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UNIQUENESS OF THE COEFFICIENT RING IN SOME GROUP RINGS

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1. Let $\langle x\rangle$ be an infinite cyclic group and $R_i\langle x\rangle$ its group ring over a ring (with identity) R_i, for $i=1$ and 2. Let $J(R_i)$ be the Jacobson radical of R_i. In this note we study the question of whether or not $R_1\langle x\rangle\simeq R_2\langle x\rangle$ implies $R_1\simeq R_2$. We prove that this is so if Z_i, the centre of R_i, is semi-perfect and $J(Z_i\langle x\rangle)=J(Z_i)\langle x\rangle$ for $i=1$ and 2. In particular, when Z_i is perfect the second condition is satisfied and the isomorphism of group rings $R_i\langle x\rangle$ implies the isomorphism of R_i. The corresponding problem for polynomial rings was considered by Coleman and Enochs [2]. We like to thank the referee for pointing out that some of the techniques used in the proof of Theorem 1 were also used by Gilmer [3] in a different context.

2. Some lemmas.

LEMMA 1. *Let G be a group. Then* $(R_1\oplus R_2)G\simeq R_1G\oplus R_2G$

Proof. Define $\sigma:(R_1\oplus R_2)G\to R_1G\oplus R_2G$ by

$$\sigma\left(\sum_i (r_i, s_i)g_i\right) = \left(\sum_i r_ig_i, \sum_i s_ig_i\right).$$

It is clear that σ is an isomorphism.

REMARK. We shall identify the two isomorphic rings of this lemma whenever it is convenient to do so.

LEMMA 2. *Let F and K be fields such that* $\sigma:F\langle x\rangle\to K\langle x\rangle$ *is an isomorphism. Then* $\sigma(F)=K$.

Proof. Let $f\neq 0, -1$ be an element of F. Since $\sigma(f)$ is a unit of $K\langle x\rangle$, we have $\sigma(f)=kx^i$, $k\in K$. Therefore, $\sigma(1+f)=1+kx^i$. Since $1+f$ is a unit, $i=0$ and $\sigma(F)\subset K$. By using σ^{-1}, we conclude that $\sigma(F)=K$.

LEMMA 3. *Let R and S be finite direct sums of fields such that* $\sigma:R\langle x\rangle\to S\langle x\rangle$ *is an isomorphism. Then* $\sigma(R)=S$.

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Proof. Let $R=F_1\oplus\cdots\oplus F_n$, $S=K_1\oplus\cdots\oplus K_m$, direct sums of fields F_i and K_j. With the identification of Lemma 1, $R\langle x\rangle=F_1\langle x\rangle\oplus\cdots\oplus F_n\langle x\rangle$ and $S\langle x\rangle=K_1\langle x\rangle\oplus\cdots\oplus K_m\langle x\rangle$. The only primitive idempotents in $R\langle x\rangle$ are $e_i=(0,\ldots,0,1,0,\ldots,0)$ and similarly for $S\langle x\rangle$. Hence $\sigma(F_i\langle x\rangle)=K_j\langle x\rangle$ for some j and $m=n$. By Lemma 2, $\sigma(F_i)=K_j$. Hence, it follows that $\sigma(R)=S$.

We need a few facts about the Jacobson radical of $R\langle x\rangle$. The proof of the next lemma is found in [1].

LEMMA 4 (Amitsur). $J(R[x])=N[x]$, *where N is a nil ideal of R.*

LEMMA 5. $J(R\langle x\rangle)\subseteq N\langle x\rangle$, *where N is the same ideal as in the last lemma.*

Proof. Let $a=\sum a_ix^i\in J(R\langle x\rangle)$. We can assume that $a=a_1x+\cdots+a_nx^n$ by multiplying by a suitable power of x. Now a has a right quasi-inverse, say, $b=\sum b_jx^j\in R\langle x\rangle$. Since $a+b+ab=0$, we have $b_m=0$ for $m<0$. Thus $b=\sum_0^s b_jx^j$. Let A be the ideal of $R[x]$ generated by a. Then $A\subset J(R\langle x\rangle)$; so if $y\in A$, then y has a right quasi-inverse z in $R\langle x\rangle$. As above, we see that $z\in R[x]$. Hence A is a right quasi-regular ideal of $R[x]$ and we have $A\subseteq J(R[x])=N[x]$.

COROLLARY. $J(R\langle x\rangle)\subseteq J(R)\langle x\rangle$.

LEMMA 6. *Let R be a perfect commutative ring. Then $J(R\langle x\rangle)=J(R)\langle x\rangle$.*

Proof. Since R is perfect, $J(R)$ is T-nilpotent. Since R is commutative, $J(R)\langle x\rangle$ is nil. Hence $J(R)\langle x\rangle\subseteq J(R\langle x\rangle)$. Together with the corollary above, we have $J(R)\langle x\rangle=J(R\langle x\rangle)$.

