Computing indefinite integrals by difference equations

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1. *Introduction*

In teaching mathematics to first-year undergraduates, and thus in the appropriate calculus textbooks, the task of calculating an integral that satisfies a specific first-order or second-order recurrence relation often appears. These relations are obtained mainly by applying the method of integration by parts. Calculating such integrals is usually tedious, especially for an integer $n > 2$, time-consuming, and presents the possibility of making a large number of errors when computing involves multiple iterative steps. In [1], it is shown that in two cases (Theorems 2.1. and 2.3), the process of calculating integrals satisfying first-order recurrence relations can be performed quickly using easily memorised closed-form formulas for corresponding primitive functions. The question can rightly be asked whether there is a faster way to calculate other integrals of this type. In this paper, our goal is to give an affirmative answer to such a question, though without convering all situations. Since each recurrence relation is equivalent to a difference equation of the same order, the calculation of integrals mentioned above can be reduced to solving the corresponding difference equations. Since every first-order or second-order linear difference equation is solvable, it follows that for every integral which can be reduced to a first-order or second-order recurrence formula, it is possible to find corresponding primitive functions directly. Sometimes such a procedure is much faster than iterative solving of the integral. Closed-form formulas for the integrals discussed in the following sections are not unknown (see [2]). However, here our goal is to present the idea of computing indefinite integrals using difference equations. We will discuss it in more detail in Section 2. In Section 3, we discuss the application of the results obtained to calculate several improper integrals and the application of some of them in different sciences. An exciting example of such an application is the integral $\int_0^\infty x^{(2n+1)/2} e^{-ax} dx$, which in the case $n = 1$ is used in the kinetic theory of gases, particularly in the Maxwell-Boltzmann distribution of gas molecules by energies (see Remark 4). Also, we compare the formulas obtained by the method of difference equations with the formulas obtained using Wolfram Alpha software (see Remark 5).

2. *Formulas in closed form for repeated integration by parts*

In this section, we will consider indefinite integrals satisfying a first-order or second-order recurrence relation. Recall that each recurrence relation is equivalent to the corresponding difference equation of the same order. Therefore, we will consider closed-form formulas that represent solutions of first-order and second-order linear difference equations in the following subsections.

https://doi.org/10.1017/mag.2023.99 Published online by Cambridge University Press

2.1. *Integrals that satisfy first-order reccurence relations*

The first-order linear difference equation has the following form:

$$x_{n+1} = a_n x_n + b_n, \qquad n \geqslant n_0 \geqslant 0, \tag{1}$$

where $\{a_n\}_{n_0}^{\infty}$ and $\{b_n\}_{n_0}^{\infty}$ are given sequences. By using mathematical induction it can be shown that its general solution is of the form

$$x_n = x_{n_0} \prod_{i=n_0}^{n-1} a_i + \sum_{k=n_0}^{n-1} b_k \prod_{i=k+1}^{n-1} a_k, \tag{2}$$

for all $n \geqslant n_0 \geqslant 0$ (see, e.g. [3, 4, 5, 6]). By including the so-called initial condition x_0, a particular solution of (1) is obtained. Therefore the following theorem holds.

Theorem 1

Let integrals I_n satisfy the recursive relation

$$I_{n+1} = a_n I_n + b_n, \tag{3}$$

for $n \geqslant n_0 \geqslant 0$. Then, apart from an arbitrary constant,

$$I_n = I_{n_0} \prod_{i=n_0}^{n-1} a_i + \sum_{k=n_0}^{n-1} b_k \prod_{i=k+1}^{n-1} a_i. \tag{4}$$

Hence, to solve integrals that satisfy a recursive relation (3), it is enough to apply an easy-to-remember formula (4).

Let us demonstrate this procedure in the proof of the results of the following theorem.

Theorem 2

Let n be an integer. Then, apart from an arbitrary additive constant, we have:

(a) ([1], Theorems 2.3 and 2.1)

$$\int x^n e^{-\alpha x} dx = -\frac{e^{-\alpha x}}{\alpha}\left[x^n + \frac{(x^n)^{(1)}}{\alpha} + \frac{(x^n)^{(2)}}{\alpha^2} + \dots + \frac{(x^n)^{(n-1)}}{\alpha^{n-1}} + \frac{(x^n)^{(n)}}{\alpha^n}\right], \text{for } n > 0 \text{ and } \alpha \neq 0.$$

In particular,

$$\begin{aligned}\int x^n e^{-x} dx &= -e^{-x}\left[x^n + (x^n)^{(1)} + (x^n)^{(2)} + \dots + (x^n)^{(n-1)} + (x^n)^{(n)}\right]\\ &= -e^{-x}\left[x^n + nx^{n-1} + n(n-1)x^{n-2} + \dots + n!\,x + n!\right].\end{aligned}$$

(b)

$$\int \frac{dx}{(1+x^2)^n} = \frac{1.3.\cdots.(2n-3)}{2^{n-1}(n-1)!}\left[\arctan x + \sum_{k=1}^{n-1} \frac{2^{k-1}(k-1)!}{1.3.\cdots.(2k-1)} \cdot \frac{x}{(1+x^2)^k}\right], \tag{5}$$

for $n > 1$.

