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RESEARCH ARTICLE

On the cohomology of reciprocity sheaves

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Abstract

In this paper, we show the existence of an action of Chow correspondences on the cohomology of reciprocity sheaves. In order to do so, we prove a number of structural results, such as a projective bundle formula, a blow-up formula, a Gysin sequence and the existence of proper pushforward. In this way, we recover and generalise analogous statements for the cohomology of Hodge sheaves and Hodge-Witt sheaves.

We give several applications of the general theory to problems which have been classically studied. Among these applications, we construct new birational invariants of smooth projective varieties and obstructions to the existence of zero cycles of degree 1 from the cohomology of reciprocity sheaves.

Contents

Introduction

0.1. Overview

It is a well-known fact that a large class of cohomology theories for algebraic varieties can be equipped with an exceptional, covariant functoriality, despite the fact that they are naturally contravariant. The existence of this kind of 'trace' or 'Gysin' morphism associated to projective (or even proper) maps of smooth schemes is usually manifesting the existence of some Poincaré duality theory for the cohomology one is interested in; if one replaces cohomology with homology, which is naturally covariant, the exceptional functoriality is conversely represented by the existence of a pullback along a certain class of maps. The construction of cohomological Gysin morphisms has occupied vast literature, stemming from Grothendieck's trace formalism for coherent cohomology [Har66].

A classical instance in the homological setting is represented by the Chow groups. If X is a smooth quasi-projective variety over a field k, the Chow groups $\mathrm{CH}_*(X)$ are naturally covariant for proper maps and admit contravariant Gysin maps for quasi-projective local complete intersection morphisms [Ful98]. Fulton's construction of the Gysin morphism was later promoted by Voevodsky in the context of his triangulated category of mixed motives $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{Nis}}(k)$ over a perfect field k. Associated to a codimension n closed immersion of smooth k-schemes, $i\colon Z \to X$, Voevodsky [Voe00b] constructed a distinguished triangle:

$$M(X-Z) \to M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\partial_{X,Z}} M(X-Z)[1],$$

where i^* is the Gysin morphism and $\partial_{X,Z}$ is a residue map. Combining it with a projective bundle formula for motives, also provided by Voevodsky, the classical method of Grothendieck allows one to define exceptional functoriality along an arbitrary projective morphism between smooth k-varieties, factoring it as a closed immersion followed by a projection of a projective bundle. This as well as the naturality properties of Voevodsky's Gysin maps have been studied in detail by Déglise [Dég08], [Dég12].

In more recent times, Gysin morphisms for generalised cohomology theories have been constructed in the context of $\mathbf{A}^1$-homotopy theory, making use of the full six functor formalism as developed by [Ayo07a], [Ayo07b] and [CD19] (see [DJK18] for more history and updated developments in that direction).

From the Gysin sequence, the projective bundle formula and the blow-up formula (the latter being also an ingredient in the construction of the first one) in the triangulated category of Voevodsky's motives, it is possible to get corresponding formulas for every cohomology theory which is representable in $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{Nis}}(k)$. This is the case of the sheaf cohomology of any complex of (strictly) $\mathbf{A}^1$-invariant Nisnevich sheaves with transfers.

However, $\mathbf{A}^1$-invariant Nisnevich sheaves do not encompass all of the phenomena that one would like to study. Interesting examples of sheaves which fail to satisfy this property are given by the sheaves

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of (absolute and relative) differential forms, $\Omega^i_{-/\mathbb{Z}}, \Omega^i_{-/k}$, the p-typical de Rham-Witt sheaves of Bloch-Deligne-Illusie, $W_m\Omega^i$, smooth commutative k-groups schemes with a unipotent part (seen as sheaves with transfer) or the complexes $R\varepsilon_*\mathbb{Z}/p^r(n)$, where $\mathbb{Z}/p^r(n)$ is the étale motivic complex of weight n with $\mathbb{Z}/p^r$ coefficients, ε is the change of site functor from the étale to the Nisnevich topology and $p > 0$ is the characteristic of k. For some of the above examples, instances of an exceptional functoriality have been studied before, with results scattered in the literature. In the case of the sheaves of differential forms, the existence of the pushforward is of course a consequence of general Grothendieck duality (e.g. [Har66], [Nee96]). In this paper, we offer a unified approach to treat the cohomology of arbitrary *reciprocity sheaves*, a notion that includes all of the above examples: this is a particular abelian[1] subcategory $\mathbf{RSC}_{\mathrm{Nis}}$ of the category of Nisnevich sheaves with transfers on the category $\mathbf{Sm}_k$ of smooth and separated k-schemes. Its objects satisfy, roughly speaking, the property that for any $X \in \mathbf{Sm}_k$, each section $a \in F(X)$ 'has bounded ramification', that is, that the corresponding map $a\colon \mathbb{Z}_{tr}(X) \to F$ factors through a quotient $h_0(\mathcal{X})$ of $\mathbb{Z}_{tr}(X)$, associated to a pair $\mathcal{X} = (\overline{X}, X_\infty)$, where $\overline{X}$ is a proper scheme over k and X_∞ is an effective Cartier divisor on $\overline{X}$, such that $X = \overline{X} - |X_\infty|$ (see 1.6 for more details). The category of reciprocity sheaves has been introduced by Kahn-Saito-Yamazaki in [KSY22] (see also its precursor [KSY16]) and is based on a generalisation of the idea of Rosenlicht and Serre of the modulus of a rational map from a curve to a commutative algebraic group [Ser84, Chapter III].

Voevodsky's category of homotopy invariant Nisnevich sheaves, $\mathbf{HI}_{\mathrm{Nis}}$, is an abelian subcategory of $\mathbf{RSC}_{\mathrm{Nis}}$. Heuristically, $\mathbf{A}^1$-invariant sheaves are special reciprocity sheaves with the property that every section $a \in F(X)$ has 'tame' ramification at infinity. Slightly more exotic examples of reciprocity sheaves are given by the sheaves Conn^1 (in characteristic zero), whose sections over X are rank 1-connections, or Lisse^1_ℓ (in characteristic $p > 0$), whose sections on X are the lisse $\overline{\mathbb{Q}}_\ell$-sheaves of rank 1. Since $\mathbf{RSC}_{\mathrm{Nis}}$ is abelian, and it is equipped with a lax[2] symmetric monoidal structure [RSY22], many more interesting examples can be manufactured by taking kernels, quotients and tensor products (see 11.1 for even more examples).

0.2. Cohomology of cube invariant sheaves

In order to formulate our main results, we need a bit of extra notation. In [KMSY21a], the authors introduced the category $\underline{\mathbf{M}}\mathbf{Cor}$ of *modulus correspondences*, whose objects are pairs $\mathcal{X} = (\overline{X}, X_\infty)$, called *modulus pairs*, where $\overline{X}$ is a separated scheme of finite type over k equipped with an effective Cartier divisor X_∞ (the case $X_\infty = \emptyset$ is allowed), such that the *interior* $\overline{X} - |X_\infty| = X$ is smooth. The morphisms are finite correspondences on the interiors satisfying some admissibility and properness conditions (see 1.1). The category $\underline{\mathbf{M}}\mathbf{Cor}$ admits a symmetric monoidal structure, denoted $\otimes$. Let $\underline{\mathbf{M}}\mathbf{PST}$ be the category of additive presheaves of abelian groups on $\underline{\mathbf{M}}\mathbf{Cor}$. Given $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$ and $F \in \underline{\mathbf{M}}\mathbf{PST}$, we write $F_{\mathcal{X}}$ for the presheaf on the small étale site $\overline{X}_{\text{ét}}$ given by $U \mapsto F(U, U \times_{\overline{X}} X_\infty)$. We say that F is a Nisnevich sheaf if, for every $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$, the restriction $F_{\mathcal{X}}$ is a Nisnevich sheaf; the full subcategory of Nisnevich sheaves of $\underline{\mathbf{M}}\mathbf{PST}$ is denoted $\underline{\mathbf{M}}\mathbf{NST}$. Thanks to [KMSY21a], the inclusion $\underline{\mathbf{M}}\mathbf{NST} \subset \underline{\mathbf{M}}\mathbf{PST}$ has an exact left adjoint (the sheafification functor). Among the objects of $\underline{\mathbf{M}}\mathbf{PST}$, we are interested in a special class, namely, those which satisfy the properties of being *cube invariant*, *semipure* and with *M-reciprocity* (see 1.4). The first two properties are easy to explain. Let $\overline{\square} = (\mathbf{P}^1, \infty) \in \underline{\mathbf{M}}\mathbf{Cor}$. Then $F \in \underline{\mathbf{M}}\mathbf{PST}$ is cube invariant, if for any $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$ the natural map:

$$F(\mathcal{X}) \to F(\mathcal{X} \otimes \overline{\square})$$

induced by the projection $\overline{X} \times \mathbf{P}^1 \to \overline{X}$ is an isomorphism. We have that F is semipure if the natural map:

$$F(\mathcal{X}) \to F(X, \emptyset), \quad (X = \overline{X} - |X_\infty|)$$

[1]The fact that the category of reciprocity sheaves is abelian is a nontrivial result (see [Sai20a]).
[2]In the sense that only a weak form of associativity is known to hold (cf. [RSY22, Theorem 1.5])

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is injective. The last condition of M-reciprocity is slightly more technical, and we refer the reader to the body of the paper. We write $\mathbf{CI}^{\tau,sp}$ for the category of cube invariant, semipure presheaves with M-reciprocity and $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ for $\mathbf{CI}^{\tau,sp} \cap \underline{\mathbf{M}}\mathbf{NST}$. It is possible to show (see [MS20, Section 1.6], [KSY22, Section 2.3.7]) that there is a fully faithful functor:

$$\underline{\omega}^{\mathbf{CI}} \colon \mathbf{RSC}_{\mathrm{Nis}} \to \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$$

admitting an exact[3] left adjoint, so that one can, in particular, specialise Theorem 0.1 below on cube invariant sheaves to the case of reciprocity sheaves. If $G \in \mathbf{RSC}_{\mathrm{Nis}}$, we write $\widetilde{G}$ for $\underline{\omega}^{\mathbf{CI}}(G)$, and for $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, $n \geq 1$, let us write:

$$\gamma^n F = \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\widetilde{\mathbf{G}_m^{\otimes_{\mathbf{HI}_{\mathrm{Nis}}} n}}, F) \cong \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\widetilde{\mathcal{K}^M_n}, F).$$

This is a form of (negative) twist (see 4.4, called *contraction* in Voevodsky's theory [MVW06, Chapter 23]). The tensor product with subscript **HI** is the tensor product for homotopy invariant Nisnevich sheaves with transfers from [MVW06, Chapter 8], $\mathcal{K}^M_n$ is the sheaf of improved Milnor K-theory introduced in [Ker10] and the isomorphism follows from a result of Voevodsky [RSY22, Section 5.5]. See Theorems 11.1 and 11.8 for some computations of the twists. The Bloch formula implies that for any family of supports Φ and any cycle $\alpha \in CH^i_\Phi(X)$ (see 5.1), there is a natural cupping map:

$$c_\alpha \colon (\gamma^i F)_{\mathcal{X}}[-i] \to R\underline{\Gamma}_\Phi F_{\mathcal{X}} \quad \text{in } D(X_{\mathrm{Nis}}),$$

which is compatible with refined intersection and pullback, see 5.8.

The following theorem summarises parts of our results. Write $\underline{\mathbf{M}}\mathbf{Cor}_{ls}$ for the subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$, whose objects $\mathcal{X} = (X, D)$ satisfy the additional condition that $X \in \mathbf{Sm}$ and $|D|$ is a simple normal crossing divisor.

Theorem 0.1. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, and let $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$.*

(1) *(Projective bundle formula, Theorem 6.3) Let V be a locally free $\mathcal{O}_X$-module of rank $n+1$, and let $P = \mathbf{P}(V) \xrightarrow{\pi} X$ be the corresponding projective bundle. Let $\mathcal{P} = (P, \pi^* D)$. Then there is a natural isomorphism in $D(X_{\mathrm{Nis}})$:*

$$\sum_{i=0}^{n} \lambda^i_V : \bigoplus_{i=0}^{n} (\gamma^i F)_{\mathcal{X}}[-i] \to R\pi_* F_{\mathcal{P}},$$

where λ^i_V is induced by c_{ξ^i} for the i-fold power $\xi^i \in \mathrm{CH}^i(X)$ of the first Chern class ξ of V.

(2) *(Gysin sequence, Theorem 7.16) Let $i \colon Z \hookrightarrow X$ be a smooth closed subscheme of codimension j intersecting D transversally (Definition 2.11), and set $\mathcal{Z} = (Z, D_{|Z})$. Then there is a canonical distinguished triangle in $D(X_{\mathrm{Nis}})$:*

$$i_* \gamma^j F_{\mathcal{Z}}[-j] \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} F_{\mathcal{X}} \xrightarrow{\rho^*} R\rho_* F_{(\tilde{X}, D_{|\tilde{X}}+E)} \xrightarrow{\partial} i_* \gamma^j F_{\mathcal{Z}}[-j+1], \tag{0.1.1}$$

where $\rho : \tilde{X} \to X$ is the blow-up of X along Z and $E = \rho^{-1}(Z)$. The Gysin map $g_{\mathcal{Z}/\mathcal{X}}$ satisfies an excess intersection formula (7.9.1), it is compatible with smooth base change (Proposition 7.9) and the cup product with Chow classes (Proposition 7.8).

We stress the fact that, in constrast to the $\mathbf{A}^1$-invariant setting, our Gysin sequence does not involve the cohomology of the open complement of $Z \subset X$ but rather, the cohomology of a modulus pair constructed by taking the blow-up of X along Z. When $F = \widetilde{G}$ and $G \in \mathbf{HI}_{\mathrm{Nis}}$, one can in fact verify that (0.1.1) gives back the classical Gysin sequence of Déglise and Voevodsky. For

[3] That is, commuting with finite limits and colimits.

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non-$\mathbf{A}^1$-invariant sheaves, the existence of the Gysin map is new essentially in all of the above-mentioned examples: for instance, it does not follow from the work of Gros [Gro85] for the de Rham-Witt sheaves. Other interesting cases are given by $F = \widetilde{\text{Conn}}^1$ or $\widetilde{\text{Lisse}}^1$ (see Corollary 11.6). We may also apply (0.1.1) for $D = \emptyset$ and F the whole de Rham-Witt complex and obtain in this way a Gysin sequence for the crystalline cohomology $Ru_{X*}\mathcal{O}_{X/W_n}$, where $u_X : (X/W_n)_{\text{crys}} \to X_{\text{Nis}}$ is the natural map of sites, which generalises to higher codimension the classical sequence induced by the residue map along a smooth closed divisor(see Corollary 11.10 and the following remark).

The key computation leading to the above results is the vanishing $H^i(Y, F_{(Y,\rho^* L)}) = 0$, for $i \geq 1$, where $\rho : Y \to \mathbf{A}^n$ is the blow-up in the origin and $L \subset \mathbf{A}^n$ a hyperplane passing through the origin (see Theorem 2.12). The proof of this theorem occupies almost all of Section 2 and relies deeply on the theory of modulus sheaves with transfers.

By factoring any projective morphism as a closed embedding followed by a projection from a projective bundle, we can use Theorem 0.1 to construct pushforward maps (in fact, we construct the pushforward with proper support along a quasi-projective morphism; see Definition 8.5 and Proposition 8.6 for the main properties). Note that the pushforward is compatible with composition, smooth base change and cup product with Chow classes (see 9.5 and Theorem 9.7).

For $F = \underline{\omega}^{\mathbf{CI}} W_m\Omega^i$, the construction gives even a refinement of the pushforward map for cohomology of Hodge-Witt differentials constructed by Gros [Gro85] (see Corollary 0.6 below).

0.3. Chow correspondences

When a cohomology theory is equipped with pushforward with proper support and a cup product with cycles, it is possible, with a bit of extra work, to produce an action of Chow correspondences. Let S be a separated k-scheme of finite type, and let C_S be the category whose objects are maps $(f : X \to S)$ with the property that the induced map $X \to \text{Spec}(k)$ is smooth and quasi-projective. As for morphisms, we set (if Y is connected):

$$C_S(X, Y) = \text{CH}^{\dim Y}_{\Phi^{\text{prop}}_{X\times_S Y}}(X \times Y),$$

where $\Phi^{\text{prop}}_{X\times_S Y}$ is the family of supports on $X \times Y$ consisting of closed subsets which are contained in $X \times_S Y$, and that are proper over X. Composition is given by the usual composition of correspondences using the refined intersection product [Ful98, Chapter 16]. If $F^\bullet$ is a bounded below complex of reciprocity sheaves and $(f : X \to S)$ and $(g : Y \to S)$ are objects of C_S, we can define for $\alpha \in C_S(X, Y)$ a morphism:

$$\alpha^* : Rg_* F^\bullet_Y \to Rf_* F^\bullet_X \text{ in } D^+(S_{\text{Nis}})$$

that is compatible with the composition of correspondences, satisfies a projection formula and gives back the pushforward for reciprocity sheaves when $\alpha = [\Gamma^t_h]$ is the transpose of the graph of a proper S-morphism $h : X \to Y$ (see Proposition 9.10).

For homotopy invariant sheaves, the existence of the action of Chow correspondences follows from work of Rost [Ros96] and Déglise [Dég12] (although, to our knowledge, this has not been spelled out explicitly in the literature).

Previous instances of constructions of an action of Chow correspondences on the cohomology of Hodge and Hodge-Witt differentials can be found in [CR11] and [CR12]. However, we remark that the approach followed in this paper is conceptually different: in [CR11] and [CR12], the existence of the whole de Rham and de Rham-Witt complex, with its structure of graded algebra, was used. In contrast, here, the projective pushforward is directly constructed starting from a single reciprocity sheaf F (and its twists). Our statements are also finer, since we get morphisms defined at the level of derived categories, rather than just between the cohomology groups.

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0.4. Applications

Let us now discuss how we can apply the formulas established so far to get new interesting invariants.

0.4.1. Obstructions to the existence of zero cycles of degree 1

In Section 10.1, we explain how to use the proper correspondence action on the cohomology of an arbitrary reciprocity sheaf to construct very general obstructions of Brauer-Manin type to the existence of zero cycles on smooth projective varieties over function fields, recovering the classical obstruction as a special case.

Here is the main result (see Theorem 10.1):

Theorem 0.2. *Let $f: Y \to X$ be a dominant quasi-projective morphism between connected smooth k-schemes. Assume that there are integral subschemes $V_i \subset Y$, which are proper, surjective and generically finite over X of degree n_i, $i = 1, \dots, s$. Set $N = \gcd(n_1, \dots, n_s)$. Let $F^\bullet \in \mathrm{Comp}^+(\mathbf{RSC}_{\mathrm{Nis}})$ be a bounded below complex of reciprocity sheaves. Then there exists a morphism $\sigma : Rf_* F_Y^\bullet \to F_X^\bullet$ in $D(X_{\mathrm{Nis}})$, such that the composition:*

$$F_X^\bullet \xrightarrow{f^*} Rf_* F_Y^\bullet \xrightarrow{\sigma} F_X^\bullet$$

is multiplication with N.

In particular, if f is proper and $f^*: H^i(X, F_X^\bullet) \to H^i(Y, F_Y^\bullet)$ is not split injective, then the generic fibre of f cannot have index 1, that is, there cannot exist a zero cycle of degree 1. It is then possible to assemble the morphisms σ in order to produce a generalisation of the classical Brauer-Manin obstruction in the case of the function field of a curve (see (10.2.3) and the references there for more details). This is explained in Corollary 10.4.

See also the end of Section 10.1 for a comprehensive list of references to previous works where unramified cohomology groups have been used to study obstructions to the local-global principle for rational points, rather than for zero cycles, over special types of global fields.

0.4.2. Birational invariants

Once we have established an action of Chow correspondences on the cohomology of reciprocity sheaves, this can be used to find birational invariants.

Let us fix again a separated k-scheme of finite type S. We say that $(f: X \to S)$ and $(g: Y \to S) \in C_S$, with X and Y integral, are *properly birational over S* if there exists an integral scheme Z (that we call *proper birational correspondence*) over S and two proper birational S-morphisms $Z \to X$, $Z \to Y$ (note that we don't assume that f or g is proper). If we let $Z_0 \subset X \times Y$ be the image of $Z \to X \times Y$, we can then look at the composition $[Z_0]^* \circ [Z_0^t]^*$ and get, for example, the following result.

Theorem 0.3 (see Theorem 10.10). *Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$, and assume that $F(\xi) = 0$, for all points ξ which are finite and separable over a point of X or Y of codimension ≥ 1. Then any proper birational correspondence between X and Y induces an isomorphism:*

$$Rg_* F_Y \xrightarrow{\simeq} Rf_* F_X.$$

If $Y = S$ in the statement of Theorem 0.3, we get a vanishing $R^i f_* F_X = 0$ for $i \geq 1$ and for any projective birational morphism $f: X \to Y$ and F as in the theorem. The prototype example of a sheaf satisfying the condition $F(\xi) = 0$ is the sheaf of top differential forms, $\Omega^{\dim X}_{/k}$. For this, the birational invariance is classical in characteristic zero and follows from Hironaka's resolution of singularities. In positive characteristic, it was proven in [CR11] by using a similar action of Chow correspondences (although the statements in *loc.cit.* were for the cohomology groups, not for the whole complexes in the derived category; see also [Kov17]). On the other hand, Theorem 0.3 provides a very general class of birational invariants, many of which are new to us: for example, using results of Geisser-Levine [GL00],

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we can consider the cohomology of the étale motivic complexes $R^i\varepsilon_*(\mathbb{Z}/p^n(d))$ (for all i and n, if $\mathrm{char}(k) = p > 0$), where $d = \dim X = \dim Y$; see Corollary 11.16 for a more extensive list). Among the other applications, we can use Theorem 0.3 to generalise parts of [Pir12, Theorem 3.3] (which generalises [CTV12, Proposition 3.4]; see Corollary 11.19 for more details).

We remark that the global sections of reciprocity sheaves enjoy a general invariance under proper (stable) birational correspondences, without assuming $F(\xi) = 0$ for ξ as above (see Theorem 10.7 and the notations there).

As a byproduct of 0.2, we also get (stably) proper birational invariance (see Definition 10.2) for the n-torsion of the relative Picard scheme, $\mathrm{Pic}_{X/S}[n]$, for all n and any flat, geometrically integral and projective morphism $X \to S$ between smooth connected k-schemes, such that the generic fibre has index 1. This is classical and known to the experts if S is the spectrum of an algebraically closed field, but it is new for general S (see Corollary 11.24).

0.4.3. Decomposition of the diagonal

In section 10.3, we investigate the implications of the cycle action in case we have a decomposition of the diagonal, a method which was first employed in [BS83]. For example, we obtain:

Theorem 0.4 (see Theorem 10.13). *Let $f\colon X \to S$ be a smooth projective morphism, where S is the henselisation of a smooth k-scheme in a 1-codimensional point or a regular connected affine scheme of dimension ≤ 1 and of finite type over a function field K over k. Assume that the diagonal cycle $[\Delta_{X_\eta}]$ of the generic fibre X_η of f has an integral decomposition. Then, for any $F \in \mathbf{RSC}_{\mathrm{Nis}}$, the pullback along f induces an isomorphism:*

$$H^0(X, F) \cong H^0(S, F).$$

See Remark 10.14 for some conditions under which the diagonal decomposes. Note that in the case $F = R^i\varepsilon_*\mathbb{Z}/p^n(j)$, with $(i, j) \neq (0, 0)$, and X is defined over an algebraically closed field of characteristic $p > 0$ and admits an integral decomposition of the diagonal, we obtain $H^0(X, R^i\varepsilon_*\mathbb{Z}/p^n(j)) = H^0(\mathrm{Spec}\, k, R^i\varepsilon_*\mathbb{Z}/p^n(j)) = 0$. (The vanishing follows from [GL00].) This immediately implies a positive answer to Problem 1.2 of [ABBvB19] and reproves Theorem 1 in *loc. cit.* (see Corollary 11.21; also see the recent work [Ota20], for a different approach).

In the case $S = \mathrm{Spec}\, k$ and F is $\mathbf{A}^1$-invariant, Theorem 0.4 is classical; Totaro proved that it also holds for $F = \Omega^i_{-/k}$ (see [Tot16, Lemma 2.2]) and – building on ideas of Voisin and Colliot-Thélène-Pirutka – used this to find many new examples of hypersurfaces that are not stably rational. It is an interesting question, whether the flexibility in the choice of the sheaf F coming from Theorem 0.4 — for example, F can be any quotient of $\Omega^i_{-/k}$, say $F = \Omega^N_{-/k}/\mathrm{dlog}\, K^M_N$ from Corollary 11.16 — can be used to find new examples of nonstably rational varieties.

Results for higher cohomology groups are also obtained if F satisfies certain extra assumptions (see Theorems 10.15 and 10.16 and Corollary 11.22 for examples).

0.4.4. Cohomology of ordinary varieties

Following Bloch-Kato [BK86] and Illusie-Raynaud [IR83], we say that a variety X over a perfect field k of characteristic $p > 0$ is *ordinary* if $H^m(X, B^r_X) = 0$ for all m and r, where $B^r_X = \mathrm{Im}(d\colon \Omega^{r-1}_X \to \Omega^r_X)$. It is equivalent to ask that the Frobenius $F\colon H^q(X, W\Omega^r_X) \to H^q(X, W\Omega^r_X)$ is bijective for all q and r. If X is an abelian variety A, this recovers the property that the p-rank of A is the maximum possible, namely, equal to its dimension. For them, we have the following result.

Corollary 0.5 (see Corollary 11.14). *Let $f\colon X \to S$ be a surjective morphism between smooth projective connected k-schemes. Assume that the generic fibre has index prime to p. Then:*

$$X \text{ is ordinary} \implies S \text{ is ordinary.}$$

Note that the assumption on the generic fibre is of course guaranteed if $X_{k(S)}$ has a zero cycle of degree prime to p (for example, when $X_{k(S)}$ is an abelian variety). Similar implications

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hold for the properties 'X is Hodge-Witt' or 'the crystalline cohomology of X is torsion-free' (see Remark 11.15).

In connection to ordinary varieties, let us also mention the following result (see Corollary 11.12):

Corollary 0.6. *Let $f: Y \to X$ be a morphism of relative dimension $r \geq 0$ between smooth projective k-schemes. Assume that X is ordinary. Then the Ekedahl-Grothendieck pushfoward (see [Gro85, Chapter II, 1]) factors via:*

$$R\Gamma(Y, W_n\Omega^q_Y)[r] \to R\Gamma(Y, W_n\Omega^q_Y/B^q_{n,\infty})[r] \xrightarrow{f_*} R\Gamma(X, W_n\Omega^{q-r}_X), \tag{0.6.1}$$

where $B^q_{n,\infty} = \bigcup_s F^{s-1}dW_{n+s-1}\Omega^{q-1}$ (see [IR83, Chapter IV, (4.11.2)]) and f_* is induced by the pushforward from 9.5.

Note that this is an essentially immediate consequence of the fact that the sheaves $B^q_{n,\infty}$ are reciprocity sheaves, our general formalism and the computation of the twists of Theorem 11.8. In fact, even when X is not ordinary, we always obtain a factorisation in top degree:

$$R\Gamma(Y, W_n\Omega^r_Y)[r] \to R\Gamma(Y, W_n\Omega^r_Y/B^r_{n,\infty})[r] \xrightarrow{f_*} R\Gamma(X, W_n\mathcal{O}_X)$$

as a byproduct of the proof of Corollary 11.12.

0.4.5. Relationship with logarithmic motives

In [BPØ22], Park, Østvær and the first author recently introduced a triangulated category of *logarithmic motives* over a field k. Similar in spirit to Voevodsky's construction, the starting point is the category lSm/k of log smooth (fs)-log schemes over k, promoted then to a category of correspondences. The localisation with respect to a new Grothendieck topology, called the dividing-Nisnevich topology, and with respect to the log scheme $\overline{\square}$, the log compactification of $\mathbf{A}^1_k$, produces the category denoted by $\mathbf{logDM}^{\mathrm{eff}}_{dNis}(k)$.

A theorem of Saito (see [Sai20b]) shows that there exists a fully faithful exact functor:

$$\mathcal{L}og\colon \mathbf{RSC}_{\mathrm{Nis}} \to \mathbf{Shv}^{\mathrm{ltr}}_{dNis}(k, \mathbb{Z}),$$

such that $\mathcal{L}og(F)$ is strictly $\overline{\square}$-invariant in the sense of [BPØ22, Definition 5.2.2], where the target is the category of dividing Nisnevich sheaves with log transfers on lSm/k (see [BPØ22, Section 2.4]). This shows that Nisnevich cohomology of reciprocity sheaves is representable in $\mathbf{logDM}^{\mathrm{eff}}_{dNis}(k)$. Formulas like the projective bundle formula, the blow-up formula, the existence of the Gysin sequence and so on in $\mathbf{logDM}^{\mathrm{eff}}_{dNis}(k)$ can then be used to rededuce *a posteriori* some of the results in the present paper, under some auxiliary assumptions. We warn the reader that in the proof of the main result of [Sai20b], one needs in an essential way the formalism of pushforward maps along projective morphisms that we show in the present work.

Moreover, note that the motivic formulas given in [BPØ22] cannot be used to deduce results involving higher modulus, that we do instead systematically in the present paper, and that the projective bundle formula, the blow-up formula and the Gysin triangle (using the identification of the log Thom space) in [BPØ22] are only proved under the assumptions of resolution of singularities, which we don't need. Finally, a general theory of log motives over a base (not just over a field) would be necessary to get the full strength of the sheaf-theoretic version of the results in this work.

Warning. The content of Theorem 0.1 and of other main results in this paper (namely, Corollary 2.19 and Theorem 3.1) are a sheaf theoretic analogue to some of the results on motives with modulus in [KMSY20], more precisely to [KMSY20, Theorem 7.3.2], [KMSY20, Theorem 7.4.3] and [KMSY20, Theorem 7.4.4] (the latter being in fact a theorem of Keiho Matsumoto [Mat22], proved only for the inclusion of a smooth divisor Z in X, whereas we consider the case of Z being a smooth closed subscheme of any codimension). We warn the reader that our results *cannot be recovered* from the existing literature:

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for this to be the case, it would be necessary to show that the cohomology of $\overline{\square}$-invariant sheaves is representable in the category of motives with modulus $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}(k)$ constructed in [KMSY20]. In view of [KMSY20, Theorem 5.2.4], one would require a positive answer to the following two questions.

Question 0.7.

(1) Is the Nisnevich cohomology of $\overline{\square}$-invariant sheaves invariant under blow-up with centre contained in the support of the modulus?
(2) Is a $\overline{\square}$-invariant sheaf F equivalent (in the derived category of sheaves) to its derived Suslin complex $RC_*^{\overline{\square}}(F)$ defined in [KMSY20, Definition 5.2.3]?

Both questions seem out of reach for general $\overline{\square}$-invariant sheaves: note that (1) would amount to answering affirmatively to [KMSY21a, Question 1, p.4], and that a (weaker) version of it is the content of Theorem 2.12, which is one of the crucial technical results of this paper.

Question (2) is equivalent to asking whether the cohomology of a $\overline{\square}$-invariant sheaf with transfers is again $\overline{\square}$-invariant. For $\mathbb{A}^1$-invariant sheaves with transfers, this is a deep theorem of Voevodsky and boils down to studying a nontrivial interaction between the Nisnevich sheafification functor and the localisation functor $L_{\mathbb{A}^1}(-)$. For semipure sheaves (cf. 1.4 below), this is shown in [Sai20a], but the general case is wide open (the first and third author once claimed the general case in characteristic 0, but a gap was found in its proof). We hope that the main results of this paper are useful in attempts to answer the above open questions.

Moreover, even if both questions are answered positively, in order to get the full statement of Theorem 0.1 from the motivic point of view, it would be necessary to develop the whole theory of motives with modulus over a base, which is not available at the moment.

0.5. Organisation of the paper

We conclude this introduction with a quick presentation of the structure of the paper.

In §1, we discuss some preliminaries and fix the notation. Nothing in this section is new, and it can be found in [KMSY21a], [KMSY21b], [MS20]. In §2, we prove a key 'descent' property for $\overline{\square}$-invariant sheaves, namely, Proposition 2.5. This is a crucial technical result that allows us to prove the invariance of the cohomology of cube invariant sheaves along a certain class of blow-ups (see Theorem 2.12). Once this is established, we proceed to prove that the cohomology of cube invariant sheaves is also invariant with respect to the product with the modulus pair $(\mathbf{P}^n, \mathbf{P}^{n-1})$, Theorem 2.18. In §3, we prove a smooth blow-up formula; and in §4, we introduce the twist and prove some of its basic properties. In §5, we use Rost's theory of cycle modules together with a formula for the tensor product of reciprocity sheaves to construct the cup product with Chow classes. In §6, we prove the projective bundle formula, and in §7, we construct the Gysin sequence: for this, we essentially follow the steps of Voevodsky's construction in [Voe00b], but we also get a finer theory with supports (the local Gysin map). In §8, we assemble the Gysin maps and the morphisms induced by the projective bundle formula to construct general pushforwards. In this section, we make use also of the cancellation theorems of [MS20]. In §9, we explain the construction of the action of Chow correspondences on reciprocity sheaves (and complexes of sheaves). Finally, in §10 and §11, we collect the main applications and a list of examples of reciprocity sheaves. The reader who is mostly interested in examples and applications may read the last two sections without having precise knowledge of modulus sheaves with transfers.

In the paper, we use frequently the results from [Sai20a], which plays a fundamental role for us.

1. Preliminaries

1.1. Notations and conventions

In the whole paper, we fix a perfect base field k. We denote by $\mathbf{Sm}$ the category of smooth separated k-schemes. We write $\mathbf{P}^1 = \mathbf{P}^1_k$ etc. and $X \times Y = X \times_k Y$ for k-schemes X, Y. For a function field K/k,

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we denote by $K\{x_1,\dots,x_n\}$ the henselisation of $K[x_1,\dots,x_n]_{(x_1,\dots,x_n)}$. Let R be a regular noetherian k-algebra. By [Pop86, Theorem 1.8] and [AGV72, Exp I, Proposition 8.1.6], we can write $R = \varinjlim_i R_i$, where $(R_i)_i$ is a directed system of smooth k-algebras, and we use the notation $F(R) = \varinjlim_i F(\operatorname{Spec} R_i)$, for any presheaf F on **Sm**. If X is a scheme and F is a Nisnevich sheaf on X, we will denote by $H^i(X,F) = H^i(X_{\mathrm{Nis}},F)$ the ith cohomology group of F on the small Nisnevich site of X, similar with higher direct images. We denote by $X_{(n)}$ (respectively, $X^{(n)}$) the set of n (respectively, co-) dimensional points in X.

1.2. A recollection on modulus sheaves with transfers

We recall some terminology and notations from the theory of modulus sheaves with transfers (see [KMSY21a], [KMSY21b], [KSY22] and [Sai20a] for details).

1.1. A modulus pair $\mathcal{X} = (\overline{X}, X_\infty)$ consists of a separated k-scheme of finite type $\overline{X}$ and an effective (or empty) Cartier divisor X_∞, such that $X := \overline{X} \setminus |X_\infty|$ is smooth; it is called *proper* if $\overline{X}$ is proper over k. Given two modulus pairs $\mathcal{X} = (\overline{X}, X_\infty)$ and $\mathcal{Y} = (\overline{Y}, Y_\infty)$, with opens $X := \overline{X} \setminus |X_\infty|$ and $Y := \overline{Y} \setminus |Y_\infty|$, an admissible left proper prime correspondence from $\mathcal{X}$ to $\mathcal{Y}$ is given by an integral closed subscheme $Z \subset X \times Y$ which is finite and surjective over a connected component of X, such that the normalisation of its closure $\overline{Z}^N \to \overline{X} \times \overline{Y}$ is proper over $\overline{X}$ and satisfies:

$$X_{\infty|\overline{Z}^N} \geq Y_{\infty|\overline{Z}^N}, \tag{1.1.1}$$

as Weil divisors on $\overline{Z}^N$, where $X_{\infty|\overline{Z}^N}$ (respectively, $Y_{\infty|\overline{Z}^N}$) denotes the pullback of X_∞ (respectively, Y_∞) to $\overline{Z}^N$. The free abelian group generated by such correspondences is denoted by $\underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X},\mathcal{Y})$. By [KMSY21a, Propositions 1.2.3 and 1.2.6], modulus pairs and left proper admissible correspondences define an additive category that we denote by $\underline{\mathbf{M}}\mathbf{Cor}$. We write **MCor** for the full subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$, whose objects are proper modulus pairs. We denote by τ the inclusion functor $\tau\colon\ \mathbf{MCor} \to \underline{\mathbf{M}}\mathbf{Cor}$. The induced category of additive presheaves of abelian groups is denoted by $\underline{\mathbf{MPST}}$ (respectively, **MPST**). We have functors:

$$\omega\colon\ \mathbf{MCor} \to \mathbf{Cor}, \quad \underline{\omega}\colon\ \underline{\mathbf{M}}\mathbf{Cor} \to \mathbf{Cor}$$

given by $(\underline{X}, X_\infty) \mapsto \underline{X} \setminus |X_\infty|$, where **Cor** is the category of finite correspondences introduced by Suslin-Voevodsky (see, e.g. [MVW06]). Note that there is also a fully faithful functor:

$$\mathbf{Cor} \to \underline{\mathbf{M}}\mathbf{Cor}, \quad X \mapsto (X, \emptyset).$$

We will abuse notation by writing:

$$X = (X,\emptyset) \in \underline{\mathbf{M}}\mathbf{Cor}, \quad \text{for } X \in \mathbf{Sm}\,. \tag{1.1.2}$$

Write τ^* for the restriction functor along τ, and write $\tau_!$ for its left Kan extension. Similarly, write ω^* (respectively, $\underline{\omega}^*$) for the restriction functor along ω (respectively, $\underline{\omega}$) and $\omega_!$ (respectively, $\underline{\omega}_!$) for its left Kan extension. We have the following commutative diagrams at our disposal:

$$\begin{array}{ccc} \underline{\mathbf{MPST}} & \xleftarrow{\ \tau_!\ } & \mathbf{MPST} \\ & {\scriptstyle \underline{\omega}_!}\searrow \quad \swarrow{\scriptstyle \omega_!} & \\ & \mathbf{PST} & \end{array} \qquad \begin{array}{ccc} \underline{\mathbf{MPST}} & \underset{\tau_!}{\overset{\tau^*}{\rightleftarrows}} & \mathbf{MPST} \\ & {\scriptstyle \underline{\omega}^*}\swarrow \quad \searrow{\scriptstyle \omega^*} & \\ & \mathbf{PST}. & \end{array} \tag{1.1.3}$$

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Here, **PST** is the category of presheaves of abelian groups on **Cor**, the functors in the left triangle are left adjoint to the functors in the right triangle, all the functors are exact, the diagrams commute and we have $\tau^*F(\mathcal{X}) = F(\mathcal{X})$, $\underline{\omega}^*F(\mathcal{X}) = F(X)$ and:

$$\underline{\omega}_!F(X) = F(X, \emptyset) \overset{(1.1.2)}{=:} F(X) \tag{1.1.4}$$

for $\mathcal{X} = (\overline{X}, X_\infty)$ and $X = \overline{X} \setminus |X_\infty|$.

We denote by $\mathbb{Z}_{\mathrm{tr}}(\mathcal{X})$ the presheaf on $\underline{\mathbf{MCor}}$ (respectively, **MCor**) represented by $\mathcal{X}$ in $\underline{\mathbf{MCor}}$ (respectively, in **MCor**). We have $\tau_!\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) = \mathbb{Z}_{\mathrm{tr}}(\mathcal{X})$ and $\underline{\omega}_!\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) = \mathbb{Z}_{\mathrm{tr}}(X)$.

Let $\mathcal{X} = (\overline{X}, X_\infty)$, $\mathcal{Y} = (\overline{Y}, Y_\infty) \in \underline{\mathbf{MCor}}$. We set:

$$\mathcal{X} \otimes \mathcal{Y} := (\overline{X} \times \overline{Y}, p^*X_\infty + q^*Y_\infty),$$

where p and q are the projections from $\overline{X} \times \overline{Y}$ to $\overline{X}$ and $\overline{Y}$, respectively. In fact, this defines a symmetric monoidal structure on $\underline{\mathbf{MCor}}$ (respectively, **MCor**) which extends (via Yoneda) uniquely to a right exact monoidal structure $\otimes$ on $\underline{\mathbf{MPST}}$ (respectively, **MPST**). Similarly, there is a monoidal structure on **PST**. The functors $\underline{\omega}_!$, $\omega_!$, $\tau_!$ are monoidal, since they are all defined as left Kan extensions of the functors $\underline{\omega}$, ω and τ, which are clearly monoidal. For $F \in \underline{\mathbf{MPST}}$, the functor $(-) \otimes F : \underline{\mathbf{MPST}} \to \underline{\mathbf{MPST}}$ admits a right adjoint denoted by $\underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(F, -)$; similar with $F \in \mathbf{MPST}$ (see, e.g. [MVW06, Chapter 8]).

1.2. For $F \in \underline{\mathbf{MPST}}$ and $\mathcal{X} = (\overline{X}, X_\infty) \in \underline{\mathbf{MCor}}$, denote by $F_{\mathcal{X}}$ the presheaf:

$$(\text{ét}/\overline{X})^{\mathrm{op}} \ni U \mapsto F_{\mathcal{X}}(U) := F(U, X_{\infty|U}), \tag{1.2.1}$$

where $(\text{ét}/\overline{X})$ denotes the category of all étale maps $U \to \overline{X}$. We say F is a Nisnevich sheaf if $F_{\mathcal{X}}$ is a Nisnevich sheaf, for all $\mathcal{X} \in \underline{\mathbf{MCor}}$. We denote by $\underline{\mathbf{MNST}}$ the full subcategory of $\underline{\mathbf{MPST}}$ consisting of Nisnevich sheaves.

We say $F \in \mathbf{MPST}$ is a Nisnevich sheaf if $\tau_!F$ is and denote the corresponding full subcategory by **MNST**. The functors in (1.1.3) restrict to Nisnevich sheaves and have the same adjointness and exactness properties (see [KMSY21b, 4.2.5, 5.1.1, 6.2.1]). Furthermore, there are Nisnevich sheafification functors:

$$\underline{a}_{\mathrm{Nis}} : \underline{\mathbf{MPST}} \to \underline{\mathbf{MNST}}, \quad a_{\mathrm{Nis}} : \mathbf{MPST} \to \mathbf{MNST},$$
$$a^V_{\mathrm{Nis}} : \mathbf{PST} \to \mathbf{NST},$$

which are left adjoint to the forgetful functors, restrict to the identity on Nisnevich sheaves and satisfy:

$$\underline{\omega}_!\underline{a}_{\mathrm{Nis}} = a^V_{\mathrm{Nis}}\underline{\omega}_!, \quad \omega_!a_{\mathrm{Nis}} = a^V_{\mathrm{Nis}}\omega_!, \quad \tau_!a_{\mathrm{Nis}} = \underline{a}_{\mathrm{Nis}}\tau_!, \tag{1.2.2}$$

and:

$$a_{\mathrm{Nis}}\omega^* = \omega^*a^V_{\mathrm{Nis}}, \quad \underline{a}_{\mathrm{Nis}}\underline{\omega}^* = \underline{\omega}^*a^V_{\mathrm{Nis}} \tag{1.2.3}$$

(see [KMSY21a, Theorem 2], [KMSY21b, Theorems 4.2.4, 4.2.5 and 6.2.1]; a^V_{Nis} was constructed by Voevodsky). It follows that **NST**, $\underline{\mathbf{MNST}}$ and **MNST** are Grothendieck abelian categories and that the sheafification functors are exact. For $F \in \underline{\mathbf{MPST}}$ and $\mathcal{X} = (\overline{X}, X_\infty) \in \underline{\mathbf{MCor}}$, we have:

$$\underline{a}_{\mathrm{Nis}}(F)(\mathcal{X}) = \varinjlim_{f : \overline{Y} \to \overline{X}} F_{(\overline{Y}, f^*X_\infty), \mathrm{Nis}}(\overline{Y}), \tag{1.2.4}$$

where the limit is over all proper morphisms $f : \overline{Y} \to \overline{X}$ which restrict to an isomorphism over $X = \overline{X} \setminus |X_\infty|$ and $F_{(\overline{Y}, f^*X_\infty), \mathrm{Nis}}$ denotes the Nisnevich sheafification of the presheaf $F_{(\overline{Y}, f^*X_\infty)}$ on the

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site $\overline{Y}_{\mathrm{Nis}}$ (see [KMSY21a, Theorem 2(1)]. In the following, we will use the notation:

$$F_{\mathrm{Nis}} := \underline{a}_{\mathrm{Nis}}(F), \quad H_{\mathrm{Nis}} := a^V_{\mathrm{Nis}}(H), \quad F \in \underline{\mathbf{M}}\mathbf{PST},\ H \in \mathbf{PST}.$$

Lemma 1.3. *A morphism $\varphi : F \to G$ in $\underline{\mathbf{M}}\mathbf{NST}$ is surjective (i.e. has vanishing cokernel) if for all $\mathcal{X} = (\overline{X}, X_\infty) \in \underline{\mathbf{M}}\mathbf{Cor}$, with $\overline{X}$ normal, and all $x \in \overline{X}$, the morphism:*

$$F(\mathcal{X}_{(x)}) \to G(\mathcal{X}_{(x)})$$

is surjective, where $\mathcal{X}_{(x)} = (\overline{X}_{(x)}, X_{\infty|\overline{X}_{(x)}})$ and $\overline{X}_{(x)} = \operatorname{Spec} \mathcal{O}^h_{\overline{X},x}$ is the spectrum of the henselisation of the local ring $\mathcal{O}_{\overline{X},x}$.

Proof. Let C be the cokernel of φ in $\underline{\mathbf{M}}\mathbf{PST}$. We want to show $\underline{a}_{\mathrm{Nis}}(C) = 0$. For $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$, set $C_{\mathcal{X}} = \operatorname{Coker}(\varphi_{\mathcal{X}} : F_{\mathcal{X}} \to G_{\mathcal{X}})$ in the category of presheaves on $(\text{ét}/\overline{X})$; denote by $C_{\mathcal{X},\mathrm{Nis}}$ its Nisnevich sheafification. By (1.2.4), it suffices to show $C_{\mathcal{X},\mathrm{Nis}} = 0$, if $\overline{X}$ is normal. The latter is equivalent to the surjectivity of $\varphi_{\mathcal{X}}$ in the category of Nisnevich sheaves on $\overline{X}$, which is equivalent to the statement. □

1.4. Set $\overline{\square} := (\mathbf{P}^1, \infty) \in \mathbf{MCor}$. For $F \in \underline{\mathbf{M}}\mathbf{PST}$, we say that:

(1) F is *cube invariant* if the map $F(\mathcal{X}) \to F(\mathcal{X} \otimes \overline{\square})$ induced by the pullback along the projection is an isomorphism.
(2) F has *M-reciprocity* if the counit map $\tau_! \tau^* F \to F$ is an isomorphism.
(3) F is *semipure* if the unit map $F \to \underline{\omega}^* \underline{\omega}_! F$ is injective.

We denote by $\underline{\mathbf{M}}\mathbf{PST}^\tau$ the full subcategory of $\underline{\mathbf{M}}\mathbf{PST}$ consisting of the objects with M-reciprocity. Note that for $\mathcal{X}$, a proper modulus pair, we have $\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) \in \underline{\mathbf{M}}\mathbf{PST}^\tau$. We denote by $\mathbf{CI}^{\tau,sp}$ the full subcategory of $\underline{\mathbf{M}}\mathbf{PST}$ consisting of the cube invariant semipure objects with M-reciprocity. We set:

$$\underline{\mathbf{M}}\mathbf{NST}^\tau := \underline{\mathbf{M}}\mathbf{PST}^\tau \cap \underline{\mathbf{M}}\mathbf{NST} \quad \text{and} \quad \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}} := \mathbf{CI}^{\tau,sp} \cap \underline{\mathbf{M}}\mathbf{NST}\,.$$

By [Sai20a, Theorem 10.1], the sheafification functor $\underline{a}_{\mathrm{Nis}}$ restricts to:

$$\underline{a}_{\mathrm{Nis}} : \mathbf{CI}^{\tau,sp} \to \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}\,. \tag{1.4.1}$$

The natural inclusion $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}} \hookrightarrow \underline{\mathbf{M}}\mathbf{PST}^\tau$ has a left adjoint:

$$h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{PST}^\tau \to \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}} \tag{1.4.2}$$

given by:

$$h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(F) = \underline{a}_{\mathrm{Nis}}(\underline{h}^{\overline{\square}}_0(F)^{\mathrm{sp}}),$$

where for $G \in \underline{\mathbf{M}}\mathbf{PST}$:

(1) $\underline{h}^{\overline{\square}}_0(G) \in \underline{\mathbf{M}}\mathbf{PST}$ is the maximal cube invariant quotient of G defined by:

$$\underline{h}^{\overline{\square}}_0(G)(\mathcal{X}) = \operatorname{Coker}(G(\mathcal{X} \otimes \overline{\square}) \xrightarrow{i_0^* - i_1^*} G(\mathcal{X})), \tag{1.4.3}$$

where $i_\varepsilon : \{\varepsilon\} \to \overline{\square}$, $\varepsilon \in \{0,1\}$, are induced by the natural closed immersions,
(2) $G^{\mathrm{sp}} = \operatorname{Im}(G \to \underline{\omega}^* \underline{\omega}_! G)$ denotes the *semipurification* of F.

The left adjointness of (1.4.2) to the natural inclusion follows from [MS20, Lemma 1.14(i)] and the adjunction $\tau_! \dashv \tau^*$. We note that for any $F \in \underline{\mathbf{M}}\mathbf{PST}$, the presheaf $h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(F)$ is defined and is in fact a cube invariant, semipure Nisnevich sheaf on $\underline{\mathbf{M}}\mathbf{Cor}$.

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For $\mathcal{X}$ a proper modulus pair, we set:

$$h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\mathcal{X}) := h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X})) \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}. \tag{1.4.4}$$

Lemma 1.5. *Let $F \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}$ and $G, H \in \underline{\mathbf{M}}\mathbf{PST}^{\tau}$. Assume there is a surjection $\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) \twoheadrightarrow G$, for some $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$. We have:*

(1) $\underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(G, F) \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}$;

(2) $\mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(H \otimes G, F) = \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(H), \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(G, F))$.

Proof. (1). First assume $G = \mathbb{Z}_{\mathrm{tr}}(\mathcal{X})$, for some $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$. In this case, $\underline{\mathrm{Hom}}(G, F)(\mathcal{Y}) = F(\mathcal{X} \otimes \mathcal{Y})$. Clearly, this defines a cube invariant Nisnevich sheaf. It has M-reciprocity by [Sai20a, Lemma 1.27(2)] and has semipurity by [Sai20a, Lemma 1.29(2)]. Hence, $\underline{\mathrm{Hom}}(G, F) \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}$ in this case. In the general, case consider a resolution:

$$\bigoplus_j \mathbb{Z}_{\mathrm{tr}}(\mathcal{Y}_j) \to \bigoplus_i \mathbb{Z}_{\mathrm{tr}}(\mathcal{X}_i) \to G \to 0.$$

We obtain an exact sequence:

$$0 \to \underline{\mathrm{Hom}}(G, F) \to \prod_i \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}_i), F) \to \prod_j \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{Y}_j), F). \tag{1.5.1}$$

This directly implies cube invariance and semipurity. The sheaf property holds since $\underline{i}_{\mathrm{Nis}}\underline{a}_{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{PST} \to \underline{\mathbf{M}}\mathbf{PST}$ is left exact, where $\underline{i}_{\mathrm{Nis}}$ is the forgetful functor. In general, M-reciprocity won't hold since $\tau_!\tau^*$ does not commute with infinite products; however, it clearly holds if the first product in (1.5.1) is finite and by assumption we find such a resolution. (2) follows from (1) and adjunction. □

1.6. The full subcategory of **PST** given by $\mathbf{RSC} := \underline{\omega}_! \mathbf{CI}^{\tau,\mathrm{sp}}$ is called the category of *reciprocity presheaves*. The full subcategory of **NST** given by $\mathbf{RSC}_{\mathrm{Nis}} := \underline{\omega}_! \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}$ is called the category of *reciprocity sheaves*. It is direct to see that **RSC** is an abelian category, closed under subobjects and quotients in **PST**. On the other hand, it is a theorem [Sai20a, Theorem 0.1] that $\mathbf{RSC}_{\mathrm{Nis}}$ is also abelian. We use the following notation for a proper modulus pair $\mathcal{X}$:

$$h_0(\mathcal{X}) := \underline{\omega}_!(h_0^{\overline{\square}}(\mathcal{X})) = \underline{\omega}_!(h_0^{\overline{\square},\mathrm{sp}}(\mathcal{X})) \in \mathbf{RSC},$$

and:

$$h_{0,\mathrm{Nis}}(\mathcal{X}) := \underline{\omega}_!(h_{0,\mathrm{Nis}}^{\overline{\square}}(\mathcal{X})) = \underline{\omega}_!(h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\mathcal{X})) \in \mathbf{RSC}_{\mathrm{Nis}}.$$

Note that $h_{0,\mathrm{Nis}}(\mathcal{X}) = h_0(\mathcal{X})_{\mathrm{Nis}}$. By [MS20, (1.13)] (see also [KSY22, Proposition 2.3.7]), there is an adjunction:

$$\mathbf{CI}_{\mathrm{Nis}}^{\tau,sp} \underset{\underline{\omega}_!}{\overset{\underline{\omega}^{\mathbf{CI}}}{\leftarrow\!\!\rightarrow}} \mathbf{RSC}_{\mathrm{Nis}}, \tag{1.6.1}$$

where $\underline{\omega}^{\mathbf{CI}}$ is right adjoint to $\underline{\omega}_!$ and is given by:

$$\underline{\omega}^{\mathbf{CI}}(F) = \tau_! \,\mathrm{Hom}_{\mathbf{MPST}}(h_0^{\overline{\square}}(-), \omega^* F).$$

In the notation of [KSY22], we have $\underline{\omega}^{\mathbf{CI}} = \tau_! \omega^{\mathbf{CI}}$.

