is surjective and it means that $L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$ satisfies (S).

We also have that if $L$ satisfies (LE), then $L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$ satisfies (LE). Let $\alpha\in\Delta$ and decompose $L^\alpha$ as $L^\alpha \simeq \bigoplus_{\Omega\in W'_\alpha\setminus\mathcal{A}} L^{(\Omega)}$ such that $\operatorname{supp} L^{(\Omega)} \subset \Omega$. Then, $(L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))^\alpha \simeq \bigoplus_{\Omega\in W'_\alpha\setminus\mathcal{A}} L^{(\Omega)}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$ and it gives a desired decomposition in (LE).

Let $K_1 \subset K_2 \subset \mathcal{A}$ be closed subsets and suppose that $L \in \widetilde{\mathcal{K}}_\Delta$. Since $L\in\widetilde{\mathcal{K}}_\Delta$, $L/L_{K_1}$ and $L/L_{K_2}$ are both graded free. Hence, $L_{K_1}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \simeq (L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_{K_1} \subset (L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_{K_2} \simeq L_{K_2}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$. By the right exactness of the tensor product, we have $(L_{K_2}/L_{K_1})\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \simeq (L_{K_2}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))/(L_{K_1}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})) \simeq (L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_{K_2}/(L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_{K_1}$. Therefore, for any locally closed subset $K\subset\mathcal{A}$, we have $L_K\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \simeq (L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_K$. In particular, $L\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \in \widetilde{\mathcal{K}}_{\Delta,\mathbb{K}/\mathfrak{m}}$.

We return to the proof of the lemma. We have $M\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \in \widetilde{\mathcal{K}}_{\Delta,\mathbb{K}/\mathfrak{m}}$ as we have proved. We have $M_{\{A_i,A_is\}}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \simeq (M\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_{\{A_i,A_is\}}$. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} (N_{I_i}/N_{I_{i-1}})\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) & \longrightarrow & M_{\{A_i,A_is\}}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \\ \downarrow & & \downarrow\wr \\ (N\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_{I_i}/(N\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_{I_{i-1}} & \xrightarrow{\sim} & (M\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_{\{A_i,A_is\}}. \end{array}$$

Note that the bottom homomorphism is an isomorphism since the lemma is proved if $\mathbb{K}$ is a field.

We prove that the left vertical map is an isomorphism by backward induction on $i$. By inductive hypothesis, $N_{I_{i'}}/N_{I_{i'-1}} \simeq M_{\{A_{i'},A_{i'}s\}}$ for any $i' > i$ and, in particular, it is graded free. Hence, $N/N_{I_i}$ is also graded free. Therefore, we have $N_{I_i}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \simeq (N\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}))_{I_i}$. Now we get the desired result by applying the five lemmas to the following commutative diagram with exact columns:

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$$
\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
N_{I_{i-1}} \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) & \twoheadrightarrow & (N \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))_{I_{i-1}} \\
\downarrow & & \downarrow \\
N_{I_i} \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) & \xrightarrow{\sim} & (N \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))_{I_i} \\
\downarrow & & \downarrow \\
(N_{I_i}/N_{I_{i-1}}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) & \longrightarrow & (N \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))_{I_i}/(N \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))_{I_{i-1}} \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
$$

□

**Lemma 2.28.** *Set $N = M * B_s$. Then, for any two closed subsets $I_1, I_2$ with $I_1 \supset I_2$, $N_{I_1}/N_{I_2}$ is a graded free $S$-module.*

**Proof.** Take $A_0, A_1 \in \mathcal{A}$ such that $\operatorname{supp}_{\mathcal{A}} N \subset [A_0, A_1]$. Replacing $I_1$ with $I_1 \cap \{A \in \mathcal{A} \mid A \geq A_0\}$ and $I_2$ with $I_2 \cup \{A \in \mathcal{A} \mid A \not\leq A_1\}$, we may assume $I_1 \setminus I_2$ is finite. We can take a sequence of closed subsets $I_2 = I'_0 \subset I'_1 \subset \cdots \subset I'_r = I_1$ such that $\#(I'_i \setminus I'_{i-1}) = 1$. Let $A_i$ such that $I'_i = I'_{i-1} \cup \{A_i\}$. Then by Lemma 2.27, $N_{I'_i}/N_{I'_{i-1}} \simeq M_{\{A_i, A_i s\}}(\varepsilon(A_i))$, where $\varepsilon(A_i) \in \{\pm 1\}$ is as in the proof of Lemma 2.27. In particular, this is graded free and therefore, $M_{I_1}/M_{I_2} = M_{I'_r}/M_{I'_0}$ is also graded free. □

**Proof of Proposition 2.24.** Set $N = M * B_s$. We prove that $N$ satisfies (S). Let $I_1, I_2$ be closed subsets, and we prove the surjectivity of $N_{I_1}/N_{I_1 \cap I_2} \hookrightarrow N_{I_1 \cup I_2}/N_{I_2}$. For each $\nu \in X^\vee_{\mathbb{K}}$, let $S_{(\nu)}$ be the localization at the prime ideal $(\nu)$. Then, $S_{(\nu)}$ is an $S^\alpha$-algebra for some $\alpha \in \Delta^+$. Since $N_{(\nu)} = S_{(\nu)} \otimes_S N$ satisfies (LE), $N_{(\nu)}$ satisfies (S) by Lemma 2.10. Hence, this embedding is surjective after applying $S_{(\nu)} \otimes_S$. Put $L_{(\nu)} = S_{(\nu)} \otimes_S L$ for a left $S$-module $L$. Since $N_{I_1}/N_{I_1 \cap I_2}$ is graded free by Lemma 2.28, we have $N_{I_1}/N_{I_1 \cap I_2} = \bigcap_\nu (N_{I_1}/N_{I_1 \cap I_2})_{(\nu)}$. Hence, $N_{I_1}/N_{I_1 \cap I_2} = \bigcap_\nu (N_{I_1}/N_{I_1 \cap I_2})_{(\nu)} \simeq \bigcap_\nu (N_{I_1 \cup I_2}/N_{I_2})_{(\nu)} \supset N_{I_1 \cup I_2}/N_{I_2}$. We get the surjectivity.

Now $N_{\{A\}}$ is well-defined and isomorphic to $M_{\{A, As\}}(\varepsilon(A))$, where $\varepsilon(A) \in \{\pm 1\}$ is as in the proof of Lemma 2.27. Hence, $N_{\{A\}}$ is graded free; namely, $N$ admits a standard filtration. □

As a consequence of Lemma 2.27, we get the following corollary.

**Corollary 2.29.** *If $M \in \widetilde{\mathcal{K}}_\Delta$, then we have*

$$
(M * B_s)_{\{A\}} \simeq \begin{cases} M_{\{A, As\}}(-1) & (As < A), \\ M_{\{A, As\}}(1) & (As > A). \end{cases}
$$

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*Therefore, we have*

$$\operatorname{grk}((M*B_s)_{\{A\}}) = \begin{cases} v^{-1}(\operatorname{grk}(M_{\{A\}})+\operatorname{grk}(M_{\{As\}})) & (As<A),\\ v(\operatorname{grk}(M_{\{A\}})+\operatorname{grk}(M_{\{As\}})) & (As>A)\end{cases}$$

*for each* $A\in\mathcal{A}$ *and* $s\in S_{\mathrm{aff}}$.

The action of $\mathcal{S}\mathrm{Bimod}$ preserves $\widetilde{\mathcal{K}}_P$ too.

**Proposition 2.30.** *We have* $\widetilde{\mathcal{K}}_P*\mathcal{S}\mathrm{Bimod}\subset\widetilde{\mathcal{K}}_P$.

**Proof.** Let $M\in\widetilde{\mathcal{K}}_P$ and $s\in S_{\mathrm{aff}}$. We prove $M*B_s\in\widetilde{\mathcal{K}}_P$. We have already proved that $M*B_s\in\widetilde{\mathcal{K}}_\Delta$.

Assume that a sequence $M_1\to M_2\to M_3$ in $\widetilde{\mathcal{K}}_\Delta$ satisfies (ES). By Lemma 2.17, $0\to(M_1)_{\{A,As\}}\to(M_2)_{\{A,As\}}\to(M_3)_{\{A,As\}}\to 0$ is also exact for any $A\in\mathcal{A}$. Hence, $0\to(M_1*B_s)_{\{A\}}\to(M_2*B_s)_{\{A\}}\to(M_3*B_s)_{\{A\}}\to 0$ is exact (i.e., $M_1*B_s\to M_2*B_s\to M_3*B_s$ also satisfies (ES)). Since $M\in\widetilde{\mathcal{K}}_P$, the sequence $0\to\operatorname{Hom}^\bullet(M,M_1*B_s)\to\operatorname{Hom}^\bullet(M,M_2*B_s)\to\operatorname{Hom}^\bullet(M,M_3*B_s)\to 0$ is exact. By Proposition 2.22, $M*B_s\in\widetilde{\mathcal{K}}_P$. □

### 2.7. Indecomposable objects

Assume that $\mathbb{K}$ is complete local Noetherian integral domain. For $M,N\in\widetilde{\mathcal{K}}'$, $\operatorname{Hom}^\bullet_S(M,N)$ is finitely generated as an $S$-module since $M,N$ are finitely generated and $S$ is Noetherian. Hence, $\operatorname{Hom}^\bullet_{\widetilde{\mathcal{K}}'}(M,N)\subset\operatorname{Hom}^\bullet_S(M,N)$ is also finitely generated. Therefore, $\operatorname{Hom}_{\widetilde{\mathcal{K}}'}(M,N)$ is finitely generated $\mathbb{K}$-module. Hence, $\widetilde{\mathcal{K}}'$ has Krull-Schmidt property. This is also true for $\widetilde{\mathcal{K}}_P$.

Let $(\mathbb{R}\Delta)_{\mathrm{int}}=\{\lambda\in\mathbb{R}\Delta\mid\langle\lambda,\Delta^\vee\rangle\subset\mathbb{Z}\}$ be the set of integral weights. For $\lambda\in(\mathbb{R}\Delta)_{\mathrm{int}}$, let $\Pi_\lambda$ be the set of alcoves $A$ such that $\langle\lambda,\alpha^\vee\rangle-1<\langle a,\alpha^\vee\rangle<\langle\lambda,\alpha^\vee\rangle$ for any $a\in A$ and simple root $\alpha$. The set $\Pi_\lambda$ is called a box and each $A\in\mathcal{A}$ is contained in a box. Each $\Pi_\lambda$ has the unique maximal element $A_\lambda^-$. Let $W'_\lambda=\operatorname{Stab}_{W'_{\mathrm{aff}}}(\lambda)$ be the stabilizer. Then, $A_\lambda^-$ is the minimal element in $W'_\lambda A_\lambda^-$. The set $W'_\lambda A_\lambda^-$ is the set of alcoves whose closure contains $\lambda$.

We define $Q_\lambda\in\widetilde{\mathcal{K}}$ as follows. Consider the orbit $W'_\lambda A_\lambda^-$ through $A_\lambda^-$. As an $(S,R)$-bimodule, it is given by

$$Q_\lambda=\{(z_A)\in S^{W'_\lambda A_\lambda^-}\mid z_A\equiv z_{s_{\alpha,\langle\lambda,\alpha^\vee\rangle}A}\pmod{\alpha^\vee}\text{ for }\alpha\in\Delta\text{ and }A\in W'_\lambda A_\lambda^-\}$$

where the right action of $R$ is given by $(z_A)f=(f_Az_A)$. We have $Q_\lambda^\emptyset=(S^\emptyset)^{W'_\lambda A_\lambda^-}$. The module $(Q_\lambda)_A^\emptyset$ is the $A$-component if $A\in W'_\lambda A_\lambda^-$, and 0 otherwise.

The definition of $Q_\lambda$ comes from the structure sheaf of the moment graph associated to $W_{\mathrm{f}}$. The structure sheaf is defined by

$$\mathcal{Z}=\{(z_x)_{x\in W_{\mathrm{f}}}\in S^{W_{\mathrm{f}}}\mid z_x\equiv z_{s_\alpha x}\pmod{\alpha^\vee}\}.$$

The natural map $W'_\lambda\hookrightarrow W'_{\mathrm{aff}}\to W_{\mathrm{f}}$ is an isomorphism. The map $W_{\mathrm{f}}\simeq W'_\lambda\xrightarrow{w\mapsto w(A_\lambda^-)}W'_\lambda A_\lambda^-$ is a bijection which preserves orders and, by this bijection, we have $\mathcal{Z}\simeq Q_\lambda$.

The following are well-known. (See [Abe20b] for example.)

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- The map $S \otimes_{S^{W_{\mathrm{f}}}} S \to \mathcal{Z}$ defined by $f \otimes g \mapsto (x^{-1}(f)g)_{x\in W_{\mathrm{f}}}$ is an isomorphism.
- Let $K \subset W_{\mathrm{f}}$ be a closed subset and $w \in K$ such that $K \setminus \{w\}$ is closed. Put $\mathcal{Z}_K = \{(z_x) \in \mathcal{Z} \mid z_x = 0 \text{ for } x \notin K\}$ and the same for $\mathcal{Z}_{K\setminus\{w\}}$. Then, $\mathcal{Z}_K/\mathcal{Z}_{K\setminus\{w\}} \simeq S(-2\ell(w_0w))$ as a left $S$-module.

Let $d\colon \mathcal{A}\times\mathcal{A} \to \mathbb{Z}$ be the function defined in [Lus80, 1.4]. From the second property, we get the following.

**Lemma 2.31.** *Let $A \in W'_\lambda A^-_\lambda$ and $I \subset \mathcal{A}$ is a closed subset such that $A \in I$ and $I \setminus \{A\}$ is closed. Then, we have $(Q_\lambda)_I/(Q_\lambda)_{I\setminus\{A\}} \simeq S(2d(A, A^-_\lambda))$.*

It is easy to see that $Q_\lambda$ satisfies (LE) and the argument of the proof of Proposition 2.24 with the above lemma implies that $Q_\lambda$ also satisfies (S). Hence, we have $Q_\lambda \in \mathcal{K}_\Delta$.

**Lemma 2.32.** *Let $S_0$ be a flat commutative graded $S$-algebra. We have $\mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}'(S_0)}(S_0\otimes_S Q_\lambda, M) \simeq M_{\{A'\in\mathcal{A}\mid A'\geq A^-_\lambda\}}$ for $M \in \widetilde{\mathcal{K}}'(S_0)$. Therefore, $S_0\otimes_S Q_\lambda \in \widetilde{\mathcal{K}}_P(S_0)$.*

**Proof.** Since $S_0$ is flat, we have

$$S_0\otimes_S Q_\lambda = \{(z_A) \in S_0^{W_{\mathrm{f}}A_0^-} \mid z_A \equiv z_{s_{\alpha,\langle\lambda,\alpha^\vee\rangle}A} \pmod{\alpha^\vee} \text{ for } \alpha\in\Delta \text{ and } A \in W_{\mathrm{f}}A^-_\lambda\}.$$

Put $I = \{A' \in \mathcal{A} \mid A' \geq A^-_\lambda\}$ and $q = (1)_{A\in W_{\mathrm{f}}A^-_\lambda} \in S_0\otimes_S Q_\lambda$.

Any $(S_0,R)$-bimodule is regarded as an $S_0\otimes R$-module. Let $M \in \widetilde{\mathcal{K}}_\Delta(S_0)$ and $m \in M$. According to the decomposition $M^\emptyset = \bigoplus_{A\in\mathcal{A}} M^\emptyset_A$, $m$ can be written as $m = \sum_{A\in\mathcal{A}} m_A$ with $m_A \in M^\emptyset_A$. Consider $S^{W_{\mathrm{f}}} = \{f \in S \mid w(f) = f \text{ for all } w \in W_{\mathrm{f}}\}$. Then, we have the following.

- For $A \in \mathcal{A}$ and $f \in S^{W_{\mathrm{f}}}$, $f^A$ does not depend on $A$.
- For $f \in S$, we have $fm = \sum fm_A = \sum m_A f^A$.