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(c)

$$\int \frac{dx}{(x^2 - a^2)^n} = \left(-\frac{1}{2a^2}\right)^{n-1} \frac{1.3.\cdots.(2n-3)}{(n-1)!}$$
$$\times\left[\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + \sum_{k=1}^{n-1}\left(-2a^2\right)^{k-1}\frac{(k-1)!}{1.3.\cdots.(2k-1)}\frac{x}{(x^2-a^2)^k}\right], \quad (6)$$

for $n > 1$ and $a \neq 0$.

(d)

$$\int \ln^n x\, dx = (-1)^n n!x\left[1 - \ln x + \frac{1}{2!}\ln^2 x - \frac{1}{3!}\ln^3 x + \cdots + \frac{(-1)^n}{n!}\ln^n x\right], \quad (7)$$

for $n \geqslant 0$ and $x > 0$.

(e)

$$\int \sqrt{x}\,\ln^n x\, dx = (-1)^n\left(\frac{2}{3}\right)^{n+1} n!\,x^{3/2}\sum_{k=0}^{n}(-1)^k\left(\frac{3}{2}\right)^k\frac{\ln^k x}{k!}$$
$$= (-1)^n\left(\frac{2}{3}\right)^{n+1} n!x^{3/2}\left[1 - \frac{3}{2}\frac{\ln x}{1!} + \left(\frac{3}{2}\right)^2\frac{\ln^2 x}{2!} - \ldots + (-1)^n\left(\frac{3}{2}\right)^2\frac{\ln^n x}{n!}\right], \quad (8)$$

for $n \geqslant 0$ and $x > 0$.

Proof:

(a) Setting $I_n = \int x^n e^{-\alpha x}dx$ and using integration by parts, we obtain

$$I_{n+1} = \frac{n+1}{\alpha}I_n - \frac{e^{-\alpha x}}{\alpha}x^{n+1},$$

which is a difference equation of the form (3), where $a_n = \dfrac{n+1}{\alpha}$, $b_n = -\dfrac{e^{-\alpha x}}{\alpha}x^{n+1}$, $n_0 = 0$ and $I_0 = \int e^{-\alpha x}dx = -\dfrac{e^{-\alpha x}}{\alpha}$, apart from an arbitrary additive constant. By solving this equation using (4), we get

$$I_n = \left(\prod_{i=0}^{n-1}\frac{i+1}{\alpha}\right)I_0 + \sum_{k=0}^{n-1}\left(\prod_{i=k+1}^{n-1}\frac{i+1}{\alpha}\right)\left(-\frac{e^{-\alpha x}}{\alpha}x^{k+1}\right)$$
$$= \frac{n!}{\alpha^n}\left(-\frac{e^{-\alpha x}}{\alpha}\right) - e^{-\alpha x}\sum_{k=0}^{n-1}\frac{(k+2)\cdots n}{\alpha^{n-k}}x^{k+1}$$
$$= \frac{n!}{\alpha^n}\left(-\frac{e^{-\alpha x}}{\alpha}\right) - e^{-\alpha x}\sum_{k=0}^{n-1}\frac{n!}{\alpha^{n-k}(k+1)!}x^{k+1}$$
$$= -\frac{e^{-\alpha x}}{\alpha}\left[\frac{n!}{\alpha^n} + \frac{n!x}{\alpha^{n-1}} + \ldots + \frac{n(n-1)x^{n-2}}{\alpha^2} + \frac{nx^{n-1}}{\alpha} + x^n\right]$$
$$= -\frac{e^{-\alpha x}}{\alpha}\left[x^n + \frac{(x^n)^{(1)}}{\alpha} + \frac{(x^n)^{(2)}}{\alpha^2} + \ldots + \frac{(x^n)^{(n-1)}}{\alpha^{n-1}} + \frac{(x^n)^{(n)}}{\alpha}\right].$$

For $\alpha = 1$, the result for the integral $\int x^n e^{-x}\,dx$ is obtained.