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Recall that Voevodsky's category of homotopy invariant Nisnevich sheaves, $\mathbf{HI}_{\mathrm{Nis}}$, is an abelian subcategory of $\mathbf{RSC}_{\mathrm{Nis}}$, and thanks to [Voe00a, Theorem 5.6], the natural inclusion $\mathbf{HI}_{\mathrm{Nis}} \to \mathbf{NST}$ has a left adjoint:

$$h^{\mathbf{A}^1}_{0,\mathrm{Nis}} : \mathbf{NST} \to \mathbf{HI}_{\mathrm{Nis}} . \tag{1.6.2}$$

By [KSY22, Proposition 2.3.2], we have:

$$h^{\mathbf{A}^1}_{0,\mathrm{Nis}}(h_{0,\mathrm{Nis}}(\mathcal{X})) = h^{\mathbf{A}^1}_{0,\mathrm{Nis}}(\mathbb{Z}_{\mathrm{tr}}(\omega\mathcal{X})). \tag{1.6.3}$$

2. Cohomology of blow-ups and invariance properties

2.1. A lemma on modulus descent

Notation 2.1. For $m, n \geq 1$, we use the following notation:

$$\overline{\square}^{(m,n)} := (\mathbf{P}^1, m \cdot 0 + n \cdot \infty), \quad \overline{\square}^{(n)} := \overline{\square}^{(n,n)}.$$

In particular,

$$\overline{\square}^{(1)} = (\mathbf{P}^1, 0 + \infty).$$

Lemma 2.2. *Let R be an integral regular k-algebra. For all $m, n \geq 1$, there is an isomorphism:*

$$\theta_{m,n} : h_0(\overline{\square}^{(m,n)})(R) \xrightarrow{\simeq} ((R[t]/t^m)^\times \oplus (R[z]/z^n)^\times)/R^\times \oplus \mathbb{Z},$$

where $R^\times$ acts diagonally on the direct sum. If $Z \in \underline{\omega}_!\mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{(m,n)})(R)$ is a prime correspondence which we can write as $Z = V(g)$, for an irreducible polynomial $g = a_r t^r + \ldots + a_1 t + a_0 \in R[t]$ with $a_r, a_0 \in R^\times$, and $r \geq 1$, then:

$$\theta_{m,n}(Z) = (g(t)/(t-1)^r, g_\infty(z)/(1-z)^r, r),$$

where $g_\infty(z) = a_0 z^r + \ldots + a_{r-1} z + a_r$. Furthermore, if $m' \leq m$ and $n' \leq n$, then we obtain a commutative diagram:

$$\begin{array}{ccc} h_0(\overline{\square}^{(m',n')})(R) & \xrightarrow{\theta_{m',n'}} & ((R[t]/t^{m'})^\times \oplus (R[z]/z^{n'})^\times)/R^\times \oplus \mathbb{Z} \\ \downarrow & & \downarrow \\ h_0(\overline{\square}^{(m,n)})(R) & \xrightarrow{\theta_{m,n}} & ((R[t]/t^{m})^\times \oplus (R[z]/z^{n})^\times)/R^\times \oplus \mathbb{Z}, \end{array}$$

where the vertical map on the left-hand side is induced by $\overline{\square}^{(m',n')} \to \overline{\square}^{(m,n)}$ in $\underline{\mathbf{M}}\mathbf{Cor}$ and the vertical map on the right is the natural quotient map.

Proof. The map $\theta_{m,n}$ is the composition of the two isomorphisms:

$$\begin{aligned} h_0(\overline{\square}^{(m,n)})(R) &\xrightarrow{\simeq (*)} \mathrm{Pic}(\mathbf{P}^1_R, m \cdot 0 + n \cdot \infty) \\ &\xrightarrow{\simeq (**)} ((R[t]/t^m)^\times \oplus (R[z]/z^n)^\times)/R^\times \oplus \mathbb{Z}, \end{aligned}$$

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which are defined as follows. We denote by $F_R := m \cdot 0_R + n \cdot \infty_R \subset \mathbf{P}^1_R$ the closed subscheme: (*) is induced by the classical map from Weil to Cartier divisors:

$$\mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{(m,n)})(R) \ni D \mapsto (\mathcal{O}(D), \mathrm{id}_{\mathcal{O}_{F_R}}) \in \mathrm{Pic}(\mathbf{P}^1_R, F_R),$$

where $\mathcal{O}(D)$ is the line bundle on $\mathbf{P}^1_R$ given by $\mathcal{O}(D)(U) = \{f \in R(t)^\times \mid \mathrm{div}_U(f) \geq D\}$; it is an isomorphism by [RY16, Theorem 1.1]. For (**), consider the exact sequence:

$$H^0(\mathbf{P}^1_R, \mathcal{O}^\times) \to H^0(F_R, \mathcal{O}^\times) \to \mathrm{Pic}(\mathbf{P}^1_R, F_R) \to \mathrm{Pic}(\mathbf{P}^1_R) \to \mathrm{Pic}(F_R).$$

The last map decomposes as $\mathrm{Pic}(R) \oplus \mathrm{Pic}(\mathbf{P}^1) \to \mathrm{Pic}(F_R)$ given by:

$$(M, \mathcal{O}(\{1\})^{\otimes n}) \mapsto (M_{|\mathbf{P}^1_R} \otimes (\mathcal{O}(\{1\})^{\otimes n})_{|F_R} = M_{|F_R}.$$

Since $F_R \to \mathrm{Spec}\, R$ has a section, the map $\mathrm{Pic}(R) \to \mathrm{Pic}(F_R)$ is injective. Hence, the above sequence yields an exact sequence:

$$H^0(\mathbf{P}^1_R, \mathcal{O}^\times) \to H^0(F_R, \mathcal{O}^\times) \to \mathrm{Pic}(\mathbf{P}^1_R, F_R) \xrightarrow{d} \mathbb{Z} \to 0,$$

where $d(L, \alpha) := d(L) := \deg(L_{|\mathbf{P}^1_{\mathrm{Frac}(R)}})$; we can choose a splitting of d by $r \mapsto (\mathcal{O}_{\mathbf{P}^1_R}(\{1\})^{\otimes r}, \mathrm{id}_{\mathcal{O}_{F_R}})$; the map in the middle sends $u \in H^0(F_R, \mathcal{O}^\times)$ to $(\mathcal{O}_{\mathbf{P}^1_R}, u\cdot : \mathcal{O}_{F_R} \xrightarrow{\simeq} \mathcal{O}_{F_R})$, where $u\cdot$ is the isomorphism given by multiplication by u. Let (L, α) be a pair with L a line bundle on $\mathbf{P}^1_R$ with $d(L) = r$ and $\alpha : \mathcal{O}_{F_R} \xrightarrow{\simeq} L_{|F_R}$ an isomorphism; we find an isomorphism $\varphi : L \otimes \mathcal{O}(\{1\})^{\otimes -r} \xrightarrow{\simeq} \mathcal{O}_{\mathbf{P}^1_R}$ and define the isomorphism α' as the composition:

$$\alpha' = (\alpha'_{m\cdot 0}, \alpha'_{n\cdot\infty}) : \mathcal{O}_{F_R} \xrightarrow{\simeq \alpha} L_{|F_R} = (L \otimes \mathcal{O}(\{1\})^{\otimes -r})_{|F_R} \xrightarrow{\varphi_{|F_R}} (\mathcal{O}_{\mathbf{P}^1})_{|F_R},$$

where the equality follows from the fact that we have a canonical identification $\mathcal{O}(\{1\})_{|F_R} = \mathcal{O}_{|F_R}$. Hence, φ induces an isomorphism $(L \otimes \mathcal{O}(\{1\})^{\otimes -r}, \alpha) \cong (\mathcal{O}_{\mathbf{P}^1_R}, \alpha')$; the isomorphism (**) is given by:

$$(L, \alpha) \mapsto (\alpha'_{m\cdot 0}(1), \alpha'_{n\cdot\infty}(1), d(L)).$$

Let $Z = V(g) \in \mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{(m,n)})(R)$ be a prime correspondence as in the statement. Write $t = T_0/T_1$, and let $G \in R[T_0, T_1]$ be the homogenisation of g. We have an isomorphism:

$$\mathcal{O}(Z) \otimes \mathcal{O}(\{1\})^{\otimes -r} = \mathcal{O}_{\mathbf{P}^1_R} \cdot \tfrac{(T_0 - T_1)^r}{G} \xrightarrow{\simeq} \mathcal{O}_{\mathbf{P}^1_R},$$

where the second isomorphism is given by multiplication with $G/(T_0 - T_1)^r$. Thus, $\theta_{m,n}$ admits the description from the statement, where $z = 1/t$. The commutativity of the diagram follows directly from this. □

Remark 2.3. Denote by $\mathbb{W}_m$ the ring scheme of big Witt vectors of length m. If A is a ring, we can identify the A-rational points of the underlying group scheme with:

$$\mathbb{W}_m(A) = (1 + tA[t])^\times / (1 + t^{m+1}A[t])^\times.$$

Then the maps $\theta_{m,n}$ from Lemma 2.2, $m, n \geq 1$, induce isomorphisms in **NST**:

$$\theta_{m,n} : h_{0,\mathrm{Nis}}(\overline{\square}^{(m,n)}) \xrightarrow{\simeq} \mathbb{W}_{m-1} \oplus \mathbb{W}_{n-1} \oplus \mathbf{G}_m \oplus \mathbb{Z}.$$

Indeed, it follows immediately from Lemma 2.2 that we have such an isomorphism of Nisnevich sheaves. To check the compatibility with transfers, it suffices to check the compatibility with transfers of the limit

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$\varprojlim_{m,n} \theta^{m,n}$ (since the transition maps are surjective). Since $\mathbb{W} \oplus \mathbb{W} \oplus \mathbf{G}_m \oplus \mathbb{Z}$ is a $\mathbb{Z}$-torsion-free sheaf on $\mathbf{Sm}_{\mathrm{Nis}}$ for which the pullback along dominant étale maps is injective, the compatibility with transfers follows automatically from [MS20, Lemma 1.1].

Lemma 2.4. *The unit map:*

$$h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\overline{\square}^{(1)}) \xrightarrow{\simeq} \underline{\omega}^* \underline{\omega}_! h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\overline{\square}^{(1)}) \cong \underline{\omega}^*(\mathbf{G}_m \oplus \mathbb{Z}) \tag{2.4.1}$$

is an isomorphism in $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. *Furthermore, the natural maps:*

$$h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\overline{\square}^{(m,n)}) \to h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\overline{\square}^{(1)}) \tag{2.4.2}$$

are surjective, for all $m, n \geq 1$, *and there exists a splitting in* $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$:

$$s_{m,n} : \underline{\omega}^*(\mathbf{G}_m \oplus \mathbb{Z}) \to h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\overline{\square}^{(m,n)})$$

of (2.4.2), such that the following diagram is commutative for integers $m' \geq m$ *and* $n' \geq n$:

$$\begin{array}{ccc} \underline{\omega}^*(\mathbf{G}_m \oplus \mathbb{Z}) & \xrightarrow{s_{m',n'}} & h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\overline{\square}^{(m',n')}) \\ & \searrow^{s_{m,n}} & \downarrow \\ & & h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\overline{\square}^{(m,n)}). \end{array} \tag{2.4.3}$$

Proof. The second isomorphism in (2.4.1) holds by Lemma 2.2 and Remark 2.3; the unit map is injective by semipurity. We show the surjectivity of the composite map:

$$h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\overline{\square}^{(m,n)}) \to h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\overline{\square}^{(1)}) \to \underline{\omega}^*(\mathbf{G}_m \oplus \mathbb{Z}) \tag{2.4.4}$$

for $m, n \geq 1$. By Lemma 1.3, it suffices to show the surjectivity on $(\mathrm{Spec}\, R, (f))$, where R is an integral normal local k-algebra and $f \in R \setminus \{0\}$, such that R_f is regular. Denote by:

$$\psi : \mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{(m,n)})(R, f) \to R^\times_f \oplus \mathbb{Z}$$

the precomposition of (2.4.4) evaluated at (R, f) with the quotient map:

$$\mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{(m,n)})(R, f) \to h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\overline{\square}^{(m,n)})(R, f).$$

By Lemma 2.2:

$$\psi(V(a_0 + a_1 t + \ldots + a_r t^r)) = ((-1)^r a_0/a_r, r), \tag{2.4.5}$$

provided that $Z = V(a_0 + a_1 t + \ldots + a_r t^r)$ is an admissible prime correspondence and $a_i \in R_f$. We claim that ψ is surjective. To this end, observe that for $a \in R^\times_f$, we find $N \geq 0$ and $b \in R$, such that:

$$ab = f^{nN}, \quad \text{and} \quad af^{mN} \in R. \tag{2.4.6}$$

Set $W := V(t^{mnN} + (-1)^{mnN} a) \subset \mathrm{Spec}\, R_f\, [t, 1/t]$ and $K = \mathrm{Frac}(R)$. Let $t^{mnN} + (-1)^{mnN} a = \prod_i h_i$ be the decomposition into monic irreducible factors in $K[t, 1/t]$, and denote by $W_i \subset \mathrm{Spec}\, R_f\, [t, 1/t]$ the closure of $V(h_i)$ (note that $W_i = W_j$ for $i \neq j$ is allowed). The W_i correspond to the components of W which are dominant over R_f; since W is finite (the polynomial defining W is monic) and surjective

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over R_f, so are the W_i. We claim:

$$W_i \in \mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{(m,n)})(R,f). \tag{2.4.7}$$

Indeed, let I_i (respectively, J_i) be the ideal of the closure of W_i in $\operatorname{Spec} R[t]$ (respectively, $\operatorname{Spec} R[z]$ with $z = 1/t$). By (2.4.6):

$$bt^{nmN} + (-1)^{mnN} f^{nN} \in I_i \quad \text{and} \quad f^{mN} + (-1)^{mnN} f^{mN} a z^{mnN} \in J_i.$$

Hence, $(f/t^m)^{nN} \in R[t]/I_i$ and $(f/z^n)^{mN} \in R[z]/J_i$. It follows that f/t^m (respectively, f/z^n) is integral over $R[t]/I_i$ (respectively, $R[z]/J_i$); thus, (2.4.7) holds. Put:

$$W_a = \sum_i W_i \in \mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{(m,n)})(R,f).$$

We claim:

$$\psi(W_a) = (a, mnN) \in R_f^\times \oplus \mathbb{Z}. \tag{2.4.8}$$

Indeed, it suffices to show this after restriction to the generic point of R, in which case, it follows directly from the definition of the W_i and (2.4.5). This implies the surjectivity of ψ and that of (2.4.4). Next, we show that (2.4.4) has a splitting. Let $\omega_a^{m,n} \in h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\overline{\square}^{(m,n)})(R,f)$ be the class of W_a and $\lambda_a^{m,n} = \omega_a^{m,n} - \omega_1^{m,n}$, where $\omega_1^{m,n}$ is defined as $\omega_a^{m,n}$ replacing a by 1 (and using the same N). By (2.4.8), the image of $\lambda_a^{m,n}$ under the map (2.4.4):

$$h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\overline{\square}^{(m,n)})(R,f) \to R_f^\times \oplus \mathbb{Z}$$

is $(a, 0)$.

Claim 2.4.1. *$\lambda_a^{m,n}$ is independent of the choice of N, and we have:*

$$\lambda_{ab}^{m,n} = \lambda_a^{m,n} + \lambda_b^{m,n} \quad \text{for } a, b \in R_f^\times. \tag{2.4.9}$$

Moreover, for $m' \geq m$ and $n' \geq n$, the image of $\lambda_a^{m',n'}$ under:

$$h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\overline{\square}^{(m',n')}) \to h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\overline{\square}^{(m,n)})$$

coincides with $\lambda_a^{m,n}$.

By the semipurity of $h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\overline{\square}^{(m,n)})$ and [Sai20a, Theorem 3.1], we have an injective homomorphism:

$$h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\overline{\square}^{(m,n)})(R,f) \hookrightarrow \underline{\omega}_! h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\overline{\square}^{(m,n)})(K) = h_0(\overline{\square}^{(m,n)})(K). \tag{2.4.10}$$

By Lemma 2.2, the isomorphism:

$$\theta_{m,n} : h_0(\overline{\square}^{(m,n)})(K) \xrightarrow{\simeq} ((K[t]/t^m)^\times \oplus (K[z]/z^n)^\times)/K^\times \oplus \mathbb{Z}$$

sends $\omega_a^{m,n}$ to:

$$\theta_{m,n}(\omega_a^{m,n}) = \left(\frac{(-1)^{mnN} a}{(t-1)^{mnN}}, \frac{1}{(1-z)^{mnN}}, mnN \right).$$

Thus, $\theta_{m,n}(\lambda_a^{m,n}) = (a, 1, 0)$, which is independent of N. By the injectivity of (2.4.10), this implies the first two assertions of the claim; similarly, the final assertion of the claim follows from the commutative

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diagram in Lemma 2.2. Since $\lambda_a^{m,n}$ does not change if we replace f by uf with $u \in R^\times$, the map $a \to \lambda_a^{m,n}$ glues to give a global morphism of Nisnevich sheaves which induces the splitting $s_{m,n}$ from the statement. It remains to check that $s_{m,n}$ is compatible with transfers. To this end, it suffices to check that $\underline{\omega}_!(s_{m,n})$ is compatible with transfers, and since the transition maps are surjective, it further suffices to show that:

$$\varprojlim_{m,n} \underline{\omega}_!(s_{m,n}) : \mathbf{G}_m \oplus \mathbb{Z} \to \varprojlim_{m,n} \underline{\omega}_! h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\overline{\square}^{(m,n)})$$

is compatible with transfers. Since we can identify the target with $\mathbb{W} \oplus \mathbb{W} \oplus \mathbf{G}_m \oplus \mathbb{Z}$ by Remark 2.3, the compatibility holds automatically by [MS20, Lemma 1.1]. □

Proposition 2.5. *Denote by $\psi : \mathbf{A}^1_y \times \mathbf{A}^1_s \to \mathbf{A}^1_x \times \mathbf{A}^1_s$ the morphism induced by the $k[s]$-algebra morphism $k[x,s] \to k[y,s]$, $x \mapsto ys$. We denote by the same symbol, the induced morphism in $\underline{\mathbf{M}}\mathbf{Cor}$:*

$$\psi : \overline{\square}_y^{(1)} \otimes \overline{\square}_s^{(2)} \to \overline{\square}_x^{(1)} \otimes \overline{\square}_s^{(1)}. \tag{2.5.1}$$

Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$. Then ψ^* factors as follows:

$$\begin{array}{ccc} & & F(\overline{\square}_y^{(1)} \otimes \overline{\square}_s^{(1)} \otimes \mathcal{X}) \\ & \nearrow & \downarrow \\ F(\overline{\square}_x^{(1)} \otimes \overline{\square}_s^{(1)} \otimes \mathcal{X}) & \xrightarrow{\psi^*} & F(\overline{\square}_y^{(1)} \otimes \overline{\square}_s^{(2)} \otimes \mathcal{X}), \end{array}$$

where the vertical map is induced by the natural morphism $\overline{\square}_s^{(2)} \to \overline{\square}_s^{(1)}$.

Proof. It is direct to check that ψ induces a morphism (2.5.1). To check the factorisation statement, we may replace F by $\underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), F)$ to reduce to the case $\mathcal{X} = (\mathrm{Spec}\, k, \emptyset)$ (see Lemma 1.5(1)). By Yoneda and (1.4.2), we are reduced to show that we have a factorisation as follows:

$$\begin{array}{ccc} & & h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\overline{\square}_y^{(1)} \otimes \overline{\square}_s^{(1)}) \\ & \swarrow & \uparrow a \\ h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\overline{\square}_x^{(1)} \otimes \overline{\square}_s^{(1)}) & \xleftarrow{\psi} & h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\overline{\square}_y^{(1)} \otimes \overline{\square}_s^{(2)}). \end{array} \tag{2.5.2}$$

By [MS20, Lemma 1.14(iii)] and Lemma 2.4, the map a is surjective. Thus, we have to show $\psi(\mathrm{Ker}\, a) = 0$. By semipurity, it suffices to show that we have a factorisation as in (2.5.2) after applying $\underline{\omega}_!$. By [RSY22, Proposition 5.6], we have:

$$H := \underline{\omega}_!(h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(\overline{\square}_x^{(1)} \otimes \overline{\square}_s^{(1)})) = h_{0,\mathrm{Nis}}(\overline{\square}_x^{(1)} \otimes \overline{\square}_s^{(1)}) = \mathcal{K}_2^M \oplus \mathbf{G}_m \oplus \mathbf{G}_m \oplus \mathbb{Z},$$

where $\mathcal{K}_2^M$ is the (improved) Milnor K-theory sheaf; in particular, H is $\mathbf{A}^1$-invariant. Thus, $\underline{\omega}_!(\psi)$ and $\underline{\omega}_!(a)$ factor via $h_{0,\mathrm{Nis}}^{\mathbf{A}^1}(h_{0,\mathrm{Nis}}(\overline{\square}_y^{(1)} \otimes \overline{\square}_s^{(2)}))$ (cf. (1.6.2)). Thus, we obtain solid arrows in **NST**:

$$\begin{array}{ccc} & & H \\ & \swarrow & \uparrow \bar{a} \\ H & \xleftarrow[\bar{\psi}]{} & h_{0,\mathrm{Nis}}^{\mathbf{A}^1}(h_{0,\mathrm{Nis}}(\overline{\square}_y^{(1)} \otimes \overline{\square}_s^{(2)})). \end{array} \tag{2.5.3}$$

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Since $\bar{a}$ is the composition of the natural isomorphisms (cf. (1.6.3)):

$$h^{\mathbf{A}^1}_{0,\mathrm{Nis}}(h_{0,\mathrm{Nis}}(\overline{\square}^{(1)}_y \otimes \overline{\square}^{(2)}_s)) \cong h^{\mathbf{A}^1}_{0,\mathrm{Nis}}(\mathbb{Z}_{\mathrm{tr}}(\mathbf{A}^1_y \setminus \{0\}) \otimes_{\mathbf{PST}} \mathbb{Z}_{\mathrm{tr}}(\mathbf{A}^1_s \setminus \{0\})) \cong H$$

the dotted arrow exists, which completes the proof. □

Remark 2.6. Going through the definitions, one can check that the map $H \to H$ induced by $\bar{\psi}$ in (2.5.3) is on a regular local ring R given by:

$$(\{a,b\}, c, d, n) \mapsto (\{a,b\} + \{d,-1\}, cd, d, n),$$

where we use the identification $H(R) = K^M_2(R) \oplus R^\times \oplus R^\times \oplus \mathbb{Z}$.

2.2. Cohomology of a blow-up centred in the smooth part of the modulus

The goal of this section is to prove Theorem 2.12 below, giving the invariance of the cohomology of cube invariant sheaves along a certain class of blow-ups. This plays a fundamental role in what follows, and it is used in the proof of the $(\mathbf{P}^n, \mathbf{P}^{n-1})$-invariance of the cohomology.

Recall the following definition from [Sai20a, Section 5].

Definition 2.7. Let $F \in \mathbf{CI}^{\tau,sp}$. We define the modulus presheaf $\sigma^{(n)}(F)$ by:

$$\sigma^{(n)}(F)(\mathcal{Y}) = \mathrm{Coker}(F(\mathcal{Y}) \xrightarrow{\mathrm{pr}^*} F(\mathcal{Y} \otimes (\mathbf{P}^1, n0 + \infty))),$$

where pr^* is the pullback along the projection $\mathrm{pr}\colon \mathcal{Y} \otimes (\mathbf{P}^1, n0+\infty) \to \mathcal{Y}$. Note that pr^* is split injective, with left inverse given by the inclusion $i_1\colon \mathrm{Spec}\, k \hookrightarrow \mathbf{P}^1$ of the 1-section. Hence, we have an isomorphism, natural in $\mathcal{Y}$:

$$F(\mathcal{Y} \otimes (\mathbf{P}^1, n0+\infty)) \cong \sigma^{(n)}(F)(\mathcal{Y}) \oplus F(\mathcal{Y}).$$

Following [Sai20a, Definition 5.6], we write $F^{(n)}_{-1}$ for $\sigma^{(n)}(F)$ when F is moreover in $\underline{\mathbf{M}}\mathbf{NST}$. Note that we have a natural identification:

$$F^{(n)}_{-1} = \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}((\mathbf{P}^1, n\cdot 0+\infty)/1, F) = F(- \otimes (\mathbf{P}^1, n0+\infty))/F(-),$$

where $(\mathbf{P}^1, n\cdot 0+\infty)/1 = \mathrm{Coker}(\mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}\, k, \emptyset) \xrightarrow{i_1} \mathbb{Z}_{\mathrm{tr}}(\mathbf{P}^1, n\cdot 0+\infty))$ in $\underline{\mathbf{M}}\mathbf{PST}$. By Lemma 1.5(1), we have $F^{(n)}_{-1} \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ if $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, so that the association $F \mapsto F^{(n)}_{-1}$ gives an endofunctor of $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. This construction is the modulus version of Voevodsky's contraction functor (see [MVW06, p.191].

Notation 2.8. We denote by $\underline{\mathbf{M}}\mathbf{Cor}_{ls}$ the full subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ consisting of 'log smooth' modulus pairs, that is, objects $\mathcal{X} = (X, D)$, where $X \in \mathbf{Sm}$ and $|D|$ is a simple normal crossing divisor (in particular, each irreducible component of $|D|$ is a smooth divisor in X). Note that $\otimes$ restricts to a monoidal structure on $\underline{\mathbf{M}}\mathbf{Cor}_{ls}$.

Lemma 2.9. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Let $H \hookrightarrow X$ be a smooth divisor, such that $|D| + H$ is SNCD, and denote by $j : U := X \setminus H \hookrightarrow X$ the inclusion of the complement. Then:*

$$R^i j_* F_{(U, D_{|U})} = 0, \quad \textit{for all } i \geq 1,$$

where $F_{(U,D_{|U})}$ denotes the Nisnevich sheaf on U defined in (1.2.1).

Proof. This is an immediate consequence of [Sai20a, Corollary 8.6(3)]. □

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Lemma 2.10. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Let $E_i \subset \mathbf{A}^1$, $i = 1, \dots, n$, be effective (or empty) divisors, and denote by $\pi : \mathbf{A}^n_X \to X$ the projection. Then:*

$$R^i\pi_*(F_{(\mathbf{A}^1,E_1)\otimes\dots\otimes(\mathbf{A}^1,E_n)\otimes\mathcal{X}}) = 0, \quad \textit{for all } i \geq 1,\ n \geq 0.$$

Proof. First consider the case $n = 1$. Set $E := E_1$ and $\overline{\square}^{(E,r)} := (\mathbf{P}^1, E + r \cdot \infty)$, for $r \geq 1$. The natural morphism $\overline{\square}^{(E,r)} \to \overline{\square}$ induces a map $F_{\mathcal{X}\otimes\overline{\square}} \to F_{\mathcal{X}\otimes\overline{\square}^{(E,r)}}$. The cohomology sheaves of the cone C of this map are supported in $X \times |E + \infty|$, whence $R^i\overline{\pi}_*C = 0$, $i \geq 1$, where $\overline{\pi} : \mathbf{P}^1_X \to X$ is the projection. We obtain surjections:

$$R^i\overline{\pi}_*F_{\mathcal{X}\otimes\overline{\square}} \to R^i\overline{\pi}_*F_{\mathcal{X}\otimes\overline{\square}^{(E,r)}} \to 0, \quad \text{for all } i \geq 1.$$

By the cube invariance of cohomology (see [Sai20a, Theorem 9.3]), the left term vanishes. Thus, M-reciprocity (see [Sai20a, Lemma 1.27(1)]) yields:

$$0 = \varinjlim_r R^i\overline{\pi}_*F_{\mathcal{X}\otimes\overline{\square}^{(E,r)}} = R^i\overline{\pi}_*j_*F_{(\mathbf{A}^1,E)\otimes\mathcal{X}},$$

where $j : \mathbf{A}^1_X \hookrightarrow \mathbf{P}^1_X$ is the open immersion. Together with Lemma 2.9, we obtain:

$$R^i\overline{\pi}_*Rj^k_*F_{(\mathbf{A}^1,E)\otimes\mathcal{X}} = 0, \quad \text{for all } i \geq 1, k \geq 0.$$

Thus, the vanishing $R^i\pi_*F_{(\mathbf{A}^1,E)\otimes\mathcal{X}} = 0$ follows from the Leray spectral sequence.

The general case follows by induction (by factoring π as $\mathbf{A}^n_X \xrightarrow{\pi_1} \mathbf{A}^{n-1}_X \xrightarrow{\pi_{n-1}} X$ and observing):

$$\pi_{1*}(F_{(\mathbf{A}^1,E_1)\otimes\dots\otimes(\mathbf{A}^1,E_n)\otimes\mathcal{X}}) = F_{1,(\mathbf{A}^1,E_2)\otimes\dots\otimes(\mathbf{A}^1,E_n)\otimes\mathcal{X}},$$

where $F_1 := \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(\mathbf{A}^1, E_1), F)$ lies in $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ by Lemma 1.5(1). □

2.11. We recall some standard terminology. Let $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$, $Y \in \mathbf{Sm}$, and let $f : Y \to X$ be a k-morphism of finite type. We say *D is transversal to f*, if for any number of irreducible components $D_1, \dots, D_r$ of the SNCD $|D|$, the morphism f intersects the scheme-theoretic intersection $D_1 \cap \dots \cap D_r$ transversally (i.e. the scheme-theoretic inverse image $f^{-1}(D_1 \cap \dots \cap D_r)$ is smooth over k and of codimension r in Y). Note that f is always transversal to the empty divisor.

If f is a closed immersion, we also say Y and D intersect transversally. Since X is of finite type over a perfect field, this is equivalent to say, that for any point $x \in Y \cap D$, we find a regular sequence of parameters $t_1, \dots, t_n \in \mathcal{O}_{X,x}$, such that $\mathcal{O}_{Y,x} = \mathcal{O}_{X,x}/(t_1, \dots, t_s)$ and the irreducible components of $|D|$ containing x are in $\operatorname{Spec} \mathcal{O}_{X,x}$ given by $V(t_{s+1}), \dots, V(t_r)$, with $1 \leq s \leq r \leq n$.

Theorem 2.12. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Assume there is a smooth irreducible component D_0 of $|D|$ which has multiplicity 1 in D. Let $Z \subset X$ be a smooth closed subscheme which is contained in D_0 and intersects $|D - D_0|$ transversally. Let $\rho : Y \to X$ be the blow-up in Z. Then the natural map:*

$$F_{\mathcal{X}} \xrightarrow{\simeq} R\rho_*F_{(Y,\rho^*D)}$$

is an isomorphism in the derived category of abelian Nisnevich sheaves on X.

The proof is given in 2.16. The key point is to understand the case of the blow-up of $\mathbf{A}^2$ in the origin with D_0 a line, which is established in the next Lemma. Here, after some preliminary steps, we are reduced to prove the vanishing of the cohomology of the pushforward of F along the projection from the blow-up to the exceptional divisor. This is where the modulus descent, in that, Proposition 2.5, is crucially used.

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Lemma 2.13. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Let $\rho : Y \to \mathbf{A}^2$ be the blow-up in the origin $0 \in \mathbf{A}^2$, and let L be a line containing 0. Then:*

$$R^i \rho_{X*} F_{(Y,\rho^* L)\otimes\mathcal{X}} = 0, \quad \textit{for all } i \geq 1,$$

where $\rho_X := \rho \times \mathrm{id}_X : Y \times X \to \mathbf{A}^2 \times X$ is the base change of ρ.

Proof. We can assume X is henselian local and:

$$L = V(x) \subset \mathbf{A}^2 = \operatorname{Spec} k[x, y].$$

Set:

$$\mathcal{F} := F_{(Y,\rho^* L)\otimes\mathcal{X}};$$

it is a Nisnevich sheaf on $Y \times X$. For $i \geq 1$, the higher direct images $R^i \rho_{X*}\mathcal{F}$ are supported in $0 \times X$, whence:

$$H^j(\mathbf{A}^2_X, R^i \rho_{X*}\mathcal{F}) = 0, \quad \text{for all } i, j \geq 1,$$

and:

$$R^i \rho_{X*}\mathcal{F} = 0 \iff H^0(\mathbf{A}^2_X, R^i \rho_{X*}\mathcal{F}) = 0.$$

Furthermore, $\rho_{X*}\mathcal{F} = F_{(\mathbf{A}^2,L)\otimes\mathcal{X}}$, since $(Y, \rho^* L)\otimes\mathcal{X} \cong (\mathbf{A}^2, L)\otimes\mathcal{X}$ in $\underline{\mathbf{M}}\mathbf{Cor}$ (see [KMSY21a, Section 1.7]). Hence, by Lemma 2.10:

$$H^i(\mathbf{A}^2_X, \rho_{X*}\mathcal{F}) = H^i(\mathbf{A}^2_X, F_{(\mathbf{A}^2,L)\otimes\mathcal{X}}) = 0.$$

Thus, the Leray spectral sequence yields:

$$H^0(\mathbf{A}^2_X, R^i \rho_{X*}\mathcal{F}) = H^i(Y \times X, \mathcal{F}), \quad i \geq 0, \tag{2.13.1}$$

and we have to show that this group vanishes for $i \geq 1$. Write:

$$Y = \operatorname{Proj} k[x, y][S, T]/(xT - yS) \subset \mathbf{A}^2 \times \mathbf{P}^1,$$

and denote by:

$$\pi : Y \times X \hookrightarrow \mathbf{A}^2 \times \mathbf{P}^1_X \to \mathbf{P}^1_X = \operatorname{Proj} \mathcal{O}_X[S, T]$$

the morphism induced by projection. In order to show that (2.13.1) vanishes, we can project along π and use the Leray spectral sequence:

$$H^{i-j}(\mathbf{P}^1_X, R\pi^j_* \mathcal{F}) \Rightarrow H^i(Y \times X, \mathcal{F})$$

to reduce the problem to showing that:

$$H^i(\mathbf{P}^1_X, R^j \pi_* \mathcal{F}) = 0, \quad i \geq 1, j \geq 0. \tag{2.13.2}$$

The terms $R^j \pi_* \mathcal{F}$ for $j \geq 1$ are easy to handle using Lemma 2.10. Indeed, set $s = S/T$, and write:

$$\mathbf{P}^1 \setminus \{\infty\} = \mathbf{A}^1_s := \operatorname{Spec} k[s], \quad \mathbf{P}^1 \setminus \{0\} = \operatorname{Spec} k[\tfrac{1}{s}].$$

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Set $U := \mathbf{A}^1_s \times X$ and $V := (\mathbf{P}^1 \setminus \{0\}) \times X$ and:

$$\mathcal{U} := (\mathbf{A}^1_s, 0) \otimes \mathcal{X}, \quad \mathcal{V} := (\mathbf{P}^1 \setminus \{0\}) \otimes \mathcal{X}.$$

We have:

$$\pi^{-1}(U) = \mathbf{A}^1_y \times U, \quad \pi^{-1}(V) = \mathbf{A}^1_x \times V,$$

and the restriction of π to these open subsets is given by projection. Furthermore by construction,

$$\mathcal{F}_{|\pi^{-1}(U)} = F_{(\mathbf{A}^1_y,0)\otimes\mathcal{U}}, \quad \mathcal{F}_{|\pi^{-1}(V)} = F_{(\mathbf{A}^1_x,0)\otimes\mathcal{V}}. \tag{2.13.3}$$

Thus, Lemma 2.10 (in the case $n = 1$) yields:

$$R^j\pi_*\mathcal{F} = 0, \quad j \geq 1.$$

It remains to show:

$$H^i(\mathbf{P}^1_X, \pi_*\mathcal{F}) = 0, \quad i \geq 1. \tag{2.13.4}$$

Set:

$$F_1 := \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(\mathbf{A}^1_x, 0), F). \tag{2.13.5}$$

Note that $F_1 \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ by Lemma 1.5(1). Let $j : V \hookrightarrow \mathbf{P}^1_X$ be the open immersion. Its base change along π induces a morphism:

$$\iota : (\mathbf{A}^1_x, 0) \otimes \mathcal{V} \to (Y, \rho^*L) \otimes \mathcal{X} \quad \text{in } \underline{\mathbf{M}}\mathbf{Cor}\,. \tag{2.13.6}$$

This yields an exact sequence of Nisnevich sheaves on $\mathbf{P}^1_X$:

$$0 \to \pi_*\mathcal{F} \xrightarrow{\pi_*(\iota^*)} j_*F_{1,\mathcal{V}} \to \Gamma \to 0,$$

defining Γ; here, the first map is injective by the semipurity of F. Since Γ is supported on $0 \times X$, we obtain for $i \geq 2$:

$$\begin{aligned} H^i(\mathbf{P}^1_X, \pi_*\mathcal{F}) &= H^i(\mathbf{P}^1_X, j_*F_{1,\mathcal{V}}) \\ &= H^i(V, F_{1,\mathcal{V}}), && \text{by Lemma 2.9,} \\ &= 0, && \text{by Lemma 2.10.} \end{aligned}$$

It remains to prove the vanishing (2.13.4) for $i = 1$. This will occupy the rest of the proof. Let:

$$a\colon Y \times X \to \mathbf{A}^1_x \times \mathbf{P}^1 \times X$$

be induced by the base change of the closed immersion $Y \hookrightarrow \mathbf{A}^2 \times \mathbf{P}^1$ followed by the base change of the projection $\mathbf{A}^2 \to \mathbf{A}^1_x$. The map a induces a morphism:

$$\alpha : (Y, \rho^*L) \otimes \mathcal{X} \to (\mathbf{A}^1_x, 0) \otimes \mathbf{P}^1_{\mathcal{X}} \quad \text{in } \underline{\mathbf{M}}\mathbf{Cor}, \tag{2.13.7}$$

where $\mathbf{P}^1_{\mathcal{X}} := \mathbf{P}^1 \otimes \mathcal{X}$ and which precomposed with ι from (2.13.6) yields the morphism:

$$\alpha\iota : (\mathbf{A}^1_x, 0) \otimes \mathcal{V} \to (\mathbf{A}^1_x, 0) \otimes \mathbf{P}^1_{\mathcal{X}} \tag{2.13.8}$$

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induced by the open immersion $\mathbf{A}^1_x \times V \hookrightarrow \mathbf{A}^1_x \times \mathbf{P}^1_X$. This gives a factorisation:

$$\begin{array}{ccc} F_{1,\mathbf{P}^1_{\mathcal{X}}} & \xrightarrow{\pi_*(\alpha^*)} & \pi_*\mathcal{F} \\ & \searrow & \downarrow{\scriptstyle \pi_*(\iota^*)} \\ & & j_*F_{1,\mathcal{V}}, \end{array}$$

where the diagonal morphism is injective by [Sai20a, Theorem 3.1(2)] and the semipurity of F_1. This implies that the morphism labeled $\pi_*(\alpha^*)$ is injective too. Similarly, the embedding $\mathcal{V} \to (\mathbf{P}^1, 0) \otimes \mathcal{X}$ induces another injective morphism $F_{1,(\mathbf{P}^1,0)\otimes\mathcal{X}} \to j_*F_{1,\mathcal{V}}$. In total, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{1,\mathbf{P}^1_{\mathcal{X}}} & \xrightarrow{\pi_*(\alpha^*)} & \pi_*\mathcal{F} & \longrightarrow & \underline{\Sigma} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \pi_*(\iota^*)} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & F_{1,\mathbf{P}^1_{\mathcal{X}}} & \xrightarrow{\pi_*((\alpha\iota)^*)} & j_*F_{1,\mathcal{V}} & \longrightarrow & \underline{\Lambda} & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & F_{1,\mathbf{P}^1_{\mathcal{X}}} & \longrightarrow & F_{1,(\mathbf{P}^1,0)\otimes\mathcal{X}} & \longrightarrow & \underline{\Lambda}(0) & \longrightarrow & 0, \end{array} \tag{2.13.9}$$

with exact rows, defining the cokernels $\underline{\Sigma}$, $\underline{\Lambda}$ and $\underline{\Lambda}(0)$, as well as the map φ. Applying $R\Gamma(\mathbf{P}^1_X, -)$ yields:

$$\begin{array}{ccccccc} \Sigma & \xrightarrow{\partial_1} & H^1(\mathbf{P}^1_X, F_{1,\mathbf{P}^1_{\mathcal{X}}}) & \longrightarrow & H^1(\mathbf{P}^1_X, \pi_*\mathcal{F}) & \to & 0 \\ \downarrow{\scriptstyle \varphi} & & \| & & \downarrow & & \\ \Lambda & \xrightarrow{\partial_2} & H^1(\mathbf{P}^1_X, F_{1,\mathbf{P}^1_{\mathcal{X}}}) & \longrightarrow & H^1(\mathbf{P}^1_X, j_*F_{1,\mathcal{V}}) & \to & 0 \\ \uparrow & & \| & & \uparrow & & \\ \Lambda(0) & \twoheadrightarrow & H^1(\mathbf{P}^1_X, F_{1,\mathbf{P}^1_{\mathcal{X}}}) & \longrightarrow & H^1(\mathbf{P}^1_X, F_{1,(\mathbf{P}^1,0)\otimes\mathcal{X}}) = 0 & & \end{array}$$

with exact rows and in which the ∂_i are the connecting homomorphisms and where:

$$\Sigma := H^0(\mathbf{P}^1_X, \underline{\Sigma}), \quad \Lambda := H^0(\mathbf{P}^1_X, \underline{\Lambda}), \quad \Lambda(0) := H^0(\mathbf{P}^1_X, \underline{\Lambda}(0)).$$

The group $H^1(\mathbf{P}^1, F_{1,(\mathbf{P}^1,0)\otimes\mathcal{X}})$ vanishes by the cube invariance of cohomology (see [Sai20a, Theorem 9.3]), thus, $\partial_{2|\Lambda(0)}$ is surjective, the vanishing (2.13.4) for $i = 1$ will follow, if we can show:

$$\Lambda(0) \subset \varphi(\Sigma). \tag{2.13.10}$$

Note that $\underline{\Sigma}$, $\underline{\Lambda}$ and $\underline{\Lambda}(0)$ have support in $0 \times X \subset U$, so we can compute the global sections on U instead of $\mathbf{P}^1$ to show (2.13.10). Now, since $H^1(U, F_{1,\mathcal{U}}) = 0$, by Lemma 2.10, unravelling the definitions, we obtain from (2.13.3) and (2.13.9) with $G := F(- \otimes \mathcal{X})$ and $\mathbf{A}^1_s = \mathbf{P}^1 \setminus \{\infty\}$ the following descriptions:

$$\Sigma = \frac{G((\mathbf{A}^1_y, 0) \otimes (\mathbf{A}^1_s, 0))}{\alpha^* G((\mathbf{A}^1_x, 0) \otimes \mathbf{A}^1_s)},$$

$$\Lambda = \frac{G((\mathbf{A}^1_x, 0) \otimes (\mathbf{A}^1_s \setminus \{0\}, \emptyset))}{G((\mathbf{A}^1_x, 0) \otimes \mathbf{A}^1_s)},$$

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$$\Lambda(0) = \frac{G((\mathbf{A}^1_x, 0) \otimes (\mathbf{A}^1_s, 0))}{G((\mathbf{A}^1_x, 0) \otimes \mathbf{A}^1_s)}. \tag{2.13.11}$$

By [Sai20a, Lemma 5.9], we have isomorphisms (see Notation 2.1):

$$\frac{G(\overline{\square}^{(1)}_x \otimes (\mathbf{A}^1_s, 0))}{G((\mathbf{P}^1_x, \infty) \otimes (\mathbf{A}^1_s, 0))} \xrightarrow{\simeq} \frac{G((\mathbf{A}^1_x, 0) \otimes (\mathbf{A}^1_s, 0))}{G(\mathbf{A}^1_x \otimes (\mathbf{A}^1_s, 0))}, \tag{2.13.12}$$

$$\frac{G(\overline{\square}^{(1)}_x \otimes \overline{\square}^{(1)}_s)}{G(\overline{\square}^{(1)}_x \otimes (\mathbf{P}^1_s, \infty))} \xrightarrow{\simeq} \frac{G(\overline{\square}^{(1)}_x \otimes (\mathbf{A}^1_s, 0))}{G(\overline{\square}^{(1)}_x \otimes \mathbf{A}^1_s)}. \tag{2.13.13}$$

Write j for the open immersion $(\mathbf{A}^1_x, 0) \hookrightarrow \overline{\square}^{(1)}_x$. The base change of j^* induces a commutative diagram:

$$\begin{array}{ccccc} G((\mathbf{A}^1_x, 0) \otimes \mathbf{A}^1_s) & \longrightarrow & G((\mathbf{A}^1_x, 0) \otimes (\mathbf{A}^1_s, 0)) & \twoheadrightarrow & \Lambda(0) \\ \uparrow{\scriptstyle j^*} & & \uparrow{\scriptstyle j^*} & \nearrow & \\ G(\overline{\square}^{(1)}_x \otimes \mathbf{A}^1_s) & \longrightarrow & G(\overline{\square}^{(1)}_x \otimes (\mathbf{A}^1_s, 0)). & & \end{array} \tag{2.13.14}$$

The horizontal composite morphism is zero by (2.13.11), hence, the kernel of the diagonal arrow contains $G(\overline{\square}^{(1)}_x \otimes \mathbf{A}^1_s)$. Next, note that from (2.13.12), we get the surjective morphism:

$$G(\overline{\square}^{(1)}_x \otimes (\mathbf{A}^1_s, 0)) \oplus G(\mathbf{A}^1_x \otimes (\mathbf{A}^1_s, 0)) \to G((\mathbf{A}^1_x, 0) \otimes (\mathbf{A}^1_s, 0)) \to 0. \tag{2.13.15}$$

Combining (2.13.15), (2.13.13) and (2.13.14), we get a surjection:

$$G(\mathbf{A}^1_x \otimes (\mathbf{A}^1_s, 0)) \oplus G(\overline{\square}^{(1)}_x \otimes \overline{\square}^{(1)}_s) \twoheadrightarrow \Lambda(0). \tag{2.13.16}$$

Note that the pullback of the open immersion $\pi^{-1}(V) \hookrightarrow Y \times X$ along $\pi^{-1}(U) \hookrightarrow Y \times X$ induces the open immersion:

$$\mathbf{A}^1_x \times (\mathbf{A}^1_s \setminus \{0\}) \times X \to \mathbf{A}^1_y \times \mathbf{A}^1_s \times X,$$

which is induced by base change from the $k[s]$-linear map:

$$k[y, s] \mapsto k[x, s, 1/s], \quad y \mapsto x/s.$$

It gives the following two morphisms in $\underline{\mathbf{M}}\mathbf{Cor}$:

$$\iota_1 : (\mathbf{A}^1_x, 0) \otimes (\mathbf{A}^1_s \setminus \{0\}) \otimes \mathcal{X} \to (\mathbf{A}^1_y, 0) \otimes (\mathbf{A}^1_s, 0) \otimes \mathcal{X}.$$
$$\iota_2 : (\mathbf{A}^1_x, 0) \otimes (\mathbf{A}^1_s \setminus \{0\}) \otimes \mathcal{X} \to \overline{\square}^{(1)}_y \otimes \overline{\square}^{(2)}_s \otimes \mathcal{X}.$$

Furthermore, consider the base change of the map (2.5.1):

$$\psi : \overline{\square}^{(1)}_y \otimes \overline{\square}^{(2)}_s \otimes \mathcal{X} \to \overline{\square}^{(1)}_x \otimes \overline{\square}^{(1)}_s \otimes \mathcal{X},$$

which is induced by $x \mapsto ys$; it restricts to:

$$\psi_1 : (\mathbf{A}^1_y, 0) \otimes (\mathbf{A}^1_s, 0) \to \mathbf{A}^1_x \otimes (\mathbf{A}^1_s, 0).$$

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In particular, $\psi_1 \circ \iota_1$ is induced by the open immersion $\mathbf{A}^1_x \setminus \{0\} \times \mathbf{A}^1_s \setminus \{0\} \hookrightarrow \mathbf{A}^1_x \times \mathbf{A}^1_s \setminus \{0\}$ and $\psi \circ \iota_2$ is induced by the identity on $\mathbf{A}^1_x \setminus \{0\} \times \mathbf{A}^1_s \setminus \{0\}$. Consider the following diagram:

$$\begin{array}{ccccc}
G(\mathbf{A}^1_x \otimes (\mathbf{A}^1_s, 0)) & \xrightarrow{\psi_1^*} & G((\mathbf{A}^1_y, 0) \otimes (\mathbf{A}^1_s, 0)) & \xrightarrow{r_1} & \Sigma \\
 & & \downarrow{\scriptstyle \iota_1^*} & & \downarrow{\scriptstyle \varphi} \\
 & & G((\mathbf{A}^1_x, 0) \otimes (\mathbf{A}^1_s \setminus \{0\}, \emptyset)) & \xrightarrow{r_2} & \Lambda \\
 & & \uparrow & & \uparrow \\
 & & G(\mathbf{A}^1_x \otimes (\mathbf{A}^1_s, 0)) & \xrightarrow{r_3} & \Lambda(0).
\end{array}$$

Here, the maps r_i are the natural maps into the quotients; the diagram commutes by definition of the morphisms involved. Hence:

$$\mathrm{Im}(G(\mathbf{A}^1_x \otimes (\mathbf{A}^1_s, 0)) \to \Lambda(0)) \subset \varphi(\Sigma). \tag{2.13.17}$$

Consider now the following diagram:

$$\begin{array}{ccccc}
 & & G(\overline{\square}^{(1)}_y \otimes \overline{\square}^{(1)}_s) & \xrightarrow{r_1} & \Sigma \\
 & \nearrow & \downarrow & & \\
G(\overline{\square}^{(1)}_x \otimes \overline{\square}^{(1)}_s) & \xrightarrow{\psi^*} & G(\overline{\square}^{(1)}_y \otimes \overline{\square}^{(2)}_s) & & \downarrow{\scriptstyle \varphi} \\
 & & \downarrow{\scriptstyle \iota_2^*} & & \\
 & & G((\mathbf{A}^1_x, 0) \otimes (\mathbf{A}^1_s \setminus \{0\}, \emptyset)) & \xrightarrow{r_2} & \Lambda \\
 & & \uparrow & & \uparrow \\
 & & G(\overline{\square}^{(1)}_x \otimes \overline{\square}^{(1)}_s) & \xrightarrow{r_3} & \Lambda(0).
\end{array}$$

Here, the maps r_1 and r_3 are induced by restriction followed by the quotient map using (2.13.12) and (2.13.13); the two squares and the triangle on the lower left commute by definition of the morphisms involved; the map ψ^* factors via the dotted arrow in the diagram (by Proposition 2.5). This shows:

$$\mathrm{Im}(G(\overline{\square}^{(1)}_x \otimes \overline{\square}^{(1)}_s) \to \Lambda(0)) \subset \varphi(\Sigma),$$

which together with (2.13.17) and (2.13.16) implies (2.13.10). This completes the proof of the lemma. □

Lemma 2.14. *Let the assumptions and notations be as in Theorem 2.12. Assume additionally* $\mathrm{codim}(Z, X) \leq 2$. *Then Theorem 2.12 holds.*

Proof. There is nothing to prove for $\mathrm{codim}(Z, X) = 1$, we therefore consider the case $\mathrm{codim}(Z, X) = 2$. Since $(Y, \rho^* D) \cong (X, D)$ in **MCor** we have $\rho_* F_{(Y,\rho^* D)} \cong F_{(X,D)}$. Thus, it remains to show the vanishing:

$$R^i \rho_* F_{(Y,\rho^* D)} = 0, \quad \text{for all } i \geq 1. \tag{2.14.1}$$

The question is Nisnevich local around the points in Z. Let $z \in Z$ be a point, and consider the regular henselian local ring $A = \mathcal{O}^h_{X,z}$. For $V \subset X$, set $V_{(z)} := V \times_X \mathrm{Spec}\, A$. Denote by $D' \subset X$ the closed subscheme defined by $D - D_0$. By assumption, we find a regular system of local parameters $x, y, t_1 \ldots, t_s$

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of A, such that $Z_{(z)} = V(x, y)$, $D_{0,(z)} = V(x)$ and $D'_{(z)} = V(t_1^{n_1} \cdots t_r^{n_r})$, for some $r \leq s$ and $n_i \geq 1$. Let $K \hookrightarrow A$ be a coefficient field over k; we obtain an isomorphism:

$$K\{x, y, t_1, \ldots, t_s\} \xrightarrow{\simeq} A.$$

Let $\rho_1 : \widetilde{\mathbf{A}^2} \to \mathbf{A}^2$ be the blow-up in 0. By the above, the blow-up in Z:

$$\rho : (Y, \rho^* D) \to (X, D)$$

is Nisnevich locally around z over k isomorphic to the morphism:

$$(\widetilde{\mathbf{A}^2}, \rho_1^*(x)) \otimes (\mathbf{A}^s_K, (\prod_{i=1}^r t_i^{n_i})) \to (\mathbf{A}^2, (x)) \otimes (\mathbf{A}^s_K, (\prod_{i=1}^r t_i^{n_i})),$$

which is induced by base change from ρ_1. Hence, the vanishing (2.14.1) follows from Lemma 2.13. □

Lemma 2.15. *Let X be a finite type k-scheme and $Z_0 \subset Z_1 \subset X$ closed subschemes. Let $\rho : X' \to X$ be the blow-up of X in Z_0, and let $\rho' : X'' \to X'$ be the blow-up of X' in the strict transform $\tilde{Z}_1$ of Z_1. Furthermore, let $\sigma : Y' \to X$ be the blow-up in Z_1 and let $\sigma' : Y'' \to Y'$ be the blow-up of Y' in $\sigma^{-1}(Z_0)$. Then there is an isomorphism:*

$$\begin{array}{ccc} X'' & \xrightarrow{\simeq} & Y'' \\ & {\scriptstyle \rho\rho'} \searrow \quad \swarrow {\scriptstyle \sigma\sigma'} & \\ & X. & \end{array}$$

Proof. Recall the following general fact: Let $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$ be two coherent ideal sheaves. Then the blow-up $\tilde{X} \to X$ of X in $\mathcal{I} \cdot \mathcal{J}$ is equal to the composition $X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X$, where π_1 is the blow-up in $\mathcal{I}$ and π_2 is the blow-up in $\pi_1^{-1}\mathcal{J} \cdot \mathcal{O}_{X_1}$. This is proven using the universal property of blow-ups (see, e.g. [Sta19, Tag 080A]. Here, denote by $\mathcal{I}_i \subset \mathcal{O}_X$ the ideal sheaves of Z_i. We have $\mathcal{I}_1 \subset \mathcal{I}_0$. Let $\pi : \tilde{X} \to X$ be the blow-up of X in $\mathcal{I}_1 \cdot \mathcal{I}_0$. By the remark above, π is isomorphic as X-scheme to $\sigma\sigma'$. Furthermore, note that ρ' is also equal to the blow-up of X' in $\rho^{-1}(Z_1)$. Indeed, the ideal sheaf of $\rho^{-1}(Z_1)$ is equal to $\rho^{-1}\mathcal{I}_1 \cdot \mathcal{O}_{X'} = \mathcal{I}_E \cdot \tilde{\mathcal{I}}_1$, where $\mathcal{I}_E$ is the ideal sheaf of the exceptional divisor of ρ and $\tilde{\mathcal{I}}_1$ is the ideal sheaf of $\tilde{Z}_1$; since $\mathcal{I}_E$ is invertible, the blow-ups of X' in $\tilde{\mathcal{I}}_1$ and in $\rho^{-1}\mathcal{I}_1 \cdot \mathcal{O}_{X'}$ are isomorphic. Thus, by the remark above, the X-scheme $\rho\rho'$ is isomorphic to π as well. □

2.16. *Proof of Theorem 2.12.* The proof is by induction on $c = \operatorname{codim}(Z, X)$, the induction start for $c \leq 2$ being Lemma 2.14. Assume $c > 2$. The question is local on X. Hence, we can assume $X = \operatorname{Spec} A$ and that there is a regular sequence $y_1, \ldots, y_c, t_1, \ldots, t_r \in A$, such that $Z = V(y_1, \ldots, y_c)$, $D_0 = V(y_1)$ and $D' = D - D_0 = V(t_1^{n_1} \cdots t_r^{n_r})$, for some $n_i \geq 1$. Set $Z_2 := V(y_1, y_2)$. Let $\rho : Y \to X$ be the blow-up in Z, and denote by $\tilde{Z}_2$ the strict transform of Z_2. Then $\rho^* D$ has SNC support with the strict transform $\tilde{D}_0$ of D_0 being a smooth component containing $\tilde{Z}_2$. Furthermore, $\tilde{Z}_2$ intersects $\rho^* D - \tilde{D}_0$ transversally and $\operatorname{codim}(\tilde{Z}_2, Y) = 2$. Let $\rho' : Y' \to Y$ be the blow-up in $\tilde{Z}_2$. By Lemma 2.14 we find:

$$R\rho_* F_{(Y, \rho^* D)} \cong R(\rho\rho')_* F_{(Y', (\rho\rho')^* D)}. \tag{2.16.1}$$

Let $\sigma : W \to X$ be the blow-up in Z_2, and set $Z_{c-1} := \sigma^{-1}(Z)$. Then $\sigma^* D$ has SNC support with the exceptional divisor E being a smooth component containing Z_{c-1}. Furthermore, Z_{c-1} intersects the strict transform of D transversally and $\operatorname{codim}(Z_{c-1}, W) = c - 1$. Let $\sigma' : W' \to W$ be the blow-up in

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Z_{c-1}. By Lemma 2.14 and induction, we find:

$$F_{(X,D)} \cong R\sigma_* F_{(W,\sigma^* D)} \cong R(\sigma\sigma')_* F_{W',(\sigma\sigma')^* D}.$$

Thus, the statement follows from Lemma 2.15 and (2.16.1). □

2.3. $(\mathbf{P}^n, \mathbf{P}^{n-1})$*-invariance of cohomology*

We follow the basic strategy of [KS20, Lemma 10] (see also [BPØ22, Proposition 7.3.1].