Therefore, we have an embedding $S^{W_{\mathrm{f}}} \hookrightarrow R$ naturally and any $M$ is an $S_0\otimes_{S^{W_{\mathrm{f}}}} R$-module. Then, we have a map $S\otimes_{S^{W_{\mathrm{f}}}} R \to Q_\lambda$ defined by $f\otimes g \mapsto (fg_{w(A^-_\lambda)})$, and by the property of $\mathcal{Z}$ we have remarked, this is an isomorphism. Therefore, $Q_\lambda$ is a free $S\otimes_{S^{W_{\mathrm{f}}}} R$-module of rank one with a basis $q$. We also remark that $q \in S_0\otimes_S Q_\lambda = (S_0\otimes_S Q_\lambda)_I$. Therefore, $\varphi \mapsto \varphi(q)$ gives an embedding

$$\mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}_\Delta(S_0)}(S_0\otimes_S Q_\lambda, M) \hookrightarrow M_I.$$

Let $m \in M_I$ and $\varphi\colon S_0\otimes_S Q_\lambda \to M$ be an $(S_0,R)$-bimodule homomorphism such that $\varphi(q) = m$. We prove that this is a morphism in $\widetilde{\mathcal{K}}(S_0)$. Let $A \in W'_\lambda A^-_\lambda$. Then, $\varphi(((Q_\lambda)^\emptyset_A) \subset \bigoplus_{A'\in A+\mathbb{Z}\Delta, A'\in I} M^\emptyset_{A'}$. Therefore, the lemma follows from the following lemma. □

**Lemma 2.33.** *Let $A \in W'_\lambda A^-_\lambda$. Then, $(A+\mathbb{Z}\Delta)\cap\{A'\in\mathcal{A}\mid A'\geq A^-_\lambda\} = \{A'\in A+\mathbb{Z}\Delta \mid A'\geq A\}$.*

**Proof.** Since $A^-_\lambda$ is the minimal element in $W'_\lambda A^-_\lambda$, the right-hand side is contained in the left-hand side. Let $A'$ be in the left-hand side. Take $x \in W'_\lambda$ and $\mu \in \mathbb{Z}\Delta$ such that $A = x(A^-_\lambda)$ and $A' = A+\mu$. Then, $A' = x(A^-_\lambda)+\mu$. Since $A' \geq A^-_\lambda$ and $\lambda$ is in the closure of

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$A_\lambda^-$, we have $x(\lambda)+\mu-\lambda\in\mathbb{R}_{\geq 0}\Delta^+$ by Lemma 2.2. Since $x\in W'_\lambda=\mathrm{Stab}_{W'_{\mathrm{aff}}}(\lambda)$, $x(\lambda)=\lambda$. Therefore, $\mu\in\mathbb{R}_{\geq 0}\Delta^+$. Hence, $A'=A+\mu\geq A$. □

Let $A\in\Pi_\lambda$ and take $w\in W_{\mathrm{aff}}$ such that $A=A_\lambda^- w$. As in the proof of [Lus80, Proposition 4.2], for any $x<w$ and $A'\in W'_\lambda A_\lambda^-$, we have $A'x>A_\lambda^- w$. Let $w=s_1\cdots s_l$ be a reduced expression. Then, $Q_\lambda * B_{s_1}*\cdots * B_{s_l}$ satisfies the following.

**Lemma 2.34.** *We have the following.*

(1) $(Q_\lambda * B_{s_1}*\cdots * B_{s_l})_{\{A\}}\simeq S(l)$ *as a left* $S$*-module.*

(2) $\mathrm{supp}_{\mathcal{A}}(Q_\lambda * B_{s_1}*\cdots * B_{s_l})\subset\{A'\in\mathcal{A}\mid A'\geq A\}$.

**Proof.** (2) is obvious from what we mentioned before the lemma. We prove (1) by induction on $l$. Set $M=Q_\lambda * B_{s_1}*\cdots * B_{s_{l-1}}$ and $s=s_l$. By Lemma 2.27, $(M*B_s)_{\{A\}}\simeq M_{\{A,As\}}(1)$. By (2), $A\notin\mathrm{supp}_{\mathcal{A}}(M)$. Hence, $M_{\{A,As\}}\simeq M_{\{As\}}$. Therefore, $(M*B_s)_{\{A\}}\simeq M_{\{As\}}(1)$ and the inductive hypothesis implies (1). □

**Theorem 2.35.** *We have the following.*

(1) *For any* $A\in\mathcal{A}$*, there exists an indecomposable object* $Q(A)\in\widetilde{\mathcal{K}}_P$ *such that* $\mathrm{supp}_{\mathcal{A}}(Q(A))\subset\{A'\in\mathcal{A}\mid A'\geq A\}$ *and* $Q(A)_{\{A\}}\simeq S$*. Moreover,* $Q(A)$ *is unique up to isomorphisms.*

(2) *Any object in* $\widetilde{\mathcal{K}}_P$ *is a direct sum of* $Q(A)(n)$*, where* $A\in\mathcal{A}$ *and* $n\in\mathbb{Z}$*.*

**Proof.** Fix $s_1,\dots,s_l$ as in the above. By Lemma 2.34, there is the unique indecomposable module $Q(A)$ such that $Q(A)_{\{A\}}\simeq S$ and $Q(A)(l)$ is a direct summand of $Q_\lambda * B_{s_1}*\cdots * B_{s_l}$. It is sufficient to prove that any object $M\in\widetilde{\mathcal{K}}_P$ is a direct sum of $Q(A)(n)$'s. By induction on the rank of $M$, it is sufficient to prove that $Q(A)(n)$ is a direct summand of $M$ for some $A\in\mathcal{A}$ and $n\in\mathbb{Z}$ if $M\neq 0$.

Let $M\in\widetilde{\mathcal{K}}_P$ and let $A\in\mathrm{supp}_{\mathcal{A}}(M)$ be a minimal element. Then, $M_{\{A\}}\neq 0$. Since $M$ admits a standard filtration, $M_{\{A\}}$ is graded free. Hence, there exists $n$ such that $S(n)\simeq Q(A)(n)_{\{A\}}$ is a direct summand of $M_{\{A\}}$. Let $i\colon Q(A)(n)_{\{A\}}\to M_{\{A\}}$ (resp. $p\colon M_{\{A\}}\to Q(A)(n)_{\{A\}}$) be the embedding from (resp. projection to) the direct summand.

Let $I$ be a closed subset which contains $\mathrm{supp}_{\mathcal{A}}(M)$ such that $I\setminus\{A\}$ is closed. Then, $I\supset\{A'\in\mathcal{A}\mid A'\geq A\}\supset\mathrm{supp}_{\mathcal{A}}(Q(A))$. Therefore, we have two sequences

$$M_{I\setminus\{A\}}\to M_I=M\to M_{\{A\}},$$
$$Q(A)(n)_{I\setminus\{A\}}\to Q(A)(n)_I=Q(A)(n)\to Q(A)(n)_{\{A\}},$$

which satisfy (ES). Consider the homomorphism $Q(A)(n)\to Q(A)(n)_{\{A\}}\xrightarrow{i}M_{\{A\}}$. Since $Q(A)(n)\in\widetilde{\mathcal{K}}_P$, there exists a lift $\widetilde{i}\colon Q(A)(n)\to M$ of the above homomorphism. Similarly, we have a morphism $\widetilde{p}\colon M\to Q(A)(n)$ which is a lift of $p$. The composition $\widetilde{p}\circ\widetilde{i}\in\mathrm{End}(Q(A)(n))$ induces the identity on $Q(A)(n)_{\{A\}}$. Therefore, $1-\widetilde{p}\circ\widetilde{i}$ is not a unit. Since $Q(A)(n)$ is indecomposable, the endomorphism ring of $Q(A)(n)$ is local. Therefore, $\widetilde{p}\circ\widetilde{i}$ is an isomorphism. Hence, $Q(A)(n)$ is a direct summand of $M$. □

**Corollary 2.36.** *Any object in* $\widetilde{\mathcal{K}}_P$ *is a direct summand of a direct sum of objects of a form* $Q_\lambda * B_{s_1}*\cdots * B_{s_l}(n)$*, where* $\lambda\in(\mathbb{R}\Delta)_{\mathrm{int}}$*,* $n\in\mathbb{Z}$ *and* $s_1,\dots,s_l\in S_{\mathrm{aff}}$*.*

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**Proof.** This is obvious from Theorem 2.35 and the proof of the theorem. □

**Corollary 2.37.** *Let $M,N \in \widetilde{\mathcal{K}}_P$. Then, $\mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}_P}(M,N)$ is graded free of finite rank as an $S$-module.*

**Proof.** We may assume $M = Q_\lambda * B_{s_1} * \cdots * B_{s_l}(n)$ for some $\lambda \in (\mathbb{R}\Delta)_{\mathrm{int}}$, $n \in \mathbb{Z}$ and $s_1,\dots,s_l \in S_{\mathrm{aff}}$. Hence, by Proposition 2.22, we may assume $M = Q_\lambda$. Then, $\mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}_P}(M,N) \simeq N_{\{A' \in \mathcal{A} \mid A' \geq A_\lambda^-\}}$ and this is graded free since $N$ admits a standard filtration. □

**Corollary 2.38.** *Let $M,N \in \widetilde{\mathcal{K}}_P$. Then, for any flat commutative graded $S$-algebra $S_0$, we have $S_0 \otimes_S \mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}_P}(M,N) \simeq \mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}_{P}(S_0)}(S_0 \otimes_S M, S_0 \otimes_S N)$.*

**Proof.** As in the proof of the previous corollary, we may assume $M = Q_\lambda$. Set $I = \{A' \in \mathcal{A} \mid A' \geq A_\lambda^-\}$. Then, the corollary is equivalent to $S_0 \otimes_S N_I \simeq (S_0 \otimes_S N)_I$. This is clear. □

## 2.8. The categorification

Assume that $\mathbb{K}$ is a complete local Noetherian integral domain. We follow the notation of Soergel [Soe97] for the Hecke algebra and the periodic module. The $\mathbb{Z}[v,v^{-1}]$-algebra $\mathcal{H}$ is generated by $\{H_w \mid w \in W_{\mathrm{aff}}\}$ and defined by the following relations.

- $(H_s - v^{-1})(H_s + v) = 0$ for any $s \in S_{\mathrm{aff}}$.
- If $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$ for $w_1, w_2 \in W_{\mathrm{aff}}$, we have $H_{w_1 w_2} = H_{w_1} H_{w_2}$.

It is well-known that $\{H_w \mid w \in W_{\mathrm{aff}}\}$ is a $\mathbb{Z}[v,v^{-1}]$-basis of $\mathcal{H}$.

Set $\mathcal{P} = \bigoplus_{A \in \mathcal{A}} \mathbb{Z}[v,v^{-1}]A$ and define a right action of $\mathcal{H}$ [Soe97, Lemma 4.1] on $\mathcal{P}$ by

$$AH_s = \begin{cases} As & (As > A), \\ As + (v^{-1} - v)A & (As < A). \end{cases}$$

for $s \in S_{\mathrm{aff}}$.

For an additive category $\mathcal{B}$, let $[\mathcal{B}]$ be the split Grothendieck group of $\mathcal{B}$. We have $[\mathcal{S}\mathrm{Bimod}] \simeq \mathcal{H}$[Abe21, Theorem 4.3] and under this isomorphism, $[B_s] \in [\mathcal{S}\mathrm{Bimod}]$ corresponds to $H_s + v \in \mathcal{H}$. By $[M][B] = [M * B]$, $[\mathcal{K}_P]$ is a right $[\mathcal{S}\mathrm{Bimod}]$-module. Fix a length function $\ell \colon \mathcal{A} \to \mathbb{Z}$ in the sense of [Lus80, 2.11]. Define $\mathrm{ch}\colon [\mathcal{K}_P] \to \mathcal{P}$ by

$$\mathrm{ch}(M) = \sum_{A \in \mathcal{A}} v^{\ell(A)} \mathrm{grk}(M_{\{A\}}) A.$$

Then, by Corollary 2.29, ch is an $[\mathcal{S}\mathrm{Bimod}] \simeq \mathcal{H}$-module homomorphism.

For each $\lambda \in (\mathbb{R}\Delta)_{\mathrm{int}}$, set $e_\lambda = \sum_{A \in W_\lambda' A_\lambda^-} v^{-\ell(A)} A$. We put $\mathcal{P}^0 = \sum_{\lambda \in (\mathbb{R}\Delta)_{\mathrm{int}}} e_\lambda \mathcal{H} \subset \mathcal{P}$.

**Lemma 2.39.** *We have $\mathrm{ch}(Q_\lambda) = v^{2\ell(A_\lambda^-)} e_\lambda$.*

**Proof.** It follows from Lemma 2.31. □

**Theorem 2.40.** *We have $\mathrm{ch}\colon [\widetilde{\mathcal{K}}_P] \xrightarrow{\sim} \mathcal{P}^0$.*

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**Proof.** Since $e_\lambda = v^{-2\ell(A_\lambda^-)}\mathrm{ch}(Q_\lambda) \in \mathrm{Im}(\mathrm{ch})$, the image of ch is contained in $\mathcal{P}^0$ and it surjects to $\mathcal{P}^0$. The $\mathcal{H}$-module $[\widetilde{\mathcal{K}}_P]$ has a $\mathbb{Z}[v,v^{-1}]$-basis $[Q(A)]$ by Theorem 2.35. Since $\mathrm{ch}(Q(A)) \in v^{\ell(A)}A + \sum_{A'>A}\mathbb{Z}[v,v^{-1}]A'$, $\{\mathrm{ch}(Q(A)) \mid A \in \mathcal{A}\}$ is linearly independent. Hence, ch is injective. □

### 2.9. A relation with a work of Fiebig-Lanini

Assume that $\mathbb{K}$ is a complete local Noetherian integral domain. In [FL15], Fiebig and Lanini constructed a category denoted by $\mathbf{C}$ and proved that this is an exact category. They also constructed a wall-crossing functor $\theta_s$ for $s \in S_{\mathrm{aff}}$ on $\mathbf{C}$ and proved that projective objects are preserved by wall-crossing functors. In this subsection, we prove the following. We identify $W'_{\mathrm{aff}} \simeq W_{\mathrm{aff}}$ and $S \simeq R$ by using $A_0^+$, the maximal element in $W'_0A_0^-$.

**Theorem 2.41.** *The category $\widetilde{\mathcal{K}}_P$ is equivalent to the category of projective objects in* $\mathbf{C}$*. The action of $B_s$ on $\widetilde{\mathcal{K}}_P$ corresponds to $\theta_s$ for $s \in S_{\mathrm{aff}}$.*

Let $M \in \widetilde{\mathcal{K}}_P$, and let $J \subset \mathcal{A}$ be an open subset. Then, $M_J$ is an $R$-bimodule (as we identify $S \simeq R$) and the left action of $f \in R^{W_{\mathrm{f}}}$ is equal to the right action of $f$. Hence, $M_J$ is an $R\otimes_{R^{W_{\mathrm{f}}}} R$-module. The algebra $R\otimes_{R^{W_{\mathrm{f}}}} R$ is isomorphic to the structure algebra $\mathcal{Z}$ on the moment graph attached to $W_{\mathrm{f}}$. Hence, we get a functor $F$ from $\widetilde{\mathcal{K}}_P$ to the category of $\mathcal{Z}$-coefficient presheaves on $\mathcal{A}$.

We prove that $F$ is fully faithful. Since $M = F(M)(\mathcal{A})$ is an $R$-module, $F$ induces an injective map between the space of morphisms; namely, $F$ is faithful. Let $f\colon F(M) \to F(N)$ be a morphism between sheaves. We define $\varphi\colon M \to N$ by $M = F(M)(\mathcal{A}) \to F(N)(\mathcal{A}) = N$. Then, this is an $R$-bimodule morphism. Moreover, $\varphi$ induces $M/M_{\mathcal{A}\setminus J} = F(M)(J) \to F(N)(J) = N/N_{\mathcal{A}\setminus J}$ for any open subset $J$. Hence, $\varphi(M_I) \subset N_I$ for any closed subset $I \subset \mathcal{A}$. Therefore, $\varphi$ is a morphism in $\widetilde{\mathcal{K}}_P$, and therefore, $F$ is full.

Next, we prove that $F(M * B_s) \simeq \theta_s(F(M))$ for $M \in \widetilde{\mathcal{K}}_P$. Let $s \in S_{\mathrm{aff}}$, and let $\epsilon_s$ be the functor defined in [FL15, 8.1]. Then an argument of the proof in [Abe21, Proposition 5.3] gives $\epsilon_s(M) \simeq M \otimes_R B_s$ as $\mathcal{Z}$-modules (here, in the right-hand side, we consider a $\mathcal{Z}$-module as an $R$-bimodule via $\mathcal{Z} \simeq R\otimes_{R^{W_{\mathrm{f}}}} R$). Let $J \subset \mathcal{A}$ be an open subset and $J^\flat$ (resp. $J^\sharp$) be the largest (resp. smallest) $s$-invariant open subset which is contained in (resp. contains) $J$. Then, we have morphisms

$$(M*B_s)_{J^\sharp} \xrightarrow{j^\sharp} (M*B_s)_J \xrightarrow{j^\flat} (M*B_s)_{J^\flat}$$

such that $j^\sharp, j^\flat$ are surjective. We have $(M*B_s)_{J^\sharp} \simeq M_{J^\sharp} * B_s$ and $(M*B_s)_{J^\flat} \simeq M_{J^\flat} * B_s$ by Lemma 2.25. We have $\mathrm{supp}_{\mathcal{A}}(\mathrm{Ker}\, j_1) \subset J^\sharp \setminus J$ and $\mathrm{supp}_{\mathcal{A}}(\mathrm{Ker}\, j_2) \subset J \setminus J^\flat$. Hence, by [FL15, Lemma 2.8], $(M*B_s)_J$ satisfies the condition in [FL15, 8.3], and we get $F(M * B_s)(J) \simeq \theta_s(F(M))(J)$. Therefore, we get $F(M*B_s) \simeq \theta_s(F(M))$.