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(b) Using integration by parts and substitution, (3) is obtained, where

$$I_n = \int \frac{dx}{(1 + x^2)^n},\ a_n = \frac{2n - 1}{2n},\ b_n = \frac{x}{2n(1 + x^2)^n},\ n_0 = 1$$

$$\text{and } I_1 = \int \frac{dx}{1 + x^2} = \arctan x.$$

(c) $I_n = \dfrac{dx}{(x^2 - a^2)^n},\ a_n = -\dfrac{(2n - 1)}{2na^2},\ b_n = -\dfrac{x}{2na^2(x^2 - a^2)^n},\ n_0 = 1$ and $I_1 = \displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x - a}{x + a}\right|$, apart from an arbitrary additive constant.

(d) $I_n = \int \ln^n x\, dx,\quad a_n = -(n + 1),\quad b_n = x \ln^{n+1} x,\quad n_0 = 0$ and $I_0 = \int 1\, dx = x$, apart from an arbitrary additive constant.

(e) $I_n = \int \sqrt{x} \ln^n x\, dx,\ a_n = -\frac{2}{3}(n + 1),\ b_n = \frac{2}{3}x^{3/2} \ln^{n+1} x,\ n_0 = 0$ and $I_0 = \int \sqrt{x}\, dx = \frac{2}{3}x^{3/2}$, apart from an arbitrary additive constant.

Remark 1

Notice that the indefinite integrals $\int \ln^n x\, dx$ and $\int \sqrt{x} \ln^n x\, dx$ can be calculated almost immediately by using the substitution $\ln x = t$. Its evaluation can be finished by applying Theorem 2(a).

Remark 2

Theorem 2.1 in [1] arose from an inductive process using only derivatives. Above, we showed how the same result could be obtained by solving the corresponding difference equation.

2.2 *Integrals that satisfy second-order reccurence relations*

When integration by parts leads to a recurrence relation of the second-order, solving the corresponding difference equation cannot be achieved by means of (2). Several other methods can be used in such cases (generating functions, variation of parameters, generalized series, Z transform). However, these are usually difficult to apply when the difference equation has variable coefficients and they usually end up unsuccessfully. Fortunately, closed-form formulas exist that represent solutions of inhomogeneous second-order difference equations with variable coefficients (see [7]).

Note that it is well known how to determine the general solution of a (non)homogeneous linear difference equation with constant coefficients using the roots of the corresponding characteristic equation. Hence, if the integral can be represented in the form of a second-order difference equation with constant coefficients, then it can be solved.

Now, we will demonstrate the calculation of indefinite integrals that satisfy a second-order recurrence relation with variable coefficients. For this purpose, we will use closed-form formulas to solve second-order difference equations with variable coefficients. At this point we need to recall some preliminary concepts and results (see [7]).

Definition 1

For any sequence $\{a_n\}_{n=1}^{\infty}$ and any positive integers k and m, we define the *integral of a_n between k and m as the value*

$$\int_m^k a_j \nabla j = \text{sign}(k - m) \sum_{j = \min\{k,m\}+1}^{\max\{k,m\}} a_j,$$

where sign $(0) = 0$, sign $(-k) = -1$ and sign $(k) = 1$.

Consider the second-order difference equation with variable coefficients of the form

$$a_n x_{n+1} - b_n x_n + c_{n-1} x_{n-1} = f_n, \qquad n = 1, 2, \ldots, \tag{9}$$

where $a_n > 0$, $b_n \geqslant 0$, $c_n > 0$ and $f_n \geqslant 0$ for all $n \geqslant 1$. If $f_n = 0$ for $n \geqslant 1$, then we obtain the equation

$$a_n x_{n+1} - b_n x_n + c_{n-1} x_{n-1} = 0, \qquad n = 1, 2, \ldots, \tag{10}$$

is called the *homogeneous equation* associated with (9).

The next result is taken from [7], Proposition 4.2.

Lemma 1

If we denote by S the set of solutions of the homogeneous equation (10) and by $S(f_n)$ the set of all solutions of (9) with given sequence $\{f_n\}_{n=1}^{\infty}$, then we get $S(f_n) = x_n + S$, where $x_n \in S(f_n)$. Moreover, given non-negative integer m and $\alpha, \beta \in \mathbb{R}$, if x_n is the unique solution of the initial value problem

$$a_n x_{n+1} - b_n x_n + c_{n-1} x_{n-1} = f_n; \qquad n = 1, 2, \ldots, x_m = \alpha, x_{m+1} = \beta,$$

then $x_n = y_n + z_n$, where z_n is the unique solution of the initial value problem for the homogeneous equation

$$a_n z_{n+1} - b_n z_n + c_{n-1} z_{n-1} = 0; \qquad n = 1, 2, \ldots, z_m = \alpha, z_{m+1} = \beta,$$

and y_n is the unique solution of the following initial value problem

$$a_n y_{n+1} - b_n y_n + c_{n-1} y_{n-1} = f_n; \qquad n = 1, 2, \ldots, y_m = y_{m+1} = 0.$$

It is necessary to introduce the Green function to state the solutions of y_n and z_n explicitly.