Lemma 2.17. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Let $x \in \mathbf{P}^n$ be a k-rational point and $L \subset \mathbf{P}^n$ a hyperplane. Denote by $\rho : Y \to \mathbf{P}^n$ the blow-up in x. Denote by $q : Y \times X \to E \times X$ the base change of the morphism $Y \to E$ which parametrises the lines in $\mathbf{P}^n$ through x. Then the pullback:*

$$q^* : F_{(E,L')\otimes\mathcal{X}} \xrightarrow{\simeq} Rq_* F_{(Y,\rho^* L)\otimes\mathcal{X}}$$

is an isomorphism, where $L' = \tilde{L} \cap E$, with $\tilde{L} \subset Y$ the strict transform of L (note $L' = \emptyset$, if $x \notin L$).

Proof. Note that the projection morphism $Y \to E$ makes Y into a $\mathbf{P}^1$-bundle over E and induces a morphism $(Y, \rho^* L) \otimes \mathcal{X} \to (E, L') \otimes \mathcal{X}$. The latter morphism locally over E has the form of the projection $\overline{\square} \otimes \mathcal{W} \to \mathcal{W}$, for some $\mathcal{W} \in \underline{\mathbf{M}}\mathbf{Cor}$. Indeed, over an affine neighborhood $U \subset E$ intersecting (respectively, not intersecting) L', the modulus pair $\mathcal{W}$ can be taken to be $(U, L' \cap U) \otimes \mathcal{X}$ (respectively, $(U, \emptyset) \otimes \mathcal{X}$). In both cases, the divisor $\{\infty\} \times U \times X$ on $\mathbf{P}^1 \times U \times X$ is the restriction of the exceptional divisor to $q^{-1}(U) = \mathbf{P}^1 \times U \times X$. Thus, the statement follows from the cube invariance of cohomology (see [Sai20a, Theorem 9.3]. □

Theorem 2.18. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. Let $L \subset \mathbf{P}^n$ be a hyperplane and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Then the pullback:*

$$F_{\mathcal{X}} \xrightarrow{\simeq} R\pi_* F_{(\mathbf{P}^n,L)\otimes\mathcal{X}},$$

along the projection $\pi : \mathbf{P}^n_X \to X$ is an isomorphism.

Proof. The case $n = 1$ is [Sai20a, Theorem 9.3]. Assume $n \geq 2$. Let $x \in \mathbf{P}^n$ be a k-rational point, $L \subset \mathbf{P}^n$ a hyperplane with $x \in L$ and $\rho : Y \to \mathbf{P}^n$ the blow-up in x. Then $R\rho_* F_{(Y,\rho^* L)\otimes\mathcal{X}} = F_{(\mathbf{P}^n,L)\otimes\mathcal{X}}$ by Theorem 2.12. Thus, the statement follows from Lemma 2.17 and induction. □

Corollary 2.19. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Let V be a vector bundle on X, and denote by:*

$$\pi : \mathbf{P}(V) := \mathrm{Proj}(\mathrm{Sym}^\bullet_{\mathcal{O}_X}(V)) \to X$$

the structure map. Then π^* induces an isomorphism:

$$\pi^* : F_{\mathcal{X}} \xrightarrow{\simeq} \pi_* F_{(\mathbf{P}(V),\pi^* D)}.$$

Proof. The question is local on X, hence, we can assume that V is trivial of rank $n+1$. Let $L \subset \mathbf{P}^n$ be a hyperplane and consider:

$$F_{\mathcal{X}} \xrightarrow{\pi^*} \pi_* F_{\mathbf{P}^n\otimes\mathcal{X}} \hookrightarrow \pi_* F_{(\mathbf{P}^n,L)\otimes\mathcal{X}}.$$

The second map is injective by semipurity and [Sai20a, Theorem 3.1(2)]; the composition is an isomorphism by Theorem 2.18, hence, so is the first map. □

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3. Smooth blow-up formula

Theorem 3.1. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Let $Z \subset X$ be a smooth closed subscheme which intersects D transversally. Consider the following cartesian diagram:*

$$\begin{array}{ccc} E & \xrightarrow{i_E} & \tilde{X} \\ {\scriptstyle \rho_E}\downarrow & & \downarrow{\scriptstyle \rho} \\ Z & \xrightarrow{i} & X, \end{array}$$

in which ρ is the blow-up of X along Z. Set:

$$\tilde{\mathcal{X}} = (\tilde{X}, D_{|\tilde{X}}), \quad \mathcal{Z} = (Z, D_{|Z}), \quad \mathcal{E} = (E, D_{|E}).$$

Then there is a distinguished triangle in the bounded derived category of Nisnevich sheaves of abelian groups $D^b(X_{\mathrm{Nis}})$:

$$F_{\mathcal{X}} \xrightarrow{\rho^* \oplus (-i^*)} R\rho_* F_{\tilde{\mathcal{X}}} \oplus i_* F_{\mathcal{Z}} \xrightarrow{i_E^* + \rho_E^*} i_* R\rho_{E*} F_{\mathcal{E}} \to F_{\mathcal{X}}[1].$$

Proof. The first part of the argument is similar to the proof of [Gro85, Theorem IV.1.1.5]. We have to show that the diagram:

$$\begin{array}{ccc} F_{\mathcal{X}} & \xrightarrow{\rho^*} & R\rho_* F_{\tilde{\mathcal{X}}} \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i_E^*} \\ i_* F_{\mathcal{Z}} & \xrightarrow{\rho_E^*} & i_* R\rho_{E*} F_{\mathcal{E}} \end{array}$$

is homotopy cartesian in $D^b(X_{\mathrm{Nis}})$. To this end, it suffices to show that the following maps are isomorphisms:

$$\rho_E^* : F_{\mathcal{Z}} \to \rho_{E*} F_{\mathcal{E}}, \tag{3.1.1}$$

$$\rho^* : F_{\mathcal{X}} \to \rho_* F_{\tilde{\mathcal{X}}}, \tag{3.1.2}$$

$$i_E^* : R^j \rho_* F_{\tilde{\mathcal{X}}} \to i_* R^j \rho_{E*} F_{\mathcal{E}}, \quad j \geq 1. \tag{3.1.3}$$

The map (3.1.1) is an isomorphism by Corollary 2.19, since E is a projective bundle over Z. The question for the other two isomorphisms is Nisnevich local. Since Z and D intersect transversally, we can assume that $\mathcal{X} = (\mathbf{A}^n, \emptyset) \otimes \mathcal{Z}$ with $\mathcal{Z} = (Z, D_Z) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ and that $\tilde{X}$ is the blow up of $X = \mathbf{A}^n \times Z$ at $\{0\} \times Z$ (cf. the proof of Lemma 2.14). Write $\mathbf{A}^n = \mathbf{P}^n \setminus L$, and let Y be the blow-up of $0 \in \mathbf{P}^n$ and denote by E_0 the exceptional divisor. Note that L is embedded isomorphically into Y, not intersecting E_0.

We obtain the diagram:

$$\begin{array}{ccccc} E_0 \times Z & \xrightarrow{\bar{\imath}_E} & Y \times Z & \xrightarrow{q} & E_0 \times Z \\ {\scriptstyle \bar{\rho}_E}\downarrow & & \downarrow{\scriptstyle \bar{\rho}} & & \downarrow{\scriptstyle \bar{\rho}_E} \\ Z & \xrightarrow{\bar{\imath}} & \mathbf{P}^n \times Z & \xrightarrow{\pi} & Z, \end{array}$$

where $\bar{\imath} : Z = \{0\} \times Z \hookrightarrow \mathbf{P}^n \times Z$ is the closed immersion, $\bar{\rho}$ is the base change of the blow-up, π and $\bar{\rho}_E$ are the projections and q is as in Lemma 2.17. It remains to show that the following maps are

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isomorphisms:

$$\bar{\rho}^* : F_{(\mathbf{P}^n,L)\otimes\mathcal{Z}} \to \bar{\rho}_* F_{(Y,L)\otimes\mathcal{Z}}, \tag{3.1.4}$$

$$\bar{\imath}_E^* : R^j\bar{\rho}_* F_{(Y,L)\otimes\mathcal{Z}} \to \bar{\imath}_* R^j\bar{\rho}_{E*} F_{E_0\otimes\mathcal{Z}}, \quad j \geq 1. \tag{3.1.5}$$

Indeed, the restriction of these two isomorphisms to $\mathbf{A}^n_Z = \mathbf{P}^n_Z \setminus L_Z$ yields the isomorphisms (3.1.2) and (3.1.3).

The map (3.1.4) is an isomorphism away from $0 \times Z$. Since source and target of $R\pi_*$(3.1.4) are both isomorphic to $F_{\mathcal{Z}}$ by Theorem 2.18 and Lemma 2.17, (3.1.4) is an isomorphism everywhere. Similarly, (3.1.5) is an isomorphism if $R\pi_*$(3.1.5) is. To show the latter, first observe that we have:

$$R^a\pi_* R^b\bar{\rho}_*(F_{(Y,L)\otimes Z}) = 0, \quad \text{for } a \neq 0. \tag{3.1.6}$$

Indeed, if $b \geq 1$, then $R^b\bar{\rho}_* F_{(Y,L)\otimes\mathcal{Z}}$ has support in $0 \times Z$; if $b = 0$, the cohomology for $a \geq 1$ vanishes by (3.1.4) and Theorem 2.18. Now $R\pi_*$(3.1.5) is equal to the composition:

$$\pi_* R^j\bar{\rho}_* F_{(Y,L)\otimes\mathcal{Z}} \cong R^j\bar{\rho}_{E*} Rq_* F_{(Y,L)\otimes\mathcal{Z}} \cong R^j\bar{\rho}_{E*} F_{E_0\otimes\mathcal{Z}},$$

where the first isomorphism follows from (3.1.6) and the Leray spectral sequence and the second isomorphism holds by Lemma 2.17. This completes the proof. □

4. Twists

4.1. A tensor formula for homotopy invariant sheaves

Lemma 4.1 (Bloch-Gieseker). *Assume k infinite of exponential characteristic $p \geq 1$. Let X be an integral quasi-projective k-scheme and D a Cartier divisor on X. Let $n \geq 1$ be an integer with $(n, p) = 1$. Then there exists a finite and surjective morphism $\pi : Y \to X$ and a Cartier divisor E on Y, such that the following properties hold:*

(1) *Y is integral, normal and $\pi^{-1}(X_{\mathrm{sm}})$ is a smooth open subscheme of Y, where X_{sm} is the smooth locus of X;*
(2) *$\pi^* D = nE$;*
(3) *$\deg(\pi)$ divides a power of n;*
(4) *if D is effective, then so is E.*

Proof. The proof is a slight modification of [BG71, Lemma 2.1]. First note that (4) follows from (2) and (1). Also, it suffices to prove the statement for D as a very ample divisor. Let $i : X \hookrightarrow \mathbf{P}^N := \mathbf{P}$ be an immersion, such that $\mathcal{O}(D) = i^*\mathcal{O}_{\mathbf{P}}(1)$. By Bertini's theorem (see, e.g. [Jou83, Chapter I, Corollary 6.11]), we find hyperplanes $H_0, \ldots, H_N \subset \mathbf{P}$, such that all the intersections $H_{i_0} \cap \ldots \cap H_{i_r}$ and $H_{i_0} \cap \ldots \cap H_{i_r} \cap X_{\mathrm{sm}}$ are transversal (or empty), for all $\{i_0, \ldots, i_r\} \subset \{0, \ldots, N\}$ and all $0 \leq r \leq N$. Let Y_i be a linear polynomial defining H_i, so that $\mathbf{P} = \mathrm{Proj}\, k[Y_0, \ldots, Y_N]$. Let $\Pi : \mathbf{P} \to \mathbf{P}$ be the k-morphism defined by $Y_i \mapsto Y_i^n$, $i = 0, \ldots, N$. Note that Π is finite of degree n^N and it is étale over $\mathbf{P} \setminus \cup_i H_i$. Form the cartesian diagram:

$$\begin{array}{ccc} X' & \xrightarrow{i'} & \mathbf{P} \\ {\scriptstyle\pi'}\downarrow & & \downarrow{\scriptstyle\Pi} \\ X & \xrightarrow{i} & \mathbf{P}. \end{array}$$

Then $X' \times_X X_{\mathrm{sm}}$ is smooth: this can be checked after base change to the algebraic closure of k, and then the argument is the same as in the second and third paragraph in the proof of [BG71, Lemma 2.1]

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(the choice of the H_i is crucial here). Let $X'' \subset X'$ be an irreducible component (with reduced scheme structure), and denote by Y the normalisation of X'' and by $\pi : Y \to X$ the composition:

$$Y \to X'' \hookrightarrow X' \xrightarrow{\pi'} X$$

and by $E = \mathcal{O}_{\mathbf{P}}(1)_{|Y}$ the pullback of $\mathcal{O}_{\mathbf{P}}(1)$ along:

$$Y \to X'' \hookrightarrow X' \xhookrightarrow{i'} \mathbf{P}.$$

Then $\pi : Y \to X$ and E satisfy the conditions of the statement. □

Lemma 4.2. *Let $F, G \in \mathbf{PST}$. Let:*

$$\underline{\omega}^* F \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \underline{\omega}^* G \to \underline{\omega}^*(F \otimes_{\mathbf{PST}} G) \tag{4.2.1}$$

be the morphism in $\underline{\mathbf{M}}\mathbf{PST}$, which is induced by adjunction from the isomorphism:

$$\underline{\omega}_!(\underline{\omega}^* F \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \underline{\omega}^* G) \cong (\underline{\omega}_!\underline{\omega}^* F) \otimes_{\mathbf{PST}} (\underline{\omega}_!\underline{\omega}^* G) \cong F \otimes_{\mathbf{PST}} G.$$

Then we obtain a surjection in $\underline{\mathbf{M}}\mathbf{NST}$:

$$\underline{a}_{\mathrm{Nis}}((4.2.1)) : \underline{a}_{\mathrm{Nis}}(\underline{\omega}^* F \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \underline{\omega}^* G) \twoheadrightarrow \underline{a}_{\mathrm{Nis}}(\underline{\omega}^*(F \otimes_{\mathbf{PST}} G)).$$

Proof. Denote by H_l (respectively, H_r) the source (respectively, target) of (4.2.1), and take $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$. By definition, $\otimes_{\underline{\mathbf{M}}\mathbf{PST}}$ (respectively, $\otimes_{\mathbf{PST}}$) is the Day convolution of the tensor product on $\underline{\mathbf{M}}\mathbf{Cor}$ (respectively, on $\mathbf{Cor}$), so that we have the following presentations (which also hold for general (X, D), cf. [SV00, §2]):

$$H_l(X, D) = \left(\bigoplus_{\mathcal{Y}, \mathcal{Z} \in \underline{\mathbf{M}}\mathbf{Cor}} F(\mathcal{Y}^o) \otimes_{\mathbb{Z}} G(\mathcal{Z}^o) \otimes_{\mathbb{Z}} \underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z}) \right) / R_l,$$

where for $\mathcal{Y} = (\overline{Y}, Y_\infty)$, we set $\mathcal{Y}^o = \overline{Y} \setminus Y_\infty$, and where R_l is the subgroup generated by the elements:

$$f^* a \otimes g^* b \otimes h - a \otimes b \otimes (f \otimes g) \circ h,$$

where $\mathcal{Y}, \mathcal{Y}', \mathcal{Z}, \mathcal{Z}' \in \underline{\mathbf{M}}\mathbf{Cor}$, $a \in F(\mathcal{Y}^o)$, $b \in G(\mathcal{Z}^o)$, $f \in \underline{\mathbf{M}}\mathbf{Cor}(\mathcal{Y}', \mathcal{Y})$, $g \in \underline{\mathbf{M}}\mathbf{Cor}(\mathcal{Z}', \mathcal{Z})$ and $h \in \underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X}, \mathcal{Y}' \otimes \mathcal{Z}')$. Similarly,

$$H_r(X, D) = \left(\bigoplus_{Y, Z \in \mathbf{Sm}} F(Y) \otimes_{\mathbb{Z}} G(Z) \otimes_{\mathbb{Z}} \mathbf{Cor}(X \setminus D, Y \times Z) \right) / R_r,$$

where R_r is the subgroup generated by:

$$f^* a \otimes g^* b \otimes h - a \otimes b \otimes (f \times g) \circ h,$$

where $Y, Y', Z, Z' \in \mathbf{Sm}$, $a \in F(Y)$, $b \in G(Z)$, $f \in \mathbf{Cor}(Y', Y)$, $g \in \mathbf{Cor}(Z', Z)$ and $h \in \mathbf{Cor}(X \setminus D, Y' \times Z')$.

Let $\sum_i a_i \otimes b_i \otimes \gamma_i \in H_r(X, D)$, where $a_i \in F(Y_i)$, $b_i \in G(Y_i)$ and $\gamma_i \in \mathbf{Cor}(X \setminus D, Y_i \times Z_i)$. By [KMSY21a, Theorem 1.6.2], we find a proper morphism $\rho : X' \to X$ inducing an isomorphism

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$X' \setminus |\rho^* D| \xrightarrow{\simeq} X \setminus |D|$, such that the closure of any irreducible component of γ_i in $X' \times Y_i \times Z_i$ is finite over X', for all i. By (1.2.4) and Lemma 1.3, we are reduced to show the following:

Claim 4.2.1. *Assume X is henselian local of geometric type (i.e. $X = \mathrm{Spec}(\mathcal{O}^h_{X,x})$ for X integral quasi-projective k-scheme). Let $V \in \mathbf{Cor}(X \setminus D, Y \times Z)$ be a prime correspondence, such that the closure of $\overline{V} \subset X \times Y \times Z$ of V is finite over X, and let $a \in F(Y)$, $b \in G(Z)$. Then the class of $a \otimes b \otimes V$ in $H_r(X, D)$ lies in the image of $H_l(X, D) \xrightarrow{(4.2.1)} H_r(X, D)$.*

Let V be as above. Since the closure $\overline{V} \subset X \times Y \times Z$ of V is integral and finite over X, it is local. Denote by $v \in \overline{V}$ the closed point and by $y \in Y$, $z \in Z$ the images of v, respectively. We get induced maps $\mathcal{O}_{Y,y} \to \Gamma(\overline{V}, \mathcal{O}_{\overline{V}})$ and $\mathcal{O}_{Z,z} \to \Gamma(\overline{V}, \mathcal{O}_{\overline{V}})$. Hence:

$$V \subset (X \setminus D) \times U_1 \times U_2,$$

where $j_1 : U_1 \hookrightarrow Y$ and $j_2 : U_2 \hookrightarrow Z$ are open affines containing y and z, respectively. Denote by $V' \in \mathbf{Cor}(X \setminus D, U_1 \times U_2)$ the induced prime correspondence. Then $V = (j_1 \times j_2) \circ V'$, and, thus:

$$a \otimes b \otimes V = j_1^* a \otimes j_2^* b \otimes V' \quad \text{in } H_r(X, D).$$

Hence, Claim 4.2.1 follows from the following:

Claim 4.2.2. *Let $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$, let Y, Z be smooth quasi-projective k-schemes, let $V \in \mathbf{Cor}(X \setminus D, Y \times Z)$ be a prime correspondence and $a \in F(Y)$, $b \in G(Z)$. Then the class of $a \otimes b \otimes V$ in $H_r(X, D)$ lies in the image of $H_l(X, D) \xrightarrow{(4.2.1)} H_r(X, D)$.*

We prove the claim. First we reduce to k infinite by a standard trick: If k is finite, denote by $k(\ell)$ a $\mathbb{Z}_\ell$-Galois extension of k for a prime ℓ; by a trace argument, the (diagonal) pullback $H_r(X, D) \to H_r(X_{k(\ell)}, D_{k(\ell)}) \times H_r(X_{k(\ell')}, D_{k(\ell')})$ is injective for $\ell \neq \ell'$.

In the following, we assume k infinite. By assumption, we find proper modulus pairs $\mathcal{Y} = (\overline{Y}, Y_\infty)$ and $\mathcal{Z} = (\overline{Z}, Z_\infty)$, such that $\overline{Y}$ and $\overline{Z}$ are projective and $Y = \overline{Y} \setminus |Y_\infty|$ and $Z = \overline{Z} \setminus |Z_\infty|$. Since V is closed in $X \setminus |D| \times \overline{Y} \times \overline{Z}$, we find an integer n_0, such that $V \in \underline{\mathbf{M}}\mathbf{Cor}((X, n_0 D), \mathcal{Y} \otimes \mathcal{Z})$. Choose $n \geq n_0$ with $(n, p) = 1$. By Lemma 4.1, we find a modulus pair $\mathcal{Y}' = (\overline{Y}', Y'_\infty)$ together with a finite and surjective morphism $\bar{\pi}_{Y,n} : \overline{Y}' \to \overline{Y}$, such that $\deg \bar{\pi}_{Y,n}$ divides a power of n and $\bar{\pi}^*_{Y,n}(Y_\infty) = nY'_\infty$, similarly for $\overline{Z}$. Denote by $\pi_{Y,n} : Y' \to Y$ the induced finite and surjective morphism in $\mathbf{Sm}$ and by $\pi^t_{Y,n} \in \mathbf{Cor}(Y, Y')$ the correspondence induced by the transpose of the graph. In $H_r(X, D)$, we obtain:

$$\begin{aligned} \deg(\pi_{Y,n}) \deg(\pi_{Z,n}) \cdot (a \otimes b \otimes V) &= \pi_{Y,n*} \pi^*_{Y,n} a \otimes \pi_{Z,n*} \pi^*_{Z,n} b \otimes V \\ &= (\pi^t_{Y,n})^* \pi^*_{Y,n} a \otimes (\pi^t_{Z,n})^* \pi^*_{Z,n} b \otimes V \\ &= \pi^*_{Y,n} a \otimes \pi^*_{Z,n} b \otimes (\pi^t_{Y,n} \times \pi^t_{Z,n}) \circ V. \end{aligned}$$

Observe that the components of $(\pi^t_{Y,n} \times \pi^t_{Z,n}) \circ V \in \mathbf{Cor}(X \setminus D, Y' \times Z')$ are the irreducible components of:

$$V \times_{Y \times Z} (Y' \times Z') = (\mathrm{id}_{X \setminus |D|} \times \pi_{n,Y} \times \pi_{n,Z})^{-1}(V).$$

Let W be such a component, it comes with a finite and surjective map $W \to V$. Denote by $\overline{V} \subset X \times \overline{Y} \times \overline{Z}$ and $\overline{W} \subset X \times \overline{Y}' \times \overline{Z}'$ the closure of V and W, respectively, and denote by $\tilde{V} \to \overline{V}$ and $\tilde{W} \to \overline{W}$ the normalisations. Since $\overline{W}$ is contained in $\overline{V} \times_{\overline{Y} \times \overline{Z}} (\overline{Y}' \times \overline{Z}')$, the natural maps from $\tilde{W}$ to $\overline{Y}$ and $\overline{Z}$ factor via a morphism $\tilde{W} \to \tilde{V}$. We obtain:

$$nD_{|\tilde{W}} \geq n_0 D_{|\tilde{W}} \geq Y_{\infty|\tilde{W}} + Z_{\infty|\tilde{W}} = nY'_{\infty|\tilde{W}} + nZ'_{\infty|\tilde{W}},$$

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where the second inequality follows from $V \in \underline{\mathbf{M}}\mathbf{Cor}((X, n_0 D), \mathcal{Y} \otimes \mathcal{Z})$. Hence:

$$(\pi^t_{Y,n} \times \pi^t_{Z,n}) \circ V \in \underline{\mathbf{M}}\mathbf{Cor}((X, D), \mathcal{Y}' \otimes \mathcal{Z}').$$

It follows that $\delta_n \cdot (a \otimes b \otimes V)$ lies in the image of $H_l(X, D) \to H_r(X, D)$, where $\delta_n := \deg(\pi_{Y,n}) \deg(\pi_{Z,n})$. Choose $r \geq n_0$ with $(r, p) = 1 = (r, n)$. Since δ_n divides a power of n and δ_r divides a power of r, we find integers s, t with:

$$a \otimes b \otimes V = s\delta_n \cdot (a \otimes b \otimes V) + t\delta_r \cdot (a \otimes b \otimes V).$$

This proves Claim 4.2.2, and, hence, also the lemma. □

Proposition 4.3. *For $F_1, \ldots, F_n \in \mathbf{HI}_{\mathrm{Nis}}$, consider the map:*

$$\underline{\omega}^* F_1 \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \cdots \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \underline{\omega}^* F_n \to \underline{\omega}^*(F_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} F_n) \to \underline{\omega}^*(F_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} F_n), \quad (4.3.1)$$

where the first map is induced by (4.2.1) and the associativity of $\otimes_{\underline{\mathbf{M}}\mathbf{PST}}$ and $\otimes_{\mathbf{PST}}$ and the second map is induced by the natural surjective map (cf. (1.6.2)):

$$F_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} F_n \to h^{\mathbf{A}^1}_{0,\mathrm{Nis}}(F_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} F_n) := F_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} F_n,$$

where we use the notation from 1.4 and $\otimes_{\mathbf{HI}_{\mathrm{Nis}}}$ denotes the monoidal structure on $\mathbf{HI}_{\mathrm{Nis}}$ defined by Voevodsky. Then, (4.3.1) induces an isomorphism:

$$h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\underline{\omega}^* F_1 \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \cdots \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \underline{\omega}^* F_n) \xrightarrow{\simeq} \underline{\omega}^*(F_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} F_n). \quad (4.3.2)$$

Proof. We begin by recalling from [MS20, Proposition 3.2] that for $F, G \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, the formula $F \otimes_{\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}} G = \tau_! h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\tau^* F \otimes_{\mathbf{MPST}} \tau^* G)$ defines a symmetric monoidal structure on $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. Next, note that $\underline{\omega}^* H \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ for $H \in \mathbf{HI}_{\mathrm{Nis}}$ by [KSY22, Lemma 2.3.1] and [KMSY21b, Proposition 6.2.1b)]. Moreover:

$$\begin{aligned} h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\underline{\omega}^* F_1 \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \underline{\omega}^* F_2) &= h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\tau_! \omega^* F_1 \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \tau_! \omega^* F_2) \\ &= \tau_! h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\omega^* F_1 \otimes_{\mathbf{MPST}} \omega^* F_2) \\ &= \tau_! h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\tau^*(\underline{\omega}^* F_1) \otimes_{\mathbf{MPST}} \tau^*(\underline{\omega}^* F_2)) = \underline{\omega}^* F_1 \otimes_{\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}} \underline{\omega}^* F_2, \end{aligned}$$

for every $F_1, F_2 \in \mathbf{HI}_{\mathrm{Nis}}$. Here, the isomorphisms follow from (1.1.3) and the exactness of $\tau_!$.

We now observe that the functor $\underline{\omega}^*$ is lax monoidal from **PST** to $\underline{\mathbf{M}}\mathbf{PST}$ (this follows from the fact that $\underline{\omega}^*$ is right adjoint to $\underline{\omega}_!$, which is strict monoidal by construction). By applying $h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}$ to (4.3.1), we obtain the functorial map (4.3.2), which we can rewrite for $n = 2$ as:

$$\underline{\omega}^* F_1 \otimes_{\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}} \underline{\omega}^* F_2 \to \underline{\omega}^*(F_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} F_2). \quad (4.3.3)$$

In particular, $\underline{\omega}^*$ restricts to a lax symmetric monoidal functor from $\mathbf{HI}_{\mathrm{Nis}}$ to $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, and the statement of the proposition is equivalent to the fact $\underline{\omega}^*$ is in fact (strictly) monoidal, that is, that the map (4.3.3) is an isomorphism (note that the identity for the tensor product is simply the constant sheaf $\mathbb{Z}$ and that $\underline{\omega}^* \mathbb{Z} = \mathbb{Z}$). Since the tensor products in $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathbf{HI}_{\mathrm{Nis}}$ are, in particular, associative, it is enough to prove the claim when $n = 2$.

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By Lemma 4.2, the map (4.3.3) (or, equivalently, (4.3.2)) is surjective. On the other hand, we have:

$$\begin{aligned}\underline{\omega}_! h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\underline{\omega}^* F_1 \otimes_{\underline{\mathbf{MPST}}} \underline{\omega}^* F_2) &= a^V_{\mathrm{Nis}} \underline{\omega}_! \underline{h}^{\overline{\square}}_0(\underline{\omega}^* F_1 \otimes_{\underline{\mathbf{MPST}}} \underline{\omega}^* F_2)^{\mathrm{sp}} \\ &= a^V_{\mathrm{Nis}} \underline{\omega}_! \underline{h}^{\overline{\square}}_0 \tau_! (\omega^* F_1 \otimes_{\underline{\mathbf{MPST}}} \omega^* F_2) \\ &= a^V_{\mathrm{Nis}} \underline{\omega}_! \tau_! h^{\overline{\square}}_0(\omega^* F_1 \otimes_{\underline{\mathbf{MPST}}} \omega^* F_2) \\ &= a^V_{\mathrm{Nis}} \omega_! h^{\overline{\square}}_0(\omega^* F_1 \otimes_{\underline{\mathbf{MPST}}} \omega^* F_2),\end{aligned}$$

where the first equality follows from the definition of $h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}$ (cf. (1.4.2)) and $\underline{\omega}_! \underline{a}_{\mathrm{Nis}} = a^V_{\mathrm{Nis}} \underline{\omega}_!$ (cf. (1.2.2)), the second holds by the fact $\underline{\omega}_! A^{\mathrm{sp}} = \underline{\omega}_! A$ for $A \in \underline{\mathbf{MPST}}$ and $\underline{\omega}^* = \tau_! \omega^*$ (cf. (1.1.3)) and the monoidality of $\tau_!$, the third follows from $\underline{h}^{\overline{\square}}_0(\tau_! B) = \tau_! h^{\square}_0(B)$ for $B \in \mathbf{MPST}$, where $h^{\square}_0(B) \in \mathbf{MPST}$ is the maximal cube invariant quotient of B defined by the same way as (1.4.3) and the last holds by $\underline{\omega}_! \tau_! = \omega_!$ (cf. (1.1.3)). Thus, $\underline{\omega}_!$(4.3.2) is an isomorphism by [RSY22, Theorem 5.3], in view of $\omega^* F = \tilde{F}$ (see [RSY22, (3.14.5)]) by [KSY22, Lemma 2.3.1]. Since both sides of (4.3.2) are semipure, the map (4.3.2) is injective as well. □

4.2. Definition and basic properties of twists

Definition 4.4 (see [MS20, §2]). Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. We define $\gamma^n F$ and $F(n)$, $n \geq 0$, recursively by:

$$\gamma^0 F := F, \quad \gamma^1 F := \gamma F := \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\underline{\omega}^* \mathbf{G}_m, F), \quad \gamma^n F := \gamma(\gamma^{n-1} F)$$

and:

$$F(0) := F, \quad F(1) := h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(F \otimes_{\underline{\mathbf{MPST}}} \underline{\omega}^* \mathbf{G}_m), \quad F(n) := F(n-1)(1).$$

Corollary 4.5. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. Then $\gamma^n F$, $F(n) \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, for all $n \geq 0$. Furthermore,*

$$\gamma^n F = \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}((\overline{\square}^{(1)}_{\mathrm{red}})^{\otimes_{\underline{\mathbf{MPST}}} n}, F) = \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\underline{\omega}^* K^M_n, F), \tag{4.5.1}$$

and:

$$F(n) = h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(F \otimes_{\underline{\mathbf{MPST}}} (\overline{\square}^{(1)}_{\mathrm{red}})^{\otimes_{\underline{\mathbf{MPST}}} n}) = h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(F \otimes_{\underline{\mathbf{MPST}}} \underline{\omega}^* K^M_n), \tag{4.5.2}$$

where:

$$\overline{\square}^{(1)}_{\mathrm{red}} := \mathrm{Coker}(\mathbb{Z}_{\mathrm{tr}}(\{1\}) \to \mathbb{Z}_{\mathrm{tr}}(\mathbf{P}^1, 0 + \infty)) \in \underline{\mathbf{MPST}}^{\tau} \tag{4.5.3}$$

and K^M_n is the improved Milnor K-theory from [Ker10] (there denoted by $\hat{K}^M_n$).

Proof. For a proper modulus pair $\mathcal{X}$, we have $\tau_! \tau^* \mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) = \mathbb{Z}_{\mathrm{tr}}(\mathcal{X})$. It follows that $\overline{\square}^{(1)}_{\mathrm{red}} \in \underline{\mathbf{MPST}}^{\tau}$. By Lemma 2.4, we have $h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\overline{\square}^{(1)}_{\mathrm{red}}) = \underline{\omega}^* \mathbf{G}_m$. Thus:

$$\gamma F = \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\overline{\square}^{(1)}_{\mathrm{red}}, F). \tag{4.5.4}$$

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Indeed (we drop the index **MPST** from Hom and $\underline{\mathrm{Hom}}$):

$$\begin{aligned}\underline{\mathrm{Hom}}(\overline{\square}^{(1)}_{\mathrm{red}}, F)(\mathcal{X}) &= \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) \otimes \overline{\square}^{(1)}_{\mathrm{red}}, F)\\ &= \mathrm{Hom}(\overline{\square}^{(1)}_{\mathrm{red}}, \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), F))\\ &= \mathrm{Hom}(h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\overline{\square}^{(1)}_{\mathrm{red}}), \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), F))\\ &= \underline{\mathrm{Hom}}(h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\overline{\square}^{(1)}_{\mathrm{red}}), F)(\mathcal{X})\\ &= \gamma F(\mathcal{X}),\end{aligned}$$

where the third equality holds by Lemma 1.5(1), (2). This implies the first equality in (4.5.1) and also that $\gamma^n F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, for all $n \geq 0$ (by Lemma 1.5(1)). For the second equality in (4.5.1), first note that it follows from [Ker10] and results by Voevodsky (see [RSY22, 5.5]), that we have:

$$K^M_n \cong \mathbf{G}_m^{\otimes_{\mathbf{HI}_{\mathrm{Nis}}} n} \in \mathbf{HI}_{\mathrm{Nis}}\,. \tag{4.5.5}$$

Hence, by Proposition 4.3 and [MS20, Lemma 1.14(iii)], we obtain:

$$\underline{\omega}^* K^M_n = h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}((\underline{\omega}^*\mathbf{G}^m)^{\otimes_{\underline{\mathbf{MPST}}} n}) = h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}((\overline{\square}^{(1)}_{\mathrm{red}})^{\otimes_{\underline{\mathbf{MPST}}} n}). \tag{4.5.6}$$

Thus, the second equality in (4.5.1) follows from the adjunction (1.4.2). The equalities in (4.5.2) follow similarly. □

Remark 4.6. By Corollary 4.5, the twist $\gamma^n F$ (respectively, $F(n)$) agrees with the definition in [MS20, (2.3)] (respectively, [MS20, after Proposition 3.2]).

Remark 4.7. Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} \in \underline{\mathbf{MCor}}_{ls}$. By (4.5.4) and [Sai20a, Lemma 5.9], we have:

$$\gamma^1 F(\mathcal{X}) = \frac{F((\mathbf{P}^1, 0+\infty) \otimes \mathcal{X})}{F((\mathbf{P}^1, \infty) \otimes \mathcal{X})} = \frac{F((\mathbf{A}^1, 0) \otimes \mathcal{X})}{F(\mathbf{A}^1 \otimes \mathcal{X})}.$$

4.8. For later use in section 8, we define certain maps induced by adjunction. Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. For $n \geq 0$, we have an adjunction map:

$$F \to \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\underline{\omega}^* K^M_n, F \otimes_{\underline{\mathbf{MPST}}} \underline{\omega}^* K^M_n), \tag{4.8.1}$$

which sends $a \in F(\mathcal{X})$ to (we drop the subscript **MPST**):

$$a \otimes \mathrm{id} \in \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) \otimes \underline{\omega}^* K^M_n, F \otimes \underline{\omega}^* K^M_n),$$

where we identify an element $a \in F(\mathcal{X})$ with the map $a : \mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) \to F$. Composing (4.8.1) with the map induced by the natural map $F \otimes \underline{\omega}^* K^M_n \to h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(F \otimes \underline{\omega}^* K^M_n)$, Corollary 4.5 yields a map:

$$\kappa_n : F \to \gamma^n(F(n)) =: \gamma^n F(n), \tag{4.8.2}$$

which by Remark 4.6, coincides with the morphism [MS20, (3.5)]. Note that κ_0 is the identity and that for $m, n \geq 0$ the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\kappa_{m+n}} & \gamma^{m+n}(F(m+n)) \\ {\scriptstyle \kappa_m}\downarrow & & \| \\ \gamma^m(F(m)) & \xrightarrow{\gamma^m \kappa_n} & \gamma^m\gamma^n(F(m)(n)). \end{array} \tag{4.8.3}$$

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5. Cup product with Chow cycles with support

5.1. Milnor K-theory and intersection theory with supports

Everything in this subsection is well-known, however, we give some explanations for lack of reference.

5.1. Recall that a *family of supports* on a scheme X is a nonempty collection Φ of closed subsets of X which is stable under taking finite unions and closed subsets. The main examples are the family Φ_Z, for a closed subset $Z \subset X$, which consists of all the closed subsets in Z, the family $\Phi^{\geq c}$ of all closed subsets of codimension $\geq c$ and the family $\Phi^{\mathrm{prop}}_{X/S}$, for a morphism $X \to S$, which consists of all closed subsets in X which are proper over S. If F is a sheaf on X and Φ is a family of support, then:

$$\Gamma_\Phi(X, F) = \{s \in F(X) \mid \mathrm{supp}(s) \in \Phi\} = \varinjlim_{Z \in \Phi} \Gamma_Z(X, F),$$

and $\underline{\Gamma}_\Phi(F)(U) = \Gamma_{\Phi \cap U}(U, F)$, for an open $U \subset X$. For a morphism $f : Y \to X$, we denote by $f^{-1}\Phi$ the smallest family with supports on Y containing all closed subsets of the form $f^{-1}(Z)$, $Z \in \Phi$.

Let X be k-scheme. We denote by $\mathrm{CH}_i(X)$ the Chow group of i-dimensional cycles on X. If X is equidimensional of dimension d, we denote by $\mathrm{CH}^i(X)$ the Chow group of i-codimensional cycles on X, that is, $\mathrm{CH}^i(X) = \mathrm{CH}_{d-i}(X)$. If Φ is a family of supports on X, we set:

$$\mathrm{CH}^i_\Phi(X) = \varinjlim_{Z \in \Phi} \mathrm{CH}_{d-i}(Z), \tag{5.1.1}$$

where the transition maps in the directed limit are given by pushforward along closed immersions. Note that for a closed subset $Z \subset X$, we have:

$$\mathrm{CH}^i_Z(X) := \mathrm{CH}^i_{\Phi_Z}(X) = \mathrm{CH}_{d-i}(Z),$$

in particular, $\mathrm{CH}^i_X(X) = \mathrm{CH}^i(X)$. The notation $\mathrm{CH}^i_Z(X)$ is not superfluous, since if Z is singular, the pullback along the refined Gysin homomorphism as in [Ful98, §6] relies on the embedding $Z \hookrightarrow X$.

5.2. We recall some facts on the relation between Milnor K-theory and intersection theory. Let K^M_i be the improved Milnor K-sheaf from [Ker10]. Its restriction to **Sm** is homotopy invariant, and, hence, for $X \in \mathbf{Sm}$ its restriction to $(\text{ét}/X)$ is a Nisnevich sheaf denoted by $K^M_{i,X}$, and we have:

$$R\varepsilon_* K^M_{i,X} = \varepsilon_* K^M_{i,X}, \tag{5.2.1}$$

where $\varepsilon : X_{\mathrm{Nis}} \to X_{\mathrm{Zar}}$ denotes the canonical morphism of sites (see [Voe00b, Theorem 3.1.12]. If Z is a finite-type k-scheme, we denote by $C_\bullet(i)(Z)$ the degree i (homological) Gersten complex of $K^M_{*,Z}$ (e.g. [Ros96, Section 5]), that is:

$$C_n(i)(Z) = \bigoplus_{z \in Z_{(n)}} K^M_{n+i}(z),$$

and the differentials are induced by the tame symbol (for the tame symbol, we use the sign convention from [Ros96, p.328]). Recall that the formation $Z \mapsto C_\bullet(i)(Z)$ is covariant functorial with respect to proper maps and contravariant functorial with respect to quasi-finite flat maps (see [Ros96, Proposition (4.6)]. The assignment $U \mapsto C_\bullet(i)(U)$ defines a complex of sheaves on Z_{Nis} which we denote by $C_{\bullet,Z}(i)$. If Z is equidimensional of dimension e, then we define:

$$C^n_Z(i) := C_{e-n,Z}(i - e) \tag{5.2.2}$$

and obtain the cohomological degree i Gersten complex $C^\bullet_Z(i)$, the global sections of which we also denote by $C^\bullet(i)(Z)$.

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In the following, we assume $X \in \mathbf{Sm}$ is equidimensional. By [Ker10, Proposition 10(8)], the Gersten complex is a resolution on the Nisnevich site for the sheaf $K^M_{i,X}$, that is:

$$K^M_{i,X} \xrightarrow{\simeq} C^\bullet_X(i) \quad \text{in } D(X_{\mathrm{Nis}}).$$

Note that $C^\bullet_X(i)$ sits in cohomological degree $[0, i]$. By (5.2.1) and since $\varepsilon_* C^\bullet_X(i)$ is a flasque resolution of $\varepsilon_* K^M_{i,X}$, we can use $C^\bullet_X(i)$ to compute Nisnevich cohomology with supports of $K^M_{i,X}$. If $\dim X = d$ and $\iota : Z \hookrightarrow X$ is a closed immersion, then:

$$\underline{\Gamma}_Z C^n_X(i) = \iota_* C_{d-n,Z}(i-d).$$

This gives rise to Bloch's formula (with support):

$$\mathrm{CH}^i_\Phi(X) = H^i_{\Phi,\mathrm{Zar}}(X, K^M_i) = H^i_{\Phi,\mathrm{Nis}}(X, K^M_i) =: H^i_\Phi(X, K^M_i), \tag{5.2.3}$$

where Φ is a family of supports on X.

Lemma 5.3. *Let $f : Y \to X$ be a morphism between equidimensional smooth schemes, and let Φ be a family of supports on X. The following diagram commutes:*

$$\begin{array}{ccc} H^i_\Phi(X, K^M_i) & \xrightarrow{(5.2.3)} & \mathrm{CH}^i_\Phi(X) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ H^i_{f^{-1}\Phi}(Y, K^M_i) & \xrightarrow{(5.2.3)} & \mathrm{CH}^i_{f^{-1}\Phi}(Y), \end{array} \tag{5.3.1}$$

where the pullback on the right is induced by the refined Gysin homomorphism in [Ful98, 6.6] (see also [CR11, 1.1.30]) and the pullback on the left is induced from the sheaf structure of K^M_i on the category of schemes.

Proof. In [Ros96, p. 12], a morphism of complexes:

$$I(f) : C^\bullet(i)(X) \to C^\bullet(i)(Y) \tag{5.3.2}$$

is defined, depending on the choice of a coordination of the tangent bundle TX of X (see [Ros96, §9] for the definition of a coordination).

It is compatible with the pullback $f^* : K^M_i(X) \to K^M_i(Y)$ (by [Ros96, Proposition 12.3 and Corollary 12.4]. Furthermore, if $u : U \to X$ is étale, a coordination of TX induces by pullback a coordination of TU, and, hence, it is direct to check that we have:

$$u^*_Y \circ I(f) = I(f_U) \circ u^* : C^\bullet(i)(X) \to C^\bullet(i)(Y_U),$$

where $u_Y : Y_U \to Y$ is the base change of u along f and $f_U : Y_U \to U$ is the base change of f along u. It follows that the choice of a coordination on TX allows one to promote (5.3.2) to a morphism of complexes of sheaves on X_{Nis}:

$$\mathcal{I}(f) : C^\bullet(i)_X \to f_* C^\bullet(i)_Y, \tag{5.3.3}$$

which is compatible with the pullback $f^* : K^M_{i,X} \to f_* K^M_{i,Y}$. In view of 5.2, taking sections with support $\Gamma_\Phi(X, -)$ and then cohomology, gives a map:

$$H^i(\Gamma_\Phi(X, \mathcal{I}(f))) : H^i_\Phi(X, K^M_i) \to H^i_{f^{-1}\Phi}(Y, K^M_i)$$

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that we identify with the left vertical map in (5.3.1). Consider the following diagram of solid arrows:

$$\begin{array}{ccccc} C^i(i)(X) & \hookleftarrow & C_{d-i}(i-d)(Z) & \twoheadrightarrow & \mathrm{CH}_{d-i}(Z) \\ \downarrow{\scriptstyle I(f)} & & \downarrow{\scriptstyle I_Z(f)} & & \downarrow{\scriptstyle f^!} \\ C^i(i)(Y) & \hookleftarrow & C_{e-i}(i-e)(f^{-1}(Z)) & \twoheadrightarrow & \mathrm{CH}_{e-i}(f^{-1}(Z)), \end{array} \tag{5.3.4}$$

where $Z \in \Phi$, $d = \dim X$, $e = \dim Y$ and $f^!$ is the refined Gysin map from [Ful98, 6.6]. It remains to show that there exists a dotted arrow $I_Z(f)$ making the diagram commute. Since the pushforward on $C^\bullet(i)$ is compatible with the one on Chow groups, we can assume that Z is integral with $\dim Z = d - i$, that is, $C_{d-i}(i-d)(Z) = \mathbb{Z} \cdot [Z]$. By definition of $I(f)$ and $f^!$, it suffices to consider the case where $f = i : Y \hookrightarrow X$ is a regular closed immersion defined by a coherent ideal sheaf J. Denote by $N_{Y/X} = \mathrm{Spec}(\oplus_{n\geq 0} J^n/J^{n+1})$ the normal bundle over Y, and fix a coordination τ of $N_{Y/X}$ in the sense of [Ros96, Section 9, p.371]. Set $Z' := Z \times_X Y$ and $N := N_{Y/X} \times_Y Z'$; the pullback of τ along $Z' \hookrightarrow Y$ induces a coordination τ' of N. Denote by $C_{Z'/Z}$ the normal cone of $Z' \hookrightarrow Z$ and by $\nu : C_{Z'/Z} \hookrightarrow N$ the closed immersion induced by $J \otimes_{\mathcal{O}_X} \mathcal{O}_Z \twoheadrightarrow J\mathcal{O}_Z$. Note that $C_{Z'/Z}$ has pure dimension $d - i$ (see [Ful98, B 6.6], and, thus:

$$C_{d-i}(i-d)(C_{Z'/Z}) = \bigoplus_{z \in (C_{Z'/Z})^{(0)}} \mathbb{Z}.$$

With the notation from [Ros96, 9, 11.], we define $I_Z(f)$ to be the composition:

$$\begin{aligned} I_Z(f) : C_{d-i}(i-d)(Z) &\xrightarrow{J(Z,Z')} C_{d-i}(i-d)(C_{Z'/Z}) \\ &\xrightarrow{\nu_*} C_{d-i}(i-d)(N) \xrightarrow{r(\tau')} C_{e-i}(i-e)(Z'). \end{aligned}$$

Let $D(Z, Z') \to \mathbf{A}^1 = \mathrm{Spec}\, k[t]$ be the deformation scheme from [Ros96, 10], so that $D(Z,Z')_{|\mathbf{A}^1 \setminus \{0\}} = Z \times (\mathbf{A}^1 \setminus \{0\})$ and $D(Z,Z')_{|0} = C_{Z'/Z}$. Then by definition (see [Ros96, 11]):

$$J(Z,Z')([Z]) = \mathrm{div}_{D(Z,Z')}(t)_{|C_{Z'/Z}} = [C_{Z'/Z}].$$

where $[C_{Z'/Z}]$ denotes the cycle associated to the scheme $C_{Z'/Z}$, [Ful98, Chapter 1, paragraph 5]. Thus, the map $J(Z', Z)$ corresponds to the specialisation map $\sigma : \mathrm{CH}_{d-i}(Z) \to \mathrm{CH}_{d-i}(C_{Z'/Z})$ from [Ful98, 5.2]. Therefore, the above definition of $I_Z(f)$ makes the square on the right in (5.3.4) commutative (by the alternative description of $i^!$ on the Chow side in [Ful98, 6.2, 2nd paragraph on p.98]). We subdivide the left square as follows:

$$\begin{array}{ccccc} C_{d-i}(i-d)(Z) & & \xrightarrow{\nu_* \circ J(Z,Z')} & C_{d-i}(i-d)(N) & \xrightarrow{r(\tau')} C_{e-i}(i-e)(Z') \\ \downarrow & & & \downarrow & \downarrow \\ C^i(i)(X) & \xrightarrow{J(X,Y)} & C^i(i)(N_{Y/X}) & \xrightarrow{r(\tau)} & C^i(i)(Y), \end{array} \tag{5.3.5}$$

$$\text{(curved arrow } C^i(i)(X) \to C^i(i)(Y) \text{ labelled } I(f)\text{)}$$

the vertical maps are all induced by pushforward along the respective closed immersion. It follows directly from the definition of the maps $r(\tau)$ and $r(\tau')$ in [Ros96, (9.1)–(9.4)] that the right square of (5.3.5) commutes. For the left square, note that $D(Z, Z')$ is an integral and closed subscheme of $D(X, Y)$, hence, it is the closure of $Z \times (\mathbf{A}^1 \setminus \{0\})$ in $D(X, Y)$; furthermore,

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$D(Z, Z') \cap N_{Y/X} = D(Z, Z') \cap N = C_{Z'/Z}$. Thus, by definition:

$$J(X, Y)([Z]) = [C_{Z'/Z}] \quad \text{in } C^i(i)(N_{Y/X}).$$

This yields the commutativity of the left square in (5.3.5). □

5.4. In [Ros96, 14.1], a cross product:

$$C^p(i)(X) \times C^q(j)(Y) \to C^{p+q}(i+j)(X \times Y), \quad (a, b) \mapsto a \times b$$

is defined by sending (a_x, b_y), where $a_x \in K^M_{i-p}(x)$, $x \in X^{(p)}$ and $b_y \in K^M_{j-q}(y)$, $y \in Y^{(q)}$, to:

$$\oplus_{z \in (x \times y)^{(0)}} l_z \, a_{x|z} \cdot b_{y|z}, \tag{5.4.1}$$

where $x \times y$ denotes the fibre product of k-schemes, l_z denotes the length of the local ring of $x \times y$ at z and $a_{x|z} \in K^M_{i-p}(z)$ denotes the pullback of a_x to z and similarly with $b_{y|z}$. Note that $z \in (x \times y)^{(0)}$ implies $z \in (X \times Y)^{(p+q)}$. By [Ros96, (14.4)], we have:

$$d(a \times b) = (da) \times b + (-1)^{i-p}(a \times db), \tag{5.4.2}$$

where d denotes the differential of the Gersten complex (there seems to be a typo in the formula in *loc. cit.*: the $(-1)^n$ in that formula should be a $(-1)^{n+p}$ as follows from what is said in the proof and [Ros96, R3f and R3d]; this formula is for the homological notation, if one translates to the cohomological notation via (5.2.2), one obtains (5.4.2)). We have to modify the cross product to obtain a morphism of complexes (with the usual sign convention for a tensor product of complexes). For $a \in C^p(i)(X)$ and $b \in C^q(j)(Y)$, we set:

$$a \boxtimes b := (-1)^{i(q+j)} a \times b. \tag{5.4.3}$$

Then, we obtain:

$$d(a \boxtimes b) = (da) \boxtimes b + (-1)^p \, a \boxtimes db.$$

Thus, $\boxtimes$ induces a morphism of complexes:

$$\boxtimes : \mathrm{tot}(C^\bullet(i)(X) \otimes_{\mathbb{Z}} C^\bullet(j)(Y)) \to C^\bullet(i+j)(X \times Y), \tag{5.4.4}$$

which, via the augmentation from Milnor K-theory, is compatible with:

$$K^M_i(X) \otimes_{\mathbb{Z}} K^M_j(Y) \to K^M_{i+j}(X \times Y), \quad a \otimes b \mapsto \pi_X^* a \cdot \pi_Y^* b, \tag{5.4.5}$$

with $\pi_X : X \times Y \to X$ the projection. In degree $i + j$, the map (5.4.4) is given by:

$$\Big(\bigoplus_{x \in X^{(i)}} \mathbb{Z} \cdot x\Big) \otimes_{\mathbb{Z}} \Big(\bigoplus_{y \in Y^{(j)}} \mathbb{Z} \cdot y\Big) \to \bigoplus_{z \in X \times Y^{(i+j)}} \mathbb{Z} \cdot z,$$

$$x \otimes y \mapsto \bigoplus_{z \in (x \times y)^{(0)}} (-1)^{i(j+j)} l_z \cdot z = \bigoplus_{z \in (x \times y)^{(0)}} l_z \cdot z.$$

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Hence, for families of support Φ on X and Ψ on Y, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{CH}^i_\Phi(X) \times \mathrm{CH}^j_\Psi(Y) & \xrightarrow{\boxtimes} & \mathrm{CH}^{i+j}_{\Phi\times\Psi}(X\times Y) \\ \downarrow \simeq & & \downarrow \simeq \\ H^i_\Phi(X, K^M_i) \times H^j_\Psi(Y, K^M_j) & \xrightarrow{\boxtimes} & H^{i+j}_{\Phi\times\Psi}(X\times Y, K^M_{i+j}), \end{array} \tag{5.4.6}$$

where the upper horizontal map is the exterior product of cycles (see [Ful98, 1.10]) and $\Phi \times \Psi$ denotes the smallest family of supports containing $Z_1 \times Z_2$, for all $Z_1 \in \Phi$ and $Z_2 \in \Psi$. We note that if $\tau : X \times Y \to Y \times X$ is the switching morphism, then:

$$\tau_*(a \boxtimes b) = (-1)^{ij+pq}(b \boxtimes a), \quad a \in C^p(i)(X), b \in C^q(j)(Y), \tag{5.4.7}$$

as follows directly from (5.4.1) and (5.4.3). The above and Lemma 5.3 imply that the intersection product with support:

$$\Delta^* \circ \boxtimes : \mathrm{CH}^i_\Phi(X) \times \mathrm{CH}^j_\Psi(X) \to \mathrm{CH}^{i+j}_{\Phi\cap\Psi}(X) \tag{5.4.8}$$

from [Ful98, 8] corresponds via Bloch's formula to:

$$\Delta^* \circ \boxtimes : H^i_\Phi(X, K^M_i) \times H^j_\Psi(X, K^M_j) \to H^{i+j}_{\Phi\cap\Psi}(X, K^M_{i+j}), \tag{5.4.9}$$

where $\Phi \cap \Psi = \{Z_1 \cap Z_2 \mid Z_1 \in \Phi, Z_2 \in \Psi\}$.