Finally, we prove that the image of $F$ is projective and the functor from $\widetilde{\mathcal{K}}_P$ to the category of projective objects in $\mathbf{C}$ is essentially surjective. Let $\underline{\mathcal{K}}_\lambda$ be a projective object in $\mathbf{C}$ defined in [FL15, Section 6]. From the definitions, we have $F(Q_\lambda) = \underline{\mathcal{K}}_\lambda$. Any $M \in \widetilde{\mathcal{K}}_P$ is a direct sum of direct summands of objects of the form $M * B_{s_1} * \cdots * B_{s_l}(n)$ for $s_1,\dots,s_l \in S_{\mathrm{aff}}$ and $n \in \mathbb{Z}$. Since $F(M * B_{s_1} * \cdots * B_{s_l}(n)) = \theta_{s_l}\cdots\theta_{s_1}\underline{\mathcal{K}}_\lambda$ is projective in $\mathbf{C}$

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by [FL15, Corollary 8.7], $F(M)$ is projective in $\mathbf{C}$ for any $M\in\widetilde{\mathcal{K}}_P$. Moreover, by the proof of [FL15, Theorem 8.8], any projective object in $\mathbf{C}$ is a direct sum of direct summands of objects of the form $\theta_{s_l}\cdots\theta_{s_1}\underline{\mathcal{K}}_\lambda$. Since $F$ is fully faithful, the essential image of $F$ is closed under taking a direct summand. Hence, $F$ is essentially surjective.

## 3. The category of Andersen-Jantzen-Soergel

### 3.1. Our combinatorial category

Assume that $\mathbb{K}$ is a complete local Noetherian integral domain. In this subsection, we introduce some categories using the categories introduced in the previous section. The categories will be related to the combinatorial categories of Andersen-Jantzen-Soergel.

Let $S_0$ be a flat commutative graded $S$-algebra. Let $\mathcal{K}'(S_0)$ be the category whose objects are the same as those of $\widetilde{\mathcal{K}}'(S_0)$ and the spaces of morphisms are defined by

$$\operatorname{Hom}_{\mathcal{K}'(S_0)}(M,N)=\operatorname{Hom}_{\widetilde{\mathcal{K}}'(S_0)}(M,N)/\{\varphi\in\operatorname{Hom}_{\widetilde{\mathcal{K}}'(S_0)}(M,N)\mid\varphi(M_A^\emptyset)\subset\bigoplus_{A'>A}N_{A'}^\emptyset\}.$$

We also define $\mathcal{K}(S_0)$ and $\mathcal{K}_\Delta(S_0)$ in the same way.

**Lemma 3.1.** *Let $M,N\in\widetilde{\mathcal{K}}'(S_0)$, $\varphi\colon M\to N$ and $B\in\mathcal{S}\mathrm{Bimod}$. If $\varphi(M_A^\emptyset)\subset\bigoplus_{A'>A}N_{A'}^\emptyset$ for any $A\in\mathcal{A}$, then $\varphi\otimes\mathrm{id}\colon M*B\to N*B$ satisfies $(\varphi\otimes\mathrm{id})((M*B)_A^\emptyset)\subset\bigoplus_{A'>A}(N*B)_{A'}^\emptyset$ for any $A\in\mathcal{A}$.*

**Proof.** Recall that we have $(M*B)_A^\emptyset=\bigoplus_{x\in W_{\mathrm{aff}}}M_{Ax^{-1}}^\emptyset\otimes B_x^\emptyset$. We have $\varphi(M_{Ax^{-1}}^\emptyset)\otimes B_x^\emptyset\subset\bigoplus_{A'x^{-1}\in Ax^{-1}+\mathbb{Z}\Delta,\,A'x^{-1}>Ax^{-1}}N_{A'x^{-1}}^\emptyset\otimes B_x^\emptyset$. Since $x\colon(Ax^{-1}+\mathbb{Z}\Delta)\to(A+\mathbb{Z}\Delta)$ preserves the order, $A'x^{-1}>Ax^{-1}$ if and only if $A'>A$. Therefore, $(\varphi\otimes\mathrm{id})(M*B)_A^\emptyset\subset\bigoplus_{x\in W_{\mathrm{aff}},A'>A}N_{A'x^{-1}}^\emptyset\otimes B_x^\emptyset=\bigoplus_{A'>A}(N*B)_{A'}^\emptyset$. □

Therefore, $(M,B)\mapsto M*B$ defines a bi-functor $\mathcal{K}'(S_0)\times\mathcal{S}\mathrm{Bimod}\to\mathcal{K}'(S_0)$ and also $\mathcal{K}_\Delta(S_0)\times\mathcal{S}\mathrm{Bimod}\to\mathcal{K}_\Delta(S_0)$.

**Proposition 3.2.** *Let $M,N\in\mathcal{K}'(S_0)$ and $s\in S_{\mathrm{aff}}$. Then, $\operatorname{Hom}_{\mathcal{K}'(S_0)}(M*B_s,N)\simeq\operatorname{Hom}_{\mathcal{K}'(S_0)}(M,N*B_s)$.*

**Proof.** Let $\varphi$ and $\psi$ as in the proof of Proposition 2.22. Then, the proof of Proposition 2.22 shows that $\varphi(M_A^\emptyset)\subset\bigoplus_{A'>A}(N*B_s)_{A'}^\emptyset$ for any $A\in\mathcal{A}$ if and only if $\psi((M*B_s)_A^\emptyset)\subset\bigoplus_{A'>A}N_{A'}^\emptyset$ for any $A\in\mathcal{A}$. The proposition follows. □

For each morphism $\varphi\colon M\to N$ in $\widetilde{\mathcal{K}}(S_0)$ and $A\in\mathcal{A}$, we have a homomorphism $\varphi_{\{A\}}\colon M_{\{A\}}\to N_{\{A\}}$. Note that $\varphi(M_A^\emptyset)\subset\bigoplus_{A'>A}N_{A'}^\emptyset$ if and only if $\varphi_{\{A\}}=0$. Hence, $M\mapsto M_{\{A\}}$ defines a functor from $\mathcal{K}(S_0)$ to the category of graded $S_0$-modules. Using this, we define as follows. A sequence $M_1\to M_2\to M_3$ in $\mathcal{K}(S_0)$ satisfies (ES) if the composition $M_1\to M_2\to M_3$ is zero in $\mathcal{K}(S_0)$ and $0\to(M_1)_{\{A\}}\to(M_2)_{\{A\}}\to(M_3)_{\{A\}}\to 0$ is exact for any $A\in\mathcal{A}$. Note that a sequence $M_1\to M_2\to M_3$ in $\widetilde{\mathcal{K}}$ may not satisfy (ES) even when it satisfies (ES) in $\mathcal{K}$ since the composition $M_1\to M_2\to M_3$ may be zero only in $\mathcal{K}$.

For the definition of $\mathcal{K}_P(S_0)$, we use the same condition to define $\widetilde{\mathcal{K}}_P(S_0)$. For $M\in\mathcal{K}_\Delta(S_0)$, we say $M\in\mathcal{K}_P(S_0)$ if, for any sequence $M_1\to M_2\to M_3$ in $\mathcal{K}_\Delta(S_0)$ which satisfies

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(ES), the induced homomorphism $0 \to \mathrm{Hom}^\bullet_{\mathcal{K}_\Delta(S_0)}(M,M_1) \to \mathrm{Hom}^\bullet_{\mathcal{K}_\Delta(S_0)}(M,M_2) \to \mathrm{Hom}^\bullet_{\mathcal{K}_\Delta(S_0)}(M,M_3) \to 0$ is exact. Note that this definition is not the same as that in the Introduction. We will prove that two definitions coincide with each other later (Proposition 3.7).

**Proposition 3.3.** *An indecomposable object in $\widetilde{\mathcal{K}}'(S_0)$ such that $\mathrm{supp}_{\mathcal{A}}(M)$ is finite and is also indecomposable as an object of $\mathcal{K}'(S_0)$.*

**Proof.** Let $M \in \widetilde{\mathcal{K}}'(S_0)$ and assume that $\mathrm{supp}_{\mathcal{A}}(M)$ is finite. Then, $\{\varphi \in \mathrm{End}_{\widetilde{\mathcal{K}}'(S_0)}(M) \mid \varphi(M_A^\emptyset) \subset \bigoplus_{A'>A} M_{A'}^\emptyset\ (A \in \mathcal{A})\}$ is a two-sided ideal of $\mathrm{End}_{\widetilde{\mathcal{K}}'(S_0)}(M)$ and, since $\mathrm{supp}_{\mathcal{A}}(M)$ is finite, this is nilpotent. Therefore, the idempotent lifting property implies the proposition. □

**Lemma 3.4.** *Let $K \subset \mathcal{A}$ be a locally closed subset such that for any $A \in K$, we have $(A+\mathbb{Z}\Delta) \cap K = \{A\}$. Then, we have the following.*

1. *For a morphism $\varphi\colon M \to N$ in $\widetilde{\mathcal{K}}(S_0)$ which is zero in $\mathcal{K}(S_0)$, the homomorphism $M_K \to N_K$ is zero in $\widetilde{\mathcal{K}}(S_0)$.*
2. *Let $M_1 \to M_2 \to M_3$ be a sequence in $\widetilde{\mathcal{K}}(S_0)$ and assume that the sequence $M_1 \to M_2 \to M_3$ satisfies (ES) as a seqeune in $\mathcal{K}(S_0)$. Then, $(M_1)_K \to (M_2)_K \to (M_3)_K$ satisfies (ES) as a sequence in $\widetilde{\mathcal{K}}(S_0)$. In particular, $0 \to (M_1)_K \to (M_2)_K \to (M_3)_K \to 0$ is an exact sequence of $(S_0,R)$-bimodules.*

**Proof.** (1) We have $M_K^\emptyset = \bigoplus_{A\in K} M_A^\emptyset$ and $N_K^\emptyset = \bigoplus_{A\in K} N_A^\emptyset$. Since $\varphi = 0$ in $\mathcal{K}$, we have $\varphi(M_A^\emptyset) \subset \bigoplus_{A'>A} N_{A'}^\emptyset$ for any $A \in K$. We also know that $\varphi(M_A^\emptyset) \subset \bigoplus_{A'\in A+\mathbb{Z}\Delta} N_{A'}^\emptyset$. By the assumption, there is no $A' \in A+\mathbb{Z}\Delta$ such that $A' > A$ and $A' \in K$. Hence, $\varphi(M_A^\emptyset) = 0$.

(2) By (1), the composition $(M_1)_K \to (M_2)_K \to (M_3)_K$ is zero. □

**Lemma 3.5.** *Assume that a sequence $M_1 \to M_2 \to M_3$ in $\mathcal{K}_\Delta(S_0)$ satisfies (ES). Then, $M_1 * B \to M_2 * B \to M_3 * B$ also satisfies (ES).*

**Proof.** We may assume $B = B_s$ where $s \in S_{\mathrm{aff}}$. We take lifts of $M_1 \to M_2$ and $M_2 \to M_3$ in $\widetilde{\mathcal{K}}(S_0)$, and we regard $M_1 \to M_2 \to M_3$ also as a sequence in $\widetilde{\mathcal{K}}(S_0)$. As in Corollary 2.29, we have $(M_i * B_s)_{\{A\}} \simeq (M_i)_{\{A,As\}}(\varepsilon(A))$, where $\varepsilon(A)$ is as in the proof of Lemma 2.27. By the previous lemma, $0 \to (M_1)_{\{A,As\}} \to (M_2)_{\{A,As\}} \to (M_3)_{\{A,As\}} \to 0$ is exact. Therefore, $0 \to (M_1 * B_s)_{\{A\}} \to (M_2 * B_s)_{\{A\}} \to (M_3 * B_s)_{\{A\}} \to 0$ is exact. Hence, a sequence $M_1 * B_s \to M_2 * B_s \to M_3 * B_s$ in $\mathcal{K}_\Delta(S_0)$ satisfies (ES). □

Combining Proposition 3.2, we have $\mathcal{K}_P(S_0) * \mathcal{S}\mathrm{Bimod} \subset \mathcal{K}_P(S_0)$.

**Lemma 3.6.** *Let $\lambda \in (\mathbb{R}\Delta)_{\mathrm{int}}$. The subset $W'_\lambda A_\lambda^-$ is locally closed and we have a natural isomorphism $\mathrm{Hom}^\bullet_{\mathcal{K}(S_0)}(S_0 \otimes_S Q_\lambda, M) \simeq M_{W'_\lambda A_\lambda^-}$ for $M \in \mathcal{K}_\Delta(S_0)$.*

**Proof.** Set $I = \{A' \in \mathcal{A} \mid A' \geq A_\lambda^-\}$. We prove $I \setminus W'_\lambda A_{\lambda^-}$ is closed. Let $A_1 \in W'_\lambda A_\lambda^-$ and $A_2 \in I$ satisfies $A_2 \leq A_1$. We prove $A_2 \in W'_\lambda A_\lambda^-$. This proves that $I \setminus W'_\lambda A_{\lambda^-}$ is closed. Take $A_3 \in W'_\lambda A_\lambda^-$ such that $A_2 \in A_3 + \mathbb{Z}\Delta$. Then, by Lemma 2.33, we have $A_2 \geq A_3$. Take $x \in W'_\lambda$ and $\mu \in \mathbb{Z}\Delta$ such that $A_1 = x(A_3)$ and $A_2 = A_3 + \mu$. Then, $A_1 \geq A_2 \geq A_3$ implies

$x(\lambda)-(\lambda+\mu)\in\mathbb{R}_{\geq 0}\Delta^+$ and $(\lambda+\mu)-\lambda\in\mathbb{R}_{\geq 0}\Delta^+$. As $x(\lambda)=\lambda$, we have $\mu=0$. Hence, $A_2=A_3\in W'_\lambda A_\lambda^-$.

We have $\mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}(S_0)}(Q_\lambda,M)\simeq M_I$ where $I=\{A'\in\mathcal{A}\mid A'\geq A_\lambda^-\}$ and, under this correspondence, $\{\varphi\in\mathrm{Hom}^\bullet_{\widetilde{\mathcal{K}}(S_0)}(Q_\lambda,M)\mid \varphi((Q_\lambda)_A^\emptyset)\subset\bigoplus_{A'>A}M_{A'}^\emptyset\}$ exactly corresponds to $\{m\in M_I\mid m_A=0$ for any $A\in W'_\lambda A_{\lambda^-}\}$. Since $I\setminus W'_\lambda A_{\lambda^-}$ is closed, $\{m\in M_I\mid m_A=0$ for any $A\in W'_\lambda A_{\lambda^-}\}=M_{I\setminus W'_\lambda A_\lambda^-}$. Hence, $\mathrm{Hom}^\bullet_{\mathcal{K}(S_0)}(Q_\lambda,M)\simeq M_{W'_\lambda A_\lambda^-}$. □

**Proposition 3.7.** *The objects of $\mathcal{K}_P$ are the same as those of $\widetilde{\mathcal{K}}_P$.*

**Proof.** First, we prove that any $M\in\widetilde{\mathcal{K}}_P$ belongs to $\mathcal{K}_P$. By Theorem 2.35, we may assume $M=Q_\lambda*B_{s_1}*\cdots*B_{s_l}(n)$ for some $\lambda\in(\mathbb{R}\Delta)_{\mathrm{int}}$, $s_1,\dots,s_l\in S_{\mathrm{aff}}$ and $n\in\mathbb{Z}$. By Proposition 3.2 and Lemma 3.5, we may assume $M=Q_\lambda$.

We have $\mathrm{Hom}_{\mathcal{K}}(Q_\lambda,N)\simeq N_{W'_\lambda A_\lambda^-}$ for $N\in\mathcal{K}$. Since $W'_\lambda A_\lambda^-$ satisfies the condition of Lemma 3.4, this implies $Q_\lambda\in\mathcal{K}_P$.

The object $Q(A)$ is indecomposable in $\mathcal{K}_P$ by Proposition 3.3. Using the argument in the proof of Theorem 2.35, any object in $\mathcal{K}_P$ is a direct sum of $Q(A)(n)$ where $A\in\mathcal{A},n\in\mathbb{Z}$. Hence, the proposition is proved. □

Hence, our $\mathcal{K}_P$ is the same as that in the Introduction.

**Corollary 3.8.** *Let $M\in\mathcal{K}_P$, $N\in\mathcal{K}_\Delta$ and $S_0$ a flat commutative graded $S$-algebra.*

(1) *The natural map $S_0\otimes_S\mathrm{Hom}^\bullet_{\mathcal{K}_P}(M,N)\to\mathrm{Hom}^\bullet_{\mathcal{K}_P(S_0)}(S_0\otimes_S M,S_0\otimes_S N)$ is an isomorphism.*

(2) *We have $S_0\otimes_S M\in\mathcal{K}_P(S_0)$.*

**Proof.** We may assume $M=Q_\lambda*B_{s_1}*\cdots*B_{s_l}(n)$ for some $\lambda\in(\mathbb{R}\Delta)_{\mathrm{int}}$, $s_1,\dots,s_l\in S_{\mathrm{aff}}$ and $n\in\mathbb{Z}$.

(1) By Proposition 3.2, we may assume $M=Q_\lambda$. In this case, the corollary is equivalent to $S_0\otimes_S(N_{W'_\lambda A_\lambda^-})\simeq(S_0\otimes_S N)_{W'_\lambda A_\lambda^-}$. This is clear.

(2) By Lemma 3.5, we may assume $M=Q_\lambda$. Then, $S_0\otimes_S Q_\lambda\in\mathcal{K}_P(S_0)$ by Lemma 3.4 and 3.6. □

We can define $\mathrm{ch}\colon[\mathcal{K}_P]\to\mathcal{P}^0$ by the same formula as $\mathrm{ch}\colon[\widetilde{\mathcal{K}}_P]\to\mathcal{P}^0$. By the previous proposition with Theorem 2.40, we get the following.