Definition 2

The Green function for the difference equation (9) is defined as the function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ with the following properties:

(i) $g(\cdot, j)$ is the unique solution of the homogeneous equation (10),

(ii) $g(j, j) = 0$ and $g(j + 1, j) = \dfrac{1}{a_j}$,

for all $j \in \mathbb{N}$.

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Remark 3

The sequences $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ from Lemma 1 have the forms $y_n = \int_m^n g(n,j) f(j) \nabla j$ and $z_n = \beta a_m g(n,m) - \alpha c_m g(n+1,m)$ for any $n \in \mathbb{N} \setminus \{0\}$.

Note that (9) is called *uncoupled* if $b_n = 0$ for all $n \in \mathbb{N} \setminus \{0\}$. The following result ([7], Corollary 7.6) gives us instruction on determining the Green function for the uncoupled equation.

Lemma 2

If $a_n \neq 0$ and $c_n \neq 0$ for all $n = 1, 2, \ldots$, the Green function for the uncoupled equation

$$a_n z_{n+1} + c_{n-1} z_{n-1} = 0, \qquad n = 1, 2, \ldots \tag{11}$$

is given by

$$g(2n+1, j) = (-1)^{n-j} \frac{1}{a_{2j}} \prod_{l=j+1}^{n} \frac{c_{2l-1}}{a_{2l}}, \qquad n, j \in \mathbb{N} \setminus \{0\},$$

$$g(2n, 2j+1) = (-1)^{n-j-1} \frac{1}{c_{2n}} \prod_{l=j+1}^{n} \frac{c_{2l}}{a_{2l-1}}, \qquad n, j \in \mathbb{N} \setminus \{0\},$$

$$g(m, k) = 0, \qquad \text{otherwise.}$$

We are now in position to prove the next theorem.

Theorem 3

(a) If $I_n = \int x^{n/2} e^{-ax} dx$, $a \neq 0$, then for any integer $n > 0$, apart from an arbitrary additive constant

$$I_{2n} = -\frac{n!}{a^n} e^{-ax} \sum_{k=0}^{n-1} \frac{a^k}{(k+1)!} x^{k+1},$$

$$I_{2n+1} = \frac{1 \cdot 3 \cdots (2n+1)}{(2a)^n} \left[I_1 - \frac{e^{-ax}}{a} \sum_{k=1}^{n} \frac{(2a)^k}{1 \cdot 3 \cdots (2k+1)} x^{(2k+1)/2} \right]. \tag{12}$$

(b) For any integer $n > 0$, apart from an arbitrary additive constant

$$\int x^{2n} e^{-x^2} dx = \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \left[\int e^{-x^2} dx - e^{-x^2} \sum_{k=1}^{n} \frac{2^{k-1}}{1 \cdot 3 \cdots (2k-1)} x^{2k-1} \right],$$

where $\int e^{-x^2} dx = \frac{1}{2}\sqrt{\pi} \operatorname{erf}(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(k-1)!\,(2k-1)} x^{2k-1}$.

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Proof:

(a) Using integration by parts we have

$$I_{n+1} = \int x^{(n+1)/2} e^{-ax} dx = -\frac{1}{a} x^{(n+1)/2} e^{-ax} + \frac{n+1}{2a} \int x^{(n-1)/2} e^{-ax} dx,$$

i.e. we obtain

$$I_{n+1} = -\frac{1}{a} x^{(n+1)/2} e^{-ax} + \frac{n+1}{2a} I_{n-1}, \qquad n \geqslant 1, \tag{14}$$

which is the equation of the form (11) with $z_n = I_n$, $a_n = 1$, $c_{n-1} = -\frac{1}{2a}(n+1)$ and $f(n) = -\frac{1}{a} x^{(n+1)/2} e^{-ax}$ for all $n \geqslant 1$.