5.2. Cupping

5.5. Let $F, G \in \underline{\mathbf{MNST}}$, and let X be a k-scheme and D and E effective Cartier divisors on it, such that $(X, D), (X, E) \in \underline{\mathbf{MCor}}$. We recall that there is a natural morphism of Nisnevich sheaves on X:

$$F_{(X,D)} \otimes_{\mathbb{Z}} G_{(X,E)} \to (F \otimes_{\underline{\mathbf{MNST}}} G)_{(X,D+E)}, \tag{5.5.1}$$

which is defined as follows: For $U \to X$, we have a surjection (see the proof of Lemma 4.2):

$$\pi : \bigoplus_{\mathcal{Y},\mathcal{Z}\in\underline{\mathbf{MCor}}} F(\mathcal{Y}) \otimes_{\mathbb{Z}} G(\mathcal{Z}) \otimes_{\mathbb{Z}} \underline{\mathbf{MCor}}((U, (D+E)_U), \mathcal{Y} \otimes \mathcal{Z}) \twoheadrightarrow (F \otimes_{\underline{\mathbf{MPST}}} G)(U, (D+E)_U).$$

Composition with π gives then a morphism:

$$F(U, D_U) \otimes G(U, E_U) \to (F \otimes_{\underline{\mathbf{MPST}}} G)(U, (D+E)_U), \tag{5.5.2}$$

$$a \otimes b \mapsto \pi(a \otimes b \otimes \Delta_U),$$

where $\Delta_U \in \underline{\mathbf{MCor}}((U, (D+E)_U), (U, D_U) \otimes (U, E_U))$ is the diagonal (note that it is indeed an admissible correspondence). If we now compose (5.5.2) with the value on $(U, (D+E)_U)$ of the natural map (the sheafification):

$$(F \otimes_{\underline{\mathbf{MPST}}} G) \to \underline{a}_{\mathrm{Nis}}(F \otimes_{\underline{\mathbf{MPST}}} G) = F \otimes_{\underline{\mathbf{MNST}}} G,$$

we get the desired map (5.5.1).

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Lemma 5.6. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and X, D, E as in 5.5. Assume X is connected. Consider the map:*

$$(\gamma^1 F(X, D)) \otimes_{\mathbb{Z}} \mathbf{G}_m(X \setminus E) \to F(X, D + E) \tag{5.6.1}$$

defined as composition:

$$(\gamma^1 F(X, D)) \otimes_{\mathbb{Z}} \mathbf{G}_m(X \setminus E) \xrightarrow{(5.5.1)} (\gamma^1 F \otimes_{\underline{\mathbf{MNST}}} \underline{\omega}^* \mathbf{G}_m)(X, D + E) \xrightarrow{adj.} F(X, D + E),$$

where the morphism 'adj.' is induced by the counit of the adjunction $(-) \otimes_{\underline{\mathbf{MPST}}} \underline{\omega}^* \mathbf{G}_m \dashv \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\underline{\omega}^* \mathbf{G}_m, -)$. Then the precomposition of (5.6.1) with the natural map:

$$F(\overline{\square}^{(1)} \otimes (X, D)) \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{(1)})(X, E) \to \gamma^1 F(X, D) \otimes_{\mathbb{Z}} \mathbf{G}_m(X \setminus E),$$

stemming from (4.5.4) and Lemma 2.4, is given by:

$$F(\overline{\square}^{(1)} \otimes (X, D)) \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{(1)})(X, E) \to F(X, D + E),$$
$$a \otimes f \mapsto \Delta_X^*((f - \deg(f) \cdot s_1) \otimes \mathrm{id}_{(X,D)})^* a,$$

where $s_1 \in \underline{\mathbf{MCor}}((X, E), \overline{\square}^{(1)})$ is the graph of $X \to \mathrm{Spec}\, k = \{1\} \hookrightarrow \mathbf{P}^1$ and $\Delta_X \in \underline{\mathbf{MCor}}((X, D + E), (X, E) \otimes (X, D))$ is the graph of the diagonal.

Proof. Note that under the identification $\mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{(1)}) = \overline{\square}^{(1)}_{\mathrm{red}} \oplus \mathbb{Z}$ (see (4.5.3)), the projection to the first factor is given by:

$$\mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{(1)})(X, E) \ni f \mapsto (f - \deg(f) \cdot s_1) \in \overline{\square}^{(1)}_{\mathrm{red}}(X, E).$$

Since by Lemma 2.4 we have $h^{\overline{\square}}_{0,\mathrm{Nis}}(\overline{\square}^{(1)}_{\mathrm{red}}) = \underline{\omega}^* \mathbf{G}_m$, the statement of the lemma is direct from the explicit description of (5.5.1) in 5.5. □

Lemma 5.7. *Let X be a scheme and $Z \subset X$ a closed subset X. Let $A, B \in D(X_{\mathrm{Nis}})$, and assume that the cohomology sheaves $H^i(B)$ have support in Z, for all $i \in \mathbb{Z}$. Then the natural map:*

$$R\underline{\Gamma}_Z(A \otimes^L_{\mathbb{Z}} B) \xrightarrow{\simeq} A \otimes^L_{\mathbb{Z}} B$$

is an isomorphism. In particular, for any $C \in D(X_{\mathrm{Nis}})$, we obtain the canonical morphism:

$$A \otimes^L_{\mathbb{Z}} R\underline{\Gamma}_Z C \cong R\underline{\Gamma}_Z(A \otimes^L_{\mathbb{Z}} R\underline{\Gamma}_Z C) \to R\underline{\Gamma}_Z(A \otimes^L_{\mathbb{Z}} C). \tag{5.7.1}$$

Proof. Denote by $j : U = X \setminus Z \hookrightarrow X$ the open immersion. By assumption, $H^i(B_{|U}) = H^i(B)_{|U} = 0$, that is, $B_{|U} = 0$ in $D(X_{\mathrm{Nis}})$. Therefore, the statement follows from the distinguished triangle (see [Sta19, Tag 09XP]):

$$R\underline{\Gamma}_Z(A \otimes^L_{\mathbb{Z}} B) \to A \otimes^L_{\mathbb{Z}} B \to Rj_*(A \otimes^L_{\mathbb{Z}} B)_{|U} \xrightarrow{+1}$$

and the isomorphism $(A \otimes^L_{\mathbb{Z}} B)_{|U} = A_{|U} \otimes^L_{\mathbb{Z}} B_{|U}$. □

5.8. Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. Let $\mathcal{X} = (X, D) \in \underline{\mathbf{MCor}}$ with $X \in \mathbf{Sm}$, Φ be a family of supports on X and $\alpha \in \mathrm{CH}^i_\Phi(X)$ (see 5.1). We define the morphism:

$$c_\alpha : (\gamma^i F)_{\mathcal{X}}[-i] \to R\underline{\Gamma}_\Phi F_{\mathcal{X}} \quad \text{in } D(X_{\mathrm{Nis}}) \tag{5.8.1}$$

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as follows: choose a representative $\tilde{\alpha} \in \mathrm{CH}^i_Z(X)$, $Z \in \Phi$, of α; by the identification:

$$\mathrm{CH}^i_Z(X) = H^i_Z(X, K^M_i) = \mathrm{Ext}^i_{X_{\mathrm{Nis}}}(\mathbb{Z}_X, R\underline{\Gamma}_Z K^M_{i,X}),$$

the cycle $\tilde{\alpha}$ induces a morphism in $D(X_{\mathrm{Nis}})$ (again, denoted by $\tilde{\alpha}$):

$$\tilde{\alpha} : \mathbb{Z}_X[-i] \to R\underline{\Gamma}_Z K^M_{i,X} = R\underline{\Gamma}_Z(\underline{\omega}^* K^M_i)_{(X,\emptyset)}.$$

We define c_α as the composition in $D(X_{\mathrm{Nis}})$:

$$\begin{aligned}
(\gamma^i F)_{\mathcal{X}}[-i] &\xrightarrow{\mathrm{id}\otimes\tilde{\alpha}} (\gamma^i F)_{\mathcal{X}} \otimes^L_{\mathbb{Z}} R\underline{\Gamma}_Z K^M_{i,X} \\
&\xrightarrow{(5.7.1)} R\underline{\Gamma}_Z((\gamma^i F)_{\mathcal{X}} \otimes^L_{\mathbb{Z}} K^M_{i,X}) \\
&\xrightarrow{\mathrm{els}} R\underline{\Gamma}_\Phi((\gamma^i F)_{\mathcal{X}} \otimes^L_{\mathbb{Z}} K^M_{i,X}) \\
&\to R\underline{\Gamma}_\Phi((\gamma^i F)_{\mathcal{X}} \otimes_{\mathbb{Z}} (\underline{\omega}^* K^M_i)_{(X,\emptyset)}) \\
&\xrightarrow{(5.5.1)} R\underline{\Gamma}_\Phi(\gamma^i F \otimes_{\underline{\mathbf{MNST}}} \underline{\omega}^* K^M_i)_{\mathcal{X}} \\
&\xrightarrow{\mathrm{adj}} R\underline{\Gamma}_\Phi F_{\mathcal{X}},
\end{aligned}$$

where the map *els* is the enlarge-support map, the fourth map is induced by the quotient map $A \otimes^L B \to H_0(A \otimes^L B) = A \otimes B$ and *adj* is induced by adjunction via Corollary 4.5. It is direct to check that the definition of c_α does not depend on the choice of $\tilde{\alpha}$.

The morphism c_α satisfies the following functorial properties.

Lemma 5.9. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. Let $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ with $X \in \mathbf{Sm}$, and let Φ be a family of supports on X.*

(1) *We have $c_{\alpha+\beta} = c_\alpha + c_\beta$, for $\alpha, \beta \in \mathrm{CH}^i_\Phi(X)$.*
(2) *Let Ψ be another family of supports containing Φ. Denote by the same letter ι the natural maps $\mathrm{CH}_\Phi \to \mathrm{CH}_\Psi$ and $R\underline{\Gamma}_\Phi \to R\underline{\Gamma}_\Psi$. Then $\iota c_\alpha = c_{\iota\alpha}$, for any $\alpha \in \mathrm{CH}^i_\Phi(X)$.*
(3) *Let $\mathcal{Y} = (Y, E) \in \underline{\mathbf{M}}\mathbf{Cor}$. Let $f : Y \to X$ be a morphism in $\mathbf{Sm}$, such that $E \geq f^*D$, and let $\alpha \in \mathrm{CH}^i_\Phi(X)$. Consider the pullback cycle $f^*\alpha \in \mathrm{CH}^i_{f^{-1}\Phi}(Y)$ (see Lemma 5.3). The following diagram commutes:*

$$\begin{array}{ccc}
(\gamma^i F)_{\mathcal{X}}[-i] & \xrightarrow{c_\alpha} & R\underline{\Gamma}_\Phi F_{\mathcal{X}} \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
Rf_*(\gamma^i F)_{\mathcal{Y}}[-i] & \xrightarrow{c_{f^*\alpha}} & Rf_* R\underline{\Gamma}_{f^{-1}\Phi} F_{\mathcal{Y}} = R\underline{\Gamma}_\Phi Rf_* F_{\mathcal{Y}}.
\end{array}$$

(4) *For $\alpha \in \mathrm{CH}^i_\Phi(X)$, $\beta \in \mathrm{CH}^j_\Psi(X)$ denote by $\alpha \cdot \beta \in \mathrm{CH}^{i+j}_{\Phi\cap\Psi}(X)$ the intersection product of α and β (see (5.4.8)). The following diagram commutes:*

$$\begin{array}{ccc}
\gamma^i(\gamma^j F)_{\mathcal{X}}[-i][-j] & \xrightarrow{c_\alpha[-j]} & R\underline{\Gamma}_\Phi(\gamma^j F)_{\mathcal{X}}[-j] \\
\| \ \text{Cor 4.5} & & \downarrow{\scriptstyle c_\beta} \\
(\gamma^{i+j} F)_{\mathcal{X}}[-(i+j)] & \xrightarrow[c_{\alpha\cdot\beta}]{} & R\underline{\Gamma}_{\Phi\cap\Psi} F_{\mathcal{X}}.
\end{array}$$

Proof. (1) and (2) are immediate to check. For (3), we may assume $\Phi = \Phi_Z$, for some closed subset $Z \in X$. It suffices to show the commutativity of the adjoint square, which we can decompose into the

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following two diagrams (we write $G := \gamma^i F$):

$$\begin{array}{ccccc} f^{-1}G_{\mathcal{X}}[-i] & \xrightarrow{\mathrm{id}\otimes\alpha} & f^{-1}G_{\mathcal{X}} \otimes^L f^{-1}R\underline{\Gamma}_Z K^M_{i,X} & \longrightarrow & f^{-1}R\underline{\Gamma}_Z(G_{\mathcal{X}} \otimes K^M_{i,X}) \\ \downarrow & \textcircled{1} & \downarrow & \textcircled{2} & \downarrow \\ G_{\mathcal{Y}}[-i] & \xrightarrow[\mathrm{id}\otimes f^*\alpha]{} & G_{\mathcal{Y}} \otimes^L R\underline{\Gamma}_{f^{-1}Z} K^M_{i,Y} & \longrightarrow & R\underline{\Gamma}_{f^{-1}Z}(G_{\mathcal{Y}} \otimes K^M_{i,Y}), \end{array}$$

and:

$$\begin{array}{ccccc} f^{-1}R\underline{\Gamma}_Z(G_{\mathcal{X}} \otimes K^M_{i,X}) & \xrightarrow{(5.5.1)} & f^{-1}R\underline{\Gamma}_Z(G \otimes \underline{\omega}^* K^M_i)_{\mathcal{X}} & \xrightarrow{\text{adj.}} & f^{-1}R\underline{\Gamma}_Z F_{\mathcal{X}} \\ \downarrow & \textcircled{3} & \downarrow & \textcircled{4} & \downarrow \\ R\underline{\Gamma}_{f^{-1}Z}(G_{\mathcal{Y}} \otimes K^M_{i,Y}) & \xrightarrow[(5.5.1)]{} & R\underline{\Gamma}_{f^{-1}Z}(G \otimes \underline{\omega}^* K^M_i)_{\mathcal{Y}} & \xrightarrow[\text{adj.}]{} & R\underline{\Gamma}_{f^{-1}Z} G_{\mathcal{Y}}, \end{array}$$

where the vertical maps are induced by pullback along $f : \mathcal{Y} \to \mathcal{X}$ and, for the first diagram, we use the canonical identification $f^{-1}A \otimes^L_{\mathbb{Z}} f^{-1}B = f^{-1}(A \otimes^L_{\mathbb{Z}} B)$. The identity $Rf_* R\underline{\Gamma}_{f^{-1}Z} = R\underline{\Gamma}_Z Rf_*$ and the natural map $\mathrm{id} \to Rf_* f^{-1}$ yield by adjunction a natural transformation $f^{-1}R\underline{\Gamma}_Z \to R\underline{\Gamma}_{f^{-1}Z} f^{-1}$; using this, the commutativity of square 2 is direct to check; furthermore, the proof of the commutativity of squares 3 and 4 reduces to the case without support (i.e. $Z = X$), which is immediate to check. The commutativity of square 1 follows from Lemma 5.3. For (4), we may assume $\Phi = \Phi_Z$ and $\Psi = \Phi_{Z'}$. Consider the following diagram:

$$\begin{array}{ccccc} (\gamma^i\gamma^j F \otimes^L K_{i,Z})[-j] & \longrightarrow & (\gamma^i\gamma^j F \otimes K_i)[-j]_Z & \xrightarrow{\varphi_i} & \gamma^j F[-j]_Z \\ \downarrow{\scriptstyle \mathrm{id}\otimes\beta} & & \downarrow{\scriptstyle \mathrm{id}\otimes\beta} & & \downarrow{\scriptstyle \mathrm{id}\otimes\beta} \\ \gamma^i\gamma^j F \otimes^L K_{i,Z} \otimes^L K_{j,Z'} & \longrightarrow & ((\gamma^i\gamma^j F \otimes K_i) \otimes^L K_{j,Z'})_Z & \xrightarrow{\varphi_i} & (\gamma^j F \otimes^L K_{j,Z'})_Z \\ \downarrow & & \downarrow & & \downarrow \\ \gamma^i\gamma^j F \otimes^L (K_i \otimes K_j)_{Z\cap Z'} & \longrightarrow & (\gamma^i\gamma^j F \otimes K_i \otimes K_j)_{Z\cap Z'} & \xrightarrow{\varphi_i} & (\gamma^j F \otimes K_j)_{Z\cap Z'} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \varphi_j} \\ \gamma^{i+j} F \otimes^L K_{i+j,Z\cap Z'} & \longrightarrow & (\gamma^{i+j} F \otimes K_{i+j})_{Z\cap Z'} & \xrightarrow{\varphi_{i+j}} & F_{Z\cap Z'}, \end{array}$$

in which we skip the indices $\mathcal{X}$ and X, we write K instead of K^M, we write $C_Z := R\underline{\Gamma}_Z C$, for $C \in D(X_{\mathrm{Nis}})$, the tensor products are over $\mathbb{Z}$, the map μ is induced by multiplication $K^M_i \otimes K^M_j \to K^M_{i+j}$ and Corollary 4.5, the maps φ_i are induced by the composition:

$$(\gamma^i G)_{\mathcal{X}} \otimes_{\mathbb{Z}} K^M_{i,X} \xrightarrow{(5.5.1)} (\gamma^i G \otimes_{\underline{\mathbf{MNST}}} \underline{\omega}^* K^M_i)_{\mathcal{X}} \xrightarrow{\text{adj}} G, \tag{5.9.1}$$

for $G \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and the unlabeled arrows are induced maps of the form:

$$G \otimes^L R\underline{\Gamma}_Z H \xrightarrow{(5.7.1)} R\underline{\Gamma}_Z(G \otimes^L H) \to R\underline{\Gamma}_Z(G \otimes H),$$

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for sheaves G, H. It is direct to check that this diagram commutes. Thus, it remains to show that the composition in $D(X_{\mathrm{Nis}})$:

$$\mathbb{Z}_X[-j-i] \xrightarrow{\alpha[-j]} R\underline{\Gamma}_Z K^M_{i,X}[-j] \xrightarrow{\mathrm{id}\otimes\beta} R\underline{\Gamma}_Z K^M_{i,X} \otimes^L R\underline{\Gamma}_{Z'} K^M_{i,X} \tag{5.9.2}$$
$$\to R\underline{\Gamma}_{Z\cap Z'}(K^M_{i,X} \otimes K^M_{i,X}) \to R\underline{\Gamma}_{Z\cap Z'} K^M_{i+j,X}$$

is equal to the morphism induced by the intersection product $\alpha \cdot \beta \in \mathrm{CH}^{i+j}_{Z\cap Z'}(X)$ (see (5.4.8)). Denote by $p_i : X \times X \to X$ the projection to the ith factor and by $\Delta : X \hookrightarrow X \times X$ the diagonal. Since $\Delta^{-1} p_i^{-1} = \mathrm{id}_X$, we obtain that the above composition is adjoint to:

$$\begin{aligned} p_1^{-1}\mathbb{Z}_X[-i] \otimes^L p_2^{-1}\mathbb{Z}_X[-j] &\xrightarrow{\alpha\otimes\beta} p_1^{-1} R\underline{\Gamma}_Z K^M_{i,X} \otimes^L p_2^{-1} R\underline{\Gamma}_{Z'} K^M_{j,X} \\ &\xrightarrow{p_1^*\otimes p_2^*} R\underline{\Gamma}_{Z\times X} K^M_{i,X\times X} \otimes^L R\underline{\Gamma}_{X\times Z'} K^M_{j,X\times X} \\ &\xrightarrow{\mathrm{mult}} R\underline{\Gamma}_{Z\times Z'} K_{i+j,X\times X} \\ &\xrightarrow{\Delta^*} \Delta_* R\underline{\Gamma}_{Z\cap Z'} K^M_{i+j,X}. \end{aligned}$$

By the compatibility of (5.4.4) and (5.4.5), this composition maps in $D(X_{\mathrm{Nis}})$ to the composition of complexes:

$$\begin{aligned} p_1^{-1}\mathbb{Z}_X[-i] \otimes p_2^{-1}\mathbb{Z}_X[-j] &\xrightarrow{\alpha\otimes\beta} p_1^{-1}\underline{\Gamma}_Z C^\bullet_X(i) \otimes p_2^{-1}\underline{\Gamma}_{Z'} C^\bullet_Y(j) \\ &\xrightarrow{\boxtimes} \underline{\Gamma}_{Z\times Z'} C^\bullet_{X\times X}(i+j) \xrightarrow{I(\Delta)} \Delta_*\underline{\Gamma}_{Z\cap Z'} C^\bullet_X(i+j), \end{aligned}$$

where $I(\Delta)$ is a morphism as in (5.3.3) (and it is compatible with $\Delta^* : K^M_{X\times X,i+j} \to \Delta_* K^M_{X,i+j}$, see after (5.3.3)). Thus, it follows from the compatibility of (5.4.8) and (5.4.9) that (5.9.2) is induced by the intersection product $\alpha \cdot \beta$. □

Lemma 5.10. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ with $X \in \mathbf{Sm}$. Let E be an effective Cartier divisor on X which we view as an element in $\mathrm{CH}^1_{|E|}(X)$. Denote by $j : U = X \setminus |E| \hookrightarrow X$ the open immersion, and set $\mathcal{U} = (U, D_{|U})$. Then the composition:*

$$H^1(c_E) : (\gamma^1 F)_{\mathcal{X}} \to R^1\underline{\Gamma}_{|E|} F_{\mathcal{X}} \cong j_* F_{\mathcal{U}}/F_{\mathcal{X}}$$

factors via the natural map (which is injective by the semipurity of F):

$$F_{(X,D+E)}/F_{\mathcal{X}} \hookrightarrow j_* F_{\mathcal{U}}/F_{\mathcal{X}}. \tag{5.10.1}$$

If $\nu : V \hookrightarrow X$ is an open neighborhood of $|E|$, such that $E_{|V} = \mathrm{Div}(e)$ is principal with $e \in \Gamma(V, \mathcal{O}_V)$, then the induced map:

$$(\gamma^1 F)_{\mathcal{X}} \to F_{(X,D+E)}/F_{\mathcal{X}} \cong F_{(V,(D+E)_{|V})}/F_{(V,D_{|V})} \tag{5.10.2}$$

sends an element $a \in (\gamma^1 F)(\mathcal{X})$, represented by an element in $F(\overline{\square}^{(1)} \otimes \mathcal{X})$ which restricts to $\tilde{a} \in F((\mathbf{A}^1, 0) \otimes \mathcal{X})$ to the class modulo $F(V, D_{|V})$ of:

$$\Delta_V^*(\Gamma_e \times \nu)^* \tilde{a} \in F(V, (D+E)_{|V}), \tag{5.10.3}$$

where $\Gamma_e \in \underline{\mathbf{M}}\mathbf{Cor}((V, E_{|V}), (\mathbf{A}^1, 0))$ is the graph of the morphism $e \in \mathbf{A}^1(V)$ and $\Delta_V \in \underline{\mathbf{M}}\mathbf{Cor}((V, (D+E)_{|V}), (V, E_{|V}) \otimes (V, D_{|V}))$ is induced by the diagonal.

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Proof. Set $G^1 = \bigoplus_{x\in|E|^{(0)}} i_{x*}\mathbb{Z}$, where $i_x : \bar{x} \hookrightarrow X$ is the closed immersion. The complex $G := [j_*\mathbf{G}_{m,U} \xrightarrow{\mathrm{Div}} G^1]$ is a $\underline{\Gamma}_{|E|}$-acyclic resolution of $\mathbf{G}_{m,X}$ on X_{Nis}, and, hence:

$$R\underline{\Gamma}_{|E|}\mathbf{G}_{m,X} = G^1[-1].$$

The map $\mathbb{Z}_X[-1] \to R\underline{\Gamma}_{|E|}\mathbf{G}_{m,X}$ corresponding to $E \in \mathrm{CH}^1_{|E|}(X)$ (see 5.8) is, thus, induced by the map of complexes $\varphi_E : \mathbb{Z}_X[-1] \to G^1[-1]$, which sends $1 \in \mathbb{Z}_X$ to the Weil divisor $[E]$. Set $\mathcal{X}' := (X, D+E)$. We obtain a commutative diagram:

$$\begin{array}{ccccc}
(\gamma^1 F)_{\mathcal{X}} \otimes_{\mathbb{Z}} \mathbf{G}_{m,X} & & \xrightarrow{(5.6.1)} & & F_{\mathcal{X}} \\
\downarrow & & & & \downarrow \\
(\gamma^1 F)_{\mathcal{X}} \otimes_{\mathbb{Z}} j_*\mathbf{G}_{m,U} & & \xrightarrow{(5.6.1)} & F_{\mathcal{X}'} & \hookrightarrow j_*F_{\mathcal{U}},
\end{array}$$

where the last horizontal arrow is injective by semipurity. This yields a map on the cokernels (taken vertically):

$$(\gamma^1 F)_{\mathcal{X}} \otimes_{\mathbb{Z}} G^1 \xrightarrow{\theta^1} F_{\mathcal{X}'}/F_{\mathcal{X}} \hookrightarrow j_*F_{\mathcal{U}}/F_{\mathcal{X}},$$

where the last morphism is injective again by semipurity. Set $C := [F_{\mathcal{X}'} \to F_{\mathcal{X}'}/F_{\mathcal{X}}]$, then we obtain a morphism of complexes θ which fits in the commutative diagram of complexes:

$$\begin{array}{ccc}
((\gamma^1 F)_{\mathcal{X}} \otimes_{\mathbb{Z}} \mathbf{G}_{m,X})[0] & \xrightarrow{(5.6.1)} & F_{\mathcal{X}}[0] \\
\downarrow & & \downarrow \mathrm{qis} \\
(\gamma^1 F)_{\mathcal{X}} \otimes_{\mathbb{Z}} G & \xrightarrow{\theta} & C.
\end{array} \tag{5.10.4}$$

Consider the following diagram in $D(X_{\mathrm{Nis}})$:

$$\begin{array}{ccccc}
 & & c_E & & \\
 & \overset{\curvearrowright}{} & & \overset{\curvearrowright}{} & \\
(\gamma^1 F)_{\mathcal{X}}[-1] & \longrightarrow & (\gamma^1 F)_{\mathcal{X}} \otimes^L_{\mathbb{Z}} R\underline{\Gamma}_{|E|}\mathbf{G}_{m,X} & \longrightarrow & R\underline{\Gamma}_{|E|}F_{\mathcal{X}} \\
\| & \searrow & \downarrow (*) & & \downarrow \simeq \\
 & & R\underline{\Gamma}_{|E|}((\gamma^1 F)_{\mathcal{X}} \otimes_{\mathbb{Z}} G) & \xrightarrow{\theta} & R\underline{\Gamma}_{|E|}C \\
 & & \uparrow & & \uparrow \\
(\gamma^1 F)_{\mathcal{X}}[-1] & \xrightarrow{\mathrm{id}\otimes\varphi_E} & (\gamma^1 F)_{\mathcal{X}} \otimes_{\mathbb{Z}} G^1[-1] & \xrightarrow{\theta^1} & F_{\mathcal{X}'}/F_{\mathcal{X}}[-1],
\end{array}$$

where the map $(*)$ is the composition:

$$\begin{aligned}
(\gamma^1 F)_{\mathcal{X}} \otimes^L_{\mathbb{Z}} R\underline{\Gamma}_{|E|}\mathbf{G}_{m,X} &\xrightarrow{(5.7.1)} R\underline{\Gamma}_{|E|}((\gamma^1 F)_{\mathcal{X}} \otimes^L_{\mathbb{Z}} \mathbf{G}_{m,X}) \\
&\xrightarrow{\simeq} R\underline{\Gamma}_{|E|}((\gamma^1 F)_{\mathcal{X}} \otimes^L_{\mathbb{Z}} G) \xrightarrow{\text{nat.}} R\underline{\Gamma}_{|E|}((\gamma^1 F)_{\mathcal{X}} \otimes_{\mathbb{Z}} G),
\end{aligned}$$

and, using that G^1 and $F_{\mathcal{X}'}/F_{\mathcal{X}}$ have support in $|E|$, the lower vertical maps are induced by the natural map $\underline{\Gamma}_{|E|} \to R\underline{\Gamma}_{|E|}$. The upper half of the diagram commutes by the definition of c_E (see 5.8) and the commutativity of (5.10.4); the lower half of the diagram commutes by the definition of the involved

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maps. Thus, $H^1(c_E)$ factors via (5.10.1). If V is as in (5.10.2), then we can lift $[E] \in G^1(V)$ to $e \in \Gamma(V, j_*\mathbf{G}_m)$. Therefore, formula (5.10.3) follows directly from the definition of θ^1 and Lemma 5.6, where we have to use the fact that the graph of the composition $s_1 : V \to \operatorname{Spec} k = \{1\} \hookrightarrow \mathbf{A}^1$ defines an element in $\underline{\mathbf{M}}\mathbf{Cor}((V, \emptyset), (\mathbf{A}^1, 0))$, and, hence, $\Delta_V^*(\Gamma_{s_1} \times \nu)^*\tilde{a}$ vanishes modulo $F(V, D_{|V})$. □

Remark 5.11. Let $C_*(-) : \mathrm{Comp}^+(\mathbf{NST}) \to \mathrm{Comp}^+(\mathbf{NST})$ be the classical $\mathbf{A}^1$-fibrant replacement functor given by the Suslin complex [MVW06, Chapter 2, paragraph 14]. When applied to $\mathbb{Z}_{\mathrm{tr}}(\mathbf{G}_m^{\wedge i})$, it gives an explicit model for the weight i motivic complex $\mathbb{Z}(i)[i] = C_*\mathbb{Z}_{\mathrm{tr}}(\mathbf{G}_m^{\wedge i})$ [MVW06, Chapter 3, paragraph 1]. Let $F \in \underline{\mathbf{M}}\mathbf{NST}$. By adjunction, we get an evaluation pairing:]

$$\underline{\omega}^*(\mathbb{Z}(i)[i]) \otimes^{\mathbb{L}} \underline{\mathrm{Hom}}_{D(\underline{\mathbf{M}}\mathbf{NST})}(\underline{\omega}^*(\mathbb{Z}(i)[i]), F[0]) \to F[0],$$

where we note that $\underline{\omega}^*(\mathbb{Z}(i)[i])$ is still a bounded below complex of $\underline{\mathbf{M}}\mathbf{NST}$, since the functor $\underline{\omega}^*$ is exact. By taking $\mathcal{H}^0$ in the above pairing, we get an induced map:

$$\underline{\omega}^*\mathcal{H}^0((\mathbb{Z}(i)[i])) \otimes \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{NST}}(\underline{\omega}^*\mathcal{H}^0(\mathbb{Z}(i)[i]), F) \to F$$

noting that $\underline{\omega}^*(\mathbb{Z}(i)[i])$ is concentrated in nonnegative degrees, and since there is an isomorphism $\underline{\omega}^*\mathcal{H}^0(\mathbb{Z}(i)[i]) \cong \underline{\omega}^*K_i^M$, this reads, for every $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$, as:

$$K_{i,X}^M \otimes (\gamma^i F)_{\mathcal{X}} \to F.$$

By construction, it agrees with the cup product pairing (5.9.1). We will not use this extended version of the pairing in the rest of the paper.

6. Projective bundle formula

6.1. Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Let V be a locally free $\mathcal{O}_X$-module of rank $n+1$. Denote by:

$$\pi : P = \mathbf{P}(V) = \mathrm{Proj}(\mathrm{Sym}^{\bullet}_{\mathcal{O}_X} V) \to X$$

the projection of the corresponding projective bundle, and set $\mathcal{P} := (P, \pi^*D)$. Let $\xi := c_1(\mathcal{O}_P(1)) \in \mathrm{CH}^1(P)$ be the first Chern class of the hyperplane line bundle $\mathcal{O}_P(1)$, and denote by $\xi^i \in \mathrm{CH}^i(P)$ its i-fold selfintersection. We denote by λ_V^i, the composition in $D(X_{\mathrm{Nis}})$:

$$\lambda_V^i : (\gamma^i F)_{\mathcal{X}}[-i] \xrightarrow{\pi^*} R\pi_*(\gamma^i F)_{\mathcal{P}}[-i] \xrightarrow[(5.8.1)]{c_{\xi^i}} R\pi_* F_{\mathcal{P}}, \quad 0 \le i \le n.$$

We, thus, get a map:

$$\lambda_V = \sum_{i=0}^{n} \lambda_V^i : \bigoplus_{i=0}^{n} (\gamma^i F)_{\mathcal{X}}[-i] \to R\pi_* F_{\mathcal{P}}. \tag{6.1.1}$$

Lemma 6.2. *Let F and (X, D) be as in 6.1. Consider the projection $\pi : \mathbf{P}^1 \times X \to X$. Then:*

$$H^1(\lambda_V^1) : (\gamma^1 F)_{\mathcal{X}} \xrightarrow{\simeq} R^1\pi_* F_{\mathbf{P}^1 \otimes \mathcal{X}}$$

is an isomorphism and $R^i\pi_* F_{\mathbf{P}^1 \otimes \mathcal{X}} = 0$, for all $i \ge 2$.

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Proof. Set $\mathcal{P} := \mathbf{P}^1 \otimes \mathcal{X}$ and define C^1 by the exact sequence:

$$0 \to F_{\mathcal{P}} \to \underbrace{F_{(\mathbf{P}^1,0)\otimes\mathcal{X}} \to C^1}_{=:C} \to 0, \tag{6.2.1}$$

where the first map is injective by the semipurity of F. Since C^1 has support in $0\times X$ and $R^i\pi_* F_{(\mathbf{P}^1,0)\otimes\mathcal{X}} = 0$, for all $i \geq 1$, by the cube invariance of the cohomology, [Sai20a, Theorem 9.3], it follows that C is a π_*-acylic resolution of $F_{\mathcal{P}}$ sitting in degree $[0, 1]$. This proves the vanishing statement. Furthermore, we claim that:

$$\pi_* C^1 = (\gamma^1 F)_{\mathcal{X}}.$$

Indeed, for $U \to X$ étale and $\mathcal{U} = (U, D_U)$, we have:

$$(\gamma^1 F)_{\mathcal{X}}(U) = \frac{F((\mathbf{A}^1, 0) \otimes \mathcal{U})}{F(\mathbf{A}^1 \otimes \mathcal{U})} = \pi_* C^1(U), \tag{6.2.2}$$

where the first equality holds by Remark 4.7; the second equality holds by the following two observations: since C^1 is supported on $0 \times X$, we have an exact sequence on $\mathbf{P}^1 \times X$:

$$0 \to j_* F_{\mathbf{A}^1\otimes\mathcal{X}} \to j_* F_{(\mathbf{A}^1,0)\otimes\mathcal{X}} \to C^1 \to 0,$$

where $j : \mathbf{A}^1 \times X \hookrightarrow \mathbf{P}^1 \times X$ is the open immersion, and by the Lemmas 2.9 and 2.10, we have:

$$R^1\pi_*(j_* F_{\mathbf{A}^1\otimes\mathcal{X}}) = R^1(\pi j)_* F_{\mathbf{A}^1\otimes\mathcal{X}} = 0.$$

Furthermore, the map $\pi_* F_{\mathcal{P}} \to \pi_* F_{(\mathbf{P}^1,0)\otimes\mathcal{X}}$ is an equality by Corollary 2.19, and, hence, (6.2.1), (6.2.2) and $R^1\pi_* F_{(\mathbf{P}^1,0)\otimes\mathcal{X}} = 0$ imply $R^1\pi_* F_{\mathcal{P}} = \pi_* C^1 = (\gamma^1 F)_{\mathcal{X}}$. It remains to show that $H^1(\lambda^1_V) : (\gamma^1 F)_{\mathcal{X}} \to R^1\pi_* F_{\mathcal{P}} = (\gamma^1 F)_{\mathcal{X}}$ realises such isomorphism. To this end, note that on $U \to X$, we can identify $H^1(\lambda_V)$ by Lemma 5.10 with the composition:

$$\gamma^1 F(\mathcal{U}) \xrightarrow{\pi^*} \gamma^1 F(\mathbf{P}^1 \otimes \mathcal{U}) = \frac{F((\mathbf{A}^1, 0) \otimes \mathbf{P}^1 \otimes \mathcal{U})}{F(\mathbf{A}^1 \otimes \mathbf{P}^1 \otimes \mathcal{U})} \xrightarrow{\Delta^*_{\mathbf{A}^1_U}(\Gamma\times\nu)^*} \frac{F((\mathbf{A}^1, 0) \otimes \mathcal{U})}{F(\mathbf{A}^1 \otimes \mathcal{U})} = \gamma^1 F(\mathcal{U}),$$

where $\nu : \mathbf{A}^1_U \hookrightarrow \mathbf{P}^1_U$ is the open immersion and we identify ξ on $\mathbf{P}^1_U$ with the class of the divisor $0 \times U$ and where Γ is the graph of the projection $\mathbf{A}^1 \times U \to \mathbf{A}^1$. Thus, the equality:

$$(\mathrm{id}_{\mathbf{A}^1} \times \pi) \circ (\Gamma \times \nu) \circ \Delta_{\mathbf{A}^1_U} = \mathrm{id}_{\mathbf{A}^1_U}$$

implies the statement. □

We are now ready to prove the projective bundle theorem in our setting.

Theorem 6.3. *The map (6.1.1) is an isomorphism in* $D(X_{\mathrm{Nis}})$.

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Proof. The question is local on X. Hence, the statement follows for $n = 1$ from Lemma 6.2. We now assume $n \geq 2$ and $P = \mathbf{P}^n_X$. Consider the diagram:

$$\begin{array}{ccccc} E & \xrightarrow{i_E} & Y & \xrightarrow{q} & E \\ \downarrow{\scriptstyle \rho_E} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho_E} \\ X & \xrightarrow{i} & P & \xrightarrow{\pi} & X, \end{array}$$

where π is the projection, i a section of π, ρ the blow-up of P in $i(X)$, i_E is the closed immersion of the exceptional divisor and q is the standard map, which identifies Y with $\mathbf{P}(W)$, where $W := \mathcal{O}_E \oplus \mathcal{O}_E(1)$. Denote by V (respectively, V_E) the trivial $\mathcal{O}_X$-module of rank $n+1$ (respectively, n) defining the projective bundle P (respectively, E) over X (recall, we work locally on X). We set $\mathcal{E} := (E, D_{|E})$ and $\mathcal{Y} := (Y, D_{|Y})$. By Corollary 2.19, we have:

$$R\pi_* F_{\mathcal{P}} \cong F_{\mathcal{X}} \oplus \tau^{\geq 1} R\pi_* F_{\mathcal{P}}, \tag{6.3.1}$$

where the map $F_{\mathcal{X}} = \pi_* F_{\mathcal{P}} \to R\pi_* F_{\mathcal{P}}$ is split by the section i. Thus, applying $R\pi_*$ to the exact triangle from Theorem 3.1 induced by the blow-up $Y \to P$, we can split off $F_{\mathcal{X}}$ to obtain the triangle on the bottom of the following diagram:

$$\begin{array}{ccccc} \oplus_{i=1}^n (\gamma^i F)_{\mathcal{X}}[-i] & \xrightarrow{(0,\lambda_{V_E}[-1])} & R\rho_{E*}F_{\mathcal{E}} \oplus R\rho_{E*}\gamma^1 F_{\mathcal{E}}[-1] & \xrightarrow{\mathrm{id}+0} & R\rho_{E*}F_{\mathcal{E}} \\ \downarrow{\scriptstyle \oplus_i \lambda_V^i} & & \downarrow{\scriptstyle \lambda_W} & & \| \\ \tau^{\geq 1} R\pi_* F_{\mathcal{P}} & \xrightarrow{\rho_1^*} & R\pi_* R\rho_* F_{\mathcal{Y}} & \xrightarrow{i_E^*} & R\rho_{E*}F_{\mathcal{E}}, \end{array}$$

where the map labeled by ρ_1^* is induced by (6.3.1) and ρ^*. Note that the bottom triangle is actually split since $i_E^* q^* = \mathrm{id}$. In the top left, λ_{V_E} is applied to $\mathcal{E}/\mathcal{X}$ and $(\gamma^1 F)_{\mathcal{X}}$, and it is an isomorphism by induction. Hence, the top sequence is a split triangle as well. Since λ_W is also an isomorphism by the $n = 1$ case, it remains to show that the diagram is commutative. Let $\xi = c_1(\mathcal{O}_P(1))$, $\xi_E = c_1(\mathcal{O}_E(1))$ and $\eta = c_1(\mathcal{O}_{\mathbf{P}(W)}(1))$ be the first Chern classes of the corresponding fundamental line bundles. The commutativity of the right square follows from $i_E^* \circ c_\eta = c_{i_E^*\eta} \circ i_E^* = 0$, where we use Lemma 5.9(3) for the first equality and $i_E^* \mathcal{O}_{\mathbf{P}(W)}(1) = \mathcal{O}_E$ for the second. The commutativity of the left square reduces by Lemma 5.9(3) to the equality:

$$\rho^*(\xi^i) = q^*(\xi_E^{i-1}) \cdot \eta \quad \in \mathrm{CH}^i(Y),$$

which is well-known and straightforward to check. □

7. The Gysin triangle

We begin with an elementary lemma on split exact triangles in an arbitrary triangulated category.

Lemma 7.1. *Let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{\partial} A[1]$ be an exact triangle in a triangulated category T.*

(1) *If $\tau : C \to B$ is a section of b, then there is a unique map $\sigma : B \to A$, such that $a \circ \sigma = \mathrm{id}_B - e$, where $e := \tau \circ b \in \mathrm{Hom}_T(B, B)$. Moreover, σ is a retraction of a.*
(2) *If $\sigma : B \to A$ is a retraction of a, then there is a unique map $\tau : C \to B$, such that $\tau \circ b = \mathrm{id}_B - \varepsilon$, where $\varepsilon := a \circ \sigma \in \mathrm{Hom}_T(B, B)$. Moreover, τ is a section of b.*

In (1) (respectively, (2)), we call σ (respectively, τ) the canonical retraction defined by τ (respectively, the canonical section defined by σ). Moreover, the canonical section defined by the canonical retraction of τ is equal to τ, and similarly with σ.

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Proof. (1). By the existence of the section τ, the long exact sequence stemming from applying $\mathrm{Hom}_T(B,-)$ to the exact triangle (a,b,∂) breaks up into short exact sequences; in particular, we obtain the short exact sequence:

$$0 \to \mathrm{Hom}_T(B,A) \xrightarrow{a\circ} \mathrm{Hom}_T(B,B) \xrightarrow{b\circ} \mathrm{Hom}_T(B,C) \to 0.$$

This gives a unique σ with $a\circ\sigma = \mathrm{id}_B - e$. It follows that $a\circ(\sigma\circ a) = a$. Since also $a\circ : \mathrm{Hom}_T(A,A) \to \mathrm{Hom}_T(A,B)$ is injective (by the existence of τ), we see that σ is a retraction. Similarly for (2); the other statements are clear. □

7.2. Having the projective bundle formula and the blow-up formula at our disposal, we can construct the Gysin triangle by formally following the procedure indicated by Voevodsky in [Voe00b, Section 3.5]. Note that our statement is sheaf theoretic, and, therefore, the arrows are reversed compared to *loc. cit.* We begin by setting the notation.

Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} = (X,D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ (see Notation 2.8). Let $i : Z \hookrightarrow X$ be a smooth closed subscheme intersecting D transversally (see Definition 2.11). Denote by $\rho\colon \tilde{X} \to X$ the blow-up of X along Z, and let $\rho_E\colon E = \rho^{-1}(Z) \to Z$ be the exceptional divisor. We define the modulus pairs $\tilde{\mathcal{X}} = (\tilde{X}, D_{|\tilde{X}})$, $\mathcal{Z} = (Z, D_{|Z})$ and $\mathcal{E} = (E, D_{|E})$ with the obvious convention on the divisor.

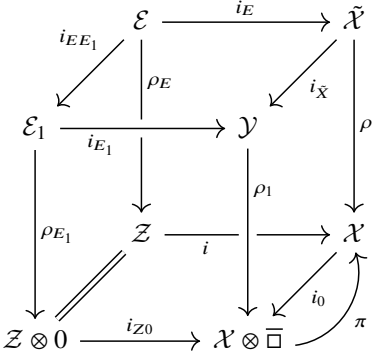
Let $\rho_1 : Y \to X\times\mathbf{P}^1$ be the blow-up of $X\times\mathbf{P}^1$ along $Z\times 0$, and let $E_1 = \rho_1^{-1}(Z\times 0)$ be the exceptional divisor. We obtain the following modulus pairs in $\underline{\mathbf{M}}\mathbf{Cor}_{ls}$:

$$\mathcal{Y} = (Y, (D\times\mathbf{P}^1 + X\times\infty)_{|Y}), \quad \mathcal{E}_1 = (E_1, (D\times 0)_{|E_1})$$

and the following obvious morphisms in $\underline{\mathbf{M}}\mathbf{Cor}$:

$$i : \mathcal{Z} \to \mathcal{X}, \quad i_{Z0} : \mathcal{Z} = \mathcal{Z}\otimes 0 \to \mathcal{X}\otimes\overline{\square},$$
$$\rho : \tilde{\mathcal{X}} \to \mathcal{X}, \quad \rho_1 : \mathcal{Y} \to \mathcal{X}\otimes\overline{\square}, \quad i_{\tilde{X}} : \tilde{\mathcal{X}} \to \mathcal{Y},$$
$$i_E : \mathcal{E} \to \tilde{\mathcal{X}}, \quad i_{E_1} : \mathcal{E}_1 \to \mathcal{Y}, \quad i_{EE_1} : \mathcal{E} \to \mathcal{E}_1,$$
$$\rho_E : \mathcal{E} \to \mathcal{Z}, \quad \rho_{E_1} : \mathcal{E}_1 \to \mathcal{Z}\otimes 0 = \mathcal{Z},$$
$$i_\varepsilon : \mathcal{X} = \mathcal{X}\otimes\varepsilon \to \mathcal{X}\otimes\overline{\square}, \quad \varepsilon \in \{0,1\}, \quad \pi : \mathcal{X}\otimes\overline{\square} \to \mathcal{X},$$

where all the i's are induced by closed immersions of the underlying schemes, in particular, $i_{\tilde{X}}$ is induced by identifying $\tilde{X}$ with the strict transform of $X\times 0$ in Y and i_{EE_1} is the pullback of $i_{\tilde{X}}$ along i_{E_1}. This gives the following commutative diagram in $\underline{\mathbf{M}}\mathbf{Cor}$:



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We will denote the underlying morphisms of schemes by the same letter. Note that $i_1 : \mathcal{X} = \mathcal{X} \otimes 1 \to \mathcal{X} \otimes \overline{\square}$ extends canonically to a morphism:

$$i_{1,Y} : \mathcal{X} \to \mathcal{Y}.$$

Finally, the morphism underlying i_{EE_1} is equal to the natural inclusion:

$$i_{EE_1} : \mathbf{P}(\mathcal{N}_{Z/X}^\vee) \hookrightarrow \mathbf{P}(\mathcal{O}_Z \oplus \mathcal{N}_{Z/X}^\vee),$$

where $\mathcal{N}_{Z/X}^\vee = \mathcal{I}/\mathcal{I}^2$ is the conormal sheaf, $\mathcal{I}$ being the ideal sheaf of $Z \hookrightarrow X$. We obtain the following diagram,

$$\begin{array}{ccc} \bigoplus_{i=0}^{j}(\gamma^i F)_{\mathcal{Z}}[-i] & \xrightarrow[\simeq]{\lambda_{\mathcal{O}_Z \oplus \mathcal{N}_{Z/X}^\vee}} & R\rho_{E_1 *}F_{\mathcal{E}_1} \\ \downarrow & & \downarrow{\scriptstyle i_{EE_1}^*} \\ \bigoplus_{i=0}^{j-1}(\gamma^i F)_{\mathcal{Z}}[-i] & \xrightarrow[\simeq]{\lambda_{\mathcal{N}_{Z/X}^\vee}} & R\rho_{E*}F_{\mathcal{E}}, \end{array} \tag{7.2.1}$$

where $j = \operatorname{codim}(Z, X)$, the horizontal maps are the isomorphisms from Theorem 6.3 and the vertical map on the left is the projection. Using Lemma 5.9, it is direct to check that (7.2.1) commutes. Thus, $i_{EE_1}^*$ in (7.2.1) has a canonical section:

$$s : R\rho_{E*}F_{\mathcal{E}} \to R\rho_{E_1 *}F_{\mathcal{E}_1},$$

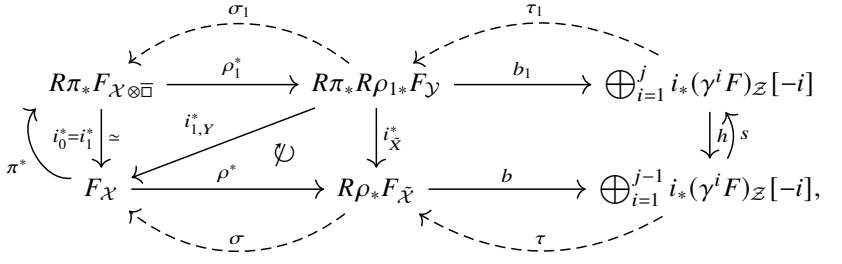
splitting off the summand $(\gamma^j F)_{\mathcal{Z}}[-j]$. Let b_1 be the morphism in $D(X_{\mathrm{Nis}})$ defined as the composition:

$$R\pi_* R\rho_{1*}F_{\mathcal{Y}} \xrightarrow{i_{E_1}^*} i_* R\rho_{E_1 *}F_{\mathcal{E}_1} \xrightarrow{\simeq} \bigoplus_{i=0}^{j} i_*(\gamma^i F)_{\mathcal{Z}}[-i] \longrightarrow \bigoplus_{i=1}^{j} i_*(\gamma^i F)_{\mathcal{Z}}[-i],$$

where the second isomorphism is the inverse of $\lambda_{\mathcal{O}_Z \oplus \mathcal{N}_{Z/X}^\vee}$, and the rightmost arrow is the canonical projection. Similarly, we define b as the composition:

$$R\rho_* F_{\tilde{\mathcal{X}}} \xrightarrow{i_E^*} i_* R\rho_{E*}F_{\mathcal{E}} \xrightarrow{\simeq} \bigoplus_{i=0}^{j-1} i_*(\gamma^i F)_{\mathcal{Z}}[-i] \longrightarrow \bigoplus_{i=1}^{j-1} i_*(\gamma^i F)_{\mathcal{Z}}[-i],$$

where the second isomorphism is the inverse of $\lambda_{\mathcal{N}_{Z/X}^\vee}$ and the last map is the canonical projection. This gives the following commutative diagram of solid arrows in $D(X_{\mathrm{Nis}})$:



(7.2.2)

where the dashed arrows are defined as follows. First, note that the bottom horizontal sequence is a distinguished triangle obtained from the distinguished blow-up triangle from Theorem 3.1 for $(\mathcal{X}, \mathcal{Z})$.

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Indeed, we have the following diagram in: $D(X_{\mathrm{Nis}})$:

$$\begin{array}{ccccccc} F_{\mathcal{X}} & \xrightarrow{\rho^*\oplus(-i^*)} & R\rho_*F_{\tilde{\mathcal{X}}}\oplus i_*F_{\mathcal{Z}} & \xrightarrow{i_E^*+\rho_E^*} & i_*R\rho_{E*}F_{\mathcal{E}} & \xleftarrow[\simeq]{\lambda_{\mathcal{N}^\vee_{Z/X}}} & \bigoplus_{i=0}^{j-1}(\gamma^iF)_{\mathcal{Z}}[-i] \\ \| & & \downarrow & & \downarrow{\scriptstyle \mathrm{can.}} & & \downarrow \\ F_{\mathcal{X}} & \xrightarrow{\rho^*} & R\rho_*F_{\tilde{\mathcal{X}}} & \xrightarrow{\mathrm{can.}\circ(i_E^*)} & i_*\tau_{\geq 1}R\rho_{E*}F_{\mathcal{E}} & \xleftarrow[\simeq]{\sum_{i=1}^{j-1}\lambda_V^i} & \bigoplus_{i=1}^{j-1}(\gamma^iF)_{\mathcal{Z}}[-i], \end{array}$$

where $V = \mathcal{N}^\vee_{Z/X}$, the unlabeled maps are the natural projections, can. is the canonical map to the truncation $\tau_{\geq 1}$ and the composition of the second and the (inverse of the) third arrow in the bottom line is b. Every square is commutative, and the middle one is clearly homotopy cartesian, so that the bottom line is part of a distinguished triangle as required. In a similar fashion, the top line of (7.2.2) is part of a distinguished triangle obtained from the blow-up triangle for $(\mathcal{X}\otimes\overline{\square}, Z\times 0)$ after applying $R\pi_*$.

Going back to (7.2.2), note that the right square is commutative thanks to the definitions of b and b_1, and the commutativity of (7.2.1). The square on the left commutes if the left vertical arrow is i_0^*. By cube invariance it is an isomorphism, with inverse π^*, and, thus, it is equal to i_1^*. Replacing i_0^* with i_1^*, we see then that the upper triangle in the left square is commutative. The lower triangle, on the other hand, does not commute. Set:

$$\sigma_1 := \pi^*\circ i_{1,Y}^*;$$

it is a retraction of ρ_1^*. We define the (dotted) map τ_1 as the canonical section defined by σ_1, see Lemma 7.1. Set:

$$\tau := i_{\tilde{X}}^*\circ\tau_1\circ s; \tag{7.2.3}$$

it is a section of b (that we can identify with $(\mathrm{can.}\circ i_E^*)$ up to the isomorphism $\tau_{\geq 1}R\rho_{E*}F_{\mathcal{E}} \simeq \bigoplus_{i=1}^{j-1}(\gamma^iF)_{\mathcal{Z}}[-i]$). We define σ as the canonical section defined by τ (again, see Lemma 7.1). The proof of the following corollary is immediate from the previous constructions.