REMARK. The above lemma can be proved for wider classes of rings but this is all we need.

The next lemma is proved in [4].

LEMMA 7. *Let R be a commutative ring with 1 such that R has no nontrivial nilpotent or idempotent elements. Then y is a unit of $R\langle x\rangle\Leftrightarrow y=ux^i$, where u is a unit of R.*

3. Rings with semi-perfect centres.

THEOREM 1. *Let R_i be a ring (with identity) for $i=1, 2$. Suppose that*

(1) *Z_i, the centre of R_i, is semi-perfect for $i=1, 2$.*

(2) *$J(Z_i\langle x\rangle)=J(Z_i)\langle x\rangle$ for $i=1, 2$.*

Then $R_1\langle x\rangle\simeq R_2\langle x\rangle\Rightarrow R_1\simeq R_2$.

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Proof. Let $\sigma: R_1\langle x\rangle \to R_2\langle x\rangle$ be the isomorphism. Then by restriction we have $\sigma: Z_1\langle x\rangle \to Z_2\langle x\rangle$. Due to the second condition of the hypothesis we have an induced isomorphism

$$\bar{\sigma}: \frac{Z_1}{J(Z_1)}\langle x\rangle \to \frac{Z_2}{J(Z_2)}\langle x\rangle.$$

By Lemma 3, it follows that $\bar{\sigma}(Z_1/J(Z_1)) = Z_2/J(Z_2)$. Also, Lemma 1 gives

$$\frac{Z_2}{J(Z_2)}\langle x\rangle = \bar{e}_1\frac{Z_2}{J(Z_2)}\langle x\rangle \oplus \cdots \oplus \bar{e}_n\frac{Z_2}{J(Z_2)}\langle x\rangle,$$

where each $\bar{e}_i$ is a primitive idempotent and each $\bar{e}_i(Z_2/J(Z_2))$ is a field F_i. Then we see that

$$\bar{\sigma}(x) = (f_1x^{i_1}, \ldots, f_nx^{i_n}) \quad \text{where} \quad i_j = \pm 1, \qquad 0 \neq f_j \in F_j \quad \text{for all } j.$$

This follows because $\bar{\sigma}(Z_1/J(Z_1))$ and $\bar{\sigma}(x)$ together must generate $(Z_2/J(Z_2))\langle x\rangle$.

Since Z_2 is semi-perfect, we can lift the idempotents $\bar{e}_i$ of $Z_2/J(Z_2)$ to primitive orthogonal idempotents e_i in Z_2 such that $1 = e_1 + e_2 + \cdots + e_n$. Then

$$R_2\langle x\rangle = e_1R_2\langle x\rangle \oplus \cdots \oplus e_nR_2\langle x\rangle.$$

Define an R_2-algebra automorphism $\beta: R_2\langle x\rangle \to R_2\langle x\rangle$ by

$$\beta(x) = (x^{i_1}, x^{i_2}, \ldots, x^{i_n}).$$

It is not too difficult to check that β is indeed an automorphism and therefore induces an automorphism $\bar{\beta}$ of $Z_2/J(Z_2)\langle x\rangle$. Notice that

$$\bar{\beta}\bar{\sigma}(x) = (f_1x, \ldots, f_nx) = (f_1, \ldots, f_n)x = ux$$

where u is a unit in $Z_2/J(Z_2)$. Since $J(Z_2\langle x\rangle) = J(Z_2)\langle x\rangle$, it follows from Lemma 5 that $J(Z_2)$ is a nil ideal. Hence we see that

$$\beta\sigma(x) = u_1x + \sum_{i\neq 1} a_ix^i$$

where u_1 is a unit of Z_2 and a_i are nilpotent elements of Z_2.

We now claim that $R_2\langle\beta\sigma(x)\rangle = R_2\langle x\rangle$. We may assume that $u_1 = 1$. Note that

$$(\beta\sigma(x))^{-1} = x^{-1}\Big(1 + \sum_{i\neq 0} a_{i+1}x^i\Big)^{-1} = x^{-1}(1 - r + r^2 + \cdots + (-1)^s r^s),$$

where $r = \sum_{i\neq 0} a_{i+1}x^i$ and $r^{s+1} = 0$. We proceed by induction on the index of nilpotency of A, the ideal of R_2 generated by $\{a_i\}$. If this index is one then $\beta\sigma(x) = x$ and we are finished. Now we can suppose that this index is greater than one. Observing that

$$(\beta\sigma(x))^i = x^i + \sum_j b_jx^j, \qquad b_j \in A$$

we obtain

$$\beta\sigma(x) - \sum_{i\neq 1} a_i(\beta\sigma(x))^i = x - \sum a_ib_jx^j, \qquad a_i, b_j \in A.$$