By Lemma 2, the Green function for (14) reads

$$\begin{aligned} g(2n+1, 2j) &= (-1)^{n-j}\left(-\frac{2j+3}{2a}\right)\left(-\frac{2j+5}{2a}\right)\dots\left(-\frac{2n+1}{2a}\right) \\ &= \frac{(2j+3)(2j+5)\dots(2n+1)}{(2a)^{n-j}}, \qquad n, j \in \{1, 2, \dots\}, \\ g(2n, 2j+1) &= (-1)^{n-j-1}\left(-\frac{a}{n+1}\right)\left(-\frac{2j+4}{2a}\right)\left(-\frac{2j+6}{2a}\right)\dots\left(-\frac{2n+2}{2a}\right) \\ &= \frac{1}{a^{n-j-1}}(j+2)(j+3)\dots n, \qquad n, j \in \{1, 2, \dots\}, \\ g(i, k) &= 0, \qquad \text{otherwise}. \end{aligned}$$

(i) Substituting n by $2n$ in (14) we see that solution of the given equation is

$$I_{2n} = J_{2n} + L_{2n}. \qquad n \geqslant 0,$$

where we determine J_{2n} and L_{2n} in the following way.

By Remark 3 (with $\alpha = I_0$ and $\beta = I_1$) the integral J_{2n} has the form

$$J_{2n} = I_1 a_0 g(2n, 0) - I_0 c_0 g(2n, 1) = 0,$$

since $g(2n, 0) = 0$ and $I_0 = 0$. Moreover, the integral L_{2n} is the solution of the following equation

$$I_{2n+2} - \frac{n+1}{a} I_{2n} = -\frac{1}{a} x^{n+1} e^{-ax} = f(2n+1), \qquad n \geqslant 0.$$

Hence, by Remark 3,

$$\begin{aligned} L_{2n} &= \int_1^{2n-1} g(2n, 2j+1) f(2j+1) \nabla(2j+1) \\ &= \sum_{2j+1=1}^{2j+1=2n-1} g(2n, 2j+1) f(2j+1) \\ &= \sum_{j=0}^{n-1} g(2n, 2j+1) f(2j+1) \\ &= -\frac{1}{a} e^{-ax} n! \sum_{j=0}^{n-1} \frac{a^j}{(j+1)!} x^{j+1}. \end{aligned}$$

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Therefore

$$I_{2n} = J_{2n} + L_{2n} = -\frac{1}{a}e^{-ax}n!\sum_{j=0}^{n-1}\frac{a^j}{(j+1)!}x^{j+1}, \qquad n \geqslant 0.$$

(ii) Analogously,

$$I_{2n+1} = J_{2n+1} + L_{2n+1}, \qquad n \geqslant 0.$$

The integral J_{2n+1} has the form

$$\begin{aligned} J_{2n+1} &= I_1 a_0 g(2n+1, 0) - I_0 c_0 g(2n+1, 1) = I_1 g(2n+1, 0) \\ &= \frac{1 \,.\, 3 \,.\, \cdots \,.\, (2n+1)}{(2a)^n} I_1, \end{aligned}$$

since $g(2n+1, 1) = 0$. The integral L_{2n+1} is the solution of the equation

$$I_{2n+1} - \frac{2n+1}{2a}I_{2n-1} = -\frac{1}{a}x^{(2n+1)/2}e^{-ax} = f(2n), \qquad n \geqslant 0$$

and by Remark 3 we have

$$\begin{aligned} L_{2n+1} &= \int_2^{2n} g(2n+1, 2j) f(2j) \nabla(2j) \\ &= \sum_{2j=2}^{2j=2n} g(2n+1, 2j) f(2j) \\ &= \sum_{j=1}^{n} g(2n+1, 2j) f(2j) \\ &= -\frac{1}{a}e^{-ax}\frac{1}{(2a)^n}\sum_{j=0}^{n-1}(2j+3)(2j+5)\ldots(2n+1)(2a)^j x^{(2j+1)/2}, \end{aligned}$$

which implies (12).

(b) Using integration by parts we have

$$\begin{aligned} \int x^{2n}e^{-x^2} &= -\frac{1}{2}\int x^{2n-1} d\left(e^{-x^2}\right) \\ &= -\frac{1}{2}x^{2n-1}e^{-x^2} + \frac{2n-1}{2}\int x^{2n-2}e^{-x^2}dx. \end{aligned}$$

It means that we obtain the following second-order recurrence relation:

$$I_{2n+2} - \frac{2n+1}{2}I_{2n} = -\frac{1}{2}x^{2n+1}e^{-x^2}, \qquad n = 0, 1, \ldots, \tag{15}$$

where $I_{2n} = \int x^{2n}e^{-x^2}dx$, $a_n = 1$, $c_{2n} = -\frac{1}{2}(2n+1)$, $f(2n+1) = -\frac{1}{2}x^{2n+1}e^{-x^2}$ $(n = 0, 1, \ldots)$. By using Lemma 2 we see that the Green function for (15) reads

$$\begin{aligned} g(2n+1, 2j) &= (-1)^{n-j}\left(-\frac{2j+2}{2}\right)\left(-\frac{2j+4}{2}\right)\ldots\left(-\frac{2n}{2}\right) \\ &= (j+1)(j+2)\ldots n, \qquad n, j \in \{0, 1, \ldots\}, \end{aligned}$$