Corollary 7.3. *Let F and $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ and $\mathcal{Z}$ be as in 7.2 and $j = \mathrm{codim}(Z, X)$. We have isomorphisms in $D(X_{\mathrm{Nis}})$:*

$$F_{\mathcal{X}} \oplus \bigoplus_{r=1}^{j-1} i_*\gamma^rF_{\mathcal{Z}}[-r] \xrightarrow[\simeq]{\rho^*+\tau} R\rho_*F_{\tilde{\mathcal{X}}}. \tag{7.3.1}$$

7.4. Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$, and let $i : Z \hookrightarrow X$ be a smooth closed subscheme of codimension $j \geq 1$ intersecting D transversally. Set $\mathcal{Z} = (Z, D_{|Z})$. Following [Voe00b, p. 220/21], we define the Gysin map:

$$g_{\mathcal{Z}/\mathcal{X}} : i_*(\gamma^jF_{\mathcal{Z}})[-j] \to F_{\mathcal{X}} \tag{7.4.1}$$

as follows: let the notation be as in (7.2.2). We set:

$$v\colon i_*(\gamma^jF)_{\mathcal{Z}}[-j] \longrightarrow R\pi_*R_{\rho_{1*}}F_{\mathcal{Y}}$$

as the composition $\tau_1\circ\nu^j$, where ν^j is the canonical inclusion $i_*(\gamma^jF)_{\mathcal{Z}}[-j] \to \bigoplus_{i=1}^{j} i_*(\gamma^iF)_{\mathcal{Z}}[-i]$. Then the Gysin map is:

$$g_{\mathcal{Z}/\mathcal{X}} := -\sigma\circ i_{\tilde{X}}^*\circ v.$$

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Remark 7.5. We can replace the top row in (7.2.2) with the equivalent one:

$$R\pi_* F_{\mathcal{X}\otimes\square} \xrightarrow[\rho_1^*]{\sigma_1} R\pi_* R\rho_{1*} F_{\mathcal{Y}} \xrightarrow[\mathrm{can}.\circ i_{E_1}^*]{\tau_1'} i_*\tau_{\geq 1} R\rho_{E_{1*}} F_{\mathcal{E}_1}., \tag{7.5.1}$$

where τ_1' is the canonical section induced by σ_1. Note that $\tau_1 = \tau_1' \circ \sum_{i=1}^{j} \lambda_{V_1}^i$, where $V_1 = \mathcal{O}_Z \oplus \mathcal{N}_{Z/X}^\vee$, and can. is the canonical map $i_* R\rho_{E_{1*}} F_{\mathcal{E}_1} \to i_*\tau_{\geq 1} R\rho_{E_{1*}} F_{\mathcal{E}_1}$. In particular, the Gysin map satisfies:

$$g_{\mathcal{Z}/\mathcal{X}} = -\sigma \circ i_{\tilde{X}}^* \circ \tau_1' \circ \lambda_{V_1}^j. \tag{7.5.2}$$

Remark 7.6. The Gysin map can be described alternatively as follows. Set:

$$\beta := i_{1,Y}^* - \sigma \circ i_{\tilde{X}}^* : R\pi_* R\rho_{1*} F_{\mathcal{Y}} \to F_{\mathcal{X}}.$$

We have $\beta \circ (\rho_1^*) = 0 : R\pi_* F_{\mathcal{X}\otimes\square} \to F_{\mathcal{X}}$, and since the top row in (7.2.2) is split exact, there exists a unique map:

$$\beta_1 : \bigoplus_{i=1}^{j} i_*(\gamma^i F)_{\mathcal{Z}}[-i] \to F_{\mathcal{X}},$$

such that $\beta = \beta_1 \circ b_1$. Then a diagram chase shows that:

$$g_{\mathcal{Z}/\mathcal{X}} = \beta_1 \circ \nu^j. \tag{7.6.1}$$

Remark 7.7. Let $i : Z \hookrightarrow X$ be as above. One can consider the class $[Z]$ of Z in the Chow group with support $\mathrm{CH}_Z^j(X)$, and the cup product construction (5.8.1) gives a morphism:

$$c_{[Z]} : (\gamma^j F)_{\mathcal{X}}[-j] \to R\underline{\Gamma}_Z F_{\mathcal{X}} \to F_{\mathcal{X}} \tag{7.7.1}$$

in $D(X_{\mathrm{Nis}})$, where the last morphism is the forget support map. It is a natural question to compare (7.7.1) and (7.4.1): this is done in Theorem 7.12 below.

Proposition 7.8. *Let $F \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}$, $\mathcal{X} = (X, D)$, $\mathcal{Z} = (Z, D_{|Z})$ be as in 7.4 above. Assume X is equidimensional, and let Φ be a family of supports on X and $\alpha \in \mathrm{CH}_\Phi^r(X)$. Then the following diagram:*

$$\begin{array}{ccc} i_*(\gamma^{j+r} F)_{\mathcal{Z}}[-j-r] & \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} & \gamma^r F_{\mathcal{X}}[-r] \\ \downarrow{\scriptstyle c_{i^*\alpha}} & & \downarrow{\scriptstyle c_\alpha} \\ i_* R\underline{\Gamma}_{Z\cap\Phi} \gamma^j F_{\mathcal{Z}}[-j] = R\underline{\Gamma}_\Phi i_* \gamma^j F_{\mathcal{Z}}[-j] & \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} & R\underline{\Gamma}_\Phi F_{\mathcal{X}} \end{array}$$

commutes, where $i^*\alpha \in \mathrm{CH}_{\Phi\cap Z}^r(Z)$ is the refined pullback of α (see Lemma 5.3).

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Proof. We use the notation from 7.2 and, in particular, (7.2.2) above. We compute:

$$\begin{aligned}
g_{\mathcal{Z}/\mathcal{X}} \circ c_{i^*\alpha} &\overset{(*)}{=} -\sigma \circ i_{\tilde{X}}^* \circ \tau_1' \circ \lambda_{V_1}^j \circ c_{i^*\alpha} \\
&\overset{(2*)}{=} -\sigma \circ i_{\tilde{X}}^* \circ \tau_1' \circ c_{\xi^j} \circ \rho_{E_1}^* \circ c_{i^*\alpha} \\
&\overset{(3*)}{=} -\sigma \circ i_{\tilde{X}}^* \circ \tau_1' \circ c_{(i\rho_{E_1})^*\alpha} \circ \lambda_{V_1}^j \\
&\overset{(4*)}{=} -\sigma \circ i_{\tilde{X}}^* \circ \tau_1' \circ c_{(i\rho_{E_1})^*\alpha} \circ ((\mathrm{can.} \circ i_{E_1}^*) \circ \tau_1') \circ \lambda_{V_1}^j \\
&\overset{(5*)}{=} -\sigma \circ i_{\tilde{X}}^* \circ \tau_1' \circ (\mathrm{can.} \circ i_{E_1}^*) \circ c_{\rho_1^*\pi^*\alpha} \circ \tau_1' \circ \lambda_{V_1}^j \\
&\overset{(6*)}{=} -\sigma \circ i_{\tilde{X}}^* \circ (\mathrm{id} - \rho_1^* \circ \sigma_1) \circ (c_{\rho_1^*\pi^*\alpha}) \circ \tau_1' \circ \lambda_{V_1}^j \\
&\overset{(7*)}{=} -\sigma \circ i_{\tilde{X}}^* \circ c_{\rho_1^*\pi^*\alpha} \circ (\mathrm{id} - \rho_1^* \circ \sigma_1) \circ \tau_1' \circ \lambda_{V_1}^j \\
&\overset{(8*)}{=} -\sigma \circ i_{\tilde{X}}^* \circ c_{\rho_1^*\pi^*\alpha} \circ \tau_1' \circ \lambda_{V_1}^j \\
&\overset{(9*)}{=} -\sigma \circ c_{\rho^*\alpha} \circ i_{\tilde{X}}^* \circ \tau_1' \circ \lambda_{V_1}^j \\
&\overset{(10*)}{=} -\sigma \circ c_{\rho^*\alpha} \circ (\rho^* \circ \sigma + \tau \circ b) \circ i_{\tilde{X}}^* \circ \tau_1' \circ \lambda_{V_1}^j \\
&\overset{(11*)}{=} -\sigma \circ \rho^* \circ c_\alpha \circ \sigma \circ i_{\tilde{X}}^* \circ \tau_1 \circ \nu^j \\
&\overset{(12*)}{=} -c_\alpha \circ \sigma \circ i_{\tilde{X}}^* \circ \tau_1 \circ \nu^j \\
&\overset{(13*)}{=} c_\alpha \circ g_{\mathcal{Z}/\mathcal{X}},
\end{aligned}$$

where:

(*) holds by (7.5.2),
(2*) holds by the definition of $\lambda_{V_1}^j$ in 6.1,
(3*) holds by Lemma 5.9(3), (4) and the definition of λ_V^j,
(4*) holds by $((\mathrm{can.} \circ i_{E_1}^*) \circ \tau_1') = \mathrm{id}$ on $i_*\tau_{\geq 1}R\rho_{E_{1*}}F_{\mathcal{E}_1}$,
(5*) holds by $i\rho_{E_1} = \pi\rho_1 i_{E_1}$ and Lemma 5.9(3),
(6*) holds since τ_1' is the section defined by the section σ_1 (see 7.1),
(7*) follows, again, from Lemma 5.9(3) and the fact that $\sigma_1 = \pi^* \circ i_{1,Y}^*$ and $\pi\rho_1 i_{1,Y} = \mathrm{id}$.
(8*) holds by $(\mathrm{id} - \rho_1^* \circ \sigma_1) \circ \tau_1' = \tau_1'$,
(9*) holds by $\pi\rho_1 i_{\tilde{X}} = \rho$ and Lemma 5.9(3),
(10*) holds by $\mathrm{id} = \rho^* \circ \sigma + \tau \circ b$,
(11*) holds by Lemma 5.9(3), $\tau_1' \circ \lambda_V^j = \tau_1 \circ \nu^j$ (see Remark 7.5) and

$$b \circ i_{\tilde{X}}^* \circ \tau_1' \circ \lambda_V^j = h \circ b_1 \circ \tau_1 \circ \nu^j = h \circ \nu^j = 0,$$

(12*) holds by $\sigma \circ \rho^* = \mathrm{id}$,
(13*) holds by definition of the Gysin map in 7.4;

whence the statement. □

Proposition 7.9. *Let $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ and $F \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}$. Let $i : Z \hookrightarrow X$ be a smooth closed subscheme of codimension j, intersecting D transversally (see Definition 2.11), and set $\mathcal{Z} = (Z, D_{|Z})$. Let $X' \in \mathbf{Sm}$, and let $f : X' \to X$ be a morphism, such that $i' : Z' := X' \times_X Z \hookrightarrow X'$ is a smooth closed subscheme of pure codimension $m \leq j$ and $|f^*D|$ is a simple normal crossing divisor intersecting Z' transversally. Set $\mathcal{X}' := (X', f^*D)$ and $\mathcal{Z}' := (Z', (f^*D)_{|Z'})$. Let $f_Z : Z' \to Z$ be the base change of f,*

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and consider the excess normal bundle:

$$\mathcal{E}xc = f_Z^*(\mathcal{N}_{Z/X})/\mathcal{N}_{Z'/X'}.$$

Then the following diagram commutes:

$$\begin{array}{ccccc} i_*\gamma^j F_{\mathcal{Z}}[-j] & & \xrightarrow{\quad g_{\mathcal{Z}/\mathcal{X}} \quad} & & F_{\mathcal{X}} \\ \downarrow{\scriptstyle f_Z^*} & & & & \downarrow{\scriptstyle f^*} \\ Rf_*(i'_*\gamma^j F_{\mathcal{Z}'}[-j]) & \xrightarrow{c_\beta} & Rf_*(i'_*\gamma^m F_{\mathcal{Z}'}[-m]) & \xrightarrow{g_{\mathcal{Z}'/\mathcal{X}'}} & Rf_*F_{\mathcal{X}'}, \end{array} \tag{7.9.1}$$

where $\beta := c_{j-m}(\mathcal{E}xc) \cap Z' \in \mathrm{CH}^{j-m}(Z')$ *and* $c_{j-m}(\mathcal{E}xc)$ *is the* $(j-m)$*th Chern class of* $\mathcal{E}xc$*. In particular, the Gysin map is compatible with smooth base change.*

Proof. First observe that given two distinguished triangles Δ and Δ' with sections σ and σ' (respectively, τ and τ') as in Lemma 7.1, and $\varphi : \Delta \to \Delta'$ is a morphism of triangles which commutes with σ and σ' (respectively, τ and τ'), then φ also commutes with the canonical sections defined by these sections. From this and the commutative diagram:

$$\begin{array}{ccccc} \mathrm{Bl}_{Z'}(X') & \hookrightarrow & \mathrm{Bl}_Z(X) \times_X X' & \to & \mathrm{Bl}_Z(X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Bl}_{Z'\times 0}(X' \times \mathbf{P}^1) & \hookrightarrow & \mathrm{Bl}_{Z\times 0}(X \times \mathbf{P}^1) \times_X X' & \to & \mathrm{Bl}_{Z\times 0}(X \times \mathbf{P}^1) \\ & \searrow & \downarrow & & \downarrow \\ & & X' \times \mathbf{P}^1 & \to & X \times \mathbf{P}^1 \end{array} \tag{7.9.2}$$

(and all the diagrams which this diagram induces), we see that the various pullbacks induced by $f : X' \to X$ commute with the section σ_1, thus, with τ_1' from (7.5.1). It is direct to check that by (7.5.2) we are reduced to show the commutativity of the following diagram:

$$\begin{array}{ccccc} i_*\gamma^j F_{\mathcal{Z}}[-j] & \xrightarrow{\lambda_V^j} & R\rho_{E_1*}F_{\mathcal{E}_1} & \xrightarrow{\tilde{f}_{E_1}^*} & Rf_{Z*}R\rho_{E_1'*}F_{\mathcal{E}_1'}, \\ \downarrow{\scriptstyle f_Z^*} & & & \nearrow{\scriptstyle \lambda_{V'}^m} & \\ Rf_*(i'_*\gamma^j F_{\mathcal{Z}'}[-j]) & \xrightarrow{c_\beta} & Rf_*(i'_*\gamma^m F_{\mathcal{Z}'}[-m]) & & \end{array} \tag{7.9.3}$$

where $V = \mathcal{N}_{Z/X}^\vee \oplus \mathcal{O}_Z$, $V' = \mathcal{N}_{Z'/X'}^\vee \oplus \mathcal{O}_{Z'}$, E_1 (respectively, E_1') is the exceptional divisor in $\mathrm{Bl}_{Z\times 0}(X \times \mathbf{P}^1)$ (respectively, in $\mathrm{Bl}_{Z'\times 0}(X' \times \mathbf{P}^1)$), the maps are as in the following commutative diagram (which is induced by (7.9.2) and is, in general, not cartesian):

$$\begin{array}{ccc} E_1' & \xrightarrow{\tilde{f}_{E_1}} & E_1 \\ \downarrow{\scriptstyle \rho_{E_1'}} & & \downarrow{\scriptstyle \rho_{E_1}} \\ Z' & \xrightarrow{f_Z} & Z, \end{array}$$

and $\mathcal{E}_1 = (E_1, D_{|E_1})$ (respectively, $\mathcal{E}_1' = (E_1', D_{|E_1'})$). In view of the definition of λ_V (see 6.1) and Lemma 5.9(3),(4), we are reduced to show:

$$\tilde{f}_{E_1}^*(\xi^j) = \zeta^m \cdot \rho_{E_1'}^*(\beta) \quad \text{in } \mathrm{CH}^j(E_1'), \tag{7.9.4}$$

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where $\xi = c_1(\mathcal{O}_{E_1}(1)) \in \mathrm{CH}^1(E_1)$ and $\zeta = c_1(\mathcal{O}_{E_1'}(1)) \in \mathrm{CH}^1(E_1')$. This follows from the excess intersection formula: Indeed, consider the cartesian diagram:

$$\begin{array}{ccccc} Z' & = & Z \times_X X' & \xrightarrow{f_Z} & Z \\ {\scriptstyle s'}\downarrow & & \downarrow & & \downarrow{\scriptstyle s} \\ E_1' & \hookrightarrow & E_1 \times_X X' & \longrightarrow & E_1, \end{array}$$

where the lower horizontal map is a factorisation of $\tilde{f}_{E_1}$ and s is the closed immersion induced by the zero section $Z \hookrightarrow N_{Z/X}$ followed by the open immersion $N_{Z/X} \hookrightarrow E_1$ and similarly with s'. We observe:

$$s'^*\mathcal{N}_{E_1'/E_1\times_X X'} = \mathcal{N}_{Z'/E_1\times_X X'}/\mathcal{N}_{Z'/E_1'} = f_Z^*\mathcal{N}_{Z/N_{Z/X}}/\mathcal{N}_{Z'/N_{Z'/X'}} = \mathcal{E}xc$$

(see, e.g. [Ful98, Example 6.3.2] for the first equality). Furthermore, we have $\xi^j = s_*Z$ and $\zeta^m = s'_*(Z')$. Thus, the projection formula [Ful98, Example 8.1.7] and the above yield:

$$\zeta^m \cdot \rho_{E_1'}^*(\beta) = s'_*(c_{j-m}(\mathcal{E}xc) \cap Z') = s'_*(c_{j-m}(s'^*\mathcal{N}_{E_1'/E_1\times_X X'}) \cap Z'),$$

which is equal to $\tilde{f}_{E_1}^*(s_*Z)$ by excess intersection [Ful98, Proposition 6.6(c)]. □

Lemma 7.10. *Let the notations be as in 7.2, and 7.4 and assume $j = \mathrm{codim}_X(Z) \geq 1$. Then the pullback maps $i_{E_1}^*$ and ρ^* induce the following isomorphism on cohomology with supports:*

$$i_{E_1}^* : R^j\pi_*R\rho_{1*}R\underline{\Gamma}_{E_1}F_{\mathcal{Y}} \xrightarrow{\simeq} i_*R^j\rho_{E_1*}F_{\mathcal{E}_1} \tag{7.10.1}$$

and:

$$\rho^* : R^j\underline{\Gamma}_Z F_{\mathcal{X}} \xrightarrow{\simeq} R^j\rho_*R\underline{\Gamma}_E F_{\tilde{\mathcal{X}}}. \tag{7.10.2}$$

Furthermore, if we define the local Gysin map $g^j_{\mathcal{Z}/\mathcal{X},Z}$ by the composition:

$$g^j_{\mathcal{Z}/\mathcal{X},Z} : i_*\gamma^j F_{\mathcal{Z}} \xrightarrow{\lambda_V^j} i_*R^j\rho_{E_1*}F_{\mathcal{E}_1} \xrightarrow{-(i_{E_1}^*)^{-1}} R^j\pi_*R\rho_{1*}R\underline{\Gamma}_{E_1}F_{\mathcal{Y}} \xrightarrow{i_{\tilde{\mathcal{X}}}^*} R^j\rho_*R\underline{\Gamma}_E F_{\tilde{\mathcal{X}}} \xrightarrow{(\rho^*)^{-1}} R^j\underline{\Gamma}_Z F_{\mathcal{X}},$$

where $V = \mathcal{N}_{Z/X}^\vee \oplus \mathcal{O}_Z$, then the Gysin map $g_{\mathcal{Z}/\mathcal{X}}$ is equal to the composition:

$$i_*\gamma^j F_{\mathcal{Z}}[-j] \xrightarrow{g^j_{\mathcal{Z}/\mathcal{X},Z}[-j]} R^j\underline{\Gamma}_Z F_{\mathcal{X}}[-j] \cong R\underline{\Gamma}_Z F_{\mathcal{X}} \to F_{\mathcal{X}},$$

where the isomorphism $R^j\underline{\Gamma}_Z F_{\mathcal{X}}[-j] \cong R\underline{\Gamma}_Z F_{\mathcal{X}}$ is [Sai20a, Corollary 8.6(3)] and the last arrow is the forget-supports map. Moreover, the Gysin map determines the local Gysin map via:

$$g^j_{\mathcal{Z}/\mathcal{X},Z} = R^j\underline{\Gamma}_Z(g_{\mathcal{Z}/\mathcal{X}})[-j].$$

Proof. We will use without further notice the isomorphism $R\underline{\Gamma}_A Rf_* = Rf_*R\underline{\Gamma}_{f^{-1}(A)}$, for a morphism $f : V \to W$ and a closed subset $A \subset W$. For (7.10.1), apply $R\pi_*R\underline{\Gamma}_{Z\times 0}$ to the blow-up sequence of $(X \otimes \overline{\square}, Z \times 0)$ (see Theorem 3.1) to obtain the following long exact sequence:

$$\ldots \to R^j\pi_*R\underline{\Gamma}_{Z\times 0}F_{\mathcal{X}\otimes\overline{\square}} \to R^j\pi_*R\rho_{1*}R\underline{\Gamma}_{E_1}F_{\mathcal{Y}} \xrightarrow{i_{E_1}^*} i_*R^j\rho_{E_1*}F_{\mathcal{E}_1} \to \ldots.$$

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Since $Z\times 0$ has codimension $j+1$ in $X\times\mathbf{P}^1$, the term on the left vanishes by [Sai20a, Corollary 8.6(3)]; by existence of the canonical section as in (7.2) (for $(\mathcal{X}\times\overline{\square}, Z\times 0)$ instead of $(\mathcal{X}, Z)$), we see that $i_{E_1}^*$ is split surjective. This yields the isomorphism (7.10.1). For (7.10.2), apply $R\underline{\Gamma}_Z$ to the blow-up sequence of $(\mathcal{X}, Z)$ to obtain the long exact sequence:

$$\ldots\to R^j\underline{\Gamma}_Z F_{\mathcal{X}}\xrightarrow{\rho^*} R^j\rho_*R\underline{\Gamma}_E F_{\tilde{\mathcal{X}}}\to i_*R^j\rho_{E*}F_{\mathcal{E}}\to\ldots.$$

By the existence of the canonical section σ as in (7.2), the map ρ^* is injective, and by the projective bundle formula (see Theorem 6.3), the right-hand side vanishes. This yields the isomorphism (7.10.2).

Since $R\underline{\Gamma}_Z = i_*Ri^!$ and $Ri^!$ is right adjoint to i_*, the Gysin morphism factors via the counit $R\underline{\Gamma}_Z F_{\mathcal{X}}\to F_{\mathcal{X}}$ and by the isomorphism $R^j\underline{\Gamma}_Z F_{\mathcal{X}}[-j]\cong R\underline{\Gamma}_Z F_{\mathcal{X}}$ also via:

$$g_{\mathcal{Z}/\mathcal{X}}: i_*\gamma^j F_{\mathcal{Z}}[-j]\xrightarrow{R^j\underline{\Gamma}_Z(g_{\mathcal{Z}/\mathcal{X}})[-j]} R^j\underline{\Gamma}_Z F_{\mathcal{X}}[-j]\cong R\underline{\Gamma}_Z F_{\mathcal{X}}\to F_{\mathcal{X}}.$$

Thus, it remains to show:

$$g^j_{\mathcal{Z}/\mathcal{X},Z} = R^j\underline{\Gamma}_Z(g_{\mathcal{Z}/\mathcal{X}}).$$

By (7.5.2), it remains to show that:

$$R^j\underline{\Gamma}_Z(\sigma) = (7.10.2)^{-1}, \tag{7.10.3}$$

and that $\tau_1^j := R^j\underline{\Gamma}_Z(\tau_1')$ is equal to the composition:

$$i_*R^j\rho_{E_1*}F_{\mathcal{E}_1}\xrightarrow{(7.10.1)^{-1}} R^j\pi_*R\rho_{1*}R\underline{\Gamma}_{\rho_1^{-1}(Z\times 0)}F_{\mathcal{Y}}\xrightarrow{e} R^j\pi_*R\rho_{1*}R\underline{\Gamma}_{\rho_1^{-1}(Z\times\mathbf{P}^1)}F_{\mathcal{Y}}, \tag{7.10.4}$$

where e is the enlarge-support map (note that $i_{\tilde{X}}^*$ from the definition of the local Gysin map factors via e). For (7.10.3), we observe that $R^j\underline{\Gamma}_Z(\sigma)$ is by definition a section of the isomorphism (7.10.2), hence, has to be the inverse of that isomorphism. To compute τ_1^j we consider the following diagram:

$$\begin{array}{ccccc} & & R^j\pi_*R\rho_{1*}R\underline{\Gamma}_{\rho_1^{-1}(Z\times 0)}F_{\mathcal{Y}} & & \\ & \overset{\sigma_1^j}{\swarrow} & \downarrow e & \overset{b''}{\searrow} & \\ R^j\underline{\Gamma}_Z R\pi_*F_{\mathcal{X}\otimes\overline{\square}} & \xrightarrow[\rho_1^*]{} & R^j\underline{\Gamma}_Z R\pi_*R\rho_{1*}F_{\mathcal{Y}} & \xrightarrow{b_1} & i_*R^j\rho_{E_1*}F_{\mathcal{E}_1} \\ \downarrow{\scriptstyle i_0^*=i_1^*}\ \simeq & \searrow^{i_{1,Y}^*} & & \overset{\tau_1^j}{\nwarrow} & \\ R^j\underline{\Gamma}_Z F_{\mathcal{X}}. & & & & \end{array} \tag{7.10.5}$$

Here, the middle line and the triangle on the left lower side are induced by applying $R^j\underline{\Gamma}_Z$ to the diagram (7.2.2) up to the isomorphism $\bigoplus_{i=1}^j(\gamma^i F)_{\mathcal{Z}}[-i]\simeq\tau_{\geq 1}R\rho_{E_1*}F_{\mathcal{E}_1}$ from (7.2.1), with the obvious notation for σ_1^j and τ_1^j, and b'' is the isomorphism (7.10.1). By definition of τ_1^j (see 7.2 and Lemma 7.1), we have:

$$\tau_1^j\circ b_1 = \mathrm{id} - \rho_1^*\sigma_1^j.$$

Thus:

$$\tau_1^j = \tau_1^j b_1 e(b'')^{-1} = e(b'')^{-1} - \rho_1^*\sigma_1^j e(b'')^{-1}. \tag{7.10.6}$$

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By definition, $\sigma_1^j = (i_0^*)^{-1} i_{1,Y}^* = \pi^* \circ i_{1,Y}^*$. We claim that $i_{1,Y}^* \circ e = 0$. Indeed, $i_{1,Y}^*$ is induced by $F_{\mathcal{Y}} \to i_{1,Y*} F_{\mathcal{X}\otimes 1}$. Consider the natural commutative diagram:

$$\begin{array}{ccc} R\underline{\Gamma}_{\rho_1^{-1}(Z\times 0)} F_{\mathcal{Y}} & \longrightarrow & R\underline{\Gamma}_{\rho_1^{-1}(Z\times 0)} (i_{1,Y*} F_{\mathcal{X}\otimes 1}) = 0 \\ \downarrow & & \downarrow \\ F_{\mathcal{Y}} & \longrightarrow & i_{1,Y*} F_{\mathcal{X}\otimes 1}, \end{array}$$

where the top right corner vanishes since $\rho_1^{-1}(Z \times 0) \cap i_{1,Y}(X \times \{1\}) = \emptyset$. The map $i_{1,Y}^* \circ e$ is induced by applying $R^j \underline{\Gamma}_Z R\pi_* R\rho_{1*}$ to this diagram and going counter-clockwise starting at the top left corner; hence, the vanishing $i_{1,Y}^* \circ e = 0$. Thus, (7.10.6) yields $\tau_1^j = e(b'')^{-1}$, which proves that τ_1^j is equal to the composition (7.10.4). □

7.11. We recall a general formula for the refined Gysin morphism of a blow-up. Let $a : V \hookrightarrow W$ be a regular closed immersion of quasi-projective k-schemes with normal sheaf $\mathcal{N}_{V/W}$, and denote by $f : \tilde{W} \to W$ the blow-up of W along V. We have then the cartesian square:

$$\begin{array}{ccc} \tilde{V} := f^{-1}(V) & \xrightarrow{b} & \tilde{W} \\ \downarrow g & & \downarrow f \\ V & \xrightarrow{a} & W. \end{array}$$

Since $\tilde{V} = \mathbf{P}(\mathcal{N}_{V/W}^\vee)$, the excess normal bundle satisfies:

$$\mathcal{E}xc = g^* \mathcal{N}_{V/W} / \mathcal{N}_{\tilde{V}/\tilde{W}} = g^* \mathcal{N}_{V/W} / \mathcal{O}_{\tilde{V}}(-1),$$

that is. it is the universal quotient bundle on $\mathbf{P}(\mathcal{N}_{V/W}^\vee)$ (see e.g. [Ful98, Section 6.7]).

Let $W' \to W$ be a finite-type morphism, and denote by $V', \tilde{W}', \tilde{V}'$ and by f', g', a', b' the base changes along $W' \to W$. The refined Gysin morphism of f is a map $f^! : \mathrm{CH}_m(W') \to \mathrm{CH}_m(\tilde{W}')$. Let $T \subset W'$ be an m-dimensional integral closed subscheme, and denote by $[T] \in \mathrm{CH}_m(W')$ its cycle class. Then by [Ful98, Example 6.7.1 and Proposition 17.5]:

$$f^![T] = [\tilde{T}] + b'_* \big\{ c(\mathcal{E}xc_{|\tilde{V}'}) \cap g'^* s(T \cap V', T) \big\}_m \quad \text{in } \mathrm{CH}_m(\tilde{W}'), \tag{7.11.1}$$

where $\tilde{T} = \mathrm{Bl}_{T\cap V'}(T) \subset \mathrm{Bl}_{V'}(W') \subset \tilde{W}'$ is the blow-up of T in $T \cap V'$, $c(\mathcal{E}xc_{|\tilde{V}'})$ is the total Chern class of the pullback of $\mathcal{E}xc$ to $\tilde{V}'$ and $s(T \cap V', T)$ is the Segre class of $T \cap V'$ in T defined in [Ful98, Section 4.2].

Theorem 7.12. *Let $F \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}$, $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ and $i : Z \hookrightarrow X$ a smooth closed subscheme of codimension j intersecting D transversally. Set $\mathcal{Z} = (Z, D_{|Z})$. Then we have the following equality of maps of sheaves on X_{Nis}:*

$$H^j(c_Z) = g^j_{\mathcal{Z}/\mathcal{X},Z} \circ i^* : \gamma^j F_{\mathcal{X}} \to i_* \gamma^j F_{\mathcal{Z}} \to R^j \underline{\Gamma}_Z F_{\mathcal{X}}, \tag{7.12.1}$$

where c_Z is the morphism (5.8.1) for Z viewed as cycle in $\mathrm{CH}^j_Z(X)$. In particular, the following diagram commutes in $D(X_{\mathrm{Nis}})$:

$$\begin{array}{ccc} \gamma^j F_{\mathcal{X}}[-j] & \xrightarrow{c_Z} & R\underline{\Gamma}_Z F_{\mathcal{X}} \\ \downarrow i^* & & \downarrow \\ i_* \gamma^j F_{\mathcal{Z}}[-j] & \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} & F_{\mathcal{X}}. \end{array}$$

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In particular, if X admits a k-morphism $q : X \to Z$, such that $q \circ i = \mathrm{id}_Z$ (locally in the Nisnevich topology, this is always possible, see Lemma 7.14 below), then:

$$g_{\mathcal{Z}/\mathcal{X}} = c_{\mathcal{Z}} \circ q^* : i_*\gamma^j F_{\mathcal{Z}}[-j] \to R\underline{\Gamma}_{\mathcal{Z}} F_{\mathcal{X}} \to F_{\mathcal{X}}. \tag{7.12.2}$$

Proof. The equivalence of the two statements follows from the isomorphism $R\underline{\Gamma}_{\mathcal{Z}} F_{\mathcal{X}} \cong R^j\underline{\Gamma}_{\mathcal{Z}} F_{\mathcal{X}}[-j]$ (see [Sai20a, Corollary 8.6(3)]) and the definition of the local Gysin map (see Lemma 7.10). We show the equality (7.12.1). This is a local question, and we can therefore assume that the normal sheaf of Z in X is trivial, that is:

$$\mathcal{N}_{Z/X} \cong \mathcal{O}_Z^{\oplus j}. \tag{7.12.3}$$

Let the notation be as in 7.2. Set $\xi = c_1(\mathcal{O}_{E_1}(1)) \in \mathrm{CH}^1(E_1)$. Note that the pullback $i_{E_1}^* : \mathrm{CH}^j_{E_1}(Y) \to \mathrm{CH}^j_{E_1}(E_1)$ is by definition (see Lemma 5.3) equal to the refined Gysin map (see [Ful98, 6.2]):

$$i_{E_1}^! : \mathrm{CH}^{j-1}(E_1) \to \mathrm{CH}^j(E_1)$$

corresponding to the diagram:

$$\begin{array}{ccc} E_1 & = & E_1 \\ \| & & \downarrow{\scriptstyle i_{E_1}} \\ E_1 & \xrightarrow{i_{E_1}} & Y. \end{array}$$

The normal sheaf of the immersion i_{E_1} is:

$$\mathcal{N}_{E_1/Y} = \mathcal{O}_Y(E_1)_{|E_1} = \mathcal{O}_{E_1}(-1).$$

Thus, by the excess intersection formula (see [Ful98, Corollary 6.3]), we find:

$$-i_{E_1}^!(\xi^{j-1}) = \xi^j. \tag{7.12.4}$$

Set $\eta := \xi^{j-1}$ viewed as an element in $\mathrm{CH}^j_{E_1}(Y)$. Consider the following diagram:

$$\begin{array}{ccccc} \gamma^j F_{\mathcal{X}}[-j] & \xrightarrow{(\pi\rho_1)^*} & R(\pi\rho_1)_*\gamma^j F_{\mathcal{Y}}[-j] & \xrightarrow{c_\eta} & R(\pi\rho_1)_* R\underline{\Gamma}_{E_1} F_{\mathcal{Y}} \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i_{E_1}^*} & & \downarrow{\scriptstyle -i_{E_1}^*} \\ i_*\gamma^j F_Z[-j] & \xrightarrow{\rho_{E_1}^*} & i_* R\rho_{E_1 *}\gamma^j F_{\mathcal{E}_1}[-j] & \xrightarrow{c_{\xi^j}} & i_* R\rho_{E_1 *} F_{\mathcal{E}_1}. \end{array}$$

$$\lambda_V^j$$

The right square commutes by (7.12.4) and Lemma 5.9(3), the left square clearly commutes, hence, so does the big outer square. Thus, by Lemma 7.10:

$$\rho^* \circ g^j_{\mathcal{Z}/\mathcal{X},Z} \circ i^* = i^*_{\tilde{X},E_1} \circ H^j(c_\eta) \circ (\pi\rho_1)^* : i_*\gamma^j F_{\mathcal{X}} \to R^j\rho_* R\underline{\Gamma}_E F_{\tilde{\mathcal{X}}}, \tag{7.12.5}$$

where $i^*_{\tilde{X},E_1} := R^j(\pi\rho_1)_* R\underline{\Gamma}_{E_1}(i_{\tilde{X}}^*)$.

Set $Z_1 := \rho_1^{-1}(Z \times \mathbf{P}^1)$. We have $E_1 \subset Z_1$ and $i_{\tilde{X}}^{-1}(E_1) = i_{\tilde{X}}^{-1}(Z_1) = E$. Denote by the same letter ι the two enlarge-support maps $\iota : R\underline{\Gamma}_{E_1} \to R\underline{\Gamma}_{Z_1}$ and $\iota : \mathrm{CH}^j_{E_1}(Y) \to \mathrm{CH}^j_{Z_1}(Y)$. We also denote by $i^*_{\tilde{X},Z_1}$

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the two maps induced by $i^*_{\tilde{X}}$:

$$i^*_{\tilde{X},Z_1} \ : \ R^j(\pi\rho_1)_* R\underline{\Gamma}_{Z_1} F_{\mathcal{Y}} \to R^j\rho_* R\underline{\Gamma}_E F_{\tilde{\mathcal{X}}}, \qquad \mathrm{CH}^j_{Z_1}(Y) \to \mathrm{CH}^j_E(\tilde{X}).$$

Clearly, we have, in both cases:

$$i^*_{\tilde{X},E_1} = i^*_{\tilde{X},Z_1} \circ \iota.$$

Thus, (7.12.5) yields:

$$\rho^* \circ g^j_{\mathcal{Z}/\mathcal{X},Z} \circ i^* = i^*_{\tilde{X},Z_1} \circ H^j(c_{\iota(\eta)}) \circ (\pi\rho_1)^* : i_*\gamma^j F_{\mathcal{X}} \to R^j\rho_* R\underline{\Gamma}_E F_{\tilde{\mathcal{X}}}. \tag{7.12.6}$$

The strict transform of $Z \times \mathbf{P}^1 \subset X \times \mathbf{P}^1$ in Y is the blow-up of $Z \times \mathbf{P}^1$ in the Cartier divisor $Z \times 0$ and is therefore isomorphic to $Z \times \mathbf{P}^1$. We obtain:

$$Z_1 = (Z \times \mathbf{P}^1) \cup E_1,$$

and $E_1 \cap (Z \times \mathbf{P}^1) = Z \times 0$ is embedded as the zero section in the normal bundle $N_{E_1}Y$, which is equal to $E_1 \setminus E$. On the other hand, $\tilde{X} \subset Y$ is the strict transform of $X \times 0$ and intersects E_1 in E (see, e.g. [Ful98, Section 5.1]). Thus, in Y, we have $\tilde{X} \cap (Z \times \mathbf{P}^1) = \emptyset$.

Claim 7.12.1. *We claim that the following equality holds in* $\mathrm{CH}^j_{Z_1}(Y)$*:*

$$(\pi\rho_1)^![Z] = \iota(\eta) + [Z \times \mathbf{P}^1],$$

where $(\pi\rho_1)^! : \mathrm{CH}_{d-j}(Z) \to \mathrm{CH}_{d+1-j}(Z_1)$ *is the refined Gysin map corresponding to the cartesian diagram:*

$$\begin{array}{ccc} Z_1 & \longrightarrow & Z \\ \downarrow & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{\pi\rho_1} & X. \end{array}$$

Assuming Claim 7.12.1, we can conclude as follows. The composition

$$\gamma^j F_{\mathcal{Y}}[-j] \xrightarrow{c_{[Z\times\mathbf{P}^1]}} R\underline{\Gamma}_{Z_1} F_{\mathcal{Y}} \xrightarrow{i^*_{\tilde{X}}} R\underline{\Gamma}_E F_{\tilde{\mathcal{X}}}$$

factors via:

$$R\underline{\Gamma}_{Z\times\mathbf{P}^1} F_{\mathcal{Y}} \xrightarrow{i^*_{\tilde{X}}} R\underline{\Gamma}_{(Z\times\mathbf{P}^1)\cap E} F_{\tilde{\mathcal{X}}}.$$

Since $\tilde{X} \cap (Z \times \mathbf{P}^1) = \emptyset$ by what was said after (7.12.6), we have:

$$i^*_{\tilde{X},Z_1} \circ c_{[Z\times\mathbf{P}^1]} = 0. \tag{7.12.7}$$

Thus, we obtain the following equality of maps $i_*\gamma^j F_{\mathcal{X}} \to R^j\rho_* R\underline{\Gamma}_E F_{\tilde{\mathcal{X}}}$:

$$\begin{aligned} \rho^* \circ g^j_{\mathcal{Z}/\mathcal{X},Z} \circ i^* &= i^*_{\tilde{X},Z_1} \circ H^j(c_{\iota(\eta)}) \circ (\pi\rho_1)^*, && \text{by (7.12.6)}, \\ &= i^*_{\tilde{X},Z_1} \circ H^j(c_{(\pi\rho_1)^*Z - [Z\times\mathbf{P}^1]}) \circ (\pi\rho_1)^*, && \text{by 7.12.1}, \\ &= i^*_{\tilde{X},Z_1}(\pi\rho_1)^* \circ H^j(c_Z), && \text{by (7.12.7), 5.9}, \\ &= \rho^* \circ H^j(c_Z). \end{aligned}$$

The statement follows since ρ^*, here, is an isomorphism, see (7.10.2).

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Proof of Claim 7.12.1. First note:

$$(\pi\rho_1)^![Z] = \rho_1^![Z \times \mathbf{P}^1].$$

To compute this expression, we apply the formula (7.11.1) in the case where $a = i_{Z0} : Z \times 0 \hookrightarrow X \times \mathbf{P}^1$, $f = \rho_1 : Y \to X \times \mathbf{P}^1$, $(W' \to W) = (Z \times \mathbf{P}^1 \hookrightarrow X \times \mathbf{P}^1)$ and $T = Z \times \mathbf{P}^1$ and $m = d + 1 - j$, where $d = \dim X$. In particular, we have:

$$\tilde{W}' = Z_1, \quad V' = V = Z \times 0, \quad \tilde{V}' = \tilde{V} = E_1,$$

and $\tilde{T} = Z \times \mathbf{P}^1$. Since the conormal bundle of $Z \times 0 \hookrightarrow Z \times \mathbf{P}^1$ is trivial, [Ful98, Proposition 4.1(a)] yields:

$$\rho_{E_1}^* s(Z \times 0, Z \times \mathbf{P}^1) = [E_1].$$

By (7.12.3), we have $\mathcal{N}_{Z\times 0/X\times\mathbf{P}^1} = \mathcal{O}_Z^{\oplus j+1}$. The Whitney formula yields:

$$c(\mathcal{E}xc_{|E_1}) = (1-\xi)^{-1} = 1 + \xi + \ldots + \xi^j,$$

where $\xi = c_1(\mathcal{O}_{E_1}(1))$. Thus, Claim 7.12.1 follows from (7.11.1). □

Corollary 7.13. *Let F, $\mathcal{X} = (X, D)$, $i : Z \hookrightarrow X$, and $\mathcal{Z}$ as in Theorem 7.12 above. Assume $D = \emptyset$ (thus, $\mathcal{X} = X$ and $\mathcal{Z} = Z$). Let Φ be a family of supports on Z and $\alpha \in \mathrm{CH}^r_\Phi(Z)$. Then:*

$$c_{i_*\alpha} = g_{Z/X} \circ c_\alpha \circ i^* : \gamma^{j+r}F_X[-j-r] \to \gamma^{j+r}F_Z[-j-r] \to R\underline{\Gamma}_\Phi \gamma^j F_Z[-j] \to R\underline{\Gamma}_\Phi F_X,$$

where we view $i_*\alpha \in \mathrm{CH}^{j+r}_\Phi(X)$.

Proof. It suffices to consider $\alpha = [V]$, with $V \subset Z$ irreducible and of codimension r, and $\Phi = \Phi_V$. By [Sai20a, Corollary 8.6(1)], we have $R\underline{\Gamma}_V F_X \cong \tau_{\geq j+r} R\underline{\Gamma}_V F_X$ (here, we need $D = \emptyset$). Hence, we have a natural map $R^{j+r}\underline{\Gamma}_V F_X[-j-r] \to R\underline{\Gamma}_V F_X$ in the derived category and the two maps in the statement are induced by composing this map with the two morphisms of sheaves:

$$H^{j+r}(c_{i_*V}),\ H^{j+r}(g_{Z/X} \circ c_V \circ i^*) : \gamma^{j+r}F_X \to R^{j+r}\underline{\Gamma}_V F_X. \tag{7.13.1}$$

Thus, it suffices to show that the two maps in (7.13.1) are equal. By [Sai20a, Corollary 8.6(1)], the restriction $R^{r+j}\underline{\Gamma}_V F_X \to \nu_* R^{r+j}\underline{\Gamma}_{V\setminus V_{\mathrm{sing}}} F_{X\setminus V_{\mathrm{sing}}}$ is injective, where $\nu : X \setminus V_{\mathrm{sing}} \hookrightarrow X$ is the open immersion. Thus, we may furthermore assume that V is smooth. The question is local on X, and we can therefore assume that there exists a closed subset $W \subset X$ of pure codimension r, such that $V = i^*W$ in $\mathrm{CH}^r_V(Z) = \mathrm{CH}^0(V)$. In this situation, we have the following equality of maps: $\gamma^{j+r}F_X[-j-r] \to R\underline{\Gamma}_V F_X$

$$g_{Z/X} \circ c_V \circ i^* = g_{Z/X} \circ c_{i^*W} \circ i^* = g_{Z/X} \circ i^* \circ c_W = c_Z \circ c_W = c_{Z\cdot W} = c_{i_*V},$$

where the second and forth equality hold by Lemma 5.9 and the third equality by Theorem 7.12. This implies the statement. □

Lemma 7.14. *Let S be an affine scheme and $Z \hookrightarrow X$ a closed immersion of affine S-schemes. Assume that X is Noetherian, integral and normal and Z is irreducible and formally smooth and of finite type over S. Then there exists a Nisnevich neighborhood $u : X' \to X$ of Z (i.e. u is étale and induces an isomorphism $u^{-1}(Z) \xrightarrow{\simeq} Z$) which admits an S-morphism $X' \to Z$, such that the composition $Z \cong u^{-1}(Z) \hookrightarrow X' \to Z$ is the identity.*

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Proof. We follow an argument in the proof of [Sai20a, Lemma 8.5]. Write $S = \operatorname{Spec} R$, $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A/I$. Set $Z_n := \operatorname{Spec} A/I^n$ and $\hat{X}_Z = \operatorname{Spec} \hat{A}_I$, where $\hat{A}_I = \varprojlim_n (A/I^n)$. Since Z is formally smooth over S, we find a compatible system of S-morphisms $\{Z_n \to Z\}$ which reduce to the identity on Z; it induces a morphism of S-schemes $\hat{\pi} : \hat{X}_Z \to Z$ of which the natural closed immersion $Z \hookrightarrow \hat{X}_Z$ is a section. We can form the closed immersion $\hat{\varepsilon} := \mathrm{id}_{\hat{X}_Z} \times \hat{\pi} : \hat{X}_Z \hookrightarrow \hat{X}_Z \times_S Z$ which restricts to the diagonal on $Z \times_S Z$. By [Elk73, Theorem 2bis], we find therefore an X_Z^h-morphism $\varepsilon^h : X_Z^h \hookrightarrow X_Z^h \times_S Z$ which restricts to the diagonal on $Z \times_S Z$, where $X_Z^h = \operatorname{Spec} A_I^h$ is the henselisation of the pair (X, Z). Composing ε^h with the projection to Z yields an S-morphism $u^h : X_Z^h \to Z$. Since X is normal and Noetherian, so is any affine étale scheme Y over X; in particular, such a Y is a disjoint union of integral normal X-schemes. Since Z is irreducible, any Nisnevich neighborhood $Y \to X$ of Z can be refined to a Nisnevich neighborhood $Y' \to X$ of Z with Y' integral. It follows that we can write $A_I^h = \varinjlim B$, where the limit is over all étale maps $A \to B$ inducing an isomorphism $A/I \to B/IB$ with B integral; the transition maps $B \to B'$ in this system are automatically étale, and, hence (since B and B' are integral), also injective; thus, also $B \to A_I^h$ is injective. Since A/I is of finite type over R, it follows that the R-algebra map $u^{h*} : A/I \to A_I^h$ factors via an R-algebra map $A/I \to B$ for some B as above. This yields the statement. □

Corollary 7.15. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Let $i : Z \hookrightarrow Z' \overset{i'}{\hookrightarrow} X$ be closed immersion of smooth schemes of codimension $a = \operatorname{codim}(Z, Z')$ and $b = \operatorname{codim}(Z', X)$, such that D intersects Z and Z' transversally. Set $\mathcal{Z} = (Z, D_{|Z})$ and $\mathcal{Z}' = (Z', D_{|Z'})$. We have the following equality:*

$$R^a\underline{\Gamma}_Z(g^b_{\mathcal{Z}'/\mathcal{X},Z'}) \circ g^a_{\mathcal{Z}/\mathcal{Z}',Z} = g^{a+b}_{\mathcal{Z}/\mathcal{X},Z} : i_*\gamma^{a+b}F_{\mathcal{Z}} \to R^{a+b}\underline{\Gamma}_Z\mathcal{F}_{\mathcal{X}}.$$

In particular, the following diagram commutes:

$$\begin{array}{ccc} i_*\gamma^{a+b}F_{\mathcal{Z}}[-a-b] & \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} & F_{\mathcal{X}} \\ & {\scriptstyle g_{\mathcal{Z}/\mathcal{Z}'}[-b]} \searrow \quad \nearrow {\scriptstyle g_{\mathcal{Z}'/\mathcal{X}}} & \\ & i'_*\gamma^b F_{\mathcal{Z}'}[-b]. & \end{array}$$

Proof. The second statement follows from the first and Lemma 7.10. The first statement is local in X, and we may therefore assume that we find a smooth closed subscheme $Z'' \subset X$ of codimension a, such that $Z = Z' \times_X Z''$. Since $i^* : \gamma^{a+b}F_{\mathcal{X}} \to i_*\gamma^{a+b}F_{\mathcal{Z}}$ is surjective by Lemma 7.14, it suffices to show the equality after precomposition with i^*. Consider the following diagram:

$$\begin{array}{ccccc} \gamma^{a+b}F_{\mathcal{X}} & \xrightarrow{c_{Z''}} & R^a\underline{\Gamma}_{Z''}\gamma^b F_{\mathcal{X}} & & \\ \downarrow{\scriptstyle i'^*} & & \downarrow{\scriptstyle i'^*} & \searrow^{c_{Z'}} & \\ i'_*\gamma^{a+b}F_{\mathcal{Z}'} & \xrightarrow{c_Z} & i'_*R^a\underline{\Gamma}_Z\gamma^b F_{\mathcal{Z}'} & \xrightarrow{g^b_{\mathcal{Z}'/\mathcal{X},Z'}} & R^{a+b}\underline{\Gamma}_Z F_{\mathcal{X}} \\ \downarrow & \nearrow {\scriptstyle g^a_{\mathcal{Z}/\mathcal{Z}',Z}} & & & \\ i_*\gamma^{a+b}F_{\mathcal{Z}}, & & & & \end{array} \qquad i^*$$

where the maps c_Z, $c_{Z'}$ and $c_{Z''}$ are defined viewing Z, Z' and Z'' as cycles in $\mathrm{CH}^a_Z(Z')$, $\mathrm{CH}^b_{Z'}(X)$ and $\mathrm{CH}^a_{Z''}(X)$, respectively. The square commutes by Lemma 5.9(3) and the triangles commute by Theorem 7.12. By definition of the refined intersection product, we have $Z' \cdot Z'' = Z$ in $\mathrm{CH}^{a+b}_Z(X)$. Thus, the statement follows from Lemma 5.9(4) and Theorem 7.12. □

Theorem 7.16. *Let $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ and $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. Let $i : Z \hookrightarrow X$ be a smooth closed subscheme of codimension j intersecting D transversally, and set $\mathcal{Z} = (Z, D_{|Z})$. Then there is a*

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canonical distinguished triangle in $D(X_{\mathrm{Nis}})$*:*

$$i_*\gamma^j F_{\mathcal{Z}}[-j] \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} F_{\mathcal{X}} \xrightarrow{\rho^*} R\rho_* F_{(\tilde{X}, D_{|\tilde{X}}+E)} \xrightarrow{\partial} i_*\gamma^j F_{\mathcal{Z}}[-j+1], \tag{7.16.1}$$

where $\rho : \tilde{X} \to X$ *is the blow-up of* X *along* Z *and* $E = \rho^{-1}(Z)$.

Proof. We first consider the case $j = 1$. In this case, $\tilde{X} = X$ and $Z = E$. Denote by $j : U = X \setminus Z \hookrightarrow X$ the open immersion, and set $\mathcal{U} = (U, D_{|U})$ and $\mathcal{X}' := (X, D + Z)$. Consider the following diagram of solid arrows of sheaves on X_{Nis}:

$$\begin{array}{ccccc} & & c_Z & & \\ & \overset{\curvearrowright}{} & & \overset{\curvearrowright}{} & \\ \gamma^1 F_{\mathcal{X}} & \xrightarrow[(5.10.2)]{} & F_{\mathcal{X}'}/F_{\mathcal{X}} & \hookrightarrow & j_* F_{\mathcal{U}}/F_{\mathcal{X}} \xrightarrow{\simeq} R^1\underline{\Gamma}_Z F_{\mathcal{X}}. \\ \downarrow{\scriptstyle i^*} & \nearrow{\scriptstyle (*)} & & & \\ i_*\gamma^1 F_{\mathcal{Z}} & & \xrightarrow[g^1_{\mathcal{Z}/\mathcal{X},Z}]{} & & \end{array} \tag{7.16.2}$$

The diagram commutes by Lemma 5.10 and Theorem 7.12, the vertical arrow is surjective by Lemma 7.14.

Claim 7.16.1. *The dotted arrow* $(*)$ *exists, makes the diagram commute and is an isomorphism (it is automatically uniquely determined).*

Indeed, the question is local around the points of Z. We may therefore assume that we have an étale morphism $u : X \to S[t]$, such that $S = \operatorname{Spec} K\{x_1, \dots, x_n\}$, with a function field K, $D = u^* \operatorname{Div}(x_1^{r_1} \cdots x_s^{r_s})$ and u induces an isomorphism $Z \cong u^{-1}(t = 0) \xrightarrow{\simeq} S$. In particular, we have a morphism $q : X \to S$, such that the composition:

$$q \circ i : Z \xrightarrow{\simeq} S \tag{7.16.3}$$

is an isomorphism. Thus, the arrow $(*)$ exists by (7.12.2) as the composition:

$$i_*\gamma^1 F_{\mathcal{Z}} \xrightarrow{q^*} \gamma^1 F_{\mathcal{X}} \xrightarrow{(5.10.2)} F_{\mathcal{X}'}/F_{\mathcal{X}}.$$

By Lemma 5.10, the map $(*)$ is induced by pullback along the composition:

$$X \xrightarrow{\Delta_X} X \times X \xrightarrow{u^*(t)\times \mathrm{id}_X} \mathbf{A}^1 \times X \xrightarrow{\mathrm{id}_{\mathbf{A}^1}\times q} \mathbf{A}^1 \times S;$$

this composition is equal to u. Set $\mathcal{S} := (S, \operatorname{Div}(x_1^{r_1} \cdots x_s^{r_s}))$. Hence, the map $(*)$ is on Z equal to (cf. Remark 4.7):

$$\frac{F((\mathbf{A}^1, 0) \otimes \mathcal{Z})}{F(\mathbf{A}^1 \otimes \mathcal{Z})} \cong \frac{F((\mathbf{A}^1, 0) \otimes \mathcal{S})}{F(\mathbf{A}^1 \otimes \mathcal{S})} \xrightarrow{u^*} \cdot \frac{F(X, D + Z)}{F(X, D)}.$$

It remains to show that u^* becomes an isomorphism if we replace X by a Nisnevich neighborhood around the point $(x_1, \dots, x_n, t)$. By the usual trace argument, we may assume that the field K is infinite. By [Sai20a, Lemma 6.7], we may therefore assume that (X, Z) is a V-pair over S (in the sense of [Sai20a, Definition 2.1]). Clearly, $(\mathbf{A}^1_S, 0_S)$ is also a V-pair over S and (7.16.3) gives an identification $Z \cong 0_S$. Thus, u^* is an isomorphism by [Sai20a, Corollary 2.21]. This proves Claim 7.16.1.

We construct the triangle from the statement in the case $j = 1$. Set:

$$\alpha := (*)^{-1} : F_{\mathcal{X}'}/F_{\mathcal{X}} \xrightarrow{\simeq} i_*\gamma^1 F_{\mathcal{Z}}.$$

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Denote by $r : F_{\mathcal{X}} \hookrightarrow F_{\mathcal{X}'}$ the inclusion. For exact triangles, we adopt the sign conventions from [Con00, Section 1.3]. Thus, the boundary map $\mathrm{cone}(r) \to F_{\mathcal{X}}[1]$ of the exact triangle determined by r, is given by $-\mathrm{id}_{F_{\mathcal{X}}}$ in degree -1. We define the boundary map ∂ as the composition:

$$\partial : F_{\mathcal{X}'} \to F_{\mathcal{X}'}/F_{\mathcal{X}} \xrightarrow{\alpha} i_*\gamma^1 F_{\mathcal{Z}},$$

and we define a quasi-isomorphism φ as the composition:

$$\varphi : \mathrm{cone}(r) \xrightarrow{\mathrm{qis}} F_{\mathcal{X}'}/F_{\mathcal{X}} \xrightarrow{\alpha} i_*\gamma^1 F_{\mathcal{Z}},$$

where the first map is induced by the quotient map in degree 0. It remains to show that the following diagram is commutative in $D(X_{\mathrm{Nis}})$:

$$\begin{array}{ccccccc} \mathrm{cone}(r)[-1] & \longrightarrow & F_{\mathcal{X}} & \xrightarrow{r} & F_{\mathcal{X}'} & \longrightarrow & \mathrm{cone}(r) \\ \downarrow{\scriptstyle \varphi[-1]} & & \| & & \| & & \downarrow{\scriptstyle \varphi} \\ i_*\gamma^1 F_{\mathcal{Z}}[-1] & \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} & F_{\mathcal{X}} & \xrightarrow{r} & \mathcal{F}_{\mathcal{X}'} & \xrightarrow{\partial} & i_*\gamma^1 F_{\mathcal{Z}}. \end{array} \tag{7.16.4}$$

By definition, the square on the right commutes; by Lemma 7.10, the square on the left is the big outer square of the following diagram:

$$\begin{array}{ccccc} \mathrm{cone}(r)[-1] & \longrightarrow & \mathrm{cone}(F_{\mathcal{X}} \to j_*F_{\mathcal{U}})[-1] & \longrightarrow & F_{\mathcal{X}} \\ \downarrow{\scriptstyle \mathrm{qis}} & & \downarrow{\scriptstyle \mathrm{qis}} & & \uparrow \\ F_{\mathcal{X}'}/F_{\mathcal{X}}[-1] & \longrightarrow & j_*F_{\mathcal{U}}/F_{\mathcal{X}}[-1] & \xrightarrow{\simeq} & R^1\underline{\Gamma}_{\mathcal{Z}}F_{\mathcal{X}}[-1] \\ \downarrow{\scriptstyle \alpha} & & & \nearrow{\scriptstyle g^1_{\mathcal{Z}/\mathcal{X},\mathcal{Z}}} & \\ i_*\gamma^1 F_{\mathcal{Z}}[-1], & & & & \end{array} \tag{7.16.5}$$

where the vertical arrow on the top right is the composition:

$$R^1\underline{\Gamma}_{\mathcal{Z}}F_{\mathcal{X}}[-1] \cong R\underline{\Gamma}_{\mathcal{Z}}F_{\mathcal{X}} \to F_{\mathcal{X}},$$

where the isomorphism comes from [Sai20a, Corollary 8.6(3)]. The lower triangle in (7.16.5) commutes by the definition of α, the left top square commutes by functoriality, the right top square commutes by the definitions of the involved maps. Thus, the square on the left in (7.16.4) commutes. We have constructed the canonical distinguished Gysin triangle in codimension 1.