**Theorem 3.9.** *We have $[\mathcal{K}_P]\simeq\mathcal{P}^0$.*

### 3.2. A formula on homomorphisms

Assume that $\mathbb{K}$ is amcomplete local Noetherian integral domain. Let $m\mapsto\overline{m}$ be a map from $\mathcal{P}^0$ to $\mathcal{P}^0$ defined in [Soe97, Theorem 4.3]. For $m\in\mathcal{P}^0$ and $m'\in\mathcal{P}$, take $c_A,d_A\in\mathbb{Z}[v,v^{-1}]$ such that $\overline{m}=\sum_{A\in\mathcal{A}}c_AA$ and $m'=\sum_{A\in\mathcal{A}}d_AA$. Set $(m,m')_{\mathcal{P}}=\sum_{A\in\mathcal{A}}c_Ad_A$. We define $\omega\colon\mathcal{H}\to\mathcal{H}$ by $\omega(\sum_{x\in W}a_x(v)H_x)=\sum_{x\in W}a_x(v^{-1})H_x^{-1}$. Then, we have

$$(mh,m')_{\mathcal{P}}=(m,m'\omega(h))_{\mathcal{P}}$$

where $m\in\mathcal{P}^0$, $m'\in\mathcal{P}$ and $h\in\mathcal{H}$. This easily follows from the definitions. Let $w_0\in W_{\mathrm{f}}$ be the longest element.

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**Theorem 3.10.** *Let $P \in \mathcal{K}_P$ and $M \in \mathcal{K}_\Delta$. Then, $\mathrm{Hom}^\bullet_{\mathcal{K}_\Delta}(P,M)$ is amgraded free left $S$-module and the graded rank is given by*

$$\mathrm{grk}\,\mathrm{Hom}^\bullet_{\mathcal{K}_\Delta}(P,M) = v^{-2\ell(w_0)}(\mathrm{ch}(P),\mathrm{ch}(M))_{\mathcal{P}}.$$

**Proof.** Since $[\mathcal{K}_P]$ is generated by elements of a form $[Q_\lambda * B_{s_1} * \cdots * B_{s_l}]$ with $\lambda \in (\mathbb{R}\Delta)_{\mathrm{int}}$ and $s_1,\ldots,s_l \in S_{\mathrm{aff}}$, we may assume $P$ has this form. Moreover, by Lemma 3.2 and the formula before the theorem, we may assume $P = Q_\lambda$. In this case, we have $\mathrm{Hom}^\bullet_{\mathcal{K}_\Delta}(P,M) \simeq M_{W'_\lambda A^-_\lambda}$, and this is graded free by the definition of $\mathcal{K}_\Delta$. Moreover, the graded rank of $M_{W'_\lambda A^-_\lambda}$ is $\sum_{A\in W'_\lambda A^-_\lambda} \mathrm{grk}(M_{\{A\}})$.

Let $S_\lambda$ be the set of reflections in $W'_\lambda$ along the walls of $A^-_\lambda$. Then, this is a generator of $W'_\lambda$, and $(W'_\lambda, S_\lambda)$ is a Coxeter system. The length function of this Coxeter system is denoted by $\ell_\lambda$.

We calculate $(\mathrm{ch}(Q_\lambda),\mathrm{ch}(M))$. We put $(\sum_{A\in\mathcal{A}} c_A A, \sum_{A\in\mathcal{A}} d_A A)' = \sum_{A\in\mathcal{A}} c_A d_A$. Let $E_\lambda \in \mathcal{P}$ be the element defined in [Soe97, 4] and $A^+_\lambda$ the maximal element in $W'_\lambda A^-_\lambda$. Then, we have $E_\lambda = \sum_{w\in W'_\lambda} v^{\ell_\lambda(w)} w A^+_\lambda$. Since $\ell(w(A^+_\lambda)) = \ell(A^+_\lambda) - \ell_\lambda(w)$, we have $e_\lambda = \sum_{w\in W'_\lambda} v^{-\ell(w(A^+_\lambda))} w(A^+_\lambda) = v^{-\ell(A^+_\lambda)} E_\lambda$. Therefore, $\mathrm{ch}(Q_\lambda) = v^{2\ell(A^-_\lambda)} e_\lambda = v^{2\ell(A^-_\lambda)-\ell(A^+_\lambda)} E_\lambda$. Since $\overline{E_\lambda} = E_\lambda$, we get $\overline{\mathrm{ch}(Q_\lambda)} = v^{-2\ell(A^-_\lambda)+\ell(A^+_\lambda)} E_\lambda = v^{-2\ell(A^-_\lambda)+2\ell(A^+_\lambda)} e_\lambda = v^{2\ell(w_0)} e_\lambda$. Hence,

$$\begin{aligned}(\mathrm{ch}(Q_\lambda),\mathrm{ch}(M))_{\mathcal{P}} &= v^{2\ell(w_0)}(e_\lambda,\mathrm{ch}(M))' \\ &= v^{2\ell(w_0)}\left(\sum_{A\in W'_\lambda A^-_\lambda} v^{-\ell(A)}A, \sum_{A\in\mathcal{A}} v^{\ell(A)}\mathrm{grk}(M_{\{A\}})A\right)' \\ &= v^{2\ell(w_0)} \sum_{A\in W'_\lambda A^-_\lambda} \mathrm{grk}(M_{\{A\}}) \\ &= v^{2\ell(w_0)}\,\mathrm{grk}\,\mathrm{Hom}^\bullet_{\mathcal{K}_P}(Q_\lambda, M).\end{aligned}$$

We get the theorem. □

### 3.3. The category $\mathcal{K}^\alpha_P$

Assume that $\mathbb{K}$ is a complete local Noetherian integral domain. In this subsection, we analyze $\mathcal{K}^\alpha_P = \mathcal{K}_P(S^\alpha)$. First, we define an object $Q_{A,\alpha}$ where $A \in \mathcal{A}$ and $\alpha \in \Delta^+$. Set $Q_{A,\alpha} = \{(a,b) \in S^2 \mid a \equiv b (\mathrm{mod}\ \alpha^\vee)\}$ and define a right action of $R$ on $Q_{A,\alpha}$ by $(x,y)f = (f_A x, s_\alpha(f_A)y)$ for $(x,y) \in Q_{A,\alpha}$ and $f \in R$. We have $Q^\emptyset_{A,\alpha} = S^\emptyset \oplus S^\emptyset$ and we set

$$(Q_{A,\alpha})^\emptyset_{A'} = \begin{cases} S^\emptyset \oplus 0 & (A' = A), \\ 0 \oplus S^\emptyset & (A' = \alpha \uparrow A), \\ 0 & (\text{otherwise}). \end{cases}$$

It is easy to see that $Q^\alpha_{A,\alpha} = S^\alpha \otimes_S Q_{A,\alpha}$ is indecomposable.

**Lemma 3.11.** *We have $Q^\alpha_{A,\alpha} \in \mathcal{K}^\alpha_P$.*

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**Proof.** It is easy to see that $Q^{\alpha}_{A,\alpha}\in\mathcal{K}^{\alpha}_{\Delta}$. Let $M\in\mathcal{K}^{\alpha}_{\Delta}$ and we analyze $\mathrm{Hom}^{\bullet}_{\mathcal{K}^{\alpha}_{\Delta}}(Q^{\alpha}_{A,\alpha},M)$. By (LE), $M\simeq\bigoplus_i M_i$ such that $\mathrm{supp}_{\mathcal{A}}(M_i)\subset W'_{\alpha,\mathrm{aff}}A_i$ for some $A_i\in\mathcal{A}$. We have $\mathrm{Hom}^{\bullet}_{\mathcal{K}^{\alpha}_{\Delta}}(Q^{\alpha}_{A,\alpha},M_i)=0$ if $A\notin W'_{\alpha,\mathrm{aff}}A_i$. Therefore, it is sufficient to prove the following: if a sequence $M_1\to M_2\to M_3$ in $\mathcal{K}^{\alpha}_{\Delta}$ satisfies (ES) and $\mathrm{supp}_{\mathcal{A}}(M_i)\subset W'_{\alpha,\mathrm{aff}}A$, then $0\to\mathrm{Hom}^{\bullet}_{\mathcal{K}^{\alpha}_{\Delta}}(Q^{\alpha}_{A,\alpha},M_1)\to\mathrm{Hom}^{\bullet}_{\mathcal{K}^{\alpha}_{\Delta}}(Q^{\alpha}_{A,\alpha},M_2)\to\mathrm{Hom}^{\bullet}_{\mathcal{K}^{\alpha}_{\Delta}}(Q^{\alpha}_{A,\alpha},M_3)\to 0$ is exact. We can apply a similar argument of the proof of Proposition 3.7. □

We can apply the argument in the proof of Theorem 2.35 and get the following proposition.

**Proposition 3.12.** *Any object in $\mathcal{K}^{\alpha}_{P}$ is a direct sum of $Q^{\alpha}_{A,\alpha}(n)$ where $A\in\mathcal{A}$ and $n\in\mathbb{Z}$.*

### 3.4. The combinatorial category of Andersen-Jantzen-Soergel

Assume that $\mathbb{K}$ is a complete local Noetherian integral domain. We recall the combinatorial category of Andersen-Jantzen-Soergel [AJS94]. We use the version introduced by Fiebig in [Fie11]. We write $\mathcal{K}_{\mathrm{AJS}}$ for this category.

Let $S_0$ be a flat commutative graded $S$-algebra and we define the category $\mathcal{K}_{\mathrm{AJS}}(S_0)$ as follows. An object of $\mathcal{K}_{\mathrm{AJS}}(S_0)$ is $\mathcal{M}=((\mathcal{M}(A))_{A\in\mathcal{A}},(\mathcal{M}(A,\alpha))_{A\in\mathcal{A},\alpha\in\Delta^+})$, where $\mathcal{M}(A)$ is a graded $(S_0)^{\emptyset}$-module and $\mathcal{M}(A,\alpha)\subset\mathcal{M}(A)\oplus\mathcal{M}(\alpha\uparrow A)$ is a graded sub-$(S_0)^{\alpha}$-module. A morphism $f\colon\mathcal{M}\to\mathcal{N}$ in $\mathcal{K}_{\mathrm{AJS}}(S_0)$ is a collection of degree zero $(S_0)^{\emptyset}$-homomorphisms $f_A\colon\mathcal{M}(A)\to\mathcal{N}(A)$ which sends $\mathcal{M}(A,\alpha)$ to $\mathcal{N}(A,\alpha)$ for any $A\in\mathcal{A}$ and $\alpha\in\Delta^+$. Put $\mathcal{K}_{\mathrm{AJS}}=\mathcal{K}_{\mathrm{AJS}}(S)$ and $\mathcal{K}^{*}_{\mathrm{AJS}}=\mathcal{K}_{\mathrm{AJS}}(S^{*})$ for $*\in\{\emptyset\}\cup\Delta$.

For each $s\in S_{\mathrm{aff}}$, the translation functor $\vartheta_s\colon\mathcal{K}_{\mathrm{AJS}}(S_0)\to\mathcal{K}_{\mathrm{AJS}}(S_0)$ is defined as

$$\vartheta_s(\mathcal{M})(A)=\mathcal{M}(A)\oplus\mathcal{M}(As)$$

and

$$\vartheta_s(\mathcal{M})(A,\alpha)=\begin{cases}\mathcal{M}(A,\alpha)\oplus\mathcal{M}(As,\alpha) & (As\notin W'_{\alpha,\mathrm{aff}}A),\\ \{(x,y)\in\mathcal{M}(A,\alpha)^2\mid x-y\in\alpha^{\vee}\mathcal{M}(A,\alpha)\} & (As=\alpha\uparrow A),\\ \alpha^{\vee}\mathcal{M}(As,\alpha)\oplus\mathcal{M}(\alpha\uparrow A,\alpha) & (As=\alpha\downarrow A).\end{cases}$$

We define $\mathcal{F}(S_0)\colon\mathcal{K}_P(S_0)\to\mathcal{K}_{\mathrm{AJS}}(S_0)$ as follows. First, we put

$$(\mathcal{F}(S_0)(M))(A)=M^{\emptyset}_A.$$

To define $(\mathcal{F}(S_0)(M))(A,\alpha)$, we take $X\in\widetilde{\mathcal{K}}_P(S_0^{\alpha})$ and an isomorphism $\varphi\colon X\to M^{\alpha}$ in $\widetilde{\mathcal{K}}_P(S_0^{\alpha})$ such that $X=\bigoplus_{\Omega\in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}}(X\cap\bigoplus_{A\in\Omega}X^{\emptyset}_A)$. Such $X$ exists since $M$ satisfies (LE). Then we have an isomorphism $X^{\emptyset}_A\simeq(X_{\geq A}/X_{>A})^{\emptyset}\simeq((M^{\alpha})_{\geq A}/(M^{\alpha})_{>A})^{\emptyset}\simeq M^{\emptyset}_A$. In general, for $Y\in\mathcal{K}_P(S_0)$, $y\in Y^{\emptyset}$ and $A\in\mathcal{A}$, write $y_A$ for the $Y^{\emptyset}_A$-component of $y$ along the decomposition $Y^{\emptyset}=\bigoplus_{A\in\mathcal{A}}Y^{\emptyset}_A$. Then, this isomorphism can be written as $x\mapsto\varphi(x)_A$. Here, we use the same letter $\varphi$ for the induced map $X^{\emptyset}\to M^{\emptyset}$.

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Now let $(\mathcal{F}(S_0)(M))(A,\alpha)$ be the image of

$$X_{\geq A} \to X_A^{\emptyset} \oplus X_{\alpha\uparrow A}^{\emptyset} \simeq M_A^{\emptyset} \oplus M_{\alpha\uparrow A}^{\emptyset}.$$

In other words, $(\mathcal{F}(S_0)(M))(A,\alpha)$ is the set of $(\varphi(x_A)_A, \varphi(x_{\alpha\uparrow A})_{\alpha\uparrow A})$ where $x \in X_{\geq A}$. We may assume $x \in \bigoplus_{A' \in W'_{\alpha,\mathrm{aff}}A} X_{A'}^{\emptyset}$. Of course, we have to prove that this space does not depend on a choice of $X$. We use the following lemma.

**Lemma 3.13.** *Let $X, Y \in \widetilde{\mathcal{K}}_P(S_0)$, $f\colon X \to Y$ be a morphism, $A \in \mathcal{A}$ and $\alpha \in \Delta^+$. Assume that $x \in X_{\geq A}^{\emptyset}$ satisfies $x_{A'} = 0$ for $A' \notin W'_{\alpha,\mathrm{aff}}A$.*

(1) *We have $f(x)_A = f(x_A)_A$ and $f(x)_{\alpha\uparrow A} = f(x_{\alpha\uparrow A})_{\alpha\uparrow A}$.*

(2) *Let $g\colon Y \to Z$ be another morphism in $\widetilde{\mathcal{K}}_P(S_0)$. Then, $g(f(x)_{A'})_{A'} = g(f(x))_{A'}$ for $A' \in \{A, \alpha\uparrow A\}$*

**Proof.** We prove (1). Let $A'' \in \mathcal{A}$. Then $f(x)_{A''} = \sum_{A' \in \mathcal{A}} f(x_{A'})_{A''}$. We have

- $x_{A'} = 0$ unless $A' \geq A$ since $x \in X_{\geq A}$.
- $x_{A'} = 0$ unless $A' \in W'_{\alpha,\mathrm{aff}}A$ from the condition on $x$.
- $f(x_{A'})_{A''} = 0$ unless $A'' \geq A'$ from the definition of morphisms in $\widetilde{\mathcal{K}}_P(S_0)$.

Therefore, in the sum $\sum_{A' \in \mathcal{A}} f(x_{A'})_{A''}$, we may assume $A'$ satisfies $A \leq A' \leq A'', A' \in W'_{\alpha,\mathrm{aff}}A$. If $A'' = A$, then $A \leq A' \leq A''$, implying $A' = A$. Hence, $f(x)_A = f(x_A)_A$. If $A'' = \alpha\uparrow A$, we have $A \leq A' \leq \alpha\uparrow A$ and $A' \in W'_{\alpha,\mathrm{aff}}A$. Thus, we have $A' = A$ or $\alpha\uparrow A$. However, by Remark 2.7, we have $f(x_A)_{\alpha\uparrow A} = 0$. Hence, $f(x)_{\alpha\uparrow A} = f(x_{\alpha\uparrow A})_{\alpha\uparrow A}$.