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$$g(2n,\ 2j+1) = (-1)^{n-j-1}\left(-\frac{1}{2n+1}\right)\left(-\frac{2j+3}{2}\right)\left(-\frac{2j+5}{2}\right)\ldots\left(-\frac{2n+1}{2}\right)$$

$$= \frac{1}{2^{n-j-1}}(2j+3)(2j+5)\ldots(2n-1), \qquad n, j \in \{0,\ 1,\ \ldots\},$$

$$g(i,\ k) = 0. \qquad \text{otherwise.}$$

The solution of (15) is

$$I_{2n} = J_{2n} + L_{2n}, \qquad n \geqslant 0,$$

where we determine J_{2n} and L_{2n} in the following way.

The integral J_{2n} has the form

$$J_{2n} = I_1 a_0 g(2n,\ 0) - I_0 c_0 g(2n,\ 1)$$

$$= -I_0\left(-\frac{1}{2}\right)\frac{1 \cdot 3 \cdots (2n-1)}{2^{n-1}} = \frac{1 \cdot 3 \cdots (2n-1)}{2^n} I_0,$$

since $g(2n,\ 0) = 0$. Also, by Remark 3

$$L_{2n} = \int_1^{2n-1} g(2n,\ 2j+1) f(2j+1) \nabla (2j+1)$$

$$= \sum_{j=0}^{n-1} g(2n,\ 2j+1) f(2j+1)$$

$$= -\frac{1 \cdot 3 \cdots (2n-1)}{2^n} e^{-x^2} \sum_{j=1}^{n} \frac{2^{j-1}}{1 \cdot 3 \cdots (2j-1)} x^{2j-1},$$

which implies (13).

3. *Applications*

In this section, we list some applications of the integrals discussed in the previous section to calculate the corresponding improper integrals using the results presented above.

Theorem 4

The following results hold:

(a) $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)^n} = \frac{1 \cdot 3 \cdots (2n-3)}{2 \cdot 4 \cdots (2n-2)} \frac{\pi}{2},$

(b) $\displaystyle\int_0^1 \ln^n x\, dx = (-1)^n n!\,,$

(c) $\displaystyle\int_0^1 \sqrt{x} \ln^n x\, dx = (-1)^n \left(\frac{2}{3}\right)^{n+1} n!\,,$

(d) $\displaystyle\int_0^\infty x^{(2n+1)/2} e^{-ax} dx = \frac{1 \cdot 3 \cdots (2n+1)}{(2a)^{n+1}} \sqrt{\frac{\pi}{a}},$

(e) $\displaystyle I_{2n} = \int_0^\infty x^{2n} e^{-x^2} dx = \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \cdot \frac{\sqrt{\pi}}{2}.$

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Proof:

(a) If we set

$$\mathcal{H}(x) = \frac{1 \cdot 3 \cdots (2n-3)}{2 \cdot 4 \cdots (2n-2)}\left[\arctan x + \frac{x}{1+x^2} + \ldots + \frac{2 \cdot 4 \cdots (2n-4)}{1 \cdot 3 \cdots (2n-3)} \cdot \frac{x}{(1+x^2)^{n-1}}\right],$$

then using Theorem 2 (b) we obtain

$$\int_0^\infty \frac{dx}{(1+x^2)^n} = \lim_{x \to \infty} \mathcal{H}(x) - \mathcal{H}(0) = \frac{1 \cdot 3 \cdots (2n-3)}{2 \cdot 4 \cdots (2n-2)} \frac{\pi}{2}. \qquad (16)$$

(b) If

$$\mathcal{F}(x) = (-1)^n n!\, x\left[1 - \ln x + \frac{1}{2!}\ln^2 x - \frac{1}{3!}\ln^3 x + \ldots + \frac{(-1)^n}{n!}\ln^n x\right],$$

then we get $\lim_{x \to 0^+} \mathcal{F}(x) = 0$ since

$$\lim_{x \to 0^+} x \ln^k x = \lim_{x \to 0^+} \frac{\ln^k x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{k\left(\ln^{k-1} x\right)\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= -k \lim_{x \to 0^+} x \ln^{k-1} x = \ldots = (-1)^k k! \lim_{x \to 0^+} x = 0.$$

According to Theorem 2 (d), the following result holds:

$$\int_0^1 \ln^n x\, dx = \mathcal{F}(1) - \lim_{x \to 0^+} \mathcal{F}(x) = (-1)^n n!\,. \qquad (17)$$