We consider the general case $j \geq 1$. Let $\rho : \tilde{X} \to X$ be the blow-up along Z and E the exceptional divisor. Set $\tilde{\mathcal{X}} = (\tilde{X}, D_{|\tilde{X}})$ and $\tilde{\mathcal{X}}' = (\tilde{X}, D_{|\tilde{X}} + E)$ (note that $|E + D_{|\tilde{X}}|$ is a SNCD), moreover, we use the notation from 7.2. Set:

$$C := \bigoplus_{r=1}^{j-1} i_*\gamma^r F_{\mathcal{Z}}[-r],$$

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and consider the following diagram in $D(X_{\mathrm{Nis}})$:

$$\begin{array}{ccc} i_*\gamma^j F_{\mathcal{Z}}[-j] \oplus C & \xrightarrow[\simeq]{(6.1.1)[-1]} & R\rho_{E*}\gamma^1 F_{\mathcal{E}}[-1] \\ \downarrow{\scriptstyle g_{\mathcal{Z}/\mathcal{X}} \oplus (-\mathrm{id})} & & \downarrow{\scriptstyle g_{\mathcal{E}/\tilde{\mathcal{X}}}} \\ F_{\mathcal{X}} \oplus C & \xrightarrow[\simeq]{(7.3.1)} & R\rho_* F_{\tilde{\mathcal{X}}} \\ \downarrow{\scriptstyle \rho^* + 0} & & \downarrow \\ R\rho_* F_{\tilde{\mathcal{X}}'} & = & R\rho_* F_{\tilde{\mathcal{X}}'} \\ \downarrow{\scriptstyle \partial_Z \oplus 0} & & \downarrow{\scriptstyle \partial_E} \\ i_*\gamma^j F_{\mathcal{Z}}[-j+1] \oplus C[1] & \xrightarrow[\simeq]{(6.1.1)} & R\rho_{E*}\gamma^1 F_{\mathcal{E}}, \end{array} \tag{7.16.6}$$

where the right column is $R\rho_*$ applied to the Gysin triangle for $E \hookrightarrow \tilde{X}$ stemming from the codimension 1 case above and the map ∂_Z is defined so that the lower square commutes. This defines the triangle (7.16.1). Note that the right column is a distinguished triangle and the left column is the direct sum of (7.16.1) and $C \xrightarrow{-\mathrm{id}} C \to 0 \to C[1]$. If the top square commutes, then (7.16.1) is therefore a distinguished triangle (by [Nee01, Proposition 1.2.3]. Thus, it remains to show:

Claim 7.16.2. *The top square in diagram (7.16.6) commutes.*

This is equivalent to the commutativity of the squares resulting from precomposition with the canonical maps $i_*\gamma^r F_{\mathcal{Z}}[-r] \to i_*\gamma^j F_{\mathcal{Z}}[-j] \oplus C$, for $r = 1, \ldots, j$. We consider two cases.

1st case: $r = 1, \ldots, j-1$. In this case, we have to show the commutativity of the following diagram:

$$\begin{array}{ccc} i_*\gamma^r F_{\mathcal{Z}}[-r] & \xrightarrow{\lambda_V^r} & i_*R\rho_* F_{\mathcal{E}} \\ \downarrow{\scriptstyle \lambda_V^{r-1}} & & \downarrow{\scriptstyle -\tau} \\ i_*R\rho_*\gamma^1 F_{\mathcal{E}}[-1] & \xrightarrow{g_{\mathcal{E}/\tilde{\mathcal{X}}}} & R\rho_* F_{\tilde{\mathcal{X}}}, \end{array} \tag{7.16.7}$$

with τ as in (7.2.3) and $V = \mathcal{N}^\vee_{Z/X}$. Note that the two compositions $-\tau \circ \lambda_V^r$ and $g_{\mathcal{E}/\tilde{\mathcal{X}}} \circ \lambda_V^{r-1}$ factor automatically via the forget-support map $R\underline{\Gamma}_Z R\rho_* F_{\tilde{\mathcal{X}}} \to R\rho_* F_{\tilde{\mathcal{X}}}$. By applying $R\underline{\Gamma}_Z$ to the second isomorphism in (7.3.1) and using the isomorphism $R^j\underline{\Gamma}_Z F_{\mathcal{X}}[-j] \cong R\underline{\Gamma}_Z F_{\mathcal{X}}$ from [Sai20a, Corollary 8.6(3)], we obtain:

$$R\underline{\Gamma}_Z R\rho_* F_{\tilde{\mathcal{X}}} \cong R^j\underline{\Gamma}_Z F_{\mathcal{X}}[-j] \oplus i_*\tau_{\geq 1} R\rho_{E*} F_{\mathcal{E}}.$$

Since $\mathrm{Hom}_{D(X_{\mathrm{Nis}})}(i_*\gamma^r F_{\mathcal{Z}}[-r], R^j\underline{\Gamma}_Z F_{\mathcal{X}}[-j]) = 0$ for $r < j$, we see that it suffices to show the equality (7.16.7) after composing with:

$$\iota_E : R\rho_* F_{\tilde{\mathcal{X}}} \xrightarrow{i_E^*} i_*R\rho_{E*} F_{\mathcal{E}} \xrightarrow{\mathrm{can.}} i_*\tau_{\geq 1} R\rho_{E*} F_{\mathcal{E}}. \tag{7.16.8}$$

Since ι_E is a section of τ, we are reduced to show:

$$R\rho_*(i_E^* \circ g_{\mathcal{E}/\tilde{\mathcal{X}}}) \circ \lambda_V^{r-1} = -\lambda_V^r : i_*\gamma^r F_{\mathcal{Z}}[-r] \to i_*R\rho_{E*} F_{\mathcal{E}}.$$

By the definition of λ_V (see (6.1.1)) and Lemma 5.9(4), it remains to check:

$$i_E^* \circ g_{\mathcal{E}/\tilde{\mathcal{X}},E} = -c_\xi : \gamma^1 F_{\mathcal{E}}[-1] \to F_{\mathcal{E}}, \tag{7.16.9}$$

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which follows from Proposition 7.9 applied to the cartesian diagram:

$$\begin{array}{ccc} E & \xrightarrow{=} & E \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle i_E} \\ E & \xrightarrow{i_E} & \tilde{X}. \end{array}$$

2nd case: $r = j$. In this case, we have to check the commutativity of the square:

$$\begin{array}{ccc} i_*\gamma^j F_{\mathcal{Z}}[-j] & \xrightarrow{\lambda_V^{j-1}} & R\rho_{E*}\gamma^1 F_{\mathcal{E}}[-1] \\ \downarrow{\scriptstyle g_{\mathcal{Z}/\mathcal{X}}} & & \downarrow{\scriptstyle R\rho_*(g_{\mathcal{E}/\tilde{\mathcal{X}}})} \\ F_{\mathcal{X}} & \xrightarrow{\rho^*} & R\rho_* F_{\tilde{\mathcal{X}}}. \end{array} \tag{7.16.10}$$

By the the second isomorphism in (7.3.1), we have the vanishing $\iota_E \circ \rho^* = 0$, with ι_E the map from (7.16.8); hence, also $\iota_E \circ \rho^* \circ g_{\mathcal{Z}/\mathcal{X}} = 0$. On the other hand, the vanishing $\iota_E \circ R\rho_*(g_{\mathcal{E}/\tilde{\mathcal{X}}}) \circ \lambda_V^{j-1} = 0$ follows from (7.16.9), Lemma 5.9(4) and the vanishing $\xi^j = 0$ in $\mathrm{CH}^j(E)$, which holds since E has relative dimension $j-1$ over Z. By Corollary 7.3 and [Sai20a, Corollary 8.6(3)], we have:

$$\begin{aligned} R\underline{\Gamma}_Z R\rho_* F_{\tilde{\mathcal{X}}} &\cong R^j\underline{\Gamma}_Z F_{\mathcal{X}}[-j] \oplus i_*\tau_{\geq 1} R\rho_{E*} F_{\mathcal{E}} \\ &\cong R^{j-1}\rho_* R^1\underline{\Gamma}_E F_{\tilde{\mathcal{X}}}[-j] \oplus i_*\tau_{\geq 1} R\rho_{E*} F_{\mathcal{E}}. \end{aligned}$$

Hence, it suffices to show the commutativity of the diagram of sheaves:

$$\begin{array}{ccc} i_*\gamma^j F_{\mathcal{Z}} & \xrightarrow{H^{j-1}(\lambda_V)} & R^{j-1}\rho_{E*}(\gamma^1 F_{\mathcal{E}}) \\ \downarrow{\scriptstyle g^j_{\mathcal{Z}/\mathcal{X},Z}} & & \downarrow{\scriptstyle g^1_{\mathcal{E}/\tilde{\mathcal{X}},E}} \\ R^j\underline{\Gamma}_Z F_{\mathcal{X}} & \xrightarrow{\rho^*} & R^{j-1}\rho_* R^1\underline{\Gamma}_E F_{\tilde{\mathcal{X}}}. \end{array} \tag{7.16.11}$$

This is a local question, and we may therefore assume that $\mathcal{N}_{Z/X} = \mathcal{O}_Z^{\oplus j}$. Thus, we are back at showing the commutativity of (7.16.10), under the additional assumption $\mathcal{N}_{Z/X} = \mathcal{O}_Z^{\oplus j}$. Hence, the statement follows from Proposition 7.9 by observing that the excess normal sheaf in question is, in this case, equal to (see [Ful98, Section 6.7]):

$$\mathcal{E}xc = \rho_E^*\mathcal{N}_{Z/X}/\mathcal{O}_E(-1) = \mathcal{O}_E^{\oplus j}/\mathcal{O}_E(-1),$$

and that the Whitney sum formula in this case yields:

$$c_{j-1}(\mathcal{E}xc) \cap E = \xi^{j-1}.$$

This shows the commutativity in the second case $r = j$ and, hence, completes the proof of Claim 7.16.2 and the theorem. □

Remark 7.17. The reader should compare Theorem 7.16 with the classical Gysin triangle in the $\mathbf{A}^1$-motivic setting. Recall that for $X \in \mathbf{Sm}$ and $i\colon Z \hookrightarrow X$ a smooth closed subscheme of codimension i, there is a distinguished triangle in $\mathbf{DM}_{gm}(k)$, called the Gysin triangle (see, e.g. [Dég12, Section 2.20]),

$$M(X-Z) \xrightarrow{j_*} M(X) \xrightarrow{i^*} M(Z)(i)[2i] \xrightarrow{\partial_{X,Z}} M(X-Z)[1], \tag{7.17.1}$$

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which gives, after applying any realisation functor $H^{*,*}(-)$, the localisation long exact sequence:

$$\cdots \to H^{n-2i,j-i}(Z) \xrightarrow{i_*} H^{n,j}(X) \xrightarrow{j^*} H^{n,j}(X-Z) \xrightarrow{\partial_{X,Z}} H^{n+1-2i,j-i}(Z) \to \cdots . \tag{7.17.2}$$

The most significant difference between our formulation, even when $D = \emptyset$, and the formulation in the $\mathbf{A}^1$-setting is that the cohomology of the open complement $U = X - Z$ of Z in X, which appears in (7.17.2) and (7.17.1), is replaced by the cohomology of the pair $(\tilde{X}, E)$, where $\tilde{X}$ is the blow-up of Z in X and E is the exceptional divisor. In the modulus setting, where smooth schemes get replaced by 'compactifications' $\mathcal{X} = (X, D)$, we need then to 'compactify' $(X - Z)$ without changing its 'homotopy type', and the pair $(\tilde{X}, E)$ does the job. For reduced modulus, the formula in Theorem 7.16 is also witnessed in the logarithmic setting (see [BPØ22, Chapter 7, Section 5]).

8. Pushforward

In this section, we construct a pushforward for $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ along projective morphisms by using the projective bundle formula from section 6 and the Gysin map from section 7. This is a classical approach which can be found, for example, in [Har66, Chapter III] (for coherent sheaves) and the dual version in [Ful98, Chapter 6] (Chow groups) and [Dég08, Section 5] (motives). In fact. we construct, the pushforward with proper support along quasi-projective morphisms, which for the Kähler (respectively, the de Rham-Witt) differentials was done in [CR11] (respectively, [CR12]).

Definition 8.1. We say a family of supports Φ on an S-scheme X is a *family of proper supports for* X/S, if Φ consists of closed subsets in X, which are proper over S.

Definition 8.2. Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Let V be a locally free $\mathcal{O}_X$-module of rank $n+1$, and denote by $\pi : P = \mathbf{P}(V) \to X$ the projection and set $\mathcal{P} = (P, \pi^* D)$. Let $j : U \hookrightarrow P$ be an open immersion, and denote by $\pi_U : U \to X$ the restriction of π and set $\mathcal{U} = (U, \pi_U^* D)$. Let Φ be a family of proper supports for U/X, and let Ψ be a family of supports on X satisfying $\Phi \subset \pi_U^{-1}\Psi$. We define the morphism in $D(X_{\mathrm{Nis}})$:

$$\mathrm{tr}_{(\mathcal{U},\Phi)/(\mathcal{X},\Psi)} : R\pi_{U*} R\underline{\Gamma}_{\Phi} F_{\mathcal{U}} \to R\underline{\Gamma}_{\Psi}(\gamma^n F)_{\mathcal{X}}[-n] \tag{8.2.1}$$

as the composition:

$$\begin{aligned} R\pi_{U*} R\underline{\Gamma}_{\Phi} F_{\mathcal{U}} \cong R\pi_* R\underline{\Gamma}_{\Phi} F_{\mathcal{P}} &\xrightarrow{\text{enlarge supp}} R\pi_* R\underline{\Gamma}_{\pi^{-1}\Psi} F_{\mathcal{P}} \\ &\xrightarrow[\simeq]{(6.1.1)^{-1}} \bigoplus_{i=0}^{n} R\underline{\Gamma}_{\Psi}(\gamma^i F)_{\mathcal{X}}[-i] \xrightarrow{\text{proj.}} R\underline{\Gamma}_{\Psi}(\gamma^n F)_{\mathcal{X}}[-n], \end{aligned}$$

where the first isomorphism is induced from the excision isomorphism:

$$Rj_* R\underline{\Gamma}_{\Phi} F_{\mathcal{U}} = R\underline{\Gamma}_{\Phi} Rj_* j^{-1} F_{\mathcal{P}} = R\underline{\Gamma}_{\Phi} F_{\mathcal{P}}$$

stemming from the fact that Φ is by assumption also a family of supports on P. If it is clear from the context which families of support we take, we also write $\mathrm{tr}_{\mathcal{U}/\mathcal{X}}$ instead of $\mathrm{tr}_{(\mathcal{U},\Phi)/(\mathcal{X},\Psi)}$. In particular, we write (see 5.1 for notation):

$$\mathrm{tr}_{\mathcal{P}/\mathcal{X}} := \mathrm{tr}_{(\mathcal{P},\Phi_P)/(\mathcal{X},\Phi_X)} : R\pi_* F_{\mathcal{P}} \to (\gamma^n F)_{\mathcal{X}}[-n],$$

which is simply the projection to the n-th component of the inverse of the projective bundle formula (6.1.1).

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Lemma 8.3. *Assumptions and notations as in Definition 8.2.*

(1) *Let* $\iota : R\pi_{U*}R\underline{\Gamma}_{\Phi}F_{\mathcal{U}} \to R\pi_*R\underline{\Gamma}_{\pi^{-1}\Psi}F_{\mathcal{P}} = R\underline{\Gamma}_{\Psi}R\pi_*F_{\mathcal{P}}$ *be the natural map (excision composed with enlarge supports). Then:*

$$\mathrm{tr}_{(\mathcal{U},\Phi)/(\mathcal{X},\Psi)} = R\underline{\Gamma}_{\Psi}(\mathrm{tr}_{\mathcal{P}/\mathcal{X}}) \circ \iota.$$

(2) *Let* $f : Y \to X$ *be a morphism in* **Sm**, *such that* $\mathcal{Y} := (Y, f^*D) \in \underline{\mathbf{MCor}}_{ls}$. *We obtain a diagram:*

$$\begin{array}{ccccc} & & \overset{\pi_{U_Y}}{\frown} & & \\ U_Y & \hookrightarrow & P_Y & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle f_U}\downarrow & & {\scriptstyle f_P}\downarrow & & \downarrow{\scriptstyle f} \\ U & \hookrightarrow & P & \xrightarrow{\pi} & X, \end{array}$$

in which the two squares are cartesian. Set $\mathcal{U}_Y = (U_Y, \pi_Y^* f^* D)$. *The following diagram commutes:*

$$\begin{array}{ccc} Rf_*R\pi_{U_Y*}R\underline{\Gamma}_{f_U^{-1}\Phi}F_{\mathcal{U}_Y} & \xrightarrow{\mathrm{tr}_{\mathcal{U}_Y/\mathcal{Y}}} & Rf_*R\underline{\Gamma}_{f^{-1}\Psi}(\gamma^n F)_{\mathcal{Y}}[-n] \\ {\scriptstyle f_U^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ R\pi_{U*}R\underline{\Gamma}_{\Phi}F_{\mathcal{U}} & \xrightarrow{\mathrm{tr}_{\mathcal{U}/\mathcal{X}}} & R\underline{\Gamma}_{\Psi}(\gamma^n F)_{\mathcal{X}}[-n]. \end{array}$$

(3) *Let* $s : X \hookrightarrow P$ *be a section of* π *which is contained in* U, *that is,* $s(X) \subset U$. *Then* $s(X)$ *defines a proper family of supports for* U/X *and the following diagram commutes:*

$$\begin{array}{ccc} R\pi_{U*}s_*\gamma^n F_{\mathcal{X}}[-n] & \xrightarrow{g_{\mathcal{X}/\mathcal{U},s(X)}} & R\pi_{U*}R\underline{\Gamma}_{s(X)}F_{\mathcal{U}} \\ & \searrow & \downarrow{\scriptstyle \mathrm{tr}_{(\mathcal{U},s(X))/\mathcal{X}}} \\ & & \gamma^n F_{\mathcal{X}}[-n], \end{array}$$

where $g_{\mathcal{X}/\mathcal{U},s(X)} = R\underline{\Gamma}_{s(X)}(g_{\mathcal{X}/\mathcal{U}}) : s_*\gamma^n F_{\mathcal{X}}[-n] \to R\underline{\Gamma}_{s(X)}F_{\mathcal{U}}$ *is induced by the Gysin map (7.4.1).*

(4) *Let* V' *be another locally free* $\mathcal{O}_X$*-module of rank* $n'+1$, *and let* $\pi' : P' := \mathbf{P}(V') \to X$ *be the projection. Let* $U' \subset P'$ *be open and* Φ' *be a family of proper supports for* U'/X. *Denote by* $\pi'_{U'}$ *the restriction of* π' *to* U', *and set* $\mathcal{U}' := (U', {\pi'_{U'}}^* D)$. *Then* $\Xi := \Phi \times_X \Phi'$ *is a proper family of supports for* $U \times_X U'/U'$ *and for* $U \times_X U'/U$ *and the following diagram commutes:*

$$\begin{array}{ccc} R(\pi_U \times_X \pi'_{U'})_*R\underline{\Gamma}_{\Xi}(F_{\mathcal{U}\otimes_{\mathcal{X}}\mathcal{U}'}) & \xrightarrow{\mathrm{tr}_{(\mathcal{U}\otimes_{\mathcal{X}}\mathcal{U}',\Xi)/(\mathcal{U},\Phi)}} & R\pi_{U*}R\underline{\Gamma}_{\Phi}(\gamma^{n'}F)_{\mathcal{U}}[-n'] \\ {\scriptstyle \mathrm{tr}_{(\mathcal{U}\otimes_{\mathcal{X}}\mathcal{U}',\Xi)/(\mathcal{U}',\Phi')}}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}_{(\mathcal{U},\Phi)/(\mathcal{X},\Psi)}} \\ R\pi'_{U'*}R\underline{\Gamma}_{\Phi'}(\gamma^n F)_{\mathcal{U}'}[-n] & \xrightarrow{\mathrm{tr}_{(\mathcal{U}',\Phi')/(\mathcal{X},\Psi)}} & R\underline{\Gamma}_{\Psi}(\gamma^{n+n'}F)_{\mathcal{X}}[-(n+n')], \end{array}$$

where $\mathcal{U} \otimes_{\mathcal{X}} \mathcal{U}' = (U \times_X U', (\pi_U \times_X \pi'_{U'})^* D)$.

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(5) *Let $i : Z \hookrightarrow X$ be a smooth closed subscheme of codimension c intersecting D transversally, and set $\mathcal{Z} := (Z, i^*D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. We obtain the diagram:*

$$\begin{array}{ccccc} & & \overset{\pi_{U_Z}}{\frown} & & \\ U_Z & \hookrightarrow & P_Z & \xrightarrow{\pi_Z} & Z \\ \downarrow{\scriptstyle i_U} & & \downarrow{\scriptstyle i_P} & & \downarrow{\scriptstyle i} \\ U & \hookrightarrow & P & \xrightarrow{\pi} & X, \end{array}$$

in which the two squares are cartesian. Set $\mathcal{U}_Z := (U_Z, \pi_{U_Z}^* i^* D)$. Then the following square commutes:

$$\begin{array}{ccc} R\pi_{U*}R\underline{\Gamma}_{\Phi} i_{U*}(\gamma^c F)_{\mathcal{U}_Z}[-c] & \xrightarrow{g_{\mathcal{U}_Z/\mathcal{U}}} & R\pi_{U*}R\underline{\Gamma}_{\Phi} F_{\mathcal{U}} \\ \downarrow{\scriptstyle \mathrm{tr}_{\mathcal{U}_Z/\mathcal{Z}}} & & \downarrow{\scriptstyle \mathrm{tr}_{\mathcal{U}/\mathcal{X}}} \\ R\underline{\Gamma}_{\Psi} i_*(\gamma^{c+n} F)_{\mathcal{Z}}[-c-n] & \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} & R\underline{\Gamma}_{\Psi}(\gamma^n F)_{\mathcal{X}}[-n], \end{array}$$

where $g_{\mathcal{U}_Z/\mathcal{U}}$ and $g_{\mathcal{Z}/\mathcal{X}}$ are the Gysin maps.

(6) *Let $i : Z \hookrightarrow X$ and $\mathcal{Z}$ be as in (5) above, and assume that i factors as $Z \hookrightarrow U \xrightarrow{\pi_U} X$, such that $\Psi \cap Z = \Phi \cap Z$. Then $Z \hookrightarrow U$ is a closed immersion of codimension $c+n$ and then following square commutes:*

$$\begin{array}{ccc} & & R\pi_{U*}R\underline{\Gamma}_{\Phi} F_{\mathcal{U}} \\ & \overset{g_{\mathcal{Z}/\mathcal{U}}}{\nearrow} & \downarrow{\scriptstyle \mathrm{tr}_{\mathcal{U}/\mathcal{X}}} \\ R\underline{\Gamma}_{\Psi} i_* \gamma^{n+c} F_{\mathcal{Z}}[-c-n] & \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} & R\underline{\Gamma}_{\Psi} \gamma^n F_{\mathcal{X}}[-n]. \end{array}$$

Proof. (1) holds by definition. For (2), first observe that $f_U^{-1}\Phi$ is a family of proper supports for U_Y/Y; by (1), we are reduced to show:

$$f^* \circ \mathrm{tr}_{\mathcal{P}/\mathcal{X}} = \mathrm{tr}_{\mathcal{P}_Y/\mathcal{Y}} \circ f_P^*.$$

By definition, and with the notation from 6.1, this follows from the equality:

$$c_{\xi_Y^i} \circ \pi_Y^* \circ f^* = f_P^* \circ c_{\xi^i} \circ \pi^*, \quad i = 0, \dots, n, \tag{8.3.1}$$

where $\xi_Y = c_1(\mathcal{O}_{P_Y}(1)) \in \mathrm{CH}^1(P_Y)$ and $\xi = c_1(\mathcal{O}_P(1)) \in \mathrm{CH}^1(P)$. Since $f_P^* \xi = \xi_Y$, the equality (8.3.1) follows from Lemma 5.9(3). For (3), it suffices to show:

$$\mathrm{tr}_{\mathcal{P}/\mathcal{X}} \circ g_{\mathcal{X}/\mathcal{P}} = \mathrm{id}_{(\gamma^n F)_{\mathcal{X}}[-n]}. \tag{8.3.2}$$

Indeed, this follows from the equality $R\underline{\Gamma}_{s(X)} F_{\mathcal{U}} = R\underline{\Gamma}_{s(X)} F_{\mathcal{P}}$, the compatibility of the Gysin with restriction along open immersions (see Proposition 7.9) and from (1). By (7.12.2), we have $g_{\mathcal{X}/\mathcal{P}} = c_{s(X)} \circ \pi^*$, where we view $s(X) \in \mathrm{CH}^n(P)$. The projective bundle formula yields:

$$s(X) = \sum_{i=0}^{n} \pi^*(\alpha_i) \cdot \xi^i, \quad \text{for certain } \alpha_i \in \mathrm{CH}^{n-i}(X).$$

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Applying π_*, we obtain $\alpha_n = X$ from [Ful98, Example 3.3.3] and the fact that s is a section of π. By Lemma 5.9(3), (4), we obtain (with the notation from 6.1):

$$g_{\mathcal{X}/\mathcal{P}} = \sum_{i=0}^{n-1} c_{\xi^i} \circ \pi^* \circ c_{\alpha_i} + c_{\xi^n} \circ \pi^* = \sum_{i=0}^{n-1} \lambda_V^i \circ c_{\alpha_i} + \lambda_V^n.$$

Thus, equality (8.3.2) follows directly from the definition of $\mathrm{tr}_{\mathcal{P}/\mathcal{X}}$. Next (4). Note that Ξ is by definition the smallest family of supports on $U \times_X U'$ containing all closed subsets of the form $Z \times_X Z'$ with $Z \in \Phi$ and $Z' \in \Phi'$. Thus, Ξ is clearly a family of proper supports over U and U', respectively, and we have $\Xi \subset (\pi_U \times_X \pi'_{U'})^{-1}(\Psi)$. Using (1), it is easy to see that the commutativity of the square in (4) is implied by the commutativity of the following diagram:

$$\begin{array}{ccc} R(\pi \times_X \pi')_* F_{\mathcal{P}\otimes_{\mathcal{X}}\mathcal{P}'} & \xrightarrow{\mathrm{tr}_{\mathcal{P}\otimes_{\mathcal{X}}\mathcal{P}'/\mathcal{P}}} & R\pi_*(\gamma^{n'}F)_{\mathcal{P}}[-n'] \\ \downarrow{\scriptstyle \mathrm{tr}_{\mathcal{P}\otimes_{\mathcal{X}}\mathcal{P}'/\mathcal{P}'}} & & \downarrow{\scriptstyle \mathrm{tr}_{\mathcal{P}/\mathcal{X}}} \\ R\pi'_*(\gamma^n F)_{\mathcal{P}'}[-n] & \xrightarrow{\mathrm{tr}_{\mathcal{P}'/\mathcal{X}}} & (\gamma^{n+n'}F)_{\mathcal{X}}[-(n+n')]. \end{array} \tag{8.3.3}$$

Let $\xi = c_1(\mathcal{O}_P(1)) \in \mathrm{CH}^1(P)$ and $\eta = c_1(\mathcal{O}_{P'}(1)) \in \mathrm{CH}^1(P')$. Denote by $p : P \times_X P' \to P$ and $q : P \times_X P' \to P'$ the projections. With the notation from 6.1, we have for $i, j = 0, \ldots, n$,

$$\begin{aligned} \lambda^j_{\pi^*V'} \circ \lambda^i_V &= c_{q^*\eta^j} \circ p^* \circ c_{\xi^i} \circ \pi^*, && \text{by defn,} \\ &= c_{q^*\eta^j} \circ c_{p^*\xi^i} \circ p^*\pi^*, && \text{by 5.9(3),} \\ &= c_{(q^*\eta^j)\cdot(p^*\xi^i)} \circ q^*\pi'^*, && \text{by 5.9(4),} \\ &= c_{p^*\xi^i} \circ c_{q^*\eta^j} \circ q^*\pi'^*, && \text{by 5.9(4),} \\ &= c_{p^*\xi^i} \circ q^* \circ c_{\eta^j} \circ \pi'^*, && \text{by 5.9(3),} \\ &= \lambda^i_{\pi'^*V} \circ \lambda^j_{V'}, && \text{by defn.} \end{aligned}$$

Now, the commutativity of the diagram (8.3.3) follows from this and the definition of tr. For (5), it suffices as above to show that the following diagram commutes:

$$\begin{array}{ccc} R\pi_* i_{P*}(\gamma^c F)_{\mathcal{P}_Z}[-c] & \xrightarrow{g_{\mathcal{P}_Z/\mathcal{P}}} & R\pi_* F_{\mathcal{P}} \\ \downarrow{\scriptstyle \mathrm{tr}_{\mathcal{P}_Z/\mathcal{Z}}} & & \downarrow{\scriptstyle \mathrm{tr}_{\mathcal{P}/\mathcal{X}}} \\ (\gamma^{c+n}F)_{\mathcal{Z}}[-c-n] & \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} & (\gamma^n F)_{\mathcal{X}}[-n]. \end{array}$$

By definition of tr, it suffices to show for all $j = 0, \ldots, n$:

$$g_{\mathcal{P}_Z/\mathcal{P}} \circ \lambda^j_{i^*V} = \lambda^j_V \circ g_{\mathcal{Z}/\mathcal{X}} : (\gamma^{c+j}F)_{\mathcal{Z}}[-c-j] \to R\pi_* F_{\mathcal{P}}.$$

Since $\lambda^j_V = c_{\xi^j} \circ \pi^*$ and $\lambda^j_{i^*V} = c_{i_P^*\xi^j} \circ \pi_Z^*$, the above equality follows from Propositions 7.8 and 7.9. Finally, (6). By considering the diagram:

$$\begin{array}{ccccc} Z & \hookrightarrow & U \times_X Z & \hookrightarrow & U \\ & \searrow & \downarrow & & \downarrow{\scriptstyle \pi_U} \\ & & Z & \xrightarrow{i} & X \end{array}$$

with cartesian square, we see that the statement follows from (5), (3) and the functoriality of the Gysin map (see Corollary 7.15). □

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8.4. Recall from [TT90, Example 2.1.2(d) and Lemma 2.1.3] that a morphism $f : Y \to X$ in **Sm** is quasi-projective in the sense of [Gro61, Definition (5.3.1)] if and only if there is a locally free $\mathcal{O}_X$-module of finite rank V, such that f factors as an immersion $Y \hookrightarrow \mathbf{P}(V)$ followed by the projection $\mathbf{P}(V) \to X$.

We say such a morphism f has *relative dimension* r, if $r = \dim Y_i - \dim X_j$ is constant, for Y_i ranging through the connected components of Y mapping to the connected component X_j of X.

Definition 8.5. Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. Let $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$, and let $f : Y \to X$ be a quasi-projective morphism in **Sm** of relative dimension $r \in \mathbb{Z}$, which is transversal to D (see Definition 2.11). Let Φ be a family of proper supports for Y/X, and let Ψ be a family of supports on X, such that $\Phi \subset f^{-1}\Psi$. Choose a factorisation:

$$f : Y \xrightarrow{i} U \xrightarrow{\pi} X, \tag{8.5.1}$$

where i is a closed immersion of codimension c and π is the composition of an open immersion into a projective bundle over X, $U \hookrightarrow P$, followed by the projection $P \to X$. Let n be the relative dimension of π, so that $r = n - c$. Set $\mathcal{Y} := (Y, f^*D)$ and $\mathcal{U} = (U, \pi_U^* D)$.

For $e \geq c = \operatorname{codim}(Y, U)$, we define the map:

$$(i, \pi)^e_* : Rf_* R\underline{\Gamma}_\Phi \gamma^e F_{\mathcal{Y}}[-e] \to R\underline{\Gamma}_\Psi \gamma^{e+r} F_{\mathcal{X}}[-e-r] \tag{8.5.2}$$

as the following composition:

$$Rf_* R\underline{\Gamma}_\Phi \gamma^e F_{\mathcal{Y}}[-e] \xrightarrow{g_{\mathcal{Y}/\mathcal{U}}} R\pi_* R\underline{\Gamma}_\Phi \gamma^{e-c} F_{\mathcal{U}}[-e+c] \xrightarrow{\mathrm{tr}_{\mathcal{U}/\mathcal{X}}} R\underline{\Gamma}_\Psi \gamma^{e+r} F_{\mathcal{X}}[-e-r].$$

Proposition 8.6. *Assumptions as in Definition 8.5.*

(1) *Let $Y \xrightarrow{i'} U' \xrightarrow{\pi'} X$ be another factorisation as in (8.5.1). Then:*

$$(i, \pi)^e_* = (i', \pi')^e_*, \quad \textit{for all } e \geq \operatorname{codim}(Y, U \times_X U').$$

(2) *Let $g : Z \to Y$ be a quasi-projective morphism in* **Sm** *of relative dimension s, which is transversal to f^*D. Let Ξ be a family of proper supports for Z/Y, such that $\Xi \subset g^{-1}\Phi$. Let $Z \xrightarrow{i'} U' \xrightarrow{\pi'} X$ be a factorisation of fg as in (8.5.1) with $\operatorname{codim}(Z, U') = c'$. Set $U'_Y = U' \times_X Y$ and $i'_Y := i' \times g : Z \hookrightarrow U'_Y$ and $\pi'_Y := p_Y : U'_Y \to Y$. Set $\mathcal{Z} := (Z, g^* f^* D)$ etc.*

Then we have $\operatorname{codim}(Z, U'_Y) = c' + r$, Ξ is also a proper family with supports for Z/X, and for $e \geq c' + c + r$, we have a commutative diagram:

$$\begin{array}{ccc} R(fg)_* R\underline{\Gamma}_\Xi \gamma^e F_{\mathcal{Z}}[-e] & \xrightarrow{(i', \pi')^e_*} & R\underline{\Gamma}_\Psi \gamma^{e+r+s} F_{\mathcal{X}}[-e-r-s] \\ & {\scriptstyle (i'_Y, \pi'_Y)^e_*} \searrow \quad \nearrow {\scriptstyle (i, \pi)^{e+s}_*} & \\ & Rf_* R\underline{\Gamma}_\Phi \gamma^{e+s} F_{\mathcal{Y}}[-e-s]. & \end{array}$$

Proof. (1). We obtain the following diagram in which $ST = S \times_X T$ and all maps are the obvious ones:

$$\begin{array}{ccccc} Y & \to & YU' & \to & Y \\ \downarrow & & \downarrow & & \downarrow {\scriptstyle i'} \\ UY & \to & UU' & \to & U' \\ \downarrow & & \downarrow & & \downarrow {\scriptstyle \pi'} \\ Y & \xrightarrow{i} & U & \xrightarrow{\pi} & X. \end{array}$$

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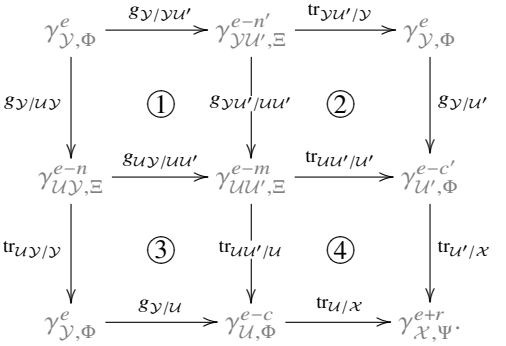
We form the modulus pairs $\mathcal{U}'$, $\mathcal{U}\mathcal{U}'$, etc. in the obvious way by pulling back the divisor from X; all these pairs are in $\underline{\mathbf{MCor}}_{ls}$, and all morphisms are transversal to the corresponding pullback of D. We can view Φ as a family of supports on Y, U, U', P, P' and $\Xi := \Phi \times_X \Phi$ as a family of supports on UY, YU', UU', PP', etc. Let $c' = \operatorname{codim}(Y, U')$, $n' = \dim(U'/X)$. We have:

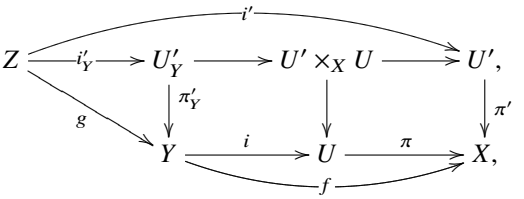
$$\operatorname{codim}(Y, U \times_X U') = n + c' = c + n' =: m.$$

We obtain the following diagram in which the grayish entries keep track of the γ-twist, the modulus pair and the support, the rest is omitted for readability:



The square ① is commutative by Corollary 7.15, the squares ② and ③ commute by Lemma 8.3(5), the square ④ commutes by Lemma 8.3(4) and finally, we have $\mathrm{tr}_{\mathcal{Y}\mathcal{U}'/\mathcal{Y}} \circ g_{\mathcal{Y}/\mathcal{Y}\mathcal{U}'} = \mathrm{id}$ and $\mathrm{tr}_{\mathcal{U}\mathcal{Y}/\mathcal{Y}} \circ g_{\mathcal{Y}/\mathcal{U}\mathcal{Y}} = \mathrm{id}$ by Lemma 8.3(3). Thus, the whole diagram commutes. It follows that going counterclockwise from the top left to the bottom right corner gives the pushforward using the factorisation (8.5.1), whereas going clockwise yields the pushforward using the primed version of this factorisation and therefore these two pushforwards agree.

(2). We have the commutative diagram:



in which the squares are cartesian. Then (2) follows directly from Lemma 8.3(4), (5), (6) and Corollary 7.15. □

8.7. Recall from 1.6 that the functor $\underline{\omega}^{\mathbf{CI}} : \mathbf{RSC}_{\mathrm{Nis}} \to \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ is right adjoint to $\underline{\omega}_!$. Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$, and set $\widetilde{F} := \underline{\omega}^{\mathbf{CI}} F$. By the weak cancellation theorem [MS20, Corollary 3.6], the natural map from 4.8:

$$\kappa_e : \widetilde{F} \xrightarrow{\simeq} \gamma^e(\widetilde{F}(e)), \quad e \geq 0 \tag{8.7.1}$$

is an isomorphism.

Let $\mathcal{X}$, $f : Y \to X$ and Φ, Ψ be as in Definition 8.5, and assume the relative dimension of f is $r = 0$. We define:

$$f_* := \kappa_e^{-1} \circ (i, \pi)^e_* \circ \kappa_e : Rf_* R\underline{\Gamma}_\Phi \widetilde{F}_\mathcal{Y} \to R\underline{\Gamma}_\Psi \widetilde{F}_\mathcal{X},$$

where $e \gg 0$. It follows from Proposition 8.6, that f_* is independent of the choice of a factorisation (8.5.2), and it follows from the commutativity of (4.8.3) that it is independent of the choice of e.

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Proposition 8.8. *Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$, and set $\widetilde{F} := \underline{\omega}^{\mathbf{CI}} F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. Let $\mathcal{X} = (X, D)$, $f : Y \to X$, $\mathcal{Y}$, Ψ and Φ be as in 8.5 above, and assume that f is of relative dimension $r = 0$.*

(1) *Let $g : Z \to Y$ be a quasi-projective morphism of relative dimension 0 in $\mathbf{Sm}$, and assume that g is transversal f^*D. Set $\mathcal{Z} := (Z, g^*f^*D)$. Let Ξ be a family of proper supports for Z/Y, such that $\Xi \subset g^{-1}\Phi$. Then Ξ is also a family of proper supports for Z/X, and we have:*

$$(f \circ g)_* = f_* g_* : Rg_* Rf_* R\underline{\Gamma}_{\Xi} \widetilde{F}_{\mathcal{Z}} \to Rf_* R\underline{\Gamma}_{\Phi} \widetilde{F}_{\mathcal{Y}} \to R\underline{\Gamma}_{\Psi} \widetilde{F}_{\mathcal{X}}.$$

(2) *Assume X and Y are connected and $\Phi = f^{-1}\Psi$. Then:*

$$\deg(Y/X) \cdot = f_* \circ f^* : R\underline{\Gamma}_{\Psi} \widetilde{F}_{\mathcal{X}} \to Rf_* R\underline{\Gamma}_{\Phi} \widetilde{F}_{\mathcal{Y}} \to R\underline{\Gamma}_{\Psi} \widetilde{F}_{\mathcal{X}},$$

where

$$\deg(Y/X) := \begin{cases} [k(Y) : k(X)] & \text{if } f \text{ is dominant} \\ 0 & \text{else.} \end{cases}$$

(3) *Assume X and Y are connected and f is proper and its restriction $f_{|Y \setminus |f^*D|} : Y \setminus |f^*D| \to X \setminus |D|$ is finite and surjective. Then:*

$$H^0(f_*) = (\Gamma_f^t)^* : f_* \widetilde{F}_{\mathcal{Y}} \to \widetilde{F}_{\mathcal{X}},$$

where $\Gamma_f^t \in \underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X}, \mathcal{Y})$ is the transpose of the graph of f.

Proof. (1). This follows from Proposition 8.6(2). (2). Choose a factorisation (8.5.1) and $e \geq c = n$, then $f_* \circ f^*$ is by Theorem 7.12 equal to the composition:

$$\begin{aligned} R\underline{\Gamma}_{\Psi} \widetilde{F}_{\mathcal{X}} &\xrightarrow{\kappa_e} R\underline{\Gamma}_{\Psi} \gamma^e (\widetilde{F}(e))_{\mathcal{X}} \\ &\xrightarrow{\pi_U^*} R\pi_{U*} R\underline{\Gamma}_{\Phi} \gamma^e (\widetilde{F}(e))_{\mathcal{U}} \\ &\xrightarrow{c_Y} R\pi_{U*} R\underline{\Gamma}_{\Phi} \gamma^{e-n} (\widetilde{F}(e))_{\mathcal{U}}[n] \\ &\xrightarrow{\mathrm{tr}_{\mathcal{U}/\mathcal{X}}} R\underline{\Gamma}_{\Psi} \gamma^e (\widetilde{F}(e))_{\mathcal{U}} \\ &\xrightarrow[\simeq]{\kappa_e^{-1}} R\underline{\Gamma}_{\Psi} \widetilde{F}_{\mathcal{X}}. \end{aligned}$$

Let $\overline{Y} \subset P$ be the closure of Y; it induces a cycle in $\mathrm{CH}^n(P)$. It remains to show:

Claim 8.8.1. *For $G \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ the composition:*

$$\gamma^n G_{\mathcal{X}} \xrightarrow{\pi^*} R\pi_* \gamma^n G_{\mathcal{P}} \xrightarrow{c_{\overline{Y}}} R\pi_* G_{\mathcal{P}}[n] \xrightarrow{\mathrm{tr}_{\mathcal{P}/\mathcal{X}}} \gamma^n G_{\mathcal{X}},$$

is equal to the multiplication with $\deg(Y/X)$.

To this end, let $\xi = c_1(\mathcal{O}_P(1)) \in \mathrm{CH}^1(P)$. By the projective bundle formula, there exist cycles $\alpha_i \in \mathrm{CH}^{n-i}(X)$, such that:

$$\overline{Y} = \sum_{i=0}^{n} \pi^* \alpha_i \cdot \xi^i, \quad \text{in } \mathrm{CH}^n(P).$$

Applying $\pi_* : \mathrm{CH}^n(P) \to \mathrm{CH}^0(X) = \mathbb{Z}$, we find $\deg(Y/X) = \alpha_n$, and, hence, Claim 8.8.1 follows from Lemma 5.9(1),(3),(4) and the definition of $\mathrm{tr}_{\mathcal{P}/\mathcal{X}}$.

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(3). By semipurity, we can assume $D = \emptyset$. Since restriction to a dense open subset is injective for $F \in \mathbf{RSC}_{\mathrm{Nis}}$ (e.g. [Sai20a, Theorem 3.1]), we can reduce to the case where X and Y are points and f is induced by a finite field extension; since both sides of the equality in (3) are transitive, we can assume that this field extension is simple, so that f factors as a closed immersion $Y \hookrightarrow \mathbf{P}^1_X$ followed by the projection $\mathbf{P}^1_X \to X$. In this situation, $H^0(f_*)$ is equal to the composition:

$$F(Y) \xrightarrow{\kappa_1} (\gamma^1 \widetilde{F}(1))(Y) \xrightarrow{g_{Y/\mathbf{P}^1_X}} H^1(\mathbf{P}^1_Y, \widetilde{F}(1)_{(\mathbf{P}^1_Y, \emptyset)}) \xrightarrow{\mathrm{tr}_{\mathbf{P}^1_X/X}} (\gamma^1 \widetilde{F}(1))(X) \xrightarrow{\kappa_1^{-1}} F(X).$$

In the following, we set:

$$\mathrm{Tr}_h := (\Gamma_h^t)^* : h_* F_{U'} \to F_U,$$

for a finite surjective morphism $h : U' \to U$ in **Sm**.

Claim 8.8.2. *Let V be a locally free $\mathcal{O}_U$-module of rank $n+1$. Set $P = \mathbf{P}(V)$ and $P' = \mathbf{P}(h^*V)$, and denote by $h' : P' \to P$ the base change of $h : U' \to U$, so that we have a commutative diagram:*

$$\begin{array}{ccc} P' & \xrightarrow{h'} & P \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ U' & \xrightarrow{h} & U. \end{array}$$

Let $G \in \mathbf{CI}^{\tau, sp}_{\mathrm{Nis}}$. Then:

$$\mathrm{Tr}_h \circ \mathrm{tr}_{P'/U'} = \mathrm{tr}_{P/U} \circ \mathrm{Tr}_{h'} : H^i(P', G_{P'}) \to H^{i-n}(U, (\gamma^n G)_U).$$

for all i.

We prove the claim. Let $\lambda^i_V = c_{\xi^i} \circ \pi^*$ be as in (6.1), where $\xi = c_1(\mathcal{O}_P(1)) \in \mathrm{CH}^1(P)$ and $\pi : P \to U$ is the projection. Then by the definition of $\mathrm{tr}_{P/U}$, it suffices to show:

$$\mathrm{Tr}_{h'} \circ \lambda^i_{h^*V} = \lambda^i_V \circ \mathrm{Tr}_h : h_*(\gamma^i G)_{U'}[-i] \to R\pi_* G_P, \quad \text{for all } i.$$

We know $\pi^* \circ \mathrm{Tr}_h = \mathrm{Tr}_{h'} \circ \pi'^*$, since the pullback is compatible with the composition of finite correspondences. Thus, we are left to show the commutativity of:

$$\begin{array}{ccc} (\gamma^i F)_P[-i] & \xrightarrow{c_{\xi^i}} & F_P \\ \uparrow{\scriptstyle \mathrm{Tr}_{h'}} & & \uparrow{\scriptstyle \mathrm{Tr}_{h'}} \\ h'_*(\gamma^i F)_{P'}[-i] & \xrightarrow[c_{h'^*\xi^i}]{} & h'_* F_{P'}. \end{array} \tag{8.8.1}$$

Using the definition of c_{ξ^i} in 5.8 and the explicit description (5.5.2) of the map (5.5.1), we see that (8.8.1) follows from the projection formula:

$$(\mathrm{Tr}_{h'}(a) \otimes \beta \otimes \Delta_P) = \mathrm{Tr}_{h'}(a \otimes h'^* \beta \otimes \Delta_{P'}) \quad \text{in } (G \otimes_{\underline{\mathbf{MPST}}} \underline{\omega}^* K^M_i)(P),$$

where $a \in G(P')$, $\beta \in K^M_i(P)$, Δ_P and $\Delta_{P'}$ are the respective diagonals. The projection formula follows from the description of $\otimes_{\underline{\mathbf{MPST}}}$ (e.g. [RSY22, Lemma 4.3]), and the equality of finite correspondences:

$$(\mathrm{id}_{P'} \times \Gamma_{h'}) \circ \Gamma_{\Delta_{P'}} \circ \Gamma^t_{h'} = (\Gamma^t_{h'} \times \mathrm{id}_P) \circ \Gamma_{\Delta_P} \in \mathbf{Cor}(P, P' \times P), \tag{8.8.2}$$

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which can be deduced from the cartesian diagram:

$$\begin{array}{ccc} P' & \xrightarrow{(\mathrm{id}_{P'}\times h')\circ\Delta_{P'}} & P'\times P \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h'\times\mathrm{id}_P} \\ P & \xrightarrow{\Delta_P} & P\times P. \end{array}$$

This completes the proof of the claim.

We come back to the proof (3). Consider the following commutative diagram:

$$\begin{array}{ccccc} Y & \hookrightarrow & \mathbf{P}^1_Y & \xrightarrow{f_!} & \mathbf{P}^1_X \\ & \searrow & \downarrow & & \downarrow \\ & & Y & \xrightarrow{f} & X, \end{array} \tag{8.8.3}$$

in which the vertical maps are the projections and the square is cartesian. Clearly, it suffices to show for $G = \widetilde{F}(1)$ and with the notation from above:

$$\mathrm{Tr}_f = \mathrm{tr}_{\mathbf{P}^1_X/X} \circ g_{Y/\mathbf{P}^1_X} : (\gamma^1 G)(Y) \to (\gamma^1 G)(X). \tag{8.8.4}$$

We compute using (8.8.3):

$$\begin{aligned} \mathrm{Tr}_f &= \mathrm{Tr}_f \circ (\mathrm{tr}_{\mathbf{P}^1_Y/Y} \circ g_{Y/\mathbf{P}^1_Y}) && \text{by Lem 8.3(3)} \\ &= \mathrm{tr}_{\mathbf{P}^1_X/X} \circ Tr_{f_!} \circ g_{Y/\mathbf{P}^1_Y} && \text{by Claim 8.8.2} \\ &= \mathrm{tr}_{\mathbf{P}^1_X/X} \circ g_{Y/\mathbf{P}^1_X} && \text{by Lem 8.9below.} \end{aligned}$$

This proves (8.8.4) and finishes the proof of the proposition. □

Lemma 8.9. *Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$ and $\widetilde{F} = \underline{\omega}^{\mathbf{CI}} F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. Let $f : X_1 \to X$ be a finite and surjective morphism in* **Sm**, *and let Z be a smooth k-scheme which comes with two closed immersions $i : Z \hookrightarrow X$ and $i_1 : Z \hookrightarrow X_1$ both of codimension 1, such that $i = f \circ i_1$. Then the following diagram commutes in $D(X_{\mathrm{Nis}})$:*

$$\begin{array}{ccc} f_* i_{1*}(\gamma^1\widetilde{F})_Z[-1] & \xrightarrow{g_{Z/X_1}} & f_*\widetilde{F}_{X_1} \\ \| & & \downarrow{\scriptstyle (\Gamma_f^t)^*} \\ i_*(\gamma^1\widetilde{F})_Z[-1] & \xrightarrow{g_{Z/X}} & \widetilde{F}_X, \end{array}$$

where Γ_f^t is the transpose of the graph of f, which we view in $\underline{\mathbf{M}}\mathbf{Cor}((X,\emptyset),(X_1,\emptyset)) = \mathbf{Cor}(X_1, X)$.

Proof. Note that $(\Gamma_f^t)^*$ also induces a morphism on the cohomology with supports:

$$f_* R^1\underline{\Gamma}_{Z_1} F_{X_1} \to f_* R^1\underline{\Gamma}_{f^{-1}(Z)} F_{X_1} = R^1\underline{\Gamma}_Z(f_* F_{X_1}) \xrightarrow{(\Gamma_f^t)^*} R^1\underline{\Gamma}_Z F_X, \tag{8.9.1}$$

where we use $R^i f_* = 0$ for all $i > 0$, which holds by the finiteness of f. Here, $Z_1 = i_1(Z)$, and the first arrow is the enlarge-support map. Using this map and the local Gysin map from Lemma 7.10, we see that the statement is local around Z. By Lemma 7.14, we can replace X by a Nisnevich neighborhood of Z to find a morphism $q : X \to Z$, such that $q \circ i = \mathrm{id}_Z$. Set $q_1 := q \circ f : X_1 \to Z$; it satisfies

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$q_1 \circ i_1 = \mathrm{id}_Z$. Whence, Theorem 7.12 yields:

$$g_{Z/X} = \iota \circ c_Z \circ q^* : i_*(\gamma^1 F)_Z[-1] \to R\underline{\Gamma}_Z F_X \to F_X,$$

$$(\Gamma_f^t)^* \circ g_{Z/X_1} = (\Gamma_f^t)^* \circ \iota \circ c_{Z_1} \circ q_1^* : f_* i_{1*}(\gamma^1 F)_{Z_1}[-1] \to f_* R\underline{\Gamma}_{Z_1} F_{X_1} \to f_* F_{X_1},$$

where c_Z (respectively, c_{Z_1}) is defined viewing $Z \in \mathrm{CH}^1_Z(X)$ (respectively, $Z_1 \in \mathrm{CH}^1_{Z_1}(X_1)$) and ι is the forget-support map in both cases. Thus, it suffices to show the equality:

$$c_Z = (8.9.1) \circ c_{Z_1} \circ f^* : (\gamma^1 F)_X[-1] \to R\underline{\Gamma}_Z F_X, \tag{8.9.2}$$

since, clearly, we also have the commutativity:

$$(\Gamma_f^t)^* \circ \iota = \iota \circ (8.9.1),$$

where ι is, again, the forget-support map.