We prove (2). We have $f(x_{A'}) \in \bigoplus_{A'' \geq A'} Y_{A''}^{\emptyset}$. Hence, $f(x_{A'}) - f(x_{A'})_{A'} \in \bigoplus_{A'' > A'} Y_{A''}^{\emptyset}$. Therefore, $g(f(x_{A'})) - g(f(x_{A'})_{A'}) \in \bigoplus_{A'' > A'} Z_{A''}^{\emptyset}$. Hence, $g(f(x_{A'}))_{A'} = g(f(x_{A'})_{A'})_{A'}$. By (1), the right-hand side is $g(f(x)_{A'})_{A'}$ and the left-hand side is $g(f(x_{A'}))_{A'} = (g \circ f)(x_{A'})_{A'} = (g \circ f)(x)_{A'} = g(f(x))_{A'}$. □

Let $\varphi'\colon X' \to M^{\alpha}$ be another isomorphism which satisfies the condition for $X$ and set $\psi = (\varphi')^{-1} \circ \varphi$. For $A' \in \{A, \alpha\uparrow A\}$, we have $\varphi(x_{A'})_{A'} = \varphi(x)_{A'} = \varphi'(\psi(x))_{A'} = \varphi'(\psi(x)_{A'})_{A'}$. Hence, $(\varphi(x_A)_A, \varphi(x_{\alpha\uparrow A})_{\alpha\uparrow A}) = (\varphi'(\psi(x)_A)_A, \varphi'(\psi(x)_{\alpha\uparrow A})_{\alpha\uparrow A})$. As $\psi$ is a morphism, $\psi(x) \in X'_{\geq A}$. Hence, the right-hand side is in $(\mathcal{F}(S_0)(M))(A,\alpha)$ determined by $X'$. Therefore, the space $(\mathcal{F}(S_0)(M))(A,\alpha)$ determined by $X$ is contained in the space $(\mathcal{F}(S_0)(M))(A,\alpha)$ determined by $X'$. By swapping $X$ with $X'$, we get the reverse inclusion and therefore, the space $(\mathcal{F}(S_0)(M))(A,\alpha)$ does not depend on the choice of $X$.

Let $f\colon M \to N$ be a morphism in $\mathcal{K}_P(S_0)$ and take a lift $\widetilde{f} \in \mathrm{Hom}_{\widetilde{\mathcal{K}}_P(S_0)}(M,N)$ of $f$. Then, we have a homomorphism $(\mathcal{F}(S_0)(f))(A)\colon M_A^{\emptyset} \to N_A^{\emptyset}$ defined by $M_A^{\emptyset} \hookrightarrow \bigoplus_{A' \geq A} M_{A'}^{\emptyset} \xrightarrow{\widetilde{f}} \bigoplus_{A' \geq A} N_{A'}^{\emptyset} \twoheadrightarrow N_A^{\emptyset}$. In other words, we put $(\mathcal{F}(S_0)(f))(A)(m) = \widetilde{f}(m)_A$. It is easy to see that this does not depend on a lift $\widetilde{f}$.

We prove that the collection $((\mathcal{F}(S_0)(f))(A))_{A \in \mathcal{A}}$ preserves $(\mathcal{F}(S_0)(M))(A,\alpha)$. Take $X \in \widetilde{\mathcal{K}}_P(S_0^{\alpha})$ and $\varphi\colon X \xrightarrow{\sim} M^{\alpha}$ as in the definition of $(\mathcal{F}(S_0)(M))(A,\alpha)$. We also take $\psi\colon Y \xrightarrow{\sim} N^{\alpha}$ where $Y \in \mathcal{K}_P(S_0^{\alpha})$ satisfies $Y = \bigoplus_{\Omega \in W'_{\alpha,\mathrm{aff}}} (Y \cap \bigoplus_{A \in \Omega} Y_A^{\emptyset})$. Let $(x_1, x_2) \in (\mathcal{F}(S_0)(M))(A,\alpha)$. There exists $x \in X_{\geq A}$ such that $(x_1, x_2) = (\varphi(x_A)_A, \varphi(x_{\alpha\uparrow A})_{\alpha\uparrow A})$.

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We may assume $x \in \bigoplus_{A' \in W'_{\alpha,\mathrm{aff}} A} X^{\emptyset}_{A'}$. We put $\widetilde{g} = \psi^{-1} \circ \widetilde{f}$. Then, $(\mathcal{F}(S_0)(f))(A)(x_1) = \widetilde{f}(\varphi(x_A)_A)_A = \psi(\widetilde{g}(\varphi(x_A)_A))_A$. By Lemma 3.13 (2) to $\varphi(x_A)_A$, we have $\psi(\widetilde{g}(\varphi(x_A)_A))_A = \psi(\widetilde{g}(\varphi(x_A)_A)_A)_A$ and again by Lemma 3.13 (1), (2), this is equal to $\psi(\widetilde{g}(\varphi(x))_A)_A$. Similarly, we have $(\mathcal{F}(S_0)(f))(\alpha \uparrow A)(x_2) = \psi(\widetilde{g}(\varphi(x))_{\alpha\uparrow A})_{\alpha\uparrow A}$. Since the element $(\psi(\widetilde{g}(\varphi(x))_A)_A, \psi(\widetilde{g}(\varphi(x))_{\alpha\uparrow A})_{\alpha\uparrow A})$ is the image of $\widetilde{g}(\varphi(x)) \in Y_{\geq A}$ under $Y_{\geq A} \to Y^{\emptyset}_A \oplus Y^{\emptyset}_{\alpha\uparrow A} \simeq N^{\emptyset}_A \oplus N^{\emptyset}_{\alpha\uparrow A}$, it is in $(\mathcal{F}(S_0)(N))(A,\alpha)$. Hence, we have proved that the collection $((\mathcal{F}(S_0)(f))(A))_{A\in\mathcal{A}}$ defines a morphism $\mathcal{F}(S_0)(M) \to \mathcal{F}(S_0)(N)$. Hence, $\mathcal{F}(S_0)$ is a functor.

Put $\mathcal{F} = \mathcal{F}(S)$ and $\mathcal{F}^* = \mathcal{F}(S^*)$ for $* \in \{\emptyset\} \cup \Delta$.

**Proposition 3.14.** *We have $\mathcal{F}(M * B_s) \simeq \vartheta_s(\mathcal{F}(M))$.*

**Proof.** Before giving a proof, we give some notation. Fix $\alpha \in \Delta$ and $M \in \mathcal{K}(S_0)$. Put $M^{(\Omega)} = M^{\alpha} \cap \bigoplus_{A\in\Omega} M^{\emptyset}_A$ for $\Omega \in W'_{\alpha,\mathrm{aff}} \backslash \mathcal{A}$. Then, if $M^{\alpha} = \bigoplus_{\Omega \in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}} (M^{\alpha} \cap \bigoplus_{A\in\Omega} M^{\emptyset}_A)$, then $(\mathcal{F}(M))(A,\alpha)$ is the image of $M^{(W'_{\alpha,\mathrm{aff}}A)}$ in $M^{\emptyset}_A \oplus M^{\emptyset}_{\alpha\uparrow A}$. As $\operatorname{supp} M^{(W'_{\alpha,\mathrm{aff}}A)} \subset W'_{\alpha,\mathrm{aff}} A$ and $W'_{\alpha,\mathrm{aff}} \cap [A, \alpha\uparrow A] = \{A, \alpha\uparrow A\}$, we have $(\mathcal{F}(M))(A,\alpha) \simeq M^{(W'_{\alpha,\mathrm{aff}}A)}_{[A,\alpha\uparrow A]}$.

Take $\delta_s \in \Lambda^{\vee}_{\mathbb{K}}$ such that $\langle \alpha_s, \delta_s \rangle = 1$ and put $b_e = (\alpha_s^{\vee})^{-1}(\delta_s \otimes 1 - 1 \otimes s(\delta_s))$ and $b_s = (\alpha_s^{\vee})^{-1}(\delta_s \otimes 1 - 1 \otimes \delta_s)$. Note that this does not depend on a choice of $\delta_s$. We fix $(B_s)^{\emptyset}_e \simeq R^{\emptyset}$ and $(B_s)^{\emptyset}_s \simeq R^{\emptyset}$ as

$$R^{\emptyset} \ni 1 \mapsto b_e \in (B_s)^{\emptyset}_e,$$
$$R^{\emptyset} \ni 1 \mapsto b_s \in (B_s)^{\emptyset}_s.$$

We have $(M * B_s)^{\emptyset}_A = M^{\emptyset}_A \otimes (B_s)^{\emptyset}_e \oplus M^{\emptyset}_{As} \otimes (B_s)^{\emptyset}_s \simeq M^{\emptyset}_A \oplus M^{\emptyset}_{As} = \vartheta_s(\mathcal{F}(M))(A)$. Here, we use the above fixed isomorphisms. We check $\mathcal{F}(M * B_s)(A,\alpha) \simeq \vartheta_s(\mathcal{F}(M))(A,\alpha)$ under this isomorphism. We may assume $M^{\alpha} = \bigoplus_{\Omega\in W'_{\alpha,\mathrm{aff}}\backslash\mathcal{A}} (M^{\alpha} \cap \bigoplus_{A\in\Omega} M^{\emptyset}_A)$.

First, we assume that $As \notin W'_{\alpha,\mathrm{aff}} A$. Then, we have $(M * B_s)^{(W'_{\alpha,\mathrm{aff}}A)} = M^{(W'_{\alpha,\mathrm{aff}}A)} \otimes b_e \oplus M^{(W'_{\alpha,\mathrm{aff}}As)} \otimes b_s$ by Lemma 2.23. As $b_e \in (B_s)^{\emptyset}_e$ (resp. $b_s \in (B_s)^{\emptyset}_s$) and $[A, \alpha\uparrow A]s \cap W'_{\alpha,\mathrm{aff}} As = [As, \alpha\uparrow As] \cap W'_{\alpha,\mathrm{aff}} As$, we have

$$(M * B_s)^{(W'_{\alpha,\mathrm{aff}}A)}_{[A,\alpha\uparrow A]} = M^{(W'_{\alpha,\mathrm{aff}}A)}_{[A,\alpha\uparrow A]} \otimes b_e \oplus M^{(W'_{\alpha,\mathrm{aff}}As)}_{[As,\alpha\uparrow As]} \otimes b_s$$

Therefore, $\mathcal{F}(M * B_s)(A,\alpha) = \mathcal{F}(M)(A,\alpha) \oplus \mathcal{F}(M)(As,\alpha) = \vartheta_s(\mathcal{F}(M))(A,\alpha)$.

Next, assume that $As = \alpha \uparrow A$. Then, we have $[A, \alpha\uparrow A] = [A, As] = \{A, As\}$. Hence, $\mathcal{F}(M * B_s)(A,\alpha) = (M * B_s)^{\alpha}_{\{A,As\}}$. Since $[A, As] = \{A, As\}$ is $s$-invariant, by Lemma 2.25, we have $(M * B_s)^{\alpha}_{[A,As]} \simeq M^{\alpha}_{[A,As]} \otimes_R B_s = \mathcal{F}(M)(A,\alpha) \otimes_R B_s$. Our claim is that the image of $M^{\alpha}_{\{A,As\}} \otimes_R B_s$ in $(M_{\{A,As\}} * B_s)^{\emptyset} \simeq (M^{\emptyset}_A \oplus M^{\emptyset}_{As}) \oplus (M^{\emptyset}_{As} \oplus M^{\emptyset}_A)$ is equal to $\{(x,y) \in M^{\alpha}_{\{A,As\}} \mid x - y \in \alpha^{\vee} M^{\alpha}_{\{A,As\}}\}$. We write the image of $m \in M$ in $M^{\emptyset}_{A'}$ by $m_{A'}$ for $A' \in \mathcal{A}$. We have $M^{\alpha}_{\{A,As\}} \otimes_R B_s = M^{\alpha}_{\{A,As\}} \otimes_{R^s} R$ and the image of $m_1 \otimes 1 + m_2 \otimes \delta_s \in M^{\alpha}_{\{A,As\}} \otimes_{R^s} R$ in $(M^{\emptyset}_A \oplus M^{\emptyset}_{As}) \oplus (M^{\emptyset}_{As} \oplus M^{\emptyset}_A)$ is

$$((m_{1,A} + \delta_s^A m_{2,A}, m_{1,As} + \delta_s^A m_{2,As}), (m_{1,As} + s(\delta_s)^A m_{2,As}, m_{1,A} + s(\delta_s)^A m_{1,A})).$$

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Therefore, we have

$$(m_{1,A}+\delta_s^A m_{2,A}, m_{1,As}+\delta_s^A m_{2,As})-(m_{1,A}+s(\delta_s)^A m_{2,A}, m_{1,As}+s(\delta_s)^A m_{2,As}) \\ =(\alpha_s^\vee)^A(m_{2,A}, m_{2,As})$$

which is in $\alpha^\vee M^\alpha_{\{A,As\}}$ since $(\alpha_s^\vee)^A\in\{\pm1\}\alpha^\vee$. From this formula it is easy to see the reverse inclusion.

Finally, we assume that $As=\alpha\downarrow A$. Note that $As<A<\alpha\uparrow A<(\alpha\uparrow A)s$. Put $N=M^{(W'_{\alpha,\mathrm{aff}}A)}$. We have $\mathcal{F}(N*B_s)(A,\alpha)\subset\mathcal{F}(N*B_s)(A)\oplus\mathcal{F}(N*B_s)(\alpha\uparrow A)=(N^\emptyset_{As}\oplus N^\emptyset_A)\oplus(N^\emptyset_{\alpha\uparrow A}\oplus N^\emptyset_{(\alpha\uparrow A)s})$. We describe the image of $(N*B_s)_{[A,\alpha\uparrow A]}$ in $(N^\emptyset_{As}\oplus N^\emptyset_A)\oplus(N^\emptyset_{\alpha\uparrow A}\oplus N^\emptyset_{(\alpha\uparrow A)s})$, or equivalently the image of $(N*B_s)_I$ where $I=\{A'\in\mathcal{A}\mid A'\geq As\}\setminus\{As\}$.

Set $I'=\{A'\in\mathcal{A}\mid A'\geq As\}$. Then, $I'\supset I$ and $I'$ is $s$-invariant. Hence, $(N*B_s)_{I'}=N_{I'}\otimes B_s=N_{I'}\otimes_{R^s}R$ by Lemma 2.25. Consider the projection $(N*B_s)_{I'}\to(N*B_s)_{As}\oplus(N*B_s)_A\oplus(N*B_s)_{\alpha\uparrow A}=(N^\emptyset_{As}\oplus N^\emptyset_A)\oplus(N^\emptyset_A\oplus N^\emptyset_{As})\oplus(N^\emptyset_{\alpha\uparrow A}\oplus N^\emptyset_{(\alpha\uparrow A)s})$. This is given by

$$\begin{array}{ccc} N_{I'}\otimes_{R^s}R & \longrightarrow & (N^\emptyset_{As}\oplus N^\emptyset_A)\oplus(N^\emptyset_A\oplus N^\emptyset_{As})\oplus(N^\emptyset_{\alpha\uparrow A}\oplus N^\emptyset_{(\alpha\uparrow A)s}) \\ \Uparrow & & \Uparrow \\ m\otimes f & \longmapsto & ((m_{As}f, m_A s(f)),(m_A f, m_{As}s(f)),(m_{\alpha\uparrow A}f, m_{(\alpha\uparrow A)s}s(f))). \end{array}$$

Any element in $N_{I'}\otimes_{R^s}R$ is written as $m_1\otimes1+m_2\otimes\delta_s$ for $m_1,m_2\in N_{I'}$. It is in $(N*B_s)_I$ if and only the projection to $(N*B_s)^\emptyset_{As}\simeq N^\emptyset_{As}\oplus N^\emptyset_A$ is zero. This projection is given by $(m_{1,As}+s_\alpha(\delta_s^A)m_{2,As}, m_{1,A}+s_\alpha(\delta_s^A)m_{2,A})$. Hence, it is sufficient to prove that the image of

$$\{m_1\otimes1+m_2\otimes\delta_s\in N_{I'}\otimes_{R^s}R\mid(m_1+s_\alpha(\delta_s^A)m_2)_{A'}=0\text{ for }A'=A,As\}$$

in $(N*B_s)^\emptyset_A\oplus(N*B_s)^\emptyset_{\alpha\uparrow A}=N^\emptyset_A\oplus N^\emptyset_{As}\oplus N^\emptyset_{\alpha\uparrow A}\oplus N^\emptyset_{(\alpha\uparrow A)s}$ is $\alpha^\vee N_{[As,A]}\oplus N_{[\alpha\uparrow A,(\alpha\uparrow A)s]}$ (note that $A=\alpha\uparrow(As)$ and $(\alpha\uparrow A)s=\alpha\uparrow(\alpha\uparrow A)$).

The image of $m_1\otimes1+m_2\otimes\delta_s$ in $N^\emptyset_A\oplus N^\emptyset_{As}\oplus N^\emptyset_{\alpha\uparrow A}\oplus N^\emptyset_{(\alpha\uparrow A)s}$ is given by

$$(m_{1,A}+\delta_s^A m_{2,A}, m_{1,As}+\delta_s^A m_{2,As}, m_{1,\alpha\uparrow A}+s_\alpha(\delta_s^A)m_{2,\alpha\uparrow A}, m_{1,(\alpha\uparrow A)s}+s_\alpha(\delta_s^A)m_{2,(\alpha\uparrow A)s}).$$

Define $\varepsilon\in\{\pm1\}$ by $\alpha_s^A=\varepsilon\alpha$. Since $m_{1,A}+s_\alpha(\delta_s^A)m_{2,A}=0$, we have $m_{1,A}+\delta_s^A m_{2,A}=(\delta_s^A-s_\alpha(\delta_s^A))m_{2,A}=\varepsilon\alpha^\vee m_{2,A}$. By the same argument, we have $m_{1,As}+\delta_s^A m_{2,As}=\varepsilon\alpha^\vee m_{2,As}$. Therefore, $(m_{1,A}+\delta_s^A m_{2,A}, m_{1,As}+\delta_s^A m_{2,As})=\alpha^\vee(\varepsilon m_{2,A},\varepsilon m_{2,As})\in\alpha^\vee N^\emptyset_{[A,As]}$. Therefore, the image is in $\alpha^\vee N_{[As,A]}\oplus N_{[\alpha\uparrow A,(\alpha\uparrow A)s]}$.