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Since A is nilpotent,

$$\beta\sigma(x)-\sum_{i\neq 1} a_i(\beta\sigma(x))^i = vx-\sum_{j\neq 1} a_ib_jx^j$$

where v is a unit in Z_2 and $a_ib_j \in A^2$ which is of smaller index of nilpotency than A. Hence,

$$R_2\left\langle\beta\sigma(x)-\sum_{i\neq 1} a_i(\beta\sigma(x))^i\right\rangle = R_2\langle x\rangle$$

and therefore,

$$R_2\langle\beta\sigma(x)\rangle = R_2\langle x\rangle.$$

Thus we have proved that the R_2-homomorphism $\alpha: R_2\langle x\rangle\to R_2\langle x\rangle$ defined by $x\xrightarrow{\alpha}\beta\sigma(x)$ is an epimorphism.

To see that α is one to one, we must show that $\sum c_i(\beta\sigma(x))^i=0\Rightarrow c_i=0$ for all i. Now let $\sum_i c_i(\beta\sigma(x))^i=0$. In $R_2/A\langle x\rangle$, we have

$$\sum \bar{c}_i\overline{(\beta\sigma(x))}^i = 0.$$

However, $\overline{\beta\sigma(x)}=\bar{u}_1x$ and therefore $\bar{c}_i\bar{u}_1^i=0$. Since u_1 is unit it follows that $\bar{c}_i=0$ and $c_i \in A$ for all i. Assume that $c_i \in A^k$, $k\geq 1$ for all i. Then in $R_2/A^{k+1}\langle x\rangle$

$$\sum \overline{c_i(\beta\sigma(x))^i} = 0.$$

However, $c_i(\beta\sigma(x))^i\equiv c_iu_1^ix^i \pmod{A^{k+1}\langle x\rangle}$ and we have

$$0 = \sum \bar{c}_i\bar{u}_1^ix^i.$$

Again it follows that $\bar{c}_i=0$ and $c_i \in A^{k+1}$ for all i. Since A is nilpotent, $c_i=0$ for all i and α is one to one. We have a ring isomorphism, $\alpha^{-1}\beta\sigma: R_1\langle x\rangle\to R_2\langle x\rangle$ such that $(\alpha^{-1}\beta\sigma)(x)=x$ and therefore $(\alpha^{-1}\beta\sigma):\Delta(R_1\langle x\rangle)\to\Delta(R_2\langle x\rangle)$, where Δ denotes the augmentation ideal. We have

$$R_1 \simeq \frac{R_1\langle x\rangle}{\Delta(R_1\langle x\rangle)} \simeq \frac{R_2\langle x\rangle}{(R_2\langle x\rangle)} \simeq R_2.$$

Corollary 1. *Let R_1, R_2 be rings with perfect centres. Then*

$$R_1\langle x\rangle \simeq R_2\langle x\rangle \Rightarrow R_1 \simeq R_2.$$

Proof. Lemma 6 and Theorem 1.
Since a left artinian ring is left perfect we also have the following.

Corollary 2. *Let R_1, R_2 be rings with artinian centres. Then*

$$R_1\langle x\rangle \simeq R_2\langle x\rangle \Rightarrow R_1 \simeq R_2.$$

Of a somewhat different nature is the next theorem.

THEOREM 2. *Let R_1 be a ring with identity such that its centre Z_1 has no nontrivial idempotent or nilpotent elements. Suppose that all units of Z_1 are algebraic over the prime subring of Z_1. Then*

$$R_1\langle x\rangle \overset{\sigma}{\simeq} R_2\langle x\rangle \Rightarrow R_1 \simeq R_2.$$

Proof. We first remark that since $Z_1\langle x\rangle$ has no nontrivial nilpotent or idempotent elements the same holds for $Z_2\langle x\rangle$. Here Z_2 is the centre of R_2. By Lemma 7, the units of $Z_i\langle x\rangle$ are of the form ux^j, $u \in Z_i$ for $i=1, 2$. If z_1 is a unit in Z_1, then we have that $\sum a_i z_1^i=0$ for some a_i in the prime subring of Z_1. Let $\sigma(z_1)=ux^j$ for some unit $u \in Z_2$. Then $\sum a_i(ux^j)^i=0$. Since u is a unit, this implies that $j=0$ and hence that $\sigma(z_1) \in Z_2$. It follows that $\sigma(x)=vx^l$ for some $v \in Z_2$, $l=\pm 1$. Define $\tau: R_2\langle x\rangle \to R_2\langle x\rangle$ by

(i) $\tau(x)=u^{-l}x^l$

and

(ii) $\tau(\sum a_j x^j)=\sum a_j \tau(x)^j$.

Then τ is an R_2-algebra automorphism and $\tau\sigma(x)=x$. Also,

$$\tau\sigma(\Delta(R_1\langle x\rangle)) = \Delta(R_2\langle x\rangle)$$

which implies that $R_1 \simeq R_2$.

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