Again, this statement is local in Z, and we can therefore assume that there are global functions $d \in H^0(X, \mathcal{O}_X)$ and $d_1 \in H^0(X_1, \mathcal{O}_{X_1})$ with:

$$Z = \mathrm{Div}_X(d) \quad \text{and} \quad Z_1 = \mathrm{Div}_{X_1}(d_1).$$

Since as cycles we have $f_* Z_1 = Z$, we can (by [Ful98, Proposition 1.4]), additionally choose d and d_1, such that:

$$\mathrm{Nm}_{X_1/X}(d_1) = d. \tag{8.9.3}$$

Note that Γ_f^t also defines an element in $\underline{\mathbf{M}}\mathbf{Cor}((X, Z), (X_1, Z_1))$, and, thus, Lemma 5.10 together with Remark 4.7 show that (8.9.2) is implied by the commutativity of the following diagram:

$$\begin{array}{ccc} \widetilde{F}((\mathbf{A}^1, 0) \otimes X_1) & \xrightarrow{(5.10.2)} & \widetilde{F}(X_1, Z_1)/F(X_1) \\ \uparrow{\scriptstyle (\mathrm{id} \times f)^*} & & \downarrow{\scriptstyle (\Gamma_f^t)^*} \\ \widetilde{F}((\mathbf{A}^1, 0) \otimes X) & \xrightarrow{(5.10.2)} & \widetilde{F}(X, Z)/F(X). \end{array}$$

We prove the commutativity of the above diagram. By (5.10.3), it suffices to show:

$$(\Gamma_f^t)^* \Delta_{X_1}^* (d_1 \times \mathrm{id}_{X_1})^* (\mathrm{id}_{\mathbf{A}^1} \times f)^* = \Delta_X^* (d \times \mathrm{id}_X)^* : \widetilde{F}((\mathbf{A}^1, 0) \times X) \to \widetilde{F}(X, Z)/F(X), \tag{8.9.4}$$

where by abuse of notation, we denote by $d : (X, Z) \to (\mathbf{A}^1, 0)$ (respectively, $d_1 : (X_1, Z) \to (\mathbf{A}^1, 0)$) the morphisms of modulus pairs induced by d (respectively, d_1) and by $\Delta_X : (X, Z) \to (X, Z) \otimes X$ (respectively, $\Delta_{X_1} : (X_1, Z) \to (X_1, Z) \otimes X_1$) the diagonal map. We have:

$$\begin{aligned} (\Gamma_f^t)^* \Delta_{X_1}^* (d_1 \times \mathrm{id}_{X_1})^* (\mathrm{id}_{\mathbf{A}^1} \times f)^* &= (\Gamma_f^t)^* \Delta_{X_1}^* (\mathrm{id}_{X_1} \times f)^* (d_1 \times \mathrm{id}_X)^* \\ &= \Delta_X^* (\Gamma_f^t \times \mathrm{id}_X)^* (d_1 \times \mathrm{id}_X)^*, \end{aligned} \tag{8.9.5}$$

where the second equality is induced by (8.8.2). Note that the graph of d_1 in $X_1 \times \mathbf{A}^1$ is given by $V(t - d_1)$, where t is the coordinate of $\mathbf{A}^1$. As in [Ful98, Propositions 1.4 and 16.1.1], we find:

$$(\Gamma_f^t \times \mathrm{id}_X)^* (d_1 \times \mathrm{id}_X)^* = ((f \times \mathrm{id}_{\mathbf{A}^1})_* (\Gamma_{d_1}) \times \mathrm{id}_X)^* = (\mathrm{Div}_{X \times \mathbf{A}^1}(P(t)) \times \mathrm{id}_X)^*, \tag{8.9.6}$$

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where $P(t) = \mathrm{Nm}_{X_1[t]/X[t]}(t - d_1) \in \mathcal{O}(X)[t]$ is the minimal polynomial of d_1 over $k(X)$. Note that by (8.9.3), we have:

$$P(t) = t^n - \mathrm{Tr}_{X_1/X}(d_1)t^{n-1} + \ldots + (-1)^n d,$$

where $n = \deg(X_1/X)$. Putting (8.9.5) and (8.9.6) together, and setting $G = \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(X), \widetilde{F}) \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, we find that (8.9.4) is implied by:

$$\mathrm{Div}_{X\times\mathbf{A}^1}(t-d)^* = \mathrm{Div}_{X\times\mathbf{A}^1}(P(t))^* : G(\mathbf{A}^1, 0) \to G(X, Z)/G(X), \tag{8.9.7}$$

where we view $\mathrm{Div}_{X\times\mathbf{A}^1}(t-d), \mathrm{Div}_{X\times\mathbf{A}^1}(P(t)) \in \underline{\mathbf{M}}\mathbf{Cor}((X, Z), (\mathbf{A}^1, 0))$. To show this we can shrink X around the generic point of Z (by purity, see [Sai20a, Corollary 8.6(1)]). Thus, in the following, we assume $Z \in X$ is a point with residue $K = k(Z)$. The map $q : X \to Z$ from the beginning of the proof induces a morphism $X \to \mathrm{Spec}\, K[d]$ which is étale. By [Sai20a, Remark 2.2(1) and Lemmas 4.2 and 4.3], we see that $(\mathbf{A}^1_K = \mathrm{Spec}\, K[t], 0_K)$ and (X, Z) are V-pairs over K in the sense of [Sai20a, Definition 2.1]. There is a canonical identification $0_K \cong Z$. We claim that $g \in \{\frac{t-d}{t-1}, \frac{P(t)}{(t-1)^n}\}$ is admissible for the pair $((\mathbf{A}^1_K, 0_K), (X, Z))$ in the sense of [Sai20a, Definition 2.3], that is, we have to show:

(1) g is regular in a neighborhood of $X \times_K 0_K$;
(2) $\mathrm{Div}_{X\times\mathbf{A}^1}(g) \times_{\mathbf{A}^1} 0_K = \Delta_{0_K}$, where $\Delta_{0_K} : 0_K \hookrightarrow X \times_K 0_K$ is the diagonal (via the identification $0_K = Z$ fixed above);
(3) g extends to an invertible function in a neighborhood of $X \times_K \infty_K$ in $X \times_K \mathbf{P}^1_K$.

All points are immediate to check. Therefore, [Sai20a, Theorem 2.10(2)] yields:

$$\mathrm{Div}_{X\times_K\mathbf{A}^1_K}(\tfrac{t-d}{t-1})^* = \mathrm{Div}_{X\times_K\mathbf{A}^1_K}(\tfrac{P(t)}{(t-1)^n})^* : G(\mathbf{A}^1_K, 0_K)/G(\mathbf{A}^1_K) \to G(X, Z)/G(X).$$

Since $\mathrm{Div}_{X\times\mathbf{A}^1}(t-1)^*(G(\mathbf{A}^1, 0)) \subset G(X)$, this implies (8.9.7) and completes the proof of the lemma. □

9. Proper correspondence action on reciprocity sheaves

In this section, we fix a reciprocity sheaf $F \in \mathbf{RSC}_{\mathrm{Nis}}$ and set $\widetilde{F} := \underline{\omega}^{\mathbf{CI}}F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, see (1.6.1).

9.1. Pushforward and cycle cupping for reciprocity sheaves

9.1. Recall the twists of reciprocity sheaves from [RSY22, Section 5.5]: For $n \geq 1$, we define recursively:

$$F\langle 0\rangle := F, \quad F\langle n\rangle := \underline{\omega}_!(\underline{\omega}^{\mathbf{CI}}(F\langle n-1\rangle)(1)), \tag{9.1.1}$$

where $(-)(1)$ denotes the twist from Definition 4.4. Thus, $F\langle n\rangle \in \mathbf{RSC}_{\mathrm{Nis}}$ and $F\langle m+n\rangle = F\langle m\rangle\langle n\rangle$, for all $m, n \geq 0$. There exists a natural surjective map:

$$\underline{\omega}_!(\widetilde{F}(n)) \twoheadrightarrow F\langle n\rangle, \tag{9.1.2}$$

which is defined as follows: Let $G \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$; twisting the adjunction map $G \to \underline{\omega}^{\mathbf{CI}}\underline{\omega}_!G$ by (1) and applying $\underline{\omega}_!$, yields a map $\underline{\omega}_!(G(1)) \to (\underline{\omega}_!G)\langle 1\rangle$. For $G = \tilde{F}$, this yields (9.1.2) for $n = 1$, and, in general, we define it recursively by:

$$\underline{\omega}_!(\widetilde{F}(n)) = \underline{\omega}_!(\widetilde{F}(n-1)(1)) \to \underline{\omega}_!(\widetilde{F}(n-1))\langle 1\rangle \to F\langle n-1\rangle\langle 1\rangle = F\langle n\rangle.$$

The surjectivity holds by [RSY22, Paragraph 5.21 (4)]. It is not known, in general, whether this is an isomorphism.

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We also define recursively for $n \geq 0$:

$$\gamma^0 F := F, \quad \gamma^n F := \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \gamma^{n-1} F). \tag{9.1.3}$$

It follows from [MS20, Proposition 2.10] that for $G \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, we have:

$$\gamma^n \underline{\omega}_! G = \underline{\omega}_! \gamma^n G \in \mathbf{RSC}_{\mathrm{Nis}}. \tag{9.1.4}$$

We obtain an isomorphism:

$$\begin{aligned} \gamma^n F\langle n\rangle \cong \gamma^{n-1}\underline{\omega}_!\gamma^1((\widetilde{F\langle n-1\rangle})(1)) \cong \gamma^{n-1}\underline{\omega}_!(\widetilde{F\langle n-1\rangle}) \\ \cong \gamma^{n-1} F\langle n-1\rangle \cong F, \end{aligned} \tag{9.1.5}$$

where the first isomorphism holds by definition and (9.1.4), the second by the weak cancellation theorem [MS20, Corollary 3.6], the third by $\underline{\omega}_!\tau_!\omega^{\mathbf{CI}} = \mathrm{id}$ and the forth by induction. It is direct to check that the composition:

$$F = \underline{\omega}_!\widetilde{F} \xrightarrow[\simeq]{\kappa_n} \underline{\omega}_!(\gamma^n \widetilde{F}(n)) \xrightarrow[\simeq]{(9.1.4)} \gamma^n \underline{\omega}_! \widetilde{F}(n) \xrightarrow{(9.1.2)} \gamma^n F\langle n\rangle \xrightarrow[\simeq]{(9.1.5)} F \tag{9.1.6}$$

is the identity.

Lemma 9.2. *The functor $\gamma^n : \mathbf{RSC}_{\mathrm{Nis}} \to \mathbf{RSC}_{\mathrm{Nis}}$ is exact, for all $n \geq 0$. Furthermore, if $\mathrm{char}(k) = 0$, and:*

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

is an exact sequence in $\underline{\mathbf{MNST}}$ with $G_i \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, then so is:

$$0 \to \gamma^n(G_1) \to \gamma^n(G_2) \to \gamma^n(G_3) \to 0.$$

Proof. First recall that $\mathbf{RSC}_{\mathrm{Nis}}$ is an abelian category (by [Sai20a, Theorem 0.1]), and that a sequence $0 \to F_1 \to F_2 \to F_3 \to 0$ in $\mathbf{RSC}_{\mathrm{Nis}}$ is exact if and only if the sequence $0 \to (F_1)_X \to (F_2)_X \to (F_3)_X \to 0$ of sheaves on X_{Nis} is exact, for any $X \in \mathbf{Sm}$. It suffices to consider the case $n = 1$. Given a short exact sequence in $\mathbf{RSC}_{\mathrm{Nis}}$ as above, we obtain for $X \in \mathbf{Sm}$ a short exact sequence on $\mathbf{P}^1_{X,\mathrm{Nis}}$:

$$0 \to (\widetilde{F}_1)_{(\mathbf{P}^1_X,\emptyset)} \to (\widetilde{F}_2)_{(\mathbf{P}^1_X,\emptyset)} \to (\widetilde{F}_3)_{(\mathbf{P}^1_X,\emptyset)} \to 0.$$

Applying $R\pi_*$, with $\pi : \mathbf{P}^1_X \to X$ the structure map, we get a short exact sequence:

$$0 \to R^1\pi_*(\widetilde{F}_1)_{(\mathbf{P}^1_X,\emptyset)} \to R^1\pi_*(\widetilde{F}_2)_{(\mathbf{P}^1_X,\emptyset)} \to R^1\pi_*(\widetilde{F}_3)_{(\mathbf{P}^1_X,\emptyset)} \to 0$$

using the fact that $\pi_*(\widetilde{F}_3)_{(\mathbf{P}^1_X,\emptyset)} = (\widetilde{F}_3)_{(X,\emptyset)}$. Applying the projective bundle formula (see Theorem 6.3) yields an exact sequence:

$$0 \to (\underline{\omega}_!\gamma^1\widetilde{F}_1)_X \to (\underline{\omega}_!\gamma^1\widetilde{F}_2)_X \to (\underline{\omega}_!\gamma^1\widetilde{F}_3)_X \to 0.$$

The first statement follows from (9.1.4).

Now assume $\mathrm{char}(k) = 0$. We show the second statement. Since γ is left exact on $\underline{\mathbf{MPST}}$, it suffices to show the surjectivity. By Lemma 1.3, Corollary 4.5 and resolution of singularities, it suffices to show $\gamma(G_2)_{\mathcal{X}} \to \gamma(G_3)_{\mathcal{X}}$ is surjective for all $\mathcal{X} \in \underline{\mathbf{MCor}}_{ls}$. This follows from the projective bundle formula, Theorem 6.3, as above. □

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Proposition 9.3. *For $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, we have $\underline{\omega}_!\gamma^n F = \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}^M_n, \underline{\omega}_! F)$. In particular, for $F \in \mathbf{RSC}_{\mathrm{Nis}}$, we have $\gamma^n F = \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}^M_n, F)$ (this is [MS20, Proposition 2.10] for $n = 1$).*

Proof. Thanks to (9.1.4) and (4.5.1), we have to show that the natural morphism:

$$\underline{\omega}_! \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\underline{\omega}^*\mathcal{K}^M_n, F) \to \underline{\mathrm{Hom}}_{\mathbf{PST}}(\underline{\omega}_!\underline{\omega}^*\mathcal{K}^M_n, \underline{\omega}_! F) = \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}^M_n, \underline{\omega}_! F) \tag{9.3.1}$$

is an isomorphism for every $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. Evaluating both sides of (9.3.1) on $X \in \mathbf{Sm}$, we see that we can replace F by $F^X = \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\mathbb{Z}_{tr}(X), F)$ and are left to show that:

$$\mathrm{Hom}_{\mathbf{PST}}(\mathcal{K}^M_n, \underline{\omega}_! F) = (\underline{\omega}_!\gamma^n F)(k) = (\gamma^n F)(k).$$

Indeed, we have:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{PST}}(\mathcal{K}^M_n, \underline{\omega}_! F) &\cong^{(1)} \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\underline{\omega}^*\mathcal{K}^M_n, \underline{\omega}^*\underline{\omega}_! F) \\
&\cong^{(2)} \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\underline{\omega}^*\mathcal{K}^M_n, \underline{\omega}^{\mathbf{CI}}\underline{\omega}_! F) \\
&\cong^{(3)} (\gamma^n \underline{\omega}^{\mathbf{CI}}\underline{\omega}_! F)(k) \\
&\cong^{(4)} H^n(\mathbf{P}^n_k, (\underline{\omega}^{\mathbf{CI}}\underline{\omega}_! F)_{(\mathbf{P}^n_k, \emptyset)}) \\
&\cong^{(5)} H^n(\mathbf{P}^n_k, F_{(\mathbf{P}^n_k, \emptyset)}) \cong^{(6)} (\gamma^n F)(k),
\end{aligned}$$

where the isomorphism (1) follows from the fact that $\underline{\omega}^*$ is fully faithful, (2) follows from the fact that $\underline{\omega}^*\mathcal{K}^M_n \in \mathbf{CI}^\tau$, by adjunction and definition of $\underline{\omega}^{\mathbf{CI}}$, (3) is (4.5.1), the isomorphisms (4) and (6) follow from Theorem 6.3 and isomorphism (5) by $\underline{\omega}_!\underline{\omega}^{\mathbf{CI}} = \mathrm{id}$. □

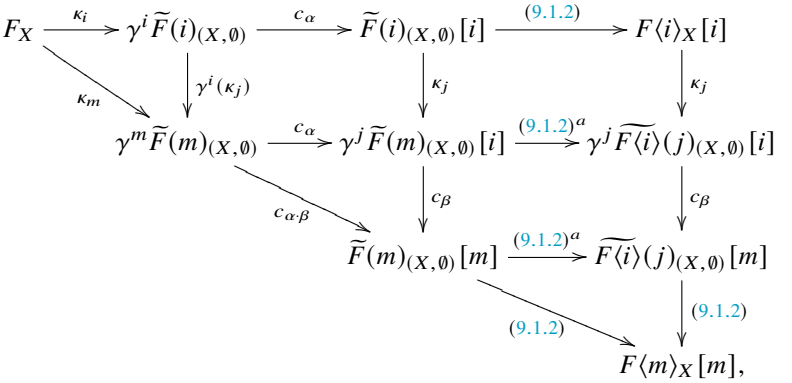
9.4. Let $X \in \mathbf{Sm}$, let Φ be a family of supports on X and let $\alpha \in \mathrm{CH}^r_\Phi(X)$. Then we define:

$$C_\alpha : F_X \to R\underline{\Gamma}_\Phi F\langle r\rangle_X[r]$$

as the composition:

$$F_X = \widetilde{F}_{(X,\emptyset)} \xrightarrow{\kappa_r} \gamma^r \widetilde{F}(r)_{(X,\emptyset)} \xrightarrow{c_\alpha} R\underline{\Gamma}_\Phi \widetilde{F}(r)_{(X,\emptyset)}[r] \xrightarrow{(9.1.2)} R\underline{\Gamma}_\Phi F\langle r\rangle_X[r].$$

We note that C_α satisfies the analogous properties of c_α listed in Lemma 5.9. This is immediate for 5.9(1)–(3). We give the argument for the analogous property of Lemma 5.9(4): Let $\alpha \in \mathrm{CH}^r_\Phi(X)$ and $\beta \in \mathrm{CH}^s_\Psi(X)$. Set $m = i + j$ and consider the following diagram in which we omit all the supports for readability:



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where (9.1.2)$^a : \widetilde{G}(n) \to \widetilde{G\langle n\rangle}$, $G \in \mathbf{RSC}_{\mathrm{Nis}}$, is induced by adjunction from (9.1.2). All the squares in the diagram commute by functoriality, the top triangle commutes by (4.8.3), the triangle in the middle commutes by Lemma 5.9(4) and the commutativity of the bottom triangle follows from the definition of the map (9.1.2). Hence, the whole diagram commutes and we obtain $C_{\alpha\cdot\beta} = C_\beta \circ C_\alpha$.

9.5. Let $f : Y \to X$ be quasi-projective in **Sm**, let Φ be a family of proper supports for Y/X and Ψ a family of supports on X, such that $\Phi \subset f^{-1}\Psi$. Set $r := \dim Y - \dim X \in \mathbb{Z}$. For $a \geq 0$ with $a + r \geq 0$ and $b \geq 0$, we define:

$$f_* : Rf_*R\underline{\Gamma}_\Phi \gamma^b(F\langle a+r\rangle)_Y[r] \to R\underline{\Gamma}_\Psi \gamma^b(F\langle a\rangle)_X$$

as follows: let $Y \xrightarrow{i} U \xrightarrow{\pi} X$ be a factorisation as in (8.5.1) with $c = \mathrm{codim}(Y, U)$ and $n = \dim(U/X)$, so that $r = n - c$; let $e \geq c$, then we define f_* as the composition:

$$\begin{aligned}
Rf_*R\underline{\Gamma}_\Phi \gamma^b(F\langle a+r\rangle)_Y[r] &\xrightarrow{(9.1.4)} Rf_*R\underline{\Gamma}_\Phi \gamma^b(\widetilde{F\langle a+r\rangle})_{(Y,\emptyset)}[r] \\
&\xrightarrow{\kappa_e} Rf_*R\underline{\Gamma}_\Phi \gamma^{b+e}(\widetilde{F\langle a+r\rangle}(e))_{(Y,\emptyset)}[r] \\
&\xrightarrow{(i,\pi)^e_*} R\underline{\Gamma}_\Psi \gamma^{b+e+r}(\widetilde{F\langle a+r\rangle}(e))_{(X,\emptyset)} \\
&\xrightarrow{(9.1.4)} R\underline{\Gamma}_\Psi \gamma^{b+e+r}\underline{\omega}_!(\widetilde{F\langle a+r\rangle}(e))_X \\
&\xrightarrow{(9.1.2)} R\underline{\Gamma}_\Psi \gamma^{b+e+r}(F\langle a+e+r\rangle)_X \\
&\xrightarrow{(9.1.5)} R\underline{\Gamma}_\Psi \gamma^b(F\langle a\rangle)_X.
\end{aligned}$$

It follows from Proposition 8.6(1) that f_* is independent of the choice of the factorisation (8.5.1), and it follows from the commutativity of (4.8.3) that it is independent of the choice of e.

Remark 9.6. By (9.1.5), we have:

$$\gamma^b(F\langle a\rangle) = \begin{cases} F\langle a-b\rangle & a \geq b \\ \gamma^{b-a}F & a \leq b. \end{cases}$$

Above, we work with $\gamma^b(F\langle a\rangle)$ so that we don't have to distinguish the two cases.

Theorem 9.7. *Let $f : Y \to X$, Φ, Ψ, a, r, b be as in 9.5.*

(1) $\mathrm{id}_{X*} : F_X \to F_X$ *is the identity.*
(2) *Let $g : Z \to Y$ be another quasi-projective morphism in* **Sm** *of relative dimension $s = \dim Z - \dim Y$, and let Ξ be a family of proper supports for Z/Y with $\Xi \subset g^{-1}\Phi$, then for $a \geq 0$ with $a + r + s \geq 0$ and $a + r \geq 0$ and $b \geq 0$, we have:*

$$(f \circ g)_* = f_* \circ g_* : R(f \circ g)_*R\underline{\Gamma}_\Xi \gamma^b(F\langle a+r+s\rangle)_Z[r+s] \to R\underline{\Gamma}_\Psi \gamma^b(F\langle a\rangle)_X.$$

(3) *Let $h : X' \to X$ be a smooth morphism. We form the base change diagram:*

$$\begin{array}{ccc} Y' & \xrightarrow{h'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{h} & X. \end{array}$$

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Set $\Psi' = h^{-1}\Psi$, $\Phi' = h'^{-1}\Phi$. *The following diagram commutes:*

$$\begin{array}{ccc} Rf_*R\underline{\Gamma}_{\Phi}\gamma^b(F\langle a+r\rangle)_Y[r] & \xrightarrow{f_*} & R\underline{\Gamma}_{\Psi}\gamma^b(F\langle a\rangle)_X \\ \downarrow{\scriptstyle h'^*} & & \downarrow{\scriptstyle h^*} \\ R(fh')_*R\underline{\Gamma}_{\Phi'}\gamma^b(F\langle a+r\rangle)_{Y'}[r] & \xrightarrow{f'_*} & Rh_*R\underline{\Gamma}_{\Psi'}\gamma^b(F\langle a\rangle)_{X'}. \end{array}$$

(4) *Let* Ξ *be some family of supports and* $\alpha \in \mathrm{CH}^s_{\Xi}(X)$. *Then* $f^*\alpha \in \mathrm{CH}^s_{f^{-1}\Xi}(Y)$ *and the following diagram commutes:*

$$\begin{array}{ccc} Rf_*R\underline{\Gamma}_{\Phi}\gamma^b(F\langle a+r\rangle)_Y[r] & \xrightarrow{C_{f^*\alpha}} & Rf_*R\underline{\Gamma}_{\Phi\cap f^{-1}\Xi}\gamma^b(F\langle a+r+s\rangle)_Y[r+s] \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ R\underline{\Gamma}_{\Psi}\gamma^b(F\langle a\rangle)_X & \xrightarrow{C_{\alpha}} & R\underline{\Gamma}_{\Psi\cap\Xi}\gamma^b(F\langle a+s\rangle)_X[s]. \end{array}$$

(5) *Let* $\beta \in \mathrm{CH}^s_{\Phi}(Y)$. *Then* $f_*\beta \in \mathrm{CH}^{s-r}_{\Psi}(X)$ *and the following diagram commutes:*

$$\begin{array}{ccc} Rf_*F_Y & \xrightarrow{C_{\beta}} & Rf_*R\underline{\Gamma}_{\Phi}F\langle s\rangle_Y[s] \\ \uparrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f_*} \\ F_X & \xrightarrow{C_{f_*\beta}} & R\underline{\Gamma}_{\Psi}F\langle s-r\rangle_X[s-r]. \end{array}$$

Proof. (1) follows from Lemma 8.3(3) and from (9.1.6) being the identity. (2) follows from Proposition 8.6(2) and (4.8.3). (3) follows from Proposition 7.9 and Lemma 8.3(2). (4) follows from Proposition 7.8 and by the definition of $\mathrm{tr}_{\mathcal{U}/\mathcal{X}}$ from the following equality for a projective bundle $\pi : P = \mathbf{P}(V) \to X$:

$$c_{\pi^*\alpha} \circ \lambda^i_V = c_{\pi^*\alpha} \circ c_{\xi^i} \circ \pi^* = \lambda^i_V \circ c_{\alpha},$$

which follows from Lemma 5.9. (5) (by Corollary 7.13), we are reduced to the case where $f = \pi : \mathbf{P}(V) \to X$ is a projective bundle with V locally free of rank $r+1$ and $\Phi = Y$, $\Psi = X$. In this case, we can write $\beta = \sum_{i=0}^{r} \xi^i \cdot \pi^*\alpha_i$, with $\alpha_i \in \mathrm{CH}^i(X)$ and $\xi^i = c_1(\mathcal{O}_{\mathbf{P}}(1))^i$. Thus, $\pi_*\beta = \alpha_r$, and, hence, the commutativity of the diagram follows, in this case, from the definition of $\mathrm{tr}_{P/X}$ and the equality:

$$c_{\beta} \circ \pi^* = \sum_{i=0}^{r} \lambda^i_V \circ c_{\alpha_i},$$

which holds by Lemma 5.9(1), (3), (4). □

9.2. Proper correspondence action

In this subsection, we fix a scheme S separated and of finite type over k.

9.8. We denote by C_S the category with objects the S-schemes $X \to S$ with the property that the induced map $X \to \mathrm{Spec}\, k$ is smooth and quasi-projective; the morphisms are given by:

$$C_S(X,Y) = \bigoplus_i \mathrm{CH}^{\dim Y_i}_{\Phi^{\mathrm{prop}}_{X\times_S Y_i}}(X \times Y_i),$$

where for simplicity we write X instead of $X \to S$, where $Y = \sqcup_i Y_i$ is the decomposition into connected components, and where $\Phi^{\mathrm{prop}}_{X\times_S Y_i}$ is the family of supports on $X \times Y_i$ consisting of those closed subsets

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which are contained in $X \times_S Y_i$ and are proper over X; the composition is defined by:

$$C_S(X_1, X_2) \times C_S(X_2, X_3) \to C_S(X_1, X_3),$$

$$(\alpha, \beta) \mapsto \beta \circ \alpha := p_{13*}(p_{12}^* \alpha \cdot p_{23}^* \beta),$$

where $p_{i,j} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ are the projections, the pullbacks p_{ij}^* are induced by flat pullback, the intersection product is given by (5.4.8) and the pushforward is well defined since p_{13} is proper along $(\Phi^{\mathrm{prop}}_{X_1 \times_S X_2} \times_k X_3) \cap (X_1 \times_k \Phi^{\mathrm{prop}}_{X_2 \times_S X_3})$ and maps this family of supports into $\Phi^{\mathrm{prop}}_{X_1 \times_S X_3}$. It follows from [CR11, Propositions 1.1.34 and 1.3.10] that C_S is a category and the identity in $C_S(X, X)$ is induced by the diagonal $\Delta \subset X \times_S X$ (cf. also [Ful98, Proposition 16.1.1]).

Note that for $S = \operatorname{Spec} k$ and $X, Y \in C_S$, we have a natural map $\mathbf{Cor}(X, Y) \to C_S(X, Y)$ which is compatible with composition.

9.9. Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$. Let S be a k-scheme, let $(f : X \to S), (g : Y \to S) \in C_S$. For $\alpha \in C_S(X, Y)$ we define a morphism in $D^+(S_{\mathrm{Nis}})$

$$\alpha^* : Rg_* F_Y \to Rf_* F_X \tag{9.9.1}$$

as follows: set $\widetilde{F} := \tau_! \omega^{\mathbf{CI}} F$; it suffices to consider the case that Y is of pure dimension d; then (9.9.1) is defined to be the composition

$$\begin{aligned} Rg_* F_Y &\xrightarrow{p_Y^*} Rg_* Rp_{Y*} F_{X \times Y} \\ &\xrightarrow{C_\alpha} Rg_* Rp_{Y*} R\underline{\Gamma}_{\Phi^{\mathrm{prop}}_{X \times_S Y}} F\langle d \rangle_{X \times Y}[d] \\ &\cong Rf_* Rp_{X*} R\underline{\Gamma}_{\Phi^{\mathrm{prop}}_{X \times_S Y}} F\langle d \rangle_{X \times Y}[d] \\ &\xrightarrow{p_{X*}} Rf_* F_X, \end{aligned}$$

where C_α is defined in 9.4, $p_X, p_Y : X \times Y \to X, Y$ denote the projections, p_{X*} is the pushforward from 9.5, and the isomorphism in the third line follows from the equality

$$g_* p_{Y*} \underline{\Gamma}_{\Phi^{\mathrm{prop}}_{X \times_S Y}}(G) = f_* p_{X*} \underline{\Gamma}_{\Phi^{\mathrm{prop}}_{X \times_S Y}}(G),$$

for any sheaf G on $X \times Y$.

Proposition 9.10. *Let the assumptions be as in 9.9 above.*

(1) *Let $(h : Z \to S) \in C_S$ and $\beta \in C_S(Y, Z)$. Then:*

$$\alpha^* \circ \beta^* = (\beta \circ \alpha)^* : Rh_* F_Z \to Rf_* F_X.$$

(2) *Let $\nu : S \to T$ be a morphism of separated and finite type k-schemes. Then ν induces a functor $\nu_* : C_S \to C_T$. Furthermore, for $\alpha \in C_S(X, Y)$, we have:*

$$(\nu_* \alpha)^* = R\nu_*(\alpha^*) : R\nu_* Rg_* F_Y \to R\nu_* Rf_* F_X.$$

(3) *Let $h : X \to Y$ be a k-morphism and denote by $[\Gamma_h] \in C_Y(X, Y)$ the class induced by the graph of h. Then:*

$$[\Gamma_h]^* = h^* : F_Y \to Rh_* F_X.$$

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(4) *Let $h : X \to Y$ be a proper S-morphism of relative dimension 0, then the transpose of the graph of h defines a class $[\Gamma_h^t] \in C_Y(Y, X)$ and:*

$$[\Gamma_h^t]^* = h_* : Rh_*F_X \to F_Y,$$

where h_* is induced by the pushforward from Proposition 8.8.

(5) *Let $V \in \mathbf{Cor}(X, Y)$ be a finite correspondence, and denote by $[V]$ its image in $C_{\mathrm{Spec}\, k}(X, Y)$. Then:*

$$V^* = [V]^* : F(Y) \to F(X).$$

Proof. For (1) it suffices to show that an equality of maps $Rh_*F_Z \to Rf_*F_X$ (with the obvious notation):

$$(p_{X*}^{XY} \circ C_\alpha \circ p_Y^{XY*}) \circ (p_{Y*}^{YZ} \circ C_\beta \circ p_Z^{YZ*}) = p_{X*}^{XZ} \circ C_{\beta\circ\alpha} \circ p_Z^{XZ*}.$$

This follows directly from Theorems 9.4 and 9.7(1)–(5) (cf. Lemma 5.9(3), (4)). The first statement of (2) follows from the fact that $X \times_S Y$ is closed in $X \times_T Y$; the second statement is direct from the definition. For (3), we first observe that:

$$\mathrm{id} = \Delta_X^* : F_X \to F_X, \tag{9.10.1}$$

where Δ_X denotes the class of the diagonal in $\mathrm{CH}^{d_X}_{\Delta_X}(X \times X)$. Indeed, this follows from Theorem 9.7(5) and the fact that $C_X : F_X \to F_X$ is the identity, where we view $X = p_{X*}(\Delta_X) \in \mathrm{CH}^0_X(X) = \mathbb{Z}$. Now (3) holds by:

$$[\Gamma_h]^* = p_{X*} \circ C_{(\mathrm{id}_X \times h)_*\Delta_X} \circ p_Y^* = p_{X*} \circ C_{\Delta_X} \circ p_X^* \circ h^* = h^*,$$

where the second equality holds by 9.7(5) and the third by (9.10.1). The proof of (4) is similar. Finally, (5). By the injectivity of the restriction map along a dense open immersion (e.g. [Sai20a, Theorem 3.1]), we can shrink X around its generic points and, henceforth, assume that X and V are smooth and irreducible. Denote by $h : V \to Y$ and $f : V \to X$ the maps induced by projection; note that f is finite and surjective. Denote by Γ_h the graph of h etc. We have $V = \Gamma_h \circ \Gamma_f^t$. By (1), (3), (4), we are reduced to show:

$$(\Gamma_f^t)^* = H^0(f_*) : F(V) \to F(X).$$

This follows from Proposition 8.8(3). □

9.11. We explain how to extend the cycle action to bounded below complexes in $\mathbf{RSC}_{\mathrm{Nis}}$. Let $F^\bullet \in \mathrm{Comp}^+(\mathbf{RSC}_{\mathrm{Nis}})$ be a bounded below complex of reciprocity sheaves. Let $(f : X \to S), (g : Y \to S) \in C_S$ and $\alpha \in C_S(X, Y)$. Then we define:

$$\alpha^* : Rg_*F_Y^\bullet \to Rf_*F_X^\bullet \qquad \text{in } D^+(S_{\mathrm{Nis}}) \tag{9.11.1}$$

as follows: Denote by Inj_S the category of injective Nisnevich sheaves on S. By (e.g. [Sta19, Tag 013V]), we have an equivalence of categories:

$$K^+(\mathrm{Inj}_S) \xrightarrow{\simeq} D^+(S_{\mathrm{Nis}}), \tag{9.11.2}$$

where K^+ denotes the homotopy category of bounded below complexes. The inverse of this equivalence induces a resolution functor $j_S : C^+(S_{\mathrm{Nis}}) \to K^+(\mathrm{Inj}_S)$, which for any bounded below complex $C^\bullet$ comes with a quasi-isomorphism of complexes $C^\bullet \to j_S(C^\bullet)$. In fact, we can choose such j_S that for a complex $C^\bullet$ we have:

$$j_S(C^\bullet) = \mathrm{tot}(\ldots \to j_S(C^i) \to j_S(C^{i+1}) \to \ldots), \tag{9.11.3}$$

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where tot(double complex) denotes the associated total complex. By construction, α induces a commutative diagram in $D^+(S_{\mathrm{Nis}})$ for all i:

$$\begin{array}{ccc} Rg_*F_Y^i & \xrightarrow{\alpha^*} & Rf_*F_X^i \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\ Rg_*F_Y^{i+1} & \xrightarrow{\alpha^*} & Rf_*F_X^{i+1}, \end{array}$$

where $d : F^i \to F^{i+1}$ is the differential in the complex. Using the resolution functors on X, Y and S, this translates into a commutative diagram in $K^+(\mathrm{Inj}_S)$:

$$\begin{array}{ccc} j_S(g_*j_Y(F_Y^i)) & \xrightarrow{\alpha^*} & j_S(f_*j_X(F_X^i)) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\ j_S(g_*j_Y(F_Y^{i+1})) & \xrightarrow{\alpha^*} & j_S(f_*j_X(F_X^{i+1})). \end{array}$$

Hence, α^* induces a morphism from the total complex of the left column (running over all i) to the total complex of the right column, using (9.11.3) (and an argument using triple complexes), it is direct to check that the latter can be identified with a morphism:

$$j_S(g_*j_Y(F_Y^\bullet)) \to j_S(f_*j_X(F_X^\bullet)),$$

which under the equivalence (9.11.2) induces the morphism (9.11.1).

It follows from Proposition 9.10 that the above construction, in fact, defines a functor:

$$C_S \to D^+(S_{\mathrm{Nis}}), \quad (f : X \to S) \mapsto Rf_*F_X^\bullet. \tag{9.11.4}$$

This functor is natural in $F^\bullet$ in the obvious sense.

Lemma 9.12. *Let $F^\bullet$, $(f : X \to S)$, $(g : Y \to S)$ and α be as in 9.11 above. Assume g is projective, Y has pure dimension d and α lies in the image of the natural map $\iota : \mathrm{CH}^d_{Z\times_S Y}(X \times Y) \to C_S(X, Y)$, for some closed subset $Z \subset X$. Then (9.11.1) factors as:*

$$Rg_*F_Y^\bullet \to Rf_*R\underline{\Gamma}_Z F_X^\bullet \to Rf_*F_X^\bullet,$$

where the second map is the forget-supports map.

Proof. We first consider the case of a sheaf $F \in \mathbf{RSC}_{\mathrm{Nis}}$. Let $\alpha_0 \in \mathrm{CH}^d_{Z\times_S Y}(X \times Y)$ with $\iota\alpha_0 = \alpha$. By Lemma 5.9(2) and the definition of C_α in 9.4, we have $\iota C_{\alpha_0} = C_\alpha$, where by abuse of notation, we denote the enlarge-support map $R\underline{\Gamma}_{Z\times_S Y} \to R\underline{\Gamma}_{\Phi^{\mathrm{prop}}_{X\times_S Y}}$ also by ι. The following diagram commutes by construction of the pushforward (see 9.5):

$$\begin{array}{ccc} Rf_*Rp_{X*}R\underline{\Gamma}_{Z\times_S Y}F\langle d\rangle_{X\times Y}[d] & \xrightarrow{\iota} & Rf_*Rp_{X*}R\underline{\Gamma}_{\Phi^{\mathrm{prop}}_{X\times_S Y}}F\langle d\rangle_{X\times Y}[d] \\ \downarrow{\scriptstyle p_{X*}} & & \downarrow{\scriptstyle p_{X*}} \\ Rf_*R\underline{\Gamma}_Z F_X & \longrightarrow & Rf_*F_X. \end{array}$$

Therefore, $p_{X*} \circ C_{\alpha_0} \circ p_Y^* : Rg_*F_Y \to Rf_*R\underline{\Gamma}_Z F_X$ induces the looked for factorisation. The case of a complex $F^\bullet$ follows directly from the sheaf case by construction of the correspondence action in 9.11. □

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The proof of the following proposition is inspired by [CL17, Lemma 8.1], where a similar result is proven for the cohomology of the de Rham-Witt complex.

Proposition 9.13. *Let $(f : X \to S)$ and $(g : Y \to S) \in C_S$. Let $V \subset X \times_S Y$ be an integral closed subscheme with $\dim V = \dim X$, which is proper over X. Assume the closure V_Y of the image of V in Y has codimension r. Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$, and assume $F(\xi) = 0$, for all points ξ which are finite and separable over the generic point of V_Y. Then:*

$$0 = [V]^* : Rg_*F_Y \to Rf_*F_X.$$

Proof. Let V and F be as in the assumption, except that V does not need to be proper over X. We can assume Y is of pure dimension d.

Claim 9.13.1. *The following composition is zero:*

$$Rg_*F_Y \xrightarrow{p_2^*} R(gp_2)_*F_{X\times Y} \xrightarrow{C_{[V]}} R(gp_2)_*R\underline{\Gamma}_V F\langle d\rangle_{X\times Y}[d]. \tag{9.13.1}$$

The claim clearly implies the statement, since $[V]^*$ factors via (9.13.1) according to (9.9.1) and Lemma 9.12. By [Sai20a, Corollary 8.6(1)], we have $R\underline{\Gamma}_V F\langle d\rangle_{X\times Y}$ is concentrated in degree $\geq d$, and, hence, $C_{[V]} : F_{X\times Y} \to R\underline{\Gamma}_V F\langle d\rangle_{X\times Y}[d]$ factors via the natural map $\mathcal{H}^d_V(F\langle d\rangle) \to R\underline{\Gamma}_V F\langle d\rangle_{X\times Y}[d]$. Thus, it suffices to prove the claim with the complex $R\underline{\Gamma}_V F\langle d\rangle_{X\times Y}[d]$ replaced by the sheaf $\mathcal{H}^d_V(F\langle d\rangle)$. For $\mathcal{U}$, a Nisnevich cover of Y and $W \to Y$ étale denote by $\mathcal{U}_W$ the induced cover of W and by $C(\mathcal{U}_W, F)$ the Čech complex of F_W. Denote by $\mathcal{C}(\mathcal{U}, F)$ the complex of sheaves on Y, given by $W \mapsto C(\mathcal{U}_W, F)$. The natural map $F_Y \to \mathcal{C}(\mathcal{U}, F)$ is a resolution (cf. [God73, Chapter II, Theorem 5.2.1]). Let $\mathcal{C}(F) = \varinjlim_{\mathcal{U}} \mathcal{C}(\mathcal{U}, F)$ be the colimit over the filtered category of Nisnevich coverings of Y with refinements as maps. Since Y is Noetherian, $\mathcal{C}(F)$ is still a complex of sheaves and defines a resolution $F_Y \to \mathcal{C}(F)$. It follows from [Sch17, Theorem 13.1] that the natural map $g_*\mathcal{C}(F) \to Rg_*F_Y$ is an isomorphism in the derived category. Note that similarly as above, we also have a natural map $(gp_2)_* \varinjlim_{\mathcal{U}} \mathcal{C}(X \times \mathcal{U}, F) \to R(gp_2)_*F_{X\times Y}$ (which is, in general, not an isomorphism). For an étale map $U \to Y$ denote by V_U the restriction of V to $X \times U$. Note that $C_{[V_U]}$ induces a map $F(X \times U) \to H^0(X \times U, \mathcal{H}^d_{V_U}(F\langle d\rangle)_{X\times U})$, which is compatible with étale pullbacks (by Lemma 5.9(3)). Therefore, $C_{[V]}$ induces the bottom right map in the following diagram, which is commutative (we set $q_2 = g \circ p_2$, $XY = X \times Y$):

$$\begin{array}{ccccc}
Rg_*F_Y & \xrightarrow{p_2^*} & Rq_{2*}F_{XY} & \xrightarrow{C_{[V]}} & Rq_{2*}\mathcal{H}^d_V(F\langle d\rangle_{XY}) \\
\uparrow \simeq & & \uparrow & & \uparrow \\
g_*\mathcal{C}(F) & \xrightarrow{p_2^*} & q_{2*}\varinjlim_{\mathcal{U}} \mathcal{C}(X\mathcal{U}, F) & \xrightarrow{C_{[V]}} & q_{2*}\varinjlim_{\mathcal{U}} \mathcal{C}(X\mathcal{U}, \mathcal{H}^d_V(F\langle d\rangle_{XY})).
\end{array}$$

This reduces Claim 9.13.1 to the following claim.

Claim 9.13.2. *Let $U \to Y$ be étale. Then the composition:*

$$F(U) \xrightarrow{p_2^*} F(X \times U) \xrightarrow{C_{[V_U]}} H^0(X \times U, \mathcal{H}^d_{V_U}(F\langle d\rangle_{X\times U})) \tag{9.13.2}$$

is zero.

We prove this claim. We may assume V_U is integral with generic point η. Since the natural restriction $\mathcal{H}^d_{V_U}(F\langle d\rangle_{X\times U}) \to \mathcal{H}^d_\eta(F\langle d\rangle_{X\times U})$ is injective by [Sai20a, Corollary 8.6(1)], we may shrink $X \times U$ around η and U around $\xi := p_2(\eta)$. The point $\xi \in U$ is finite and separable over the generic point of $p_2(V)$, and, thus, by assumption, $\dim \mathcal{O}_{U,\xi} = r$. Note that $p_2^* : \mathcal{O}_{U,\xi} \to \mathcal{O}_{X\times U,\eta}$ is essentially smooth between regular local rings, we therefore find a regular parameter sequence of $\mathcal{O}_{X\times U,\eta}$ of the form $p_2^*(s_1), \ldots, p_2^*(s_r), t_{r+1}, \ldots, t_d \in \mathcal{O}_{X\times U,\eta}$, with $s_1, \ldots, s_r$ a regular parameter sequence of $\mathcal{O}_{U,\xi}$.

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Thus, up to shrinking U, we find a neighborhood $W \subset X \times U$ of η, such that the restriction of the cycle $[V] \in \mathrm{CH}^d_V(X \times U)$ to W can be written as:

$$[V_W] = p_2^* \alpha \cdot \beta \quad \text{in } \mathrm{CH}^d_{V_W}(W),$$

with $\alpha \in \mathrm{CH}^r_A(U)$ and $\beta \in \mathrm{CH}^{d-r}_B(W)$, where $A = V(s_1, \dots, s_r)$, $B = V(t_{r+1}, \dots t_d)$. By Lemma 5.9(3), (4), the composition of (9.13.2) with the injection:

$$H^0(XU, \mathcal{H}^d_{V_U}(F\langle d\rangle_{XU})) \hookrightarrow H^0(W, \mathcal{H}^d_{V_W}(F\langle d\rangle_W)))$$

factors as:

$$F(U) \xrightarrow{C_\alpha} H^0(U, \mathcal{H}^r_A(F\langle r\rangle_Y)) \xrightarrow{C_{\beta \circ (p_{2|W})^*}} H^0(W, \mathcal{H}^d_{V_W}(F\langle d\rangle_W))). \tag{9.13.3}$$

Thus, it suffices to show that $F(U) \xrightarrow{C_\alpha} H^0(U, \mathcal{H}^r_A(F\langle r\rangle_Y))$ is the zero map. By [Sai20a, Corollary 8.6(1)], we have:

$$H^0(U, \mathcal{H}^r_A(F\langle r\rangle_Y) = H^r_A(U, F\langle r\rangle_Y) \hookrightarrow H^r_\xi(F\langle r\rangle_Y).$$

(note that $\xi = p_2(\eta)$ is the generic point of A). Hence, by Nisnevich excision and Lemma 7.14, we may assume that A is smooth over k and admits a map $U \to A$, of which the closed immersion $i : A \hookrightarrow U$ is a section. By Theorem 7.12 and the definition of C_α (see 9.4), it factors as:

$$C_\alpha : F(U) \xrightarrow{i^*} F(A) \to H^r_A(U, F\langle r\rangle_U),$$

where the second map involves the local Gysin map. Since $F(A) \subset F(\xi)$ by global injectivity, the vanishing of C_α, and, hence, of (9.13.3), follows from $F(\xi) = 0$, which holds by assumption. □

Proposition 9.14. *Let $(f : X \to S)$ and $(g : Y \to S) \in C_S$ and $F \in \mathbf{RSC}_{\mathrm{Nis}}$. Let $V \subset X \times_S Y$ be an integral closed subscheme with $\dim V = \dim X$, which is proper over X. Assume:*

(1) *there exists an integral closed subscheme $Z_0 \subset X$ of codimension $r \geq 1$, such that $V \subset Z_0 \times Y$;*
(2) *there exists a projective alteration (i.e. a generically finite, surjective and projective morphism) $Z \xrightarrow{h_0} Z_0$ with $Z \in \mathbf{Sm}$, such that $h_0 \times \mathrm{id}_Y$ induces an alteration $(h_0 \times \mathrm{id}_Y)^{-1}(V) \to V$ of degree N over V;*
(3) *$(\gamma^r F)_Z = 0$.*

Then:

$$0 = N \cdot [V]^* : Rg_* F_Y \to Rf_* F_X.$$

Proof. Set $d := \mathrm{codim}(V, X \times Y) = \dim Y$. Denote by $h := h_0 \times \mathrm{id}_Y : W := Z \times Y \to Z_0 \times Y$ and $h_1 : W \to X \times Y$ the maps induced by the alteration $h_0 : Z \to Z_0$ from (2). By assumption, $h^{-1}(V) \to V$ is a projective alteration of degree N. Let $\alpha = \sum_i m_i [A_i] \in \mathrm{CH}^{d-r}_{h^{-1}(V)}(W)$, where the A_i are those irreducible components of $h^{-1}(V)$ which are dominant and generically finite over V and where m_i is the multiplicity of A_i in the cycle $[h^{-1}(V)]$. We have $h_* \alpha = N \cdot [V] \in \mathrm{CH}^d_V(X \times Y)$. Thus, the following diagram commutes by Theorem 9.7(5):

$$\begin{array}{ccc} F_{X\times Y} & \xrightarrow{N \cdot C_V} & R\underline{\Gamma}_V F\langle d\rangle_{X\times Y}[d] \\ \downarrow{\scriptstyle h_1^*} & & \uparrow{\scriptstyle h_{1*}} \\ Rh_{1*}F_W & \xrightarrow{C_\alpha} & Rh_{1*}R\underline{\Gamma}_{h^{-1}(V)} F\langle d-r\rangle_W[d-r] \end{array} \tag{9.14.1}$$

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(see 9.4 for C_α and 9.5 for h_{1*}). By definition of the cycle action in 9.9, the map $N \cdot [V]^* : Rg_*F_Y \to Rf_*F_X$ therefore factors via:

$$R(fp_X h_1)_* R\underline{\Gamma}_{h^{-1}(V)} F\langle d-r\rangle_W [d-r] \xrightarrow{(p_X h_1)_*} Rf_*F_X, \tag{9.14.2}$$

where $p_X : X \times Y \to X$ is the projection. Denote by $h_{01} : Z \to X$ the map induced by h_0 and by $p_Z : W \to Z$ the projection. Then rel-dim$(h_{01}) = -r$, rel-dim$(p_Z) = d$ and $p_X h_1 = h_{01} p_Z$. By Theorem 9.7(2) and cancellation (9.1.5), we can rewrite (9.14.2) as the composition:

$$\begin{aligned} R(fh_{01}p_Z)_* R\underline{\Gamma}_{h^{-1}(V)} \gamma^r F\langle d\rangle_W [d-r] &\xrightarrow{p_{Z*}} R(fh_{01})_* \gamma^r F_Z[-r] \\ &\xrightarrow{h_{01*}} Rf_* \gamma^r F\langle r\rangle_X \\ &\xrightarrow{\simeq} Rf_* F_X. \end{aligned}$$

Thus, $N \cdot [V]^*$ factors via $R(fh_{01})_* \gamma^r F_Z[-r]$, which is zero by (3). □

Corollary 9.15. *Let $(f : X \to S)$ and $(g : Y \to S) \in C_S$ and $F \in \mathbf{RSC}_{\mathrm{Nis}}$. Let $V \subset X \times_S Y$ be an integral closed subscheme with* dim $V =$ dim X*, which is proper over X. Denote by $V_X \subset X$ the closure of the image of V in X. Assume* codim$(V_X, X) = r$ *and $(\gamma^r F)_Z = 0$, for all $Z \in \mathbf{Sm}$ with* dim $Z =$ dim V_X.

(1) *If the singularities of V_X can be resolved, then:*

$$0 = [V]^* : Rg_*F_Y \to Rf_*F_X.$$

(2) *If* char$(k) = p > 0$*, then there exists a number n only depending on V (not on F), such that:*

$$0 = p^n \cdot [V]^* : Rg_*F_Y \to Rf_*F_X.$$

Proof. (1) follows directly from Proposition 9.14. (2) follows from that proposition together with the Gabber-de-Jong alteration theorem (see [IT14, Theorem 2.1]). □

Remark 9.16. It would be nice to have a resolution-free proof of 9.15(1), in the spirit of Proposition 9.13. In [CR11, Proposition 3.2.2(1)], such a statement was proven for the cohomology of the Kähler differentials. But the argument relies on the Künneth decomposition for differentials, and it is not clear how to imitate this proof in the current setup.

Lemma 9.17. *Let $(f : X \to S)$, $(g : Y \to S) \in C_S$. Let $S_1 \subset Y$ be a closed integral subscheme which is finite and surjective over S, and let $\nu : \tilde{S}_1 \to S_1$ be its normalisation. Assume S and $\tilde{S}_1$ are smooth over k and f is flat. Then the cycle associated to $X \times_S S_1$ defines an element $[X \times_S S_1] \in C_S(X, Y)$ and the following diagram commutes:*

$$\begin{array}{ccc} Rg_*F_Y & \xrightarrow{[X\times_S S_1]^*} & Rf_*F_X \\ \downarrow{\scriptstyle \nu_1^*} & & \uparrow{\scriptstyle f^*} \\ (g\nu_1)_* F_{\tilde{S}_1} & \xrightarrow{(g\nu_1)_*} & F_S, \end{array}$$

where $\nu_1 : \tilde{S}_1 \to Y$ is induced by ν (note that by assumption $g\nu_1 : \tilde{S}_1 \to S$ is finite).

Proof. Let $d =$ dim Y. By Proposition 9.10(1), (3), (4), it suffices to show the following equality in $\mathrm{CH}^d_{X\times_S S_1}(X \times Y)$:

$$[\Gamma_{\nu_1}] \circ [\Gamma^t_{g\nu_1}] \circ [\Gamma_f] = [X \times_S S_1], \tag{9.17.1}$$

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where Γ_h denotes the graph of the map h and Γ_h^t its transpose. As in [Ful98, Proposition 16.1.1, (a), (c)], the left-hand side of (9.17.1) is equal to:

$$(f \times \mathrm{id}_{S_1})^*(\mathrm{id}_S \times \nu_1)_*[S \times_S \tilde{S}_1] = (f \times \mathrm{id}_{S_1})^*[S \times_S S_1] = [X \times_S S_1],$$

where we use the flatness of f for the second equality. Hence, the lemma. □

10. General applications

10.1. Obstructions to the existence of zero cycles of degree 1

We can use the existence of the proper correspondence action on the cohomology of an arbitrary reciprocity sheaf to construct new local-to-global obstruction for the existence of zero cycles of degree 1. In general, these kinds of obstructions are considered when the base is a classical global field, in that, a number field or a function field in one variable over a finite field. Instead, we have the following general result, where there is no restriction on the dimension of the base scheme.

Theorem 10.1. *Let $f : Y \to X$ be a dominant quasi-projective morphism between connected smooth k-schemes. Assume that there are integral subschemes $V_i \subset Y$ which are proper, surjective and generically finite over X of degree n_i, $i = 1, \dots, s$. Set $N = \gcd(n_1, \dots, n_s)$. Let $F^\bullet \in \mathrm{Comp}^+(\mathbf{RSC}_{\mathrm{Nis}})$ be a bounded below complex of reciprocity sheaves. Then there exists a morphism $\sigma : Rf_*F_Y^\bullet \to F_X^\bullet$ in $D(X_{\mathrm{Nis}})$, such that the composition:*

$$F_X^\bullet \xrightarrow{f^*} Rf_*F_Y^\bullet \xrightarrow{\sigma} F_X^\bullet$$

is multiplication with N. In particular, if $N = 1$, then $F_X^\bullet$ is a derived direct summand of $Rf_*F_Y^\bullet$.

Proof. Set $r = \dim Y - \dim X$. Denote by Φ the family of supports on Y generated by the V_i; Φ, therefore, is a family of proper supports for Y/X. Take $a_i \in \mathbb{Z}$, such that $N = \sum_i a_i n_i$, and set $\alpha := \sum_i a_i [V_i]$, which we can view as a cycle in $\mathrm{CH}^r_\Phi(Y)$. We define σ as the composition:

$$Rf_*F_Y^\bullet \xrightarrow{C_\alpha} Rf_*R\underline{\Gamma}_\Phi F^\bullet\langle r\rangle_Y[r] \xrightarrow{f_*} F_X^\bullet,$$

where C_α is defined as in 9.4 and f_* as in 9.5, extended to complexes as in 9.11. The statement follows from Theorem 9.7(5). □

We spell out as a corollary the implication on the index of the generic fibre of f.

Corollary 10.2. *Let $f : X \to Y$ be a projective dominant morphism between connected smooth k-schemes. Let N be the index of the generic fibre X_K over $K = k(Y)$ (i.e. N is the gcd of the residue field degrees $[K(x) : K]$, where $x \in X_K$ is running through all closed points).*

Then for any bounded below complex of reciprocity sheaves $F^\bullet$, and for any $i \geq 0$, the kernel $\mathrm{Ker}(f^* : H^i(Y, F_Y^\bullet) \to H^i(X, F_X^\bullet))$ is N-torsion. In particular, if $f^* : H^i(Y, F_Y^\bullet) \to H^i(X, F_X^\bullet)$ is not split injective for some $F^\bullet$ and some i, then the generic fibre of f cannot have index 1.

Let us now discuss how Theorem 10.1 and Corollary 10.2 can be specialised to construct new local-to-global obstructions for the existence of zero cycles of degree 1 (that give back the classical Brauer-Manin obstruction as a special case). For the reader's convenience, let us quickly review the construction of the Brauer-Manin pairing (see [Sai89, Section 8], [Wit12, Section 1.1]).

Let K be a function field in one variable over a finite field $\mathbf{F}_q$ of characteristic $p > 0$, and let S be a proper smooth model of K. Let $f : X \to S$ be a projective dominant morphism, with X smooth over k. Write X_K for the base change $X \times_S K$. For $v \in S_{(0)}$, let S_v be the henselisation of S at v, and let $K_v = k(S_v)$. Write X_{K_v} for $X \times_K K_v$ and X_{S_v} for $X \times_S S_v$.