However, let $m_1'\in N_{[As,A]}$ and $m_2'\in N_{[\alpha\uparrow A,(\alpha\uparrow A)s]}$. Take a lift $m_1\in N_{I'}$ (resp. $m_2\in M_{I''}$) of $m_1'$ (resp. $m_2'$) where $I''=\{A'\in A\mid A'\geq\alpha\uparrow A\}$. Put $n=m_2\otimes1+\varepsilon(m_1\otimes\delta_s-(s(\delta_s))^A m_1\otimes1)$. Then, since $m_2\in M_{I''}$, $m_{2,A}=0$, $m_{2,As}=0$. Now it is straightforward to see $n\in(M*B_s)_I$ and the image of $n$ is $(\alpha^\vee m'_{1,A},\alpha^\vee m'_{1,As}, m'_{2,\alpha\uparrow A}, m'_{2,(\alpha\uparrow A)s})$. We get the proposition. □

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### 3.5. Some calculations of homomorphisms

Assume that $\mathbb{K}$ is a complete local Noetherian integral domain. In this subsection, we fix a flat commutative graded $S$-algebra $S_0$. We define some morphisms as follows. These will be used only in this subsection. Let $A \in \mathcal{A}$ and $\alpha \in \Delta^+$.

$$\begin{aligned} i_0 &\colon Q_{A,\alpha} \to Q_{A,\alpha} \quad (f,g) \mapsto (0,\alpha^\vee g),\\ i_0^+ &\colon Q_{A,\alpha} \to Q_{\alpha\uparrow A,\alpha} \quad (f,g) \mapsto (g,f),\\ i_0^- &\colon Q_{A,\alpha} \to Q_{\alpha\downarrow A,\alpha} \quad (f,g) \mapsto (0,\alpha^\vee f). \end{aligned}$$

It is straightforward to see that these are morphisms in $\widetilde{\mathcal{K}}$. We use the same letter for the images of these morphisms in $\mathcal{K}$.

**Lemma 3.15.** *We have* $\mathrm{End}^\bullet_{\mathcal{K}(S_0)}(S_0 \otimes_S Q_{A,\alpha}) = \mathrm{End}^\bullet_{\widetilde{\mathcal{K}}(S_0)}(S_0 \otimes_S Q_{A,\alpha}) = S_0\,\mathrm{id} \oplus S_0 i_0$.

**Proof.** Put $M = S_0 \otimes_S Q_{A,\alpha}$. Note that $\mathrm{supp}_{\mathcal{A}}(M) = \{A, \alpha\uparrow A\}$. Let $\varphi \in \mathrm{End}_{\widetilde{\mathcal{K}}(S_0)}(S_0 \otimes_S Q_{A,\alpha})$. We have $\varphi(M_A^\emptyset) \subset \bigoplus_{A' \in A + \mathbb{Z}\Delta} M_{A'}^\emptyset = M_A^\emptyset$. By the same argument, we also have $\varphi(M_{\alpha\uparrow A}^\emptyset) \subset M_{\alpha\uparrow A}^\emptyset$. Therefore, $\varphi$ preserves $M_{A'}^\emptyset$ for any $A' \in \mathcal{A}$. Hence, we get the first equality of the lemma.

We prove $\varphi \in S_0\,\mathrm{id} + S_0 i_0$. Since $\varphi$ preserves $M_{A'}^\emptyset$, we have $\varphi(f,g) = (\varphi_1(f), \varphi_2(g))$ for some $\varphi_1, \varphi_2 \colon S_0^\emptyset \to S_0^\emptyset$. Restricting to $\{(f,g) \in M \mid g = 0\} = \alpha^\vee S_0 \oplus 0$, $\varphi_1$ sends $\alpha^\vee S_0$ to $\alpha^\vee S_0$. Therefore, it is given by $\varphi_1(f) = cf$ for some $c \in S_0$. Replacing $\varphi$ with $\varphi - c\,\mathrm{id}$, we may assume $\varphi_1 = 0$. The image of $\varphi$ is contained in $\{(f,g) \in M \mid f = 0\} = 0 \oplus \alpha^\vee S_0$. Hence, $\varphi_2(g) = \alpha^\vee d g$ for some $d \in S_0$ and we have $\varphi = d i_0$. □

**Lemma 3.16.** *We have* $\mathrm{Hom}^\bullet_{\mathcal{K}(S_0)}(S_0 \otimes_S Q_{A,\alpha}, S_0 \otimes_S Q_{\alpha\uparrow A,\alpha}) = S_0 i_0^+$.

**Proof.** Let $\varphi \colon S_0 \otimes_S Q_{A,\alpha} \to S_0 \otimes_S Q_{\alpha\uparrow A,\alpha}$ be a morphism in $\widetilde{\mathcal{K}}(S_0)$. By a similar argument of the proof of Lemma 3.15, $\varphi$ is given by $\varphi(f,g) = (\varphi_1(g), \varphi_2(f))$ for $\varphi_i \colon S_0^\emptyset \to S_0^\emptyset$ such that $\varphi_i(\alpha^\vee S_0) \subset \alpha^\vee S_0$ for $i = 1,2$. Hence, $\varphi_1(f) = cf$ for some $c \in S_0$. It is clear that $\varphi - c i_0^+$ is zero as a morphism in $\mathcal{K}(S_0)$. Hence, we get the lemma. □

**Lemma 3.17.** *We have* $\mathrm{Hom}^\bullet_{\mathcal{K}(S_0)}(S_0 \otimes_S Q_{A,\alpha}, S_0 \otimes_S Q_{\alpha\downarrow A,\alpha}) = S_0 i_0^-$.

**Proof.** Set $M = S_0 \otimes_S Q_{A,\alpha}$ and $N = S_0 \otimes_S Q_{\alpha\downarrow A,\alpha}$ and let $\varphi \colon M \to N$ be a morphism in $\widetilde{\mathcal{K}}(S_0)$. We have $\varphi(M_{\alpha\uparrow A}^\emptyset) \subset \bigoplus_{A' \geq \alpha\uparrow A} N_{A'}^\emptyset = 0$ and $\varphi(M_A^\emptyset) \subset \bigoplus_{A' \in A + \mathbb{Z}\Delta} N_{A'}^\emptyset = N_A^\emptyset$. Hence $\varphi(f,g) = (0, \varphi_1(f))$ for some $\varphi_1 \colon S_0^\emptyset \to S_0^\emptyset$. For any $f \in S_0$ we have $\varphi(f,f) = (0, \varphi_1(f)) \in N$. Hence, $\varphi_1(f) \in \alpha^\vee S_0$. Therefore, $\varphi_1(f) = c\alpha^\vee f$ for some $c \in S_0$. Hence, $\varphi = c i_0^-$. □

**Lemma 3.18.** *If* $A_1 \neq \alpha\downarrow A_2, A_2, \alpha\uparrow A_2$, *then* $\mathrm{Hom}_{\mathcal{K}(S_0)}(Q_{A_1,\alpha}, Q_{A_2,\alpha}) = 0$.

**Proof.** It follows from $\mathrm{supp}_{\mathcal{A}}(Q_{A_1,\alpha}) \cap \mathrm{supp}_{\mathcal{A}}(Q_{A_2,\alpha}) = \emptyset$. □

Next, we calculate homomorphisms in $\mathcal{K}_{\mathrm{AJS}}$. Set $\mathcal{Q}_{A,\alpha} = \mathcal{F}(Q_{A,\alpha})$.

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**Lemma 3.19.** *The object $\mathcal{Q}_{A,\alpha}$ is given by*

$$\mathcal{Q}_{A,\alpha}(A') = \begin{cases} S^{\emptyset} & (A' = A, \alpha\uparrow A), \\ 0 & (\textit{otherwise}), \end{cases}$$

$$\mathcal{Q}_{A,\alpha}(A',\beta) = \begin{cases} S^{\beta}\oplus 0 & (A' = A, \alpha\uparrow A,\ \beta\neq\alpha), \\ 0\oplus S^{\beta} & (\beta\uparrow A' = A, \alpha\uparrow A,\ \beta\neq\alpha), \\ \alpha^{\vee}S^{\alpha}\oplus 0 & (A' = \alpha\uparrow A,\ \beta=\alpha), \\ \{(f,g)\in (S^{\alpha})^2 \mid f\equiv g \pmod{\alpha^{\vee}}\} & (A' = A,\ \beta = \alpha), \\ 0\oplus S^{\alpha} & (A' = \alpha\downarrow A,\ \beta=\alpha), \\ 0 & (\textit{otherwise}). \end{cases}$$

**Proof.** The formula of $\mathcal{Q}_{A,\alpha}(A)$ is obvious. If $\beta\neq\alpha$, then $S^{\beta}\otimes_S \mathcal{Q}_{A,\alpha} = S^{\beta}\oplus S^{\beta}$. Hence, the formula of $\mathcal{Q}_{A,\alpha}(A',\beta)$ with $\beta\neq\alpha$ follows. The other formulas follow from a direct calculation. □

Set $\iota_0 = \mathcal{F}(i_0)$, $\iota_0^+ = \mathcal{F}(i_0^+)$, $\iota_0^- = \mathcal{F}(i_0^-)$. These morphisms are described as follows.

$$\begin{aligned} \iota_0\colon \mathcal{Q}_{A,\alpha}\to\mathcal{Q}_{A,\alpha} &\quad (\iota_0)_A = 0, (\iota_0)_{\alpha\uparrow A} = \alpha\,\mathrm{id}, \\ \iota_0^+\colon \mathcal{Q}_{A,\alpha}\to\mathcal{Q}_{\alpha\uparrow A,\alpha} &\quad (\iota_0^+)_A = 0, (\iota_0^+)_{\alpha\uparrow A} = \mathrm{id}, \\ \iota_0^-\colon \mathcal{Q}_{A,\alpha}\to\mathcal{Q}_{\alpha\downarrow A,\alpha} &\quad (\iota_0^-)_A = \alpha\,\mathrm{id}, (\iota_0^-)_{\alpha\uparrow A} = 0. \end{aligned}$$

**Lemma 3.20.** *We have* $\mathrm{End}^{\bullet}_{\mathcal{K}_{\mathrm{AJS}}(S_0)}(S_0\otimes_S\mathcal{Q}_{A,\alpha}) = S_0\,\mathrm{id}\oplus S_0\iota_0$.

**Proof.** Set $\mathcal{M} = S_0\otimes_S\mathcal{Q}_{A,\alpha}$ and let $\varphi\colon\mathcal{M}\to\mathcal{M}$ be a morphism. Since $\mathcal{M}(A') = 0$ for $A'\neq A,\alpha\uparrow A$, we have $\varphi_{A'} = 0$ for such $A'$. The morphism $\varphi$ preserves $\mathcal{M}(\beta\downarrow A,\beta) = 0\oplus S_0^{\beta}$ for any $\beta\in\Delta^+$. Hence, $\varphi_A(S_0^{\beta})\subset S_0^{\beta}$. Therefore, $\varphi_A(S_0)\subset S_0$ and hence, $\varphi_A = c\,\mathrm{id}$ for some $c\in S_0$. We also have $\varphi_{\alpha\uparrow A} = d\,\mathrm{id}$ for some $d\in S_0$.

We prove $\varphi\in S_0\,\mathrm{id} + S_0\iota_0$. By replacing $\varphi$ with $\varphi - c\,\mathrm{id}$, we may assume $\varphi_A = 0$. We have $(\varphi_A(f),\varphi_{\alpha\uparrow A}(g))\in\mathcal{M}(A,\alpha)$ for any $(f,g)\in\mathcal{M}(A,\alpha)$. Since $\varphi_A(f) = 0$, we have $\varphi_{\alpha\uparrow A}(g)\in\alpha^{\vee}S_0^{\alpha}$. Therefore, $d\in\alpha^{\vee}S_0^{\alpha}\cap S_0 = \alpha^{\vee}S_0$. We have $\varphi = (d/\alpha^{\vee})\iota_0$. □

**Lemma 3.21.** *We have* $\mathrm{Hom}^{\bullet}_{\mathcal{K}_{\mathrm{AJS}}(S_0)}(S_0\otimes_S\mathcal{Q}_{A,\alpha}, S_0\otimes_S\mathcal{Q}_{\alpha\uparrow A,\alpha}) = S_0\iota_0^+$.

**Proof.** Set $\mathcal{M} = S_0\otimes_S\mathcal{Q}_{A,\alpha}$ and $\mathcal{N} = S_0\otimes_S\mathcal{Q}_{\alpha\uparrow A,\alpha}$. Let $\varphi\colon\mathcal{M}\to\mathcal{N}$ be a morphism. Then, $\varphi_{A'} = 0$ for $A'\neq\alpha\uparrow A$. For $\beta\in\Delta^+\setminus\{\alpha\}$, since $\varphi$ sends $\mathcal{M}(\alpha\uparrow A,\beta) = S_0^{\beta}\oplus 0$ to $\mathcal{N}(\alpha\uparrow A,\beta) = S_0^{\beta}\oplus 0$, we have $\varphi_{\alpha\uparrow A}(S_0^{\beta})\subset S_0^{\beta}$. Since $\varphi$ sends $\mathcal{M}(A,\alpha)$ to $\mathcal{N}(A,\alpha) = 0\oplus S^{\alpha}$, $\varphi_{\alpha\uparrow A}(S^{\alpha})\subset S^{\alpha}$. Hence, $\varphi_{\alpha\uparrow A}\in S_0\,\mathrm{id}$ and we get the lemma. □

**Lemma 3.22.** *We have* $\mathrm{Hom}^{\bullet}_{\mathcal{K}_{\mathrm{AJS}}(S_0)}(S_0\otimes_S\mathcal{Q}_{A,\alpha}, S_0\otimes_S\mathcal{Q}_{\alpha\downarrow A,\alpha}) = S_0 i_0^-$.

**Proof.** Set $\mathcal{M} = S_0\otimes_S\mathcal{Q}_{A,\alpha}$ and $\mathcal{N} = S_0\otimes_S\mathcal{Q}_{\alpha\downarrow A,\alpha}$. Let $\varphi\colon\mathcal{M}\to\mathcal{N}$ be a morphism. Then, $\varphi_{A'} = 0$ for $A'\neq A$. For $\beta\in\Delta^+\setminus\{\alpha\}$, $\varphi$ sends $\mathcal{M}(A,\beta) = 0\oplus S_0^{\beta}$ to $\mathcal{N}(A,\beta) = S_0^{\beta}\oplus 0$. Hence, $\varphi_A(S_0^{\beta})\subset S_0^{\beta}$. The morphism $\varphi$ sends $\mathcal{M}(A,\alpha)$ to $\mathcal{N}(A,\alpha) = \alpha^{\vee}S^{\alpha}\oplus 0$. Hence, $\varphi_A(S_0^{\alpha})\subset\alpha^{\vee}S_0^{\alpha}$. Therefore, $\varphi_A\in\alpha^{\vee}S_0\,\mathrm{id}$ and we get the lemma. □

**Lemma 3.23.** *If $A_1\neq\alpha\downarrow A_2, A_2,\alpha\uparrow A_2$, then* $\mathrm{Hom}_{\mathcal{K}_{\mathrm{AJS}}(S_0)}(\mathcal{Q}_{A_1,\alpha},\mathcal{Q}_{A_2,\alpha}) = 0$.

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**Proof.** It follows from there is no $A$ such that $\mathcal{Q}_{A_1,\alpha}(A) \neq 0$ and $\mathcal{Q}_{A_2,\alpha}(A) \neq 0$. $\square$

Summarizing the calculations in this subsection, we get the following.

**Lemma 3.24.** *The functor $\mathcal{F}^\alpha = \mathcal{F}(S^\alpha)$ induces an isomorphism* $\mathrm{Hom}^\bullet_{\mathcal{K}(S_0)}(S_0 \otimes_S Q_{A_1,\alpha}, S_0 \otimes_S Q_{A_2,\alpha}) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathcal{K}_{\mathrm{AJS}}(S_0)}(S_0 \otimes_S \mathcal{F}^\alpha(Q_{A_1,\alpha}), S_0 \otimes_S \mathcal{F}^\alpha(Q_{A_2,\alpha}))$.

### 3.6. Equivalence

Assume that $\mathbb{K}$ is a complete local Noetherian integral domain.

**Lemma 3.25.** *The functor $\mathcal{F}^\alpha \colon \mathcal{K}_P^\alpha \to \mathcal{K}_{\mathrm{AJS}}^\alpha$ is fully faithful for $\alpha \in \Delta$.*

**Proof.** By Corollary 3.8 and Proposition 3.12, we may assume $M = Q^\alpha_{A_1,\alpha}$ and $N = Q^\alpha_{A_2,\alpha}$, where $A_1, A_2 \in \mathcal{A}$. Hence, the lemma follows from Lemma 3.24. $\square$

**Proposition 3.26.** *The functor $\mathcal{F} \colon \mathcal{K}_P \to \mathcal{K}_{\mathrm{AJS}}$ is fully faithful.*

**Proof.** Let $M, N \in \mathcal{K}_P$ and we prove that $\mathcal{F} \colon \mathrm{Hom}^\bullet_{\mathcal{K}_P}(M,N) \to \mathrm{Hom}^\bullet_{\mathcal{K}_{\mathrm{AJS}}}(\mathcal{F}(M), \mathcal{F}(N))$ is an isomorphism. By the diagram

$$\begin{array}{ccc} \mathrm{Hom}^\bullet_{\mathcal{K}_P}(M,N) & \xrightarrow{\quad\mathcal{F}\quad} & \mathrm{Hom}^\bullet_{\mathcal{K}_{\mathrm{AJS}}}(M,N) \\ & \searrow \qquad \swarrow & \\ & \prod_{A \in \mathcal{A}} \mathrm{Hom}^\bullet_{S^\emptyset}(M^\emptyset_A, N^\emptyset_A), & \end{array}$$

$\mathcal{F}$ is injective (the injectivity of two morphisms in the above diagram follows from the definitions).