(c) In Theorem 2 (e), set

$$\mathcal{G}(x) = (-1)^n \left(\frac{2}{3}\right)^{n+1} n!\, x^{3/2}\left[1 - \frac{3}{2}\frac{\ln x}{1!} + \left(\frac{3}{2}\right)^2 \frac{\ln^2 x}{2!} - \ldots + (-1)^n \left(\frac{3}{2}\right)^n \frac{\ln^n x}{n!}\right].$$

Now, since

$$\lim_{x \to 0^+} x^{3/2} \ln^k x = 0, \qquad k = 1, 2, \ldots,$$

we get $\lim_{x \to 0^+} \mathcal{G}(x) = 0$. Hence

$$\int_0^1 \sqrt{x} \ln^n x\, dx = \mathcal{G}(1) - \lim_{x \to 0^+} \mathcal{G}(x) = (-1)^n \left(\frac{2}{3}\right)^{n+1} n!\,. \qquad (18)$$

(d) If we set

$$\mathcal{K}(x) = \frac{1 \cdot 3 \cdots (2n+1)}{(2a)^n}\left[I_1 - \frac{e^{-ax}}{a} \sum_{k=1}^{n} \frac{(2a)^k}{1 \cdot 3 \cdots (2k+1)} x^{(2k+1)/2}\right],$$

then the consequence of Theorem 3 (e) is the following:

$$\int_0^\infty x^{(2n+1)/2} e^{-ax} dx = \lim_{x \to \infty} \mathcal{K}(x) - \mathcal{K}(0) = \frac{1 \cdot 3 \cdots (2n+1)}{(2a)^n} I_1$$

$$= \frac{1 \cdot 3 \cdots (2n+1)}{(2a)^{n+1}} \sqrt{\frac{\pi}{a}}, \qquad (19)$$

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since

$$I_1 = \int_0^\infty \sqrt{x} e^{-ax} dx = \frac{\sqrt{\pi}}{2a^{3/2}},$$

(e.g. by using Laplace transform, [8, 9]).

(e) If

$$\varphi(x) = \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \left[\int e^{-x^2} dx - e^{-x^2} \sum_{k=1}^{n} \frac{2^{k-1}}{1 \cdot 3 \cdots (2k-1)} x^{2n-1} \right],$$

then using Theorem 3 (b) we get

$$I_{2n} = \int_0^\infty x^{2n} e^{-x^2} dx = \lim_{x \to \infty} \varphi(x) - \varphi(0) = \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \int_0^\infty e^{-x^2} dx$$

$$= \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \cdot \frac{\sqrt{\pi}}{2}, \qquad (20)$$

since $I_0 = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

The integral $I_0 = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ has wide application in probability theory and mathematical statistics.

Remark 4

For n = 1, (19) has application in the kinetic theory of gases, particularly in the Maxwell-Boltzmann distribution of gas molecules by energies. Thus the average kinetic energy is expressed as

$$\overline{E} = \int_0^\infty \frac{EN_E}{N} dE,$$

where

$$N_E = \frac{dN}{dE} = \frac{2N}{\sqrt{\pi k^3 T^3}} \sqrt{E}\, e^{-E/(kT)}.$$

Now, using (19) with $a = \frac{1}{kT}$ ($T > 0$ is the abslolute temperature), we obtain

$$\overline{E} = \frac{2}{\sqrt{\pi k^3 T^3}} \int_0^\infty E^{3/2} e^{-E/(kT)} dE = \frac{2}{\sqrt{\pi k^3 T^3}} \cdot \frac{3}{\left(\frac{2}{kT}\right)^2} \sqrt{\frac{\pi}{1/(kT)}} = \frac{3}{2} kT$$

which means that the mean kinetic energy does not depend on the type of gas, but only on the temperature.

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Remark 5

For the integrals considered in the above sections, it is possible to obtain formulas in closed form by using *Wolfram Alpha* software. Such formulas are more general and apply to complex functions. As we will see in some cases, these formulas coincide entirely with the formulas obtained in this paper. In some cases, the Wolfram Alpha answer is given in the form of a power series. The following are formulas obtained using Wolfram Alpha.