We prove that $\mathcal{F}$ is surjective. For $\nu \in X_{\mathbb{K}}$, let $S_{(\nu)}$ be the localization at the prime ideal $(\nu) \subset S$. Since $\mathrm{Hom}^\bullet_{\mathcal{K}_P}(M,N)$ is graded free, we have $\mathrm{Im}(\mathcal{F}) = \bigcap_{\nu \in X_{\mathbb{K}}} S_{(\nu)} \otimes_S \mathrm{Im}(\mathcal{F})$. By Corollary 3.8, we have $S_{(\nu)} \otimes_S \mathrm{Im}(\mathcal{F}) = \mathrm{Im}(\mathcal{F}(S_{(\nu)}))$. Since any $S_{(\nu)}$ is an $S^\alpha$-algebra for some $\alpha \in \Delta$, by Proposition 3.26, we have $\mathrm{Im}(\mathcal{F}(S_{(\nu)})) = \mathrm{Hom}^\bullet_{\mathcal{K}_{\mathrm{AJS}}(S_{(\nu)})}(\mathcal{F}(S_{(\nu)})(S_{(\nu)} \otimes_S M), \mathcal{F}(S_{(\nu)})(S_{(\nu)} \otimes_S N))$. Therefore, $\mathcal{F}$ is surjective since $\bigcap_{\nu \in X_{\mathbb{K}}} \mathrm{Hom}^\bullet_{\mathcal{K}_{\mathrm{AJS}}(S_{(\nu)})}(\mathcal{F}(S_{(\nu)})(S_{(\nu)} \otimes_S M), \mathcal{F}(S_{(\nu)})(S_{(\nu)} \otimes_S N)) \supset \mathrm{Hom}^\bullet_{\mathcal{K}_{\mathrm{AJS}}}(\mathcal{F}(M), \mathcal{F}(N))$. $\square$

Set $\mathcal{Q}_\lambda = \mathcal{F}(Q_\lambda)$. Let $\mathcal{K}_{\mathrm{AJS},P}$ be the full subcategory of $\mathcal{K}_{\mathrm{AJS}}$ consisting of direct summands of direct sums of objects of a form $(\vartheta_{s_1} \circ \cdots \circ \vartheta_{s_l})(\mathcal{Q}_\lambda)(n)$ for $s_1, \ldots, s_l \in S_{\mathrm{aff}}$, $\lambda \in (\mathbb{R}\Delta)_{\mathrm{int}}$ and $n \in \mathbb{Z}$. By Proposition 3.14 and 3.26, we get the following theorem.

**Theorem 3.27.** *We have $\mathcal{K}_P \simeq \mathcal{K}_{\mathrm{AJS},P}$. In particular, the category $\mathcal{S}\mathrm{Bimod}$ acts on $\mathcal{K}_{\mathrm{AJS},P}$.*

### 3.7. Representation Theory

In this subsection, we assume that $\mathbb{K}$ is an algebraically closed field of $p > h$, where $h$ is the Coxeter number. Let $G$ be a connected reductive group over $\mathbb{K}$ and $T$ a maximal torus of $G$ with the root datum $(X, \Delta, X^\vee, \Delta^\vee)$. The Lie algebra $\mathfrak{g}$ of $G$ has a structure of a $p$-Lie algebra. Let $U^{[p]}(\mathfrak{g})$ be the restricted enveloping algebra. Let $\widehat{S}$ be the completion of $S$ at

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the augmentation ideal. For $S_0 = \widehat{S}$ or $\mathbb{K}$, let $\mathcal{C}_{S_0}$ be the category defined in [AJS94]. The category $\mathcal{C}_{\mathbb{K}}$ is equivalent to the category of $G_1T$-modules, where $G_1$ is the kernel of the Frobenius morphism. Let $Z_{S_0}(\lambda) \in \mathcal{C}_{S_0}$ be the baby Verma module with highest weight $\lambda$ and $P_{S_0}(\lambda) \in \mathcal{C}_{S_0}$ the indecomposable projective module such that $\mathbb{K} \otimes_{S_0} P_{S_0}(\lambda)$ is the projective cover of the irreducible module with highest weight $\lambda$. Such objects exist by [AJS94, 4.19 Theorem] when $S_0 = \widehat{S}$.

We fix an alcove $A_0 \in \mathcal{A}$ and $\lambda_0 \in X \cap (pA_0 - \rho)$, where $\rho$ is the half sum of positive roots and $pA_0 = \{pa \mid a \in A_0\}$. For $S_0 = \widehat{S}$ or $\mathbb{K}$, let $\mathcal{C}_{S_0,0}$ be the full subcategory of $\mathcal{C}_{S_0}$ consisting of quotients of modules of a form $\bigoplus_{w \in W'_{\mathrm{aff}}} P_{S_0}(w \cdot_p \lambda_0)^{n_w}$ where $w \cdot_p \lambda_0 = pw((\lambda_0 + \rho)/p) - \rho$ and $n_w \in \mathbb{Z}_{\geq 0}$. Then, the cateogory $\mathcal{C}_{S_0,0}$ is a direct summand of $\mathcal{C}_{S_0}$. Let $\mathrm{Proj}(\mathcal{C}_{S_0,0}) = \{P \in \mathcal{C}_{S_0,0} \mid P \text{ is projective}\}$.

Let $S_0$ be a flat commutative $S$-algebra which is not necessary graded. We consider the following object: $\mathcal{M} = ((\mathcal{M}(A))_{A \in \mathcal{A}}, (\mathcal{M}(A,\alpha))_{A \in \mathcal{A}, \alpha \in \Delta^+})$, where $\mathcal{M}(A)$ is an $(S_0)^{\emptyset}$-module and $\mathcal{M}(A,\alpha) \subset \mathcal{M}(A) \oplus \mathcal{M}(\alpha \uparrow A)$ is a sub-$(S_0)^{\alpha}$-module (we consider usual modules, not graded ones). Let $\mathcal{K}^{\mathrm{f}}_{\mathrm{AJS}}(S_0)$ be the category of such objects. Starting from this, we can define the functor $\vartheta_s$ and the category $\mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}(S_0)$ in a similar way. Andersen-Jantzen-Soergel proved the following (see [AJS94, 9.4. Proposition] for the full faithfulness. For the essential surjectivity, see the discussion in [AJS94, 16.5]). We modified the functor using [Fie11, Theorem 6.1].

**Theorem 3.28.** *There is an equivalence of the categories* $\mathcal{V}\colon \mathrm{Proj}(\mathcal{C}_{\widehat{S},0}) \xrightarrow{\sim} \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}(\widehat{S})$.

Note that the functor $\mathcal{V}$ is defined explicitly.

Let $\mathbb{K} \otimes_{\widehat{S}} \mathrm{Proj}(\mathcal{C}_{\widehat{S},0})$ be the category defined as follows. The objects of $\mathbb{K} \otimes_{\widehat{S}} \mathrm{Proj}(\mathcal{C}_{\widehat{S},0})$ are the same as those of $\mathrm{Proj}(\mathcal{C}_{\widehat{S},0})$, and the space of homomorphism is defined by

$$\mathrm{Hom}_{\mathbb{K} \otimes_{\widehat{S}} \mathrm{Proj}(\mathcal{C}_{\widehat{S},0})}(M,N) = \mathbb{K} \otimes_{\widehat{S}} \mathrm{Hom}_{\mathrm{Proj}(\mathcal{C}_{\widehat{S},0})}(M,N).$$

**Lemma 3.29.** *We have* $\mathbb{K} \otimes_{\widehat{S}} \mathrm{Proj}(\mathcal{C}_{\widehat{S},0}) \simeq \mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$.

**Proof.** We consider the functor $\mathbb{K} \otimes_{\widehat{S}} \mathrm{Proj}(\mathcal{C}_{\widehat{S},0}) \to \mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$ defined by $P \mapsto \mathbb{K} \otimes_{\widehat{S}} P$. This is essentially surjective by [AJS94, 4.19 Theorem] and fully faithful by [AJS94, 3.3 Proposition]. □

We also define $\mathbb{K} \otimes_{\widehat{S}} \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}(\widehat{S})$ and $\mathbb{K} \otimes_S \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}(S)$ in the same way.

**Lemma 3.30.** *We have the following.*

(1) *The category* $\mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}(S)$ *is equivalent to the category defined as follows: the objects are the same as* $\mathcal{K}_{\mathrm{AJS},P}$*, and the space of homomorphisms is defined by* $\mathrm{Hom}_{\mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}} = \mathrm{Hom}^{\bullet}_{\mathcal{K}_{\mathrm{AJS},P}}$.

(2) *We have* $\mathbb{K} \otimes_{\widehat{S}} \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}(\widehat{S}) \simeq \mathbb{K} \otimes_S \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}(S)$.

**Proof.** (1) is obvious.

For (2), define $\widehat{S} \otimes_S \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}$ in the obvious way. It is sufficient to prove $\mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}(\widehat{S}) \simeq \widehat{S} \otimes_S \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}$. The functor $F\colon \widehat{S} \otimes_S \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P} \to \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}(\widehat{S})$ is defined in an obvious way and it is fully faithful by [AJS94, 14.8 Lemma]. In particular, $F$ sends an indecomposable object to an indecomposable object. We define the category $\mathcal{K}_P^{\mathrm{f}}$ as in (1). Namely, the objects of $\mathcal{K}_P^{\mathrm{f}}$ are the same as those of $\mathcal{K}_P^{\mathrm{f}}$ and we define $\mathrm{Hom}_{\mathcal{K}_P^{\mathrm{f}}} = \mathrm{Hom}^\bullet_{\mathcal{K}_P}$. The indecomposable objects in $\mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P} \simeq \mathcal{K}_P^{\mathrm{f}}$ and $\mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}(\widehat{S}) \simeq \mathrm{Proj}(\mathcal{C}_{\widehat{S},0})$ are both parametrized by $\mathcal{A}$, and it is easy to see that $F$ gives a bijection between the set of indecomposable objects. Therefore, $F$ is essentially surjective. □

Therefore, we get

$$\mathrm{Proj}(\mathcal{C}_{\mathbb{K},0}) \simeq \mathbb{K} \otimes_{\widehat{S}} \mathrm{Proj}(\mathcal{C}_{\widehat{S},0}) \simeq \mathbb{K} \otimes_{\widehat{S}} \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P}(\widehat{S}) \simeq \mathbb{K} \otimes_S \mathcal{K}^{\mathrm{f}}_{\mathrm{AJS},P} \simeq \mathbb{K} \otimes_S \mathcal{K}^{\mathrm{f}}_P.$$

Since the action of $\mathcal{S}\mathrm{Bimod}$ on $\mathcal{K}_P$ is $S$-linear, it gives an action on $\mathbb{K} \otimes_S \mathcal{K}^{\mathrm{f}}_P$. Hence, $\mathcal{S}\mathrm{Bimod}$ acts on $\mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$. With respect to this action, $B_s$ acts as the wall-crossing functor. We write this action as $(M,B) \mapsto M * B$.

Now we prove the following theorem.

**Theorem 3.31.** *There is an action of $\mathcal{S}\mathrm{Bimod}$ on $\mathcal{C}_{\mathbb{K},0}$ such that $B_s$ acts as the wall-crossing functor for $s \in S_{\mathrm{aff}}$.*

Let $L(p\lambda) \in \mathcal{C}_{\mathbb{K}}$ be the irreducible module with highest weight $p\lambda$ for $\lambda \in \mathbb{Z}\Delta$. The category $\mathcal{C}_{\mathbb{K},0}$ has the structure of $\mathbb{Z}\Delta$-category via $M \mapsto M \otimes L(p\lambda)$ for $\lambda \in \mathbb{Z}\Delta$. Fix a projective $\mathbb{Z}\Delta$-generator $P$ of $\mathcal{C}_{\mathbb{K},0}$ and set $\mathcal{E} = \bigoplus_{\lambda \in \mathbb{Z}\Delta} \mathrm{Hom}_{\mathcal{C}_{\mathbb{K},0}}(P, P \otimes L(p\lambda))$. This is a $\mathbb{Z}\Delta$-graded algebra, and $\mathcal{C}_{\mathbb{K},0} \ni M \mapsto \bigoplus_{\lambda \in \mathbb{Z}\Delta} \mathrm{Hom}(P, M \otimes L(p\lambda))$ gives an equivalence of categories between $\mathcal{C}_{\mathbb{K},0}$ and the category of finitely generated $\mathbb{Z}\Delta$-graded right $\mathcal{E}$-modules [AJS94, E.4 Proposition]. Let $\mathrm{Mod}_{\mathbb{Z}\Delta}(\mathcal{E})$ be the the category of finitely generated $\mathbb{Z}\Delta$-graded right $\mathcal{E}$-modules and $\mathrm{Proj}_{\mathbb{Z}\Delta}(\mathcal{E})$ the category of projective objects in $\mathrm{Mod}_{\mathbb{Z}\Delta}(\mathcal{E})$.

**Lemma 3.32.** *We have $(Q * B) \otimes L(p\lambda) \simeq (Q \otimes L(p\lambda)) * B$ for $Q \in \mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$, $B \in \mathcal{S}\mathrm{Bimod}$ and $\lambda \in \mathbb{Z}\Delta$.*

**Proof.** Let $\lambda \in \mathbb{Z}\Delta$. Then, we have a functor $T_\lambda$ (resp. $T_{\mathrm{AJS},\lambda}$) on $\mathcal{K}_P$ (resp. $\mathcal{K}_{\mathrm{AJS},P}$) defined as follows.

- For $M \in \mathcal{K}_P$, $T_\lambda(M) = M$ and $T_\lambda(M)^\emptyset_A = M^\emptyset_{A+\lambda}$.
- For $\mathcal{M} \in \mathcal{K}_{\mathrm{AJS}}$, $T_{\mathrm{AJS},\lambda}(\mathcal{M})(A) = \mathcal{M}(A+\lambda)$ and $T_{\mathrm{AJS},\lambda}(\mathcal{M})(A,\alpha) = \mathcal{M}(A+\lambda,\alpha)$.

Since these functors are $S$-linear, they give functors on $\mathbb{K} \otimes_S \mathcal{K}_P$ and $\mathbb{K} \otimes_S \mathcal{K}_{\mathrm{AJS},P}$, respectively. These functors give structures of $\mathbb{Z}\Delta$-category on each category. It is easy to see that equivalences $\mathbb{K} \otimes_S \mathcal{K}_P \simeq \mathbb{K} \otimes_S \mathcal{K}_{\mathrm{AJS},P} \simeq \mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$ are $\mathbb{Z}\Delta$-functor. Therefore, it is sufficient to prove $T_\lambda(M * B) \simeq T_\lambda(M) * B$ for $M \in \mathcal{K}_P$ and $B \in \mathcal{S}\mathrm{Bimod}$. This follows from the definition. □

Therefore, the action of $B \in \mathcal{S}\mathrm{Bimod}$ on $\mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$ is compatible with the $\mathbb{Z}\Delta$-category structure, and it gives an action on $\mathrm{Proj}_{\mathbb{Z}\Delta}(\mathcal{E})$. We write this action again by $M \mapsto M * B$. For each $B \in \mathcal{S}\mathrm{Bimod}$, we define $\mathcal{E}(B)$ by $\mathcal{E}(B) = \bigoplus_{\lambda \in \mathbb{Z}\Delta} \mathrm{Hom}(P, (P * B) \otimes L(p\lambda))$. This is a $\mathbb{Z}\Delta$-graded $\mathcal{E}$-bimodule.

**Lemma 3.33.** *Let $Q$ be a projective finitely generated $\mathbb{Z}\Delta$-graded $\mathcal{E}$-module. Then, $Q \otimes_{\mathcal{E}} \mathcal{E}(B) \simeq Q * B$.*

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**Proof.** Let $Q_\nu$ be the $\nu$-th graded piece of $Q$, where $\nu \in \mathbb{Z}\Delta$. Let $p \in Q_\nu$ and let $\varphi_p$ be the corresponding element in $\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{Z}\Delta}(\mathcal{E})}(\mathcal{E}, Q(\nu))$. Here $(\nu)$ is the shift of the grading. Then, $\varphi_p * B$ gives $\mathcal{E} * B \to Q(\nu) * B$. By the definition, $\mathcal{E} * B = \mathcal{E}(B)$. Therefore, for $m \in \mathcal{E}(B)$, we have $\varphi_p(m) \in Q(\nu) * B \simeq (Q * B)(\nu)$. Hence, we get $Q \otimes_{\mathcal{E}} \mathcal{E}(B) \to Q * B$ by $p \otimes m \mapsto \varphi_p(m)$. This is an isomorphism if $Q = \mathcal{E}$. Hence, it is an isomorphism for any $Q \in \mathrm{Proj}_{\mathbb{Z}\Delta}(\mathcal{E})$. □

Now for $\mathbb{Z}\Delta$-graded right $\mathcal{E}$-module $M$, put $M * B = M \otimes_{\mathcal{E}} \mathcal{E}(B)$. By the above lemma, $\mathcal{E}(B_1) \otimes_{\mathcal{E}} \mathcal{E}(B_2) \simeq \mathcal{E} * B_1 * B_2 = \mathcal{E} * (B_1 \otimes B_2) \simeq \mathcal{E}(B_1 \otimes B_2)$. Hence, $(M * B_1) * B_2 = (M \otimes_{\mathcal{E}} \mathcal{E}(B_1)) \otimes_{\mathcal{E}} \mathcal{E}(B_2) \simeq M \otimes_{\mathcal{E}} (\mathcal{E}(B_1) \otimes_{\mathcal{E}} \mathcal{E}(B_2)) \simeq M \otimes_{\mathcal{E}} \mathcal{E}(B_1 \otimes B_2) = M * (B_1 \otimes B_2)$. It is easy to see that this gives an action of $\mathcal{S}\mathrm{Bimod}$ on $\mathrm{Mod}_{\mathbb{Z}\Delta}(\mathcal{E})$ and therefore, on $\mathcal{C}_{\mathbb{K},0}$.

### 3.8. Characters

Assume that $\mathbb{K}$ is an algebraically closed field of $p > h$, where $h$ is the Coxeter number. Any object $P \in \mathrm{Proj}(\mathcal{C}_{S,0})$ has a baby Verma flag. Let $(P : Z_S(w \cdot_p \lambda_0))$ be the the multiplicity of $Z_S(w \cdot_p \lambda_0)$ in $P$. The following lemma is obvious from the constructions.

**Lemma 3.34.** *Let $P \in \mathrm{Proj}(\mathcal{C}_{S,0})$ and $M \in \mathcal{K}_P$ such that $\mathcal{V}(P) \simeq \mathcal{F}(M)$. Then, we have $(P : Z_S(w \cdot_p \lambda_0)) = \mathrm{rank}(M_{\{wA_0\}})$ for $w \in W'_{\mathrm{aff}}$.*

The projective module $P_S(\lambda)$ is characterized by

- $P_S(\lambda)$ is indecomposable.
- $(P_S(\lambda) : Z_S(\lambda)) = 1$.
- $(P_S(\lambda) : Z_S(\mu)) = 0$ unless $\mu - \lambda \in \mathbb{Z}_{\geq 0}\Delta^+$.

The module $\mathcal{V}^{-1}(\mathcal{F}(Q(wA_0)))$ satisfies these conditions with $\lambda = w \cdot_p \lambda_0$ by the above lemma. We get the following.

**Proposition 3.35.** *Let $w \in W'_{\mathrm{aff}}$. Then $\mathcal{V}(P_S(w \cdot_p \lambda_0)) \simeq \mathcal{F}(Q(wA_0))$.*

The following corollary is obvious from the above proposition.

**Corollary 3.36.** *We have $[P_{\mathbb{K}}(w \cdot_p \lambda_0) : Z_{\mathbb{K}}(v \cdot_p \lambda_0)] = \mathrm{rank}(Q(wA_0)_{\{vA_0\}})$.*

### 3.9. Lusztig's conjecture

For $B \in \mathcal{S}\mathrm{Bimod}$ and $w \in W_{\mathrm{aff}}$, let $B^w$ be the image of $B \hookrightarrow B \otimes_R R^{\emptyset} = \bigoplus_{x \in W_{\mathrm{aff}}} B^{\emptyset}_x \twoheadrightarrow B^{\emptyset}_w$. Put $\mathrm{ch}(B) = \sum_{w \in W_{\mathrm{aff}}} v^{-\ell(w)} \mathrm{grk}(B^w)$. Then, $[B] \mapsto \mathrm{ch}(B)$ induces an isomorphism $[\mathcal{S}\mathrm{Bimod}] \simeq \mathcal{H}$. For each $w \in W_{\mathrm{aff}}$, there exists an indecomposable object $B(w) \in \mathcal{S}\mathrm{Bimod}$ unique up to isomorphism such that $\mathrm{ch}(B(w)) \in H_w + \sum_{x<w} \mathbb{Z}[v, v^{-1}] H_x$. We say that $B(w)$ satisfies the Soergel conjecture if $\mathrm{ch}(B(w))$ is a Kazhdan-Lusztig basis; namely, $\mathrm{ch}(B(w)) \in H_w + \sum_{x<w} v\mathbb{Z}[v] H_x$. It is known that the Soergel conjecture is satisfied by any $B(w)$ over a characteristic zero field. Therefore, for a fixed $w$, if $p$ is sufficiently large, $B(w)$ satisfies the Soergel conjecture (cf. [EW14]). We fix $\lambda \in (\mathbb{R}\Delta)_{\mathrm{int}}$ and $w \in W_{\mathrm{aff}}$ such that $A_\lambda^+ w \in \Pi_\lambda$. Here, $A_\lambda^+$ is the maximal element in $W'_\lambda A_\lambda^-$.

**Lemma 3.37.** *Let $w_\lambda \in W_{\mathrm{aff}}$ such that $A_\lambda^+ w_\lambda = A_\lambda^-$. Then, we have $S_{A_\lambda^+} * B(w_\lambda) \simeq Q_\lambda(\ell(w_0))$.*

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**Proof.** By the translation as in the proof of Lemma 2.31, we may assume $\lambda = 0$. Then, $W'_\lambda = W_{\mathrm{f}}$ and it is generated by $S_{\mathrm{aff}} \cap W_{\mathrm{f}}$. Moreover, the element $w_\lambda$ is equal to the longest element $w_0$.

It is sufficient to prove: $B(w_0) \simeq \{(z_w) \in R^{W_{\mathrm{f}}} \mid z_{wt} \equiv z_w (\mathrm{mod}\ \alpha_t)\}(\ell(w_0))$, where $t$ runs through the set of reflections in $W_{\mathrm{f}}$ and $\alpha_t$ the corresponding element in $\Lambda_{\mathbb{K}}$ [Abe21, 2.1]. Let $(G^\vee_{\mathbb{C}}, B^\vee_{\mathbb{C}}, T^\vee_{\mathbb{C}})$ be the reductive group over $\mathbb{C}$, the Borel subgroup and the maximal torus with the root datum $(X^\vee, \Delta^\vee, X, \Delta)$ and the positive system $\Delta^+ \subset \Delta$. Then, the category of $\mathbb{K}$-coefficient parity $B^\vee_{\mathbb{C}}$-equivariant sheaves on $G^\vee_{\mathbb{C}}/B^\vee_{\mathbb{C}}$ is equivalent to the category of Soergel bimodules attached to $(W_{\mathrm{f}}, X^\vee_{\mathbb{K}})$ [RW18]. The object $B(w_0)$ corresponds to the indecomposable parity sheaf such that the restriction to the big cell $B^\vee_{\mathbb{C}} w_0 B^\vee_{\mathbb{C}}/B^\vee_{\mathbb{C}}$ is $\mathbb{K}_{B^\vee_{\mathbb{C}} w_0 B^\vee_{\mathbb{C}}/B^\vee_{\mathbb{C}}}[\ell(w_0)]$. It is obvious that the constant sheaf $\mathbb{K}_{G^\vee_{\mathbb{C}}/B^\vee_{\mathbb{C}}}[\ell(w_0)]$ satisfies this condition, and therefore, the constant sheaf corresponds to $B(w_0)$. By the main theorem of [FW14], the corresponding Soergel bimodule is given as above. □

Recall that we took $w \in W_{\mathrm{aff}}$ and $\lambda \in (\mathbb{R}\Phi_{\mathrm{int}})$ such that $A^+_\lambda w \in \Pi_\lambda$. Define $S_{A^+_\lambda} \in \widetilde{\mathcal{K}}'(S)$ as follows: $S_{A^+_\lambda} = S$ as a left $S$-module and $R$ acts through $f \mapsto f_A$. We have $(S_{A^+_\lambda})^\emptyset_{A^+_\lambda} = S^\emptyset$ and $(S_{A^+_\lambda})^\emptyset_{A'} = 0$ for $A' \in \mathcal{A} \setminus \{A^+_\lambda\}$.

**Theorem 3.38.** *If $B(w)$ satisfies the Soergel conjecture, then $S_{A^+_\lambda} * B(w) \simeq Q(A^+_\lambda w)$.*

**Proof.** First, we prove that $S_{A^+_\lambda} * B(w) \in \mathcal{K}_P$. By the translation as in Lemma 2.31, we may assume $\lambda = 0$. Then, $W'_\lambda = W_{\mathrm{f}}$, and this is isomorphic to the subgroup of $W_{\mathrm{aff}}$ generated by $s \in S_{\mathrm{aff}}$ which contains a hyperplance through 0. We identify $W_{\mathrm{f}} \hookrightarrow W_{\mathrm{aff}}$. We have $sw < w$ for any $s \in W_{\mathrm{f}} \cap S_{\mathrm{aff}}$. Therefore, $H_s \operatorname{ch}(B(w)) = v^{-1} \operatorname{ch}(B(w))$ by [JW17, Lemma 4.3]. Hence, $H_x \operatorname{ch}(B(w)) = v^{-\ell(x)} \operatorname{ch}(B(w))$ for any $x \in W_{\mathrm{f}}$. Take $a_y = \sum_{n\in\mathbb{Z}} a_{y,n} v^n \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$ such that $\operatorname{ch}(B(w_0)) = \sum_{y \in W_{\mathrm{f}}} a_y H_y$ (one can write $a_y$ explicitly, but we do not do this here because we will not use this). Then, we have $\operatorname{ch}(B(w_0) \otimes B(w)) = \sum_{y\in W_{\mathrm{f}}} a_y v^{-\ell(y)} \operatorname{ch}(B(w))$. Hence, we get $B(w_0) \otimes B(w) \simeq \bigoplus_{y \in W_{\mathrm{f}}, n\in\mathbb{Z}} B(w)^{a_{y,n}}(n - \ell(y))$. Therefore, up to shift, $S_{A^+_0} * B(w)$ is a direct summand of $S_{A^+_0} * (B(w_0) \otimes B(w)) \simeq Q_0(\ell(w_0)) * B(w) \in \mathcal{K}_P$. Hence, $S_{A^+_0} * B(w) \in \mathcal{K}_P$.

We return to the proof of the theorem. By [Lus80, Theorem 5.2], $\operatorname{ch}(S_{A^+_\lambda} * B(w)) = A^+_\lambda \operatorname{ch}(B(w))$ is described by periodic Kazhdan-Lusztig polynomials; namely, we have $A^+_\lambda \operatorname{ch}(B(w)) = v^{-n} \underline{P}_{A_0}$ for some $A_0 \in \mathcal{A}$ and $n \in \mathbb{Z}$. Here, $\underline{P}_{A'} \in \mathcal{P}^0$ is the element given in [Soe97, Proposition 4.16]. We know $A^+_\lambda \operatorname{ch}(B(w)) \in A^+_\lambda w + \sum_{A' > A^+_\lambda w} \mathbb{Z}[v, v^{-1}]A'$. Comparing with [Soe97, Lemma 4.21], we have $n = \ell(w_0)$ and $\operatorname{ch}(S_{A^+_\lambda} * B(w)) \in A^+_\lambda w + \sum_{A' > A^+_\lambda w} v^{-1}\mathbb{Z}[v^{-1}]A'$. By the self-duality of $\underline{P}_{A_0}$, we have $\overline{\operatorname{ch}(S_{A^+_\lambda} * B(w))} = v^{\ell(w_0)} \underline{P}_{A_0} \in v^{2\ell(w_0)} A^+_\lambda w + \sum_{A' > A^+_\lambda w} v^{2\ell(w_0)-1} \mathbb{Z}[v^{-1}]A'$. Therefore, by Theorem 3.10, we have

$$\operatorname{grk} \operatorname{Hom}^\bullet_{\mathcal{K}}(S_{A^+_\lambda} * B(w), S_{A^+_\lambda} * B(w)) \in 1 + v^{-2}\mathbb{Z}[v^{-1}].$$

Hence, $\operatorname{End}_{\mathcal{K}}(S_{A^+_\lambda} * B(w))$ is one-dimensional, and therefore, 1 and 0 are only its idempotents. Therefore, $S_{A^+_\lambda} * B(w)$ is indecomposable. Since $Q(A^+_\lambda w)$ is a direct summand of $S_{A^+_\lambda} * B(w)$, we get the theorem. □

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From the above theorem and Corollary 3.36, the multiplicity of the baby Verma modules in the projective cover of an irreducible module is given by the value at 1 of the Kazhdan-Lusztig polynomial. Hence, the Lusztig's conjecture holds for sufficiently large $p$.

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## References

[Abe20a] N. Abe, A homomorphism between Bott-Samelson bimodules, arXiv:2012.09414.

[Abe20b] N. Abe, On singular Soergel bimodules, arXiv:2004.09014.

[Abe21] N. Abe, A bimodule description of the Hecke category, *Compos. Math.* **157**(10) (2021), 2133–2159.

[AJS94] H. H. Andersen, J. C. Jantzen and W. Soergel, Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic $p$: Independence of $p$, *Astérisque* **220** (1994), 321.

[AMRW19] P. N. Achar, S. Makisumi, S. Riche and G. Williamson, Koszul duality for Kac–Moody groups and characters of tilting modules, *J. Amer. Math. Soc.* **32**(1) (2019), 261–310.

[And98] H. H. Andersen, Tilting modules for algebraic groups, in R. W. Carter and J. Saxl (eds.), *Algebraic Groups and Their Representations* vol. 517 (Kluwer Acad. Publ., Dordrecht, 1998), 25–42.

[BR22] R. Bezrukavnikov and S. Riche, Hecke action on the principal block, *Compos. Math.* **158** (2022), no. 5, 953–1019.

[EW14] B. Elias and G. Williamson, The Hodge theory of Soergel bimodules, *Ann. of Math. 2* **180**(3) (2014), 1089–1136.

[EW16] B. Elias and G. Williamson, Soergel calculus, *Represent. Theory* **20** (2016), 295–374.

[Fie11] P. Fiebig, Sheaves on affine Schubert varieties, modular representations, and Lusztig's conjecture, *J. Amer. Math. Soc.* **24**(1) (2011), 133–181.

[Fie12] P. Fiebig, An upper bound on the exceptional characteristics for Lusztig's character formula, *J. Reine Angew. Math.* **673** (2012), 1–31.

[FL15] P. Fiebig and M. Lanini, Sheaves on the alcoves I: Projectivity and wall crossing functors, arXiv:1504.01699.

[FW14] P. Fiebig and G. Williamson, Parity sheaves, moment graphs and the $p$-smooth locus of Schubert varieties, *Ann. Inst. Fourier (Grenoble)* **64**(2) (2014), 489–536.

[JMW14] D. Juteau, C. Mautner and G. Williamson, Parity sheaves, *J. Amer. Math. Soc.* **27**(4) (2014), 1169–1212.

[JW17] L. T. Jensen and G. Williamson, The $p$-canonical basis for Hecke algebras in *Categorification and Higher Representation Theory (Contemporary Mathematics)* vol. 683 (Amer. Math. Soc., Providence, RI, 2017), 333–361.

[KL93] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras. I, II, *J. Amer. Math. Soc.* **6**(4) (1993), 905–947, 949–1011.

[KL94a] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras. III, *J. Amer. Math. Soc.* **7**(2) (1994), 335–381.

[KL94b] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras. IV, *J. Amer. Math. Soc.* **7**(2) (1994), 383–453.

[KT95] M. Kashiwara and T. Tanisaki, Kazhdan-Lusztig conjecture for affine Lie algebras with negative level, *Duke Math. J.* **77**(1) (1995), 21–62.

[KT96] M. Kashiwara and T. Tanisaki, Kazhdan-Lusztig conjecture for affine Lie algebras with negative level. II. Nonintegral case, *Duke Math. J.* **84**(3) (1996), 771–813.

[Lib08] N. Libedinsky, Sur la catégorie des bimodules de Soergel, *J. Algebra* **320**(7) (2008), 2675–2694.

[Lus80] G. Lusztig, Hecke algebras and Jantzen's generic decomposition patterns, *Adv. in Math.* **37**(2) (1980), 121–164.

[RW18] S. Riche and G. Williamson, Tilting modules and the $p$-canonical basis, *Astérisque* **397** (2018), ix+184.

[RW22] S. Riche and G. Williamson, Smith-Treumann theory and the linkage principle, *Math. Inst. Hautes Études Sci.* **136** (2022), 225–292.

[Sob20] P. Sobaje, On character formulas for simple and tilting modules, *Adv. Math.* **369**(8) (2020), 107172.

[Soe97] W. Soergel, Kazhdan-Lusztig polynomials and a combinatoric for tilting modules, Represent. *Theory* **1** (1997), 83–114 (electronic).

[Wil17] G. Williamson, Schubert calculus and torsion explosion, *J. Amer. Math. Soc.* **30**(4) (2017), 1023–1046. With a joint appendix with A. Kontorovich and P. J. McNamara.