$$\int \log^n(x)dx = (-\log(x))^{-n}\log^n(x)\Gamma\left(n+1, -\log(x)\right),$$

$$\int \sqrt{x}\log^n(x)dx = 2^{n+1}\cdot 3^{-n-1}(-\log(x))^{-n}\log^n(x)\Gamma\left(n+1, -\frac{3\log(x)}{2}\right),$$

$$\int \frac{dx}{(1+x^2)^n} = x\,{}_2F_1\left(\frac{1}{2}, n; \frac{3}{2}; -x^2\right),$$

$$\int \frac{dx}{(x^2-a^2)^n} = x\left(x^2-a^2\right)^{-n}\left(1-\frac{x^2}{a^2}\right)^n\ {}_2F_1\left(\frac{1}{2}, n; \frac{3}{2}; \frac{x^2}{a^2}\right),$$

$$\int x^{2n}e^{-x^2}dx = -\frac{1}{2}x^{2n+1}\left(x^2\right)^{-n-\frac{1}{2}}\Gamma\left(n+\frac{1}{2}, x^2\right),$$

where $\log(x)$ is the natural logarithm, $\Gamma(m, x)$ is the incomplete gamma function, ${}_2F_1\left(a, b; c; x\right)$ is the hypergeometric function. If n is an integer, then

$$\Gamma(n, x) = (n-1)!\,e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{k!},$$

so it is, for $x > 0$,

$$\int \ln^n x\,dx = (-1)^n n!\,e^{\ln x}\sum_{k=0}^{n}\frac{(-\ln(x))^k}{k!} = (-1)^n n!\,x\sum_{k=0}^{n}\frac{(-1)^k\ln^k(x)}{k!},$$

which is the same as formula (7).

Analogously,

$$\int \sqrt{x}\,\ln^n(x)\,dx = (-1)^n\left(\frac{2}{3}\right)^{n+1}\Gamma\left(n+1, -\frac{3\ln(x)}{2}\right)$$

$$= (-1)^n\left(\frac{2}{3}\right)^{n+1} n!\,x^{3/2}\sum_{k=0}^{n}(-1)^k\left(\frac{3}{2}\right)^k\frac{\ln^k(x)}{k!},$$

which is the same as the formula (8).

Since

$${}_2F_1\left(a, b; c; x\right) = 1 + \frac{ab}{1!\,c}x + \frac{a(a+1)b(b+1)}{2!\,c(c+1)}x^2 + \ldots = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},$$

where $(a)_n$ is a Pochhammer symbol, we obtain

$$\int \frac{dx}{(1+x^2)^n} = x\,{}_2F_1\left(\frac{1}{2}, n; \frac{3}{2}; -x^2\right) = x\left(1+\sum_{k=0}^{\infty}(-1)^k\frac{n(n+1)\ldots(n+k-1)}{k!\,(2k+1)}x^{2k}\right).$$

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Wolfram Alpha shows that for $n = 0, 1, 2$, the last formula coincides with formula (5). Therefore we can say that formula (5) is more practical to use because it has a more straightforward form; it is a finite sum. An analogous conclusion holds for integral $\int \frac{dx}{(x^2 - a^2)^n}$.

Assuming that n and x are real, we obtain

$$\int x^{2n} e^{-x^2} dx = -\frac{1}{2} \operatorname{sgn}(x)^{2n+1} \Gamma\left(n + \frac{1}{2}, x^2\right),$$

where

$$\Gamma\left(n + \frac{1}{2}, x^2\right) = \Gamma\left(n + \frac{1}{2}\right) \operatorname{erf}\left(\sqrt{x^2}\right) + (-1)^{n-1} e^{-x^2} \sqrt{x^2} \sum_{k=0}^{n-1} \left(\frac{1}{2} - n\right)_{n-k-1} \left(-x^2\right)^k, \quad (21)$$

which shows that the formula (13) is simpler than (21).

The advantage of the formulas obtained in this paper over the formulas obtained using Wolfram Alpha is that the former have a more straightforward form. They do not contain special functions such as gamma or hypergeometric functions. On the other hand, none of the integrals (16), (17), (18), (19) and (20) can be calculated using Wolfram Alpha. This points out the advantage of the formulas obtained in Section 2.

Acknowledgements

The authors thank Prof. Muharem Avdispahić for a careful reading of our manuscript and many significant suggestions.

Also, the authors thank the anonymous referee for numerous valuable comments and suggestions that allowed the paper to take a more receptive form.

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on behalf of The Mathematical Association

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3. Lydgate was not blind to the dangers of such friction, but he had plenty of confidence in his resolution to avoid it as far as possible: being seven-and-twenty, he felt himself experienced.

4. It was only by scorning all she met that she kept herself from tears, and the friction of people brushing past her was evidently painful.

5. "Friction of the air. Going too fast. Like meteorites and things. Too hot. And, Gibberne! Gibberne! I'm all over pricking and a sort of perspiration."

6. FIDDLE, n. An instrument to tickle human ears by friction of a horse's tail on the entrails of a cat.