Let $\varepsilon : \mathbf{Sm}_{\text{ét}} \to \mathbf{Sm}_{\mathrm{Nis}}$ be the change of site functor, and let $\mathbb{Q}/\mathbb{Z}(1)$ be the étale motivic complex of weight 1 with $\mathbb{Q}/\mathbb{Z}$ coefficients. By 11.1(6), $R^2\varepsilon_*\mathbb{Q}/\mathbb{Z}(1)$ defines a Nisnevich reciprocity sheaf.

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Since X is projective over S, any cycle $\alpha_v \in \mathrm{CH}_0(X_{K_v})$ defines an element of $C_{\mathrm{Spec}(K_v)}(\mathrm{Spec}\, K_v, X_{K_v})$, and since $F = R^2\varepsilon_*\mathbb{Q}/\mathbb{Z}(1)$ is in $\mathbf{RSC}_{\mathrm{Nis}}$, we can apply the proper correspondence action (9.9.1) to define a morphism:

$$\mathrm{Br}(X_{K_v})/\mathrm{Br}(X_{S_v}) \xrightarrow{\alpha^*} \mathrm{Br}(K_v) = \mathrm{Br}(K_v)/\mathrm{Br}(S_v), \tag{10.2.1}$$

where the last equality follows from the fact that $\mathrm{Br}(S_v) = 0$, since the residue field of S_v is finite. By Proposition 9.10(5), α^* agrees with the morphism induced by the transfer structure on the cohomology presheaves $H^2_{\text{ét}}(-, \mathbf{G}_m)$, which, in turn, is given by the classical norm map (see [MVW06, Example 2.4]). Taking (Nisnevich) cohomology with support, we can define a morphism by composition:

$$\bigoplus_{v\in S_{(0)}} \mathrm{Br}(K_v)(= H^0(K_v, F)) \to \bigoplus_{v\in S_{(0)}} H^1_v(S_v, F) \cong \bigoplus_{v\in S_{(0)}} H^1_v(S, F) \to H^1(S, F), \tag{10.2.2}$$

where the last map is surjective, since $H^1(K, F) = 0$ for dimension reasons. If we now compose (10.2.2) with (10.2.1) for varying α_v, and we reassemble the maps for $v \in S_{(0)}$, we get:

$$\Psi\colon \prod_{v\in S_{(0)}} \mathrm{CH}_0(X_{K_v}) \to \mathrm{Hom}\Big(\bigoplus_{v\in S_{(0)}} \mathrm{Br}(X_{K_v})/\mathrm{Br}(X_{S_v}), H^1(S, F)\Big),$$

and composing with the diagonal embedding $\mathrm{Br}(X_K) \xrightarrow{\iota} \bigoplus_{v\in S_{(0)}} \mathrm{Br}(X_{K_v})$, we finally get:

$$\iota^*\Psi\colon \prod_{v\in S_{(0)}} \mathrm{CH}_0(X_{K_v}) \to \mathrm{Hom}(\mathrm{Br}(X_K), H^1(S, F))$$

that we can further compose with the diagonal morphism from $\mathrm{CH}_0(X_K)$, giving:

$$\mathrm{CH}_0(X_K) \to \prod_{v\in S_{(0)}} \mathrm{CH}_0(X_{K_v}) \xrightarrow{\iota^*\Psi} \mathrm{Hom}(\mathrm{Br}(X_K), H^1(S, F)). \tag{10.2.3}$$

This is the Brauer-Manin sequence in disguise: in fact, the Brauer-Hasse-Noether Theorem (see, e.g. [Wei95, Section XIII] or [CTS20, Theorem 12.1.8]) implies that:

$$H^1(S, F) \simeq \mathrm{Coker}\Big(\mathrm{Br}(K) \to \bigoplus_{S_{(0)}} \frac{\mathrm{Br}(K_v)}{\mathrm{Br}(S_v)}\Big) \simeq \mathbb{Q}/\mathbb{Z}.$$

Conjecturally, the complex (10.2.3) is exact (see [CT99, Conjecture 4], [Sai89]).

We now extend the construction of the complex (10.2.3) replacing $\mathrm{Br}(-)$ with an arbitrary reciprocity sheaf. We begin with the following result. Note that S doesn't have to be of dimension 1, and that the ground field is an arbitrary perfect field.

Theorem 10.3. *Let $f\colon X \to S$ be a projective and dominant morphism between smooth connected k-schemes with $d = \dim(X) - \dim(S)$. Let $K = k(S)$ be the function field of S and $X_K = X \times_S \mathrm{Spec}\, K$. Let:*

$$\deg_K : \mathrm{CH}^d(X) \to \mathrm{CH}_0(X_K) \to \mathbb{Z},$$

where the second map is the degree map. Then, for any $F \in \mathbf{RSC}_{\mathrm{Nis}}$, there exist homomorphisms:

$$\phi : \mathrm{CH}^d(X) \to \mathrm{Hom}_{D(S_{\mathrm{Nis}})}(Rf_*F_X, F_S)$$

satisfying the following conditions:

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(i) *For any* $\alpha \in \mathrm{CH}^d(X)$ *with* $N = \deg_K(\alpha)$*, the composite:*

$$F_S \xrightarrow{f^*} Rf_*F_X \xrightarrow{\phi(\alpha)} F_S$$

is the multiplication by N.

(ii) *The map* $f_*F_X \to F_S$ *induced by* $\phi(\alpha)$ *depends only on the image* $\alpha_K \in \mathrm{CH}_0(X_K)$ *of* α*.*

Proof. For (i), it is enough to consider the case $\alpha = [x]$ for a closed point $x \in X_K$. Since the closure of x in X is projective, surjective and generically finite over S of degree $[k(x) : K]$, the statement follows directly from Theorem 10.1. As for (ii), it is enough to show that if β is a cycle in $\mathrm{CH}^d(X)$ supported on $f^{-1}(T)$ for some proper closed subscheme $T \subset S$, then the morphism $f_*F_X \to F_S$ induced by $\phi(\beta)$ is zero. But by Lemma 9.12, we have that $\phi(\beta)$ factors through $\underline{\Gamma_T} F_S \to F_S$, and since $\underline{\Gamma_T} F_S = 0$ by [Sai20a, Theorem 3.1], the claim follows. □

Let's go back to the case where $\dim(S) = 1$. For $F \in \mathbf{RSC}_{\mathrm{Nis}}$, we have a complex:

$$F(X_K) \xrightarrow{\iota} \bigoplus_{S_{(0)}} \frac{F(X_{K_v})}{F(X_{S_v})} \xrightarrow{\delta} H^1(X, F), \tag{10.3.1}$$

where the first map is the diagonal and the second is the composite:

$$\begin{aligned} \delta_v : F(X_{K_v}) = F(X_{S_v} - f^{-1}(v)) \to & H^1_{f^{-1}(v)}(X_{S_v}, F_X) \\ & \simeq H^1_{f^{-1}(v)}(X, F_X) \to H^1(X, F_X). \end{aligned} \tag{10.3.2}$$

Here, we have used excision in the displayed isomorphism of (10.3.2), and the fact that (10.3.1) is a complex follows at once from a diagram chase using the exact sequence of the cohomology with support.

By Theorem 10.3(ii), for $\alpha_v \in \mathrm{CH}_0(X_{K_v})$, we have a map:

$$\psi_v(\alpha_v) : \frac{F(X_{K_v})}{F(X_{S_v})} \xrightarrow{\phi(\tilde{\alpha}_v)} \frac{F(K_v)}{F(S_v)} \to H^1(S, F),$$

where $\tilde{\alpha}_v \in \mathrm{CH}^d(X_{S_v})$ is any lift of α_v. This gives homomorphisms:

$$\psi_v : \mathrm{CH}_0(X_{K_v}) \to \mathrm{Hom}\left(\frac{F(X_{K_v})}{F(X_{S_v})}, H^1(S, F)\right),$$

which we can assemble for varying v to get:

$$\Psi : \prod_{v \in S_{(0)}} \mathrm{CH}_0(X_{K_v}) \to \mathrm{Hom}\left(\bigoplus_{S_{(0)}} \frac{F(X_{K_v})}{F(X_{S_v})}, H^1(S, F)\right).$$

Composing this with the diagonal morphism from (10.3.1), we get:

$$\iota^*\Psi : \prod_{v \in S_{(0)}} \mathrm{CH}_0(X_{K_v}) \to \mathrm{Hom}(F(X_K), H^1(S, F)).$$

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We have a commutative diagram:

$$\begin{array}{ccc} \mathrm{CH}^d(X) & \xrightarrow{H^1(\phi)} & \mathrm{Hom}(H^1(X,F_X),H^1(S,F)) \\ \downarrow & & \downarrow \delta_v^* \\ \mathrm{CH}_0(X_{K_v}) & \xrightarrow{\psi_v} & \mathrm{Hom}(\frac{F(X_{K_v})}{F(X_{S_v})},H^1(S,F)) \end{array} , \tag{10.3.3}$$

where δ_v comes from (10.3.2). Hence, Theorem 10.3 implies the following.

Corollary 10.4. *Assume given* $\xi = (\alpha_v) \in \prod_{v \in S_{(0)}} \mathrm{CH}_0(X_{K_v})$. *If* ξ *is in the diagonal image of* $CH_0(X_K)$, *there exists* $s \in \mathrm{Hom}(H^1(X,F),H^1(S,F))$, *such that* $\Psi(\xi) = s \circ \delta$, *in particular, we have* $\iota^*\Psi(\xi) = 0$. *If* $\deg(\alpha_v) = 1$, *we can take* s *to be a splitting of* $f^* : H^1(S,F) \to H^1(X,F)$.

Remark 10.5. Note that thanks to Theorem 9.7(3) and the definition of σ in 10.1, the splitting s of Corollary 10.4 is functorial with respect to smooth base change $S' \to S$.

Remark 10.6. In Theorem 10.3, we have shown how it is possible to use sections of an arbitrary reciprocity sheaf to construct obstructions of Brauer-Manin type to the existence of zero cycles of degree 1 over nonclassical global fields. If one is interested in the (in general) finer question of finding obstructions to the existence of rational points over nonclassical global fields K, there is a vast literature in which higher unramified cohomology groups $H^n_{nr}(K(X)/K,\mathbb{Z}/\ell)$ or $H^{n+1}_{nr}(K(X)/K,\mathbb{Q}/\mathbb{Z}(n))$ (indeed, examples of global sections of reciprocity sheaves, see the list of examples 11.1) have been used, starting from [CT96]. The classical case, in that, using the Brauer group, corresponds to $H^2_{nr}(K(X)/K,\mathbb{Q}/\mathbb{Z}(1))$. Here are some examples of global fields together with the invariant used.

(1) Function field K of a curve over the real field (or a real closed field) using $H^n_{nr}(k(X)/K,\mathbb{Z}/2))$, [CT96], [Duc98a], [Duc98b], [PS20].
(2) Function field K of a curve over the complex field, using $H^1_{nr}(K(X)/K,\mathbb{Z}/n)$ [CTG04] and the appendix by O, Wittenberg to [OS20].
(3) Function field K of a curve over a p-adic field, using $H^3_{nr}(K(X)/K,\mathbb{Q}/\mathbb{Z}(2))$, [HSS15], [HS16] and [CTPS12], [Izq15].
(4) Function field K of a curve over $\mathbb{C}((t))$, [CTH15].

We thank J.-L. Colliot-Thélène for providing us with a list of references on the subject.

10.2. Birational invariants.

As observed in [CR11], cycle actions can be used to find birational invariants. In the following, S is a finite type separated k-scheme. We say that $(f : X \to S)$ and $(g : Y \to S) \in C_S$, with X and Y integral, are:

(1) *properly birational* over S, if there exists an integral scheme Z over S and two proper birational S-morphisms $Z \to X$, $Z \to Y$; in this case, we call Z a *proper birational correspondence between* X *and* Y (note that we don't assume that f, or g is proper);
(2) *stably properly birational over* S, if there exist locally free coherent $\mathcal{O}$-modules V and W on X and Y, respectively, such that the corresponding projective bundles $\mathbf{P}(V)$ and $\mathbf{P}(W)$ are properly birational over S.

Theorem 10.7. *A reciprocity sheaf* $F \in \mathbf{RSC}_{\mathrm{Nis}}$ *is a stably properly birational invariant over* S, *in that, for* $(f : X \to S)$, $(g : Y \to S) \in C_S$, *with* X, Y *integral, any proper birational correspondence between projective bundles over* X *and* Y *induces an isomorphism:*

$$f_*F_X \cong g_*F_Y .$$

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Proof. Let Z be a proper birational correspondence between $P := \mathbf{P}(V)$ and $Q := \mathbf{P}(W)$, where V and W are locally free coherent sheaves on X and Y, respectively. Let $Z_0 \subset P \times Q$ be the image of the induced map $Z \to P \times Q$, and denote by Z_0^t its transpose. We obtain induced elements $[Z_0] \in C_S(P, Q)$, $[Z_0^t] \in C_S(Q, P)$. By assumption on Z and the localisation sequence for Chow groups, the compositions $[Z_0] \circ [Z_0^t]$ and $[Z_0^t] \circ [Z_0]$ are equal to the respective diagonal plus a cycle E which maps to at least 1-codimensional subschemes in both P and Q. Since by [Sai20a, Theorem 3.1(2)] the restriction to the generic point $F_P \to j_* F_\eta$ is injective (and similarly for Q), E acts as zero in both cases, by Lemma 9.12. By Proposition 9.10(1), (3), the actions $[Z_0]^* : g_* \pi_{Y*} F_Q \to f_* \pi_{X*} F_P$ and $[Z_0^t]^* : f_* \pi_{X*} F_P \to g_* \pi_{Y*} F_Q$ are inverse to each other, where $\pi_X : P \to X$ and $\pi_Y : Q \to Y$ denote the projections. The statement follows from the projective bundle formula, Theorem 6.3. □

Remark 10.8. In case f and g are projective, the above theorem also follows directly from purity and the projective bundle formula (see also [CTHK97, Theorems 8.5.1 and 8.6.1].

Theorem 10.9. *Let p be the exponential characteristic of k. Let $(f : X \to S)$, $(g : Y \to S) \in C_S$, with X, Y integral, and let Z be a proper birational correspondence between them. Let $Z_0 \subset X \times Y$ be the image of $Z \to X \times Y$.*

Then there exists a natural number $n \geq 0$, such that for all $F \in \mathbf{RSC}_{\mathrm{Nis}}$ with $\gamma^1 F = 0$, the composition:

$$p^n \cdot [Z_0]^* \circ [Z_0^t]^* : Rf_* F_X \to Rg_* F_Y \to Rf_* F_X$$

is equal to the multiplication by p^n. If $p = 1$ or if singularities can be resolved in dimension $\dim X - 1$, then we have an isomorphism:

$$[Z_0]^* : Rg_* F_Y \xrightarrow{\simeq} Rf_* F_X.$$

Proof. The proof is similar to the one of Theorem 10.7, where we use the extra assumption on F and Corollary 9.15 to see that the cycle $p^n E$ acts as zero on $Rf_* F_X$ and $Rg_* F_Y$, respectively. □

Theorem 10.10. *Let $(f : X \to S)$, $(g : Y \to S) \in C_S$, with X, Y integral. Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$, and assume that $F(\xi) = 0$, for all points ξ which are finite and separable over a point of X or Y of codimension ≥ 1. Then any proper birational correspondence between X and Y induces an isomorphism:*

$$Rg_* F_Y \xrightarrow{\simeq} Rf_* F_X.$$

Proof. The proof is similar to the one of Theorem 10.7, where we use the extra assumption on F and Proposition 9.13 to see that the cycle E acts as zero on $Rf_* F_X$ and $Rg_* F_Y$, respectively. □

Remark 10.11.

(1) Note that taking $g = \mathrm{id}_Y : Y \to Y = S$ in Theorem 10.10 yields the vanishing $R^i f_* F_X = 0$, $i \geq 1$, for any projective birational morphism $f : X \to Y$ and any F as in the theorem.
(2) The archetype of a reciprocity sheaf which satisfies the condition $F(\xi) = 0$ is $\Omega^{\dim X}_{/k}$ (see Corollary 11.16 below for this and more examples). Also the next lemma shows that there is an ample supply of nontrivial reciprocity sheaves satisfying this condition.

Lemma 10.12. *Let p be the characteristic of the perfect base field k. Let ℓ be a prime number. If $\ell \neq p$, we additionally assume that $\dim_\ell k < \infty$, where $\dim_\ell$ denotes the ℓ-cohomological dimension. Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$ be ℓ-primary torsion. Let $X \in \mathbf{Sm}$ be integral and set:*

$$d = \begin{cases} \dim X, & \text{if } p = \ell \\ \dim_\ell k(X), & \text{if } p \neq \ell. \end{cases}$$

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Then for every point ξ which is finite over a point of X of codimension ≥ 1 we have:

$$F\langle d\rangle(\xi) = 0.$$

If furthermore, X is quasi-projective and has a zero cycle of degree prime to ℓ and $F(k) \neq 0$, then $F\langle d\rangle_X \neq 0$.

Proof. Note that the second statement is a direct consequence of the proof of Theorem 10.1. Let $\widetilde{F} = \underline{\omega}^{\mathbf{CI}} F \in \mathbf{RSC}_{\mathrm{Nis}}$. By Corollary 4.5, we have a surjection:

$$\underline{a}_{\mathrm{Nis}}(\widetilde{F} \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \underline{\omega}^* K_n^M) \twoheadrightarrow \widetilde{F}(n).$$

Since $\underline{\omega}_!$ is monoidal and exact and $\underline{\omega}_! \underline{a}_{\mathrm{Nis}} = a_{\mathrm{Nis}} \underline{\omega}_!$ (see 1.1 and 1.2), we obtain a surjection for $n \geq 1$:

$$F \otimes_{\mathbf{NST}} K_n^M \to \underline{\omega}_!(\widetilde{F}(n)) \xrightarrow{(9.1.2)} F\langle n\rangle. \tag{10.12.1}$$

We have to show $F\langle d\rangle(\xi) = 0$, for ξ as in the statement. By a colimit argument, we may assume that $F\langle d\rangle$ is ℓ^m-torsion for some m. Set $K = k(\xi)$. By (10.12.1), it suffices to show $(F \otimes_{\mathbf{PST}} K_d^M/\ell^m)(K) = 0$. By [IR17, Proposition 5.1.3], we have a surjection:

$$\bigoplus_{L/K} F(L) \otimes_{\mathbb{Z}} K_d^M(L)/\ell^m \twoheadrightarrow (F \otimes_{\mathbf{PST}} K_d^M/\ell^m)(K),$$

where L runs over all finite field extensions of K. Therefore, it suffices to show $K_d^M(L)/\ell^m = 0$, for all fields L finite over K. If $\ell = p$, we have $\mathrm{trdeg}(L/k) = \mathrm{trdeg}(K/k) \leq d-1$ and the vanishing follows from the Bloch-Kato-Gabber theorem (see [BK86, Corollary 2.8]):

$$K_d^M(L)/p^m \cong W_m\Omega^d_{L,\log} \subset W_m\Omega^d_L,$$

where $W_m\Omega^d_L$ is the de Rham-Witt sheaf in degree d and the fact that the latter group is zero, as follows from [Ill79, Chapter I, Proposition 3.11]. If $\ell \neq p$, we have an isomorphism:

$$K_d^M(L)/\ell \cong H^d_{\text{ét}}(L, \mu_\ell^{\otimes d}),$$

by the Milnor-Bloch-Kato conjecture proven by Voevodsky (see [Voe11, Theorem 6.16]). In this case, the vanishing follows from $\dim_\ell L = \dim_\ell k(\xi) \leq d-1$, which holds by assumption, and [Ser94, Chapter II, §4, Proposition 11]. □

10.3. Decomposition of the diagonal.

In the following, we will investigate the implications of the cycle action in case we have a decomposition of the diagonal, a method which was first employed in [BS83].

Theorem 10.13. *Let $F^\bullet$ be a bounded below complex of reciprocity sheaves. Let A be an integral excellent k-algebra of dimension ≤ 1, which is a directed limit $A = \varinjlim_\nu A_\nu$, such that the A_ν are smooth and of finite type over k and the transition maps $A_\nu \to A_{\nu'}$, for $\nu \leq \nu'$, are flat. Let $f : X \to S = \mathrm{Spec}\, A$ be a smooth projective morphism of relative dimension d. Let $\eta \in S$ be the generic point and X_η the generic fibre of f. Assume there exists an integer N, a zero cycle $\xi \in \mathrm{CH}_0(X_\eta)$ of degree N and a cycle $\beta \in \mathrm{CH}_d(Z \times_\eta X_\eta)$, where $i : Z \hookrightarrow X_\eta$ is a closed immersion of codimension ≥ 1, such that:*

$$N \cdot [\Delta_{X_\eta}] = p_2^* \xi + (i \times \mathrm{id})_* \beta \qquad \text{in } \mathrm{CH}^d(X_\eta \times_\eta X_\eta), \tag{10.13.1}$$

where Δ_{X_η} denotes the diagonal.

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Then there exists a strict closed subset $S_0 \subset S$, such that, for all $i \geq 1$ the cokernel:

$$\mathrm{Coker}(H^i(S_{\mathrm{Nis}}, F^\bullet) \oplus H^i_{\overline{Z} \cup X_{S_0}}(X_{\mathrm{Nis}}, F^\bullet) \xrightarrow{f^*+\mathrm{nat.}} H^i(X_{\mathrm{Nis}}, F^\bullet))$$

is N-torsion, where $X_{S_0} = X \times_S S_0$ and $\overline{Z} \subset X$ is the closure of Z. Furthermore, if $N = 1$ and $F^\bullet$ sits in degrees ≥ 0, then:

$$H^0(S, F^\bullet) = H^0(X, F^\bullet).$$

Proof. We assume $\dim A = 1$. The proof for $\dim A = 0$ is similar (and easier). For T a regular k-scheme and Y a quasi-projective T-scheme, we denote by $\mathrm{CH}_n(Y/T)$ the subgroup of $\mathrm{CH}_*(Y)$ formed by those cycles of relative dimension n over T (see [Ful98, Section 20.1]). We find a dense open subset $U \subset S$, such that the decomposition (10.13.1) extends to a decomposition in $\mathrm{CH}_d(X_U \times_U X_U/U)$, where $X_U = X \times_S U$. Using the localisation sequence for Chow groups, we find cycles $\bar{\xi} \in \mathrm{CH}_0(X/S)$ and $\bar{\beta} \in \mathrm{CH}_d(\overline{Z} \times_S X/S)$ which lift ξ and β, respectively, and a cycle $\alpha \in \mathrm{CH}_d(X_{S_0} \times_{S_0} X_{S_0}/S)$, $S_0 = S \setminus U$, such that the following equality holds in $\mathrm{CH}_d(X \times_S X/S)$:

$$N \cdot [\Delta_X] = p_2^* \bar{\xi} + (i \times \mathrm{id})_* \bar{\beta} + i_{0*}\alpha, \tag{10.13.2}$$

where $i_0 : X_{S_0} \times_{S_0} X_{S_0} \hookrightarrow X \times_S X$ denotes the closed immersion. Furthermore, since $X \to S$ is projective and $\dim S = 1$, we can write $\bar{\xi} = \sum_i m_i [T_i]$, where the T_i are integral closed subschemes of X which are finite over S (the T_i are quasi-finite over S by the dimension formula [Gro65, (5.6.5.1)]).

By assumption, we find a projective system of smooth projective maps $(f_\nu : X_\nu \to S_\nu)$ between smooth k-schemes, such that, for $\nu' \geq \nu$, the transition maps $S_{\nu'} \to S_\nu$ are affine and flat and we have $X_{\nu'} = X_\nu \times_{S_\nu} S_{\nu'}$, and, such that $f = \varprojlim_\nu f_\nu$. Hence:

$$\mathrm{CH}_d(X \times_S X/S) = \varinjlim_\nu \mathrm{CH}_d(X_\nu \times_{S_\nu} X_\nu/S_\nu),$$

where the transition maps on the right are induced by flat pullback. It follows that the decomposition (10.13.2) extends for ν large enough to the following decomposition with the obvious notation:

$$N \cdot [\Delta_{X_\nu}] = p_2^* \xi_\nu + (i_\nu \times \mathrm{id})_* \beta_\nu + i_{0\nu*}\alpha_\nu \quad \text{in } \mathrm{CH}_d(X_\nu \times_{S_\nu} X_\nu/S_\nu).$$

Additionally, we can assume that $\xi_\nu = \sum_i m_i [T_{i,\nu}]$, where $T_{i,\nu} \subset X_\nu$ are finite and surjective over S_ν and, such that $T_{i,\nu} \times_{S_\nu} S = T_i$. Since S is excellent and of dimension 1, so is T_i and, thus, the normalisation $\tilde{T}_i \to T_i$ is finite and $\tilde{T}_i$ is regular. Therefore, for ν large enough, we can assume that the normalisation $\tilde{T}_{i,\nu}$ of $T_{i,\nu}$ is smooth over k. In the limit, the action of $N \cdot [\Delta_{X_\nu}]$ is equal to $N \cdot \mathrm{id}$ on $R\Gamma(X, F^\bullet)$, and the action of $(i_\nu \times \mathrm{id})_* \beta_\nu + i_{0\nu*}\alpha_\nu$ on $R\Gamma(X, F^\bullet)$ factors by Lemma 9.12 via $R\Gamma_{\overline{Z} \cup X_{S_0}}(X, F^\bullet)$. Furthermore, by Lemma 9.17, the action of $p_2^* \xi_\nu$ factors in the limit via:

$$R\Gamma(X, F^\bullet) \xrightarrow{\oplus_i m_i \mu_{i,1}^*} \bigoplus_i R\Gamma(\tilde{T}_i, F^\bullet) \xrightarrow{\sum_i (f\mu_{i,1})_*} R\Gamma(S, F^\bullet) \xrightarrow{f^*} R\Gamma(X, F^\bullet), \tag{10.13.3}$$

where $\mu_i : \tilde{T}_i \to T_i$ is the normalisation and $\mu_{i,1} : \tilde{T}_i \to X$ is the induced map. This yields the first statement. If $N = 1$, ξ is a zero cycle of degree 1, and, hence, $\sum_i m_i (f\mu_{i,1})_* (f\mu_{i,1})^* = \mathrm{id}$, by Proposition 8.8(2). Thus, the precomposition of (10.13.3) with f^* is equal to f^*. For the second statement, we observe that if $F^\bullet$ is concentrated in degree ≥ 0, then $H^0_T(X, F^\bullet) = H^0_T(X, \mathcal{H}^0(F^\bullet))$, for $T \subset X$, where $\mathcal{H}^0(F^\bullet)$ is the zeroth cohomology sheaf of $F^\bullet$. Since $\mathcal{H}^0(F^\bullet) \in \mathbf{RSC}_{\mathrm{Nis}}$, the restriction $\mathcal{H}^0(F^\bullet) \to j_* \mathcal{H}^0(F^\bullet)_{|X \setminus T}$ is injective, for $T \subset X$ of positive codimension, by [Sai20a, Theorem 3.1(2)]. Hence, $H^0_T(X, F^\bullet) = 0$, for any such T, and the second statement follows from the above. □

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Remark 10.14. We recall some classical examples when the diagonal decomposes. Let K be a function field over k and X a smooth projective and geometrically connected K-scheme. For a field extension E/K, we set $X_E = X \otimes_K E$. Then the following implications hold:

$$\begin{aligned} X_{\overline{K}} \text{ is } & \text{ rationally chain connected over } \overline{K} \\ & \Rightarrow \deg : \mathrm{CH}_0(X_E)_{\mathbb{Q}} \xrightarrow{\simeq} \mathbb{Q}, \text{ for all } E/K \\ & \Rightarrow (10.13.1) \text{ holds for some } N \geq 1, \end{aligned}$$

for the first implication (see, e.g. [Kol96, Theorem IV.3.13]), for the second (see [BS83, Proposition 1]);

$$\begin{aligned} X \text{ is retract rational over } K \Rightarrow & \deg : \mathrm{CH}_0(X_E) \xrightarrow{\simeq} \mathbb{Z}, \text{ for all } E/K \\ & \Rightarrow (10.13.1) \text{ holds for } N = 1 \end{aligned}$$

(see [CTP16, Proposition 1.4 and Lemma 1.5] (recall that X is *retract rational* if there exists a dense open $U \subset X$ and a dense open $V \subset \mathbf{P}^n_K$ and a morphism $V \to U$ which admits a section). More generally, following [CTP16], one says that a smooth projective K-scheme is *universally* CH_0*-trivial* if the degree map $\mathrm{CH}_0(X_E) \to \mathbb{Z}$ is an isomorphism for every field extension E/K.

Theorem 10.15. *Let $f : X \to S = \operatorname{Spec} A$ be as in Theorem 10.13. We assume the diagonal of the generic fibre of f decomposes as in (10.13.1). Let p denote the exponential characteristic of k.*

Then there exists a number $n \geq 0$, such that for all bounded below complexes of reciprocity sheaves $F^\bullet$ with $\gamma F^i = 0$, for all i, the quotient $H^j(X, F^\bullet)/f^*H^j(S, F^\bullet)$ is $p^n N$-torsion. If, furthermore, $p^n N = 1$ (which, for example, happens if $N = 1$ and we can resolve singularities of all strict closed subschemes of X), then the pullback is an isomorphism:

$$f^* : R\Gamma(S, F^\bullet) \xrightarrow{\simeq} R\Gamma(X, F^\bullet).$$

Proof. The proof works as the one of Theorem 10.13, only that under the extra assumption on $F^\bullet$, we additionally find by Corollary 9.15 some $n \geq 0$, such that the correspondence $p^n \cdot ((i_\nu \times \mathrm{id})_* \beta_\nu + i_{0\nu *} \alpha_\nu)$ acts as zero. Note that we use Hironaka and Gabber-de-Jong to find p^n. For the claim in the brackets of the second statement, one has to employ the finer Proposition 9.14. □

Theorem 10.16. *Let S be a smooth, connected and separated k-scheme with generic point η. Let X be a smooth and quasi-projective k-scheme and $f : X \to S$ a flat, projective k-morphism of relative dimension d. We assume the diagonal of the generic fibre of f decomposes as in (10.13.1). Let $F^\bullet$ be a bounded below complex of reciprocity sheaves. Assume $F^i(\xi) = 0$, for all ξ which are finite and separable over a point of codimension ≥ 1 of X and for all i.*

Then the cohomology sheaves of the cone of:

$$f_* : Rf_* F^\bullet_X \to \gamma^d(F^\bullet)_S[-d] \tag{10.16.1}$$

are annihilated by N^2. If $N = 1$, then (10.16.1) is an isomorphism.

Proof. The proof is similar to the one of Theorem 10.13. The transposition induces an automorphism of $\mathrm{CH}_d(X_\eta \times_\eta X_\eta)$. Applying it to a decomposition (10.13.1) yields a decomposition of the form:

$$N \cdot [\Delta_{X_\eta}] = p_1^* \xi + (\mathrm{id} \times i)_* \beta,$$

with $\xi \in \mathrm{CH}_0(X_\eta)$, $\beta \in \mathrm{CH}_d(X_\eta \times_\eta Z)$ and $i : Z \hookrightarrow X_\eta$ a closed immersion of codimension ≥ 1. Set $e := \dim S$. Using the localisation sequence for Chow groups, we obtain a decomposition in

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$\mathrm{CH}_{d+e}(X \times_S X)$:

$$N \cdot [\Delta_X] = p_1^* \bar{\xi} + \underbrace{(\mathrm{id} \times i)_* \bar{\beta} + i_{0*}\alpha}_{=(*)},$$

with $\bar{\xi} \in \mathrm{CH}_e(X)$, $\bar{\beta} \in \mathrm{CH}_{d+e}(X \times_S \bar{Z})$, $\alpha \in \mathrm{CH}_{d+e}(X_{S_0} \times_{S_0} X_{S_0})$, with $\bar{Z}$ (respectively, S_0) strictly closed in X (respectively, S). By the assumption on $F^\bullet$ and Proposition 9.13, the cycle $(*)$ acts as zero on $Rf_*F_X^\bullet$. To understand the action of $p_1^*\bar{\xi}$, consider the following cartesian diagram:

$$\begin{array}{ccc} X \times_S X & \hookrightarrow & X \times X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_X \times f} \\ X & \xrightarrow{(\mathrm{id}_X, f)} & X \times S. \end{array}$$

It implies that the image of $p_1^*\bar{\xi}$ in $\mathrm{CH}^{d+e}_{X\times_S X}(X \times X)$ is equal to $(\mathrm{id}_X \times f)^*\xi_1$, with $\xi_1 := (\mathrm{id}_X, f)_*\bar{\xi} \in \mathrm{CH}^{d+e}_{X\times_S S}(X \times S)$. Let $\Gamma_f^t \in \mathrm{CH}^e_{S\times_S X}(S \times X)$ be induced by the transpose of the graph of f. As in [Ful98, Proposition 16.1.1], we have:

$$(\mathrm{id}_X \times f)^*\xi_1 = (((\mathrm{id}_X \times f)^*\xi_1)^t)^t = ((f \times \mathrm{id}_X)^*(\xi_1^t))^t = (\xi_1^t \circ \Gamma_f)^t = \Gamma_f^t \circ \xi_1.$$

Thus, by (the same argument as in) Proposition 9.10, the action of the cycle $p_1^*\bar{\xi} = (\mathrm{id}_X \times f)^*\xi_1$ on $Rf_*F_X^\bullet$ is equal to the following composition:

$$Rf_*F_X^\bullet \xrightarrow{f_*} \gamma^d(F^\bullet)_S[-d] \xrightarrow{\xi_1^*} Rf_*F_X^\bullet, \tag{10.16.2}$$

where ξ_1^* is defined as in 9.9 by:

$$\begin{aligned} \gamma^d(F^\bullet)_S[-d] &\xrightarrow{p_2^*} Rp_{2*}\gamma^d(F^\bullet)_{X\times S}[-d] \xrightarrow{\kappa_e} Rp_{2*}\gamma^{d+e}(\widetilde{F}^\bullet(e))_{(X\times S,\emptyset)}[-d] \\ &\xrightarrow{c_{\xi_1}} Rf_*Rp_{1*}R\underline{\Gamma}_{X\times_S S}\widetilde{F}^\bullet(e)_{(X\times S,\emptyset)}[e] \\ &\xrightarrow{(9.1.2)} Rf_*Rp_{1*}R\underline{\Gamma}_{X\times_S S}F^\bullet\langle e\rangle_{X\times S}[e] \xrightarrow{p_{1*}} Rf_*F_X^\bullet. \end{aligned}$$

Since $[\Delta_X]^*$ acts as the identity on $Rf_*F_X^\bullet$, the above yields altogether that (10.16.2) is equal to multiplication with N.

Furthermore, we claim:

$$N\cdot = f_* \circ \xi_1^* : \gamma^d(F^\bullet)_S[-d] \to \gamma^d(F^\bullet)_S[-d]. \tag{10.16.3}$$

Similarly as above, this comes down to show $\xi_1 \circ \Gamma_f^t = N \cdot [\Delta_S]$, which by [Ful98, Proposition 16.1.1] is equivalent to:

$$(f \times \mathrm{id}_S)_*\xi_1 = N \cdot [\Delta_S]. \tag{10.16.4}$$

By definition of ξ_1, we have:

$$(f \times \mathrm{id}_S)_*\xi_1 = (f \times f)_*\bar{\xi} = \delta_{S*}f_*\bar{\xi},$$

where $\delta_S : S \hookrightarrow S \times S$ is the diagonal. Since $\bar{\xi}$ is a lift of the degree N zero cycle ξ over η, we find $f_*\bar{\xi} = N \cdot [S] \in \mathrm{CH}^0(S)$, which yields (10.16.4).

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Altogether the compositions $\xi_1^* \circ f_*$ and $f_* \circ \xi_1^*$ are multiplication by N. A simple diagram chase shows that if $\mathcal{H}^i(C)$ are the cohomology sheaves of the cone C of f_*, then each section of $\mathcal{H}^i(C)$ is annihilated by N^2. □

11. Examples

For $F \in \mathbf{RSC}_{\mathrm{Nis}}$, we set $\tilde{F} = \underline{\omega}^{\mathbf{CI}} F \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}$ (see (1.6.1)). We spell out some of the results for specific examples. We will freely use the fact that the category of reciprocity sheaves $\mathbf{RSC}_{\mathrm{Nis}}$ is an abelian category (see [Sai20a, Theorem 0.1].

11.1. Examples of reciprocity sheaves

Here, we list some basic examples of reciprocity sheaves and morphisms between them. Note that a morphism of reciprocity sheaves $F \to G$ is the same as natural transformation of the underlying functors $\mathbf{Cor}^{\mathrm{op}} \to \mathbf{Ab}$; since $\mathbf{RSC}_{\mathrm{Nis}}$ is an abelian category, we obtain many more interesting examples by taking kernels and quotients. More examples can be fabricated by using the *lax* symmetric monoidal structure $(-,-)_{\mathbf{RSC}_{\mathrm{Nis}}}$ from [RSY22, Section 4] (denoted $\otimes$ in *loc.cit.*, see also [MS20, Section 1] for the notation).

(1) The category of homotopy invariant Nisnevich sheaves with transfers is an abelian subcategory of $\mathbf{RSC}_{\mathrm{Nis}}$ (see [KSY22, Corollary 2.3.4] and [Sai20a, Theorem 0.1]. For $H \in \mathbf{HI}_{\mathrm{Nis}}$, we have $\widetilde{H} = \underline{\omega}^* H$.

(2) Every smooth commutative k-group scheme is a reciprocity sheaf; every k-morphism between such group schemes is a morphism of reciprocity sheaves (see [KSY22, Corollary 3.2.5(1)]. In the case $\mathrm{char}(k) = 0$, we have for $(X, D) \in \mathbf{MCor}_{ls}$:

$$\widetilde{\mathbf{G}_a}(X, D) = H^0(X\mathcal{O}_X(D - |D|)),$$

(see [RS21, Corollary 6.8]); if $\mathrm{char}(k) = p > 0$, then we have for $(X, D) \in \mathbf{MCor}_{ls}$ with $U = X \backslash |D|$:

$$\widetilde{W_n}(X, D) = \left\{ a \in W_n(U) \;\middle|\; \begin{array}{l} \rho^* a \in \mathrm{fil}^F_{v_L(D)} W_n(L), \forall \rho \in U(L), \text{ for all} \\ \text{henselian discrete valuation rings } L \text{ of geo-} \\ \text{metric type over } k \end{array} \right\},$$

where $v_L(D)$ denotes the multiplicity of the pullback of D to $\mathrm{Spec}\, \mathcal{O}_L$ and $\mathrm{fil}^F_j W_n(L) = \sum_{s \geq 0} F^s(\mathrm{fil}_j W_n(L))$, where for $j \geq 1$:

$$\mathrm{fil}_j W_n(L) = \mathrm{fil}^{\log}_{j-1} W_n(L) + V^{n-r}(\mathrm{fil}^{\log}_j W_r),$$

with $r := \min\{n, \mathrm{ord}_p(j)\}$, and:

$$\mathrm{fil}^{\log}_j W_n(L) = \{(a_0, \ldots, a_{n-1}) \in W_n(L) \mid p^{n-1-i} v_L(a_i) \geq -j \text{all i}\},$$

(see [RS21, Theorem 7.20].

(3) Assume k has characteristic zero. Then the absolute Kähler differentials $\Omega^n = \Omega^n_{/\mathbb{Z}}$ and the relative ones $\Omega^n_{/k}$ form reciprocity sheaves. This follows from [KSY22, Corollary 3.2.2] and [KSY16, Theorem A 6.2]. Note that the proof in *loc. cit.* relies on duality theory. However, since we assume $\mathrm{char}(k) = 0$, the action of finite correspondences can be constructed in an ad hoc manner (see [LW09, Theorem 1.1]) and to show that the differentials (absolute or relative) have reciprocity can be shown by using residues on curves and the trace for finite field extensions (i.e. classical duality theory for smooth curves over a field of characteristic 0). We note that:

$$d : \Omega^n \to \Omega^{n+1}, \quad \mathrm{dlog} : K^M_n \to \Omega^n$$

are morphisms in $\mathbf{RSC}_{\mathrm{Nis}}$, as follows from [MS20, Lemma 1.1].

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(4) Assume $\mathrm{char}(k) = p > 0$. The (p-typical) de Rham-Witt sheaves $W_n\Omega^q$, $q \geq 0$, $n \geq 1$, of Bloch-Deligne-Illusie are reciprocity sheaves (see [KSY22, Corollary 3.2.5(3)]. As observed in [CR12], it follows from Grothendieck-Ekedahl duality theory that the structure maps of the de Rham-Witt complex F (Frobenius), V (Verschiebung), R (restriction) and d (differential) are morphisms of reciprocity sheaves (see also [RS21, Lemma 7.7]. Since $W\Omega^q = \varprojlim W_n\Omega^q$ has no p-torsion and:

$$\mathrm{dlog} : K^M_q \to W_n\Omega^q \tag{11.0.1}$$

factors via $W\Omega^q \to W_n\Omega^q$, it follows from [MS20, Lemma 1.1] that (11.0.1) is a morphism in $\mathbf{RSC}_{\mathrm{Nis}}$ (note, however, that $W\Omega^q \notin \mathbf{RSC}_{\mathrm{Nis}}$).

(5) Assume $\mathrm{char}(k) = p > 0$. Denote by $W_n\Omega^r_{\log}$ the subsheaf of $W_n\Omega^r$ étale locally generated by log forms (by e.g. [Ill79]) and by $\mathbb{Z}/p^n(r)$ the motivic complex of weight r with $\mathbb{Z}/p^n$-coefficients, viewed as a complex of étale sheaves. By [GL00], we have $W_n\Omega^r_{\log}[-r] \cong \mathbb{Z}/p^n(r)$ on $\mathbf{Sm}_{\text{ét}}$. There is an exact sequence on $\mathbf{Sm}_{\text{ét}}$ (see [CTSS83]):

$$0 \to W_n\Omega^r_{\log} \to W_n\Omega^r \xrightarrow{F-1} W_n\Omega^r/dV^{n-1}\Omega^{r-1} \to 0.$$

Since the two sheaves on the right of this complex admit a structure of coherent modules on $W_n(X)$, they are ε_*-acyclic, where $\varepsilon : \mathbf{Sm}_{\text{ét}} \to \mathbf{Sm}_{\mathrm{Nis}}$ denotes the morphism of sites. Hence, on $\mathbf{Sm}_{\mathrm{Nis}}$:

$$R\varepsilon_*\mathbb{Z}/p^n(r) \cong \left(W_n\Omega^r \xrightarrow{F-1} (W_n\Omega^r/dV^{n-1}\Omega^{r-1})\right)[-r], \tag{11.0.2}$$

which is obviously a complex of reciprocity sheaves.

(6) Denote by $\mathbb{Q}/\mathbb{Z}(n)$ the étale motivic complex of weight n with $\mathbb{Q}/\mathbb{Z}$-coefficients. Then:

$$R^i\varepsilon_*\mathbb{Q}/\mathbb{Z}(n) \in \mathbf{RSC}_{\mathrm{Nis}}, \quad \text{all } i, \tag{11.0.3}$$

where $\varepsilon : \mathbf{Sm}_{\text{ét}} \to \mathbf{Sm}_{\mathrm{Nis}}$ denotes the morphism of sites. Indeed, let p be the characteristic exponent of k. We can decompose $R^i\varepsilon_*\mathbb{Q}/\mathbb{Z}(n)$ into the prime-to-p torsion part, which is $\mathbf{A}^1$-invariant by [Voe00a, Corollary 5.29] and the p-primary torsion part which is a reciprocity sheaf by (5) above. In particular, taking $n = 1$ and $i = 2$, we see that the Brauer group defines a reciprocity sheaf, $\mathrm{Br} \in \mathbf{RSC}_{\mathrm{Nis}}$.

(7) Assume $\mathrm{char}(k) = p > 0$. Let G be a finite commutative k-group scheme. Denote by $H^1(G)$ the presheaf on $\mathbf{Sm}$ given by $X \mapsto H^1(G)(X) := H^1(X_{\mathrm{fppf}}, G)$. Then $H^1(G) \in \mathbf{RSC}_{\mathrm{Nis}}$ (see [RS21, Theorem 9.12].

11.2. Results with modulus

Theorem 11.1. *Let k be a field of characteristic zero. Then there is a canonical isomorphism in $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$:*

$$\widetilde{\mathbf{G}}_a(n) \xrightarrow{\simeq} \widetilde{\Omega^n},$$

where (n) denotes the twist from Definition 4.4. Furthermore, if $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ we have:

$$\widetilde{\mathbf{G}}_a(n)_{(X,D)} \cong \widetilde{\Omega^n}_{(X,D)} = \Omega^n_{X/\mathbb{Z}}(\log D)(D - |D|). \tag{11.1.1}$$

Furthermore,

$$(\widetilde{\Omega^n_{/k}})_{(X,D)} = \Omega^n_{X/k}(\log D)(D - |D|). \tag{11.1.2}$$

Proof. The equalities in (11.1.1) and (11.1.2) follow from [RS21]. Indeed, let $U \subset X$ be an open subscheme, and write D_U for $D \cap U$. Let $\mathcal{Y} = (Y, Y_\infty)$ be a log-smooth modulus compactification of

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(U, D_U), that is, Y is smooth and proper and $Y_\infty = \bar{D}_U + \Sigma$, where $\bar{D}_U$ and Σ are effective Cartier divisors on Y, such that $|\bar{D}_U + \Sigma|$ is an SNCD, $U = Y \setminus |\Sigma|$ and the restriction of $\bar{D}_U$ to U is equal to D_U. Set $Y_{N,\infty} = \bar{D}_U + N \cdot \Sigma$. Then for $R \in \{\mathbb{Z}, k\}$:

$$\begin{aligned} H^0(U, \Omega^n_{X/R}(\log D)(D - |D|)) &= \varinjlim_N H^0(Y, \Omega^n_{Y/R}(\log Y_\infty)(Y_{N,\infty} - |Y_\infty|)) \\ &= \varinjlim_N H^0(Y, (\widetilde{\Omega^n_{/R}})_{(Y, Y_{N,\infty})}) \\ &= H^0(U, (\widetilde{\Omega^n_{/R}})_{(X,D)}), \end{aligned}$$

where the second equality is [RS21, Corollary 6.8(1)] and the third equality holds by M-reciprocity (see 1.4).

By [RSY22, Theorem 5.20], we have a canonical isomorphism:

$$\underline{\omega}_!(\widetilde{\mathbf{G}_a}(n)) \xrightarrow{\simeq} \Omega^n, \tag{11.1.3}$$

which is defined in such a way that the composition with the natural map $\mathbf{G}_a \otimes_{\mathbb{Z}} K^M_n \to \underline{\omega}_!(\widetilde{\mathbf{G}_a}(n))$ (see (4.5.3), (5.5.1)) is given by:

$$\mathcal{O}_X \otimes_{\mathbb{Z}} K^M_{n,X} \to \Omega^n_X, \quad a \otimes \beta \mapsto a \operatorname{dlog} \beta.$$

By adjunction (see (1.6.1)), we obtain the canonical map from the statement; it is injective by semipurity. To prove surjectivity, it suffices by Lemma 1.3 and resolution of singularities to prove the surjectivity of $\widetilde{\mathbf{G}_a}(n)_{\mathcal{X}} \to \widetilde{\Omega^n}_{\mathcal{X}}$, for any $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. We may assume X is affine. By Lemma 4.1, we find a finite map $\pi_1 : Y_1 \to X$ and an effective divisor E_1 on Y_1, such that $(Y_1, E_1) \in \underline{\mathbf{M}}\mathbf{Cor}$ and $\pi_1^*|D| = 2E_1$. Using resolution of singularities, we find an isomorphism in $(Y, E) \cong (Y_1, E_1)$ in $\underline{\mathbf{M}}\mathbf{Cor}$ with $(Y, E) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ which is induced by a birational projective map $f : Y \to Y_1$ and $E = f^*E_1$. Set $\pi := f \circ \pi_1 : Y \to X$. Thus, $\pi^*|D| = 2E$, and we find an effective Cartier divisor E' on Y with $\pi^*D = 2E'$ and $|E'| = |E| =: E_0$. We have:

$$\widetilde{\mathbf{G}_a}_{(X,D)} = \mathcal{O}_X(D - |D|) \xrightarrow{\pi^*} \pi_* \mathcal{O}_Y(2E' - 2E) \subset \pi_* \widetilde{\mathbf{G}_a}_{(Y, \pi^*D - E_0)}, \tag{11.1.4}$$

where the equality and the inclusion follow from (11.1.1) with $n = 0$. Set $e = \deg \pi$ and $\mathcal{Y} = (Y, \pi^*D)$. Consider the following diagram:

$$\begin{array}{ccccc} \pi_*\left(\widetilde{\mathbf{G}_a}_{(Y,\pi^*D-E_0)} \otimes_{\mathbb{Z}} \widetilde{K^M_n}_{(Y,E_0)}\right) & \xrightarrow{(5.5.1)} & \pi_*\left(\widetilde{\mathbf{G}_a} \otimes_{\underline{\mathbf{MPST}}} \widetilde{K^M_n}\right)_{\mathcal{Y}} & \xrightarrow{(*)} & \pi_*(\widetilde{\Omega^n}_{\mathcal{Y}}) \\ \uparrow{\scriptstyle \pi^*} & & & & \downarrow{\scriptstyle (\Gamma^t_\pi)^*} \\ \widetilde{\mathbf{G}_a}_{\mathcal{X}} \otimes_{\mathbb{Z}} \widetilde{K^M_n}_{\mathcal{X}} & & \xrightarrow{e(\mathrm{id} \otimes \mathrm{dlog})} & & \widetilde{\Omega^n}_{\mathcal{X}}, \end{array}$$

where the map $(*)$ is induced by (11.1.3), the left vertical map exists by (11.1.4) and the fact that $(\widetilde{\mathcal{K}^M_n})_{\mathcal{X}} = (\underline{\omega}^* \mathcal{K}^M_n)_{\mathcal{X}} = j_*(\mathcal{K}^M_n)_{X-D}$, where j is the open immersion $X - D \hookrightarrow X$ and the right vertical map exists since π is projective and finite over $X \setminus |D|$. The diagram commutes by the explication of (11.1.3) above and the formula $(\Gamma^t_\pi)^* \circ \pi^* = e$ on $\widetilde{\Omega^n}_{\mathcal{X}}$. By the description of $\widetilde{\Omega^n}_{\mathcal{X}}$ above, we see that the bottom horizontal map is a surjective morphisms of sheaves. We can factor the composition $(\Gamma^t_\pi)^* \circ (*)$ also as:

$$\pi_*\left(\widetilde{\mathbf{G}_a} \otimes_{\underline{\mathbf{MPST}}} \widetilde{K^M_n}\right)_{\mathcal{Y}} \xrightarrow{(\Gamma^t_\pi)^*} (\widetilde{\mathbf{G}_a} \otimes_{\underline{\mathbf{MPST}}} \widetilde{K^M_n})_{\mathcal{X}} \to \widetilde{\Omega^n}_{\mathcal{X}}.$$

Hence, $\widetilde{\mathbf{G}_a}(n)_{\mathcal{X}} \to \widetilde{\Omega^n}_{\mathcal{X}}$ is surjective. □

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Corollary 11.2. *Let k be a field of characteristic zero. Then:*

$$\gamma^j\widetilde{\Omega^n} \cong \widetilde{\Omega^{n-j}}, \quad \gamma^j\widetilde{\Omega^n_{/k}} \cong \widetilde{\Omega^{n-j}_{/k}}$$

for every $n, j \geq 0$.

Proof. For the absolute differentials and $j \leq n$, this follows immediately from Theorem 11.1 and the weak cancellation theorem [MS20, Corollary 3.6] (see (8.7.1)). For $j > n$, this is a vanishing statement, which reduces to show:

$$\gamma(\widetilde{\mathbf{G}_a}) = 0. \tag{11.2.1}$$

By Lemma 6.2, we have $(\gamma\mathbf{G}_a)_X = R^1\pi_*\mathcal{O}_{\mathbf{P}^1_X} = 0$, where $\pi : \mathbf{P}^1_X \to X$ is the projection. Then semipurity and (9.1.4) together imply (11.2.1).

Now the relative case. Note that $\Omega^i_{k/\mathbb{Z}} \otimes_k \Omega^{n-i}$ is a reciprocity sheaf, since the choice of a basis of $\Omega^i_{k/\mathbb{Z}}$ yields an identification with a direct sum (indexed by the basis) $\oplus\Omega^{n-i}$; similarly with the relative differentials. It follows from this and [RS21, Theorems 4.15(4) and 6.4] that the natural map:

$$\widetilde{\Omega^i_{k/\mathbb{Z}} \otimes_k \Omega^{n-i}} \to \widetilde{\Omega^i_{k/\mathbb{Z}} \otimes_k \Omega^{n-i}_{/k}} \tag{11.2.2}$$

is surjective in $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. Set:

$$\mathrm{Fil}^{i,n} := \mathrm{Im}(\Omega^i_{k/\mathbb{Z}} \otimes_k \Omega^{n-i} \to \Omega^n), \quad i \in [0, n],$$

and $\mathrm{Fil}^{i,n} := 0$, for $i > n$. As is well known, we have an isomorphism $\mathrm{Fil}^{i,n}/\mathrm{Fil}^{i+1,n} \cong \Omega^i_{k/\mathbb{Z}} \otimes_k \Omega^{n-i}_{/k}$. Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{\mathrm{Fil}^{i+1,n-1}} & \longrightarrow & \widetilde{\mathrm{Fil}^{i,n-1}} & \longrightarrow & \widetilde{\Omega^i_{k/\mathbb{Z}} \otimes_k \Omega^{n-1-i}_{/k}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \gamma(\widetilde{\mathrm{Fil}^{i+1,n}}) & \longrightarrow & \gamma(\widetilde{\mathrm{Fil}^{i,n}}) & \longrightarrow & \gamma(\widetilde{\Omega^i_{k/\mathbb{Z}} \otimes_k \Omega^{n-i}_{/k}}) & \longrightarrow & 0.
\end{array} \tag{11.2.3}$$

The top row is exact for all n, i by the left exactness of $\underline{\omega}^{\mathbf{CI}}$ and the surjectivity of (11.2.2). Since we are in characteristic zero, the exactness of the bottom sequence follows from this and Lemma 9.2. The two vertical maps on the left are induced by the statement of the corollary for the absolute differentials, and the vertical map on the right is the induced morphism between the cokernels, in particular, the diagram is commutative. Note that by definition and (11.2.1), we have:

$$\gamma(\widetilde{\mathrm{Fil}^{i,n}}) = 0, \quad \text{for } i \geq n. \tag{11.2.4}$$

Therefore, for $n = 1$, the case of the relative differentials follows from the one for the absolute differentials and the diagram (11.2.3) with $i = 0$. Now assume we know $\gamma(\widetilde{\Omega^m_{/k}}) \cong \widetilde{\Omega^{m-1}_{/k}}$ for all $m < n$. Then by descending induction over $i \geq 1$, (11.2.4) and diagram (11.2.3), we have an isomorphism $\gamma(\widetilde{\mathrm{Fil}^{i,n}}) \cong \widetilde{\mathrm{Fil}^{i,n-1}}$, for all $i \geq 1$, and by the absolute case also for $i = 0$. Thus, taking $i = 0$ in diagram (11.2.3) also implies the statement in the relative case. □

Corollary 11.3. *Assume $\mathrm{char}(k) = 0$. Let $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$, and let $i : Z \hookrightarrow X$ be a smooth closed subscheme of codimension c intersecting D transversally (see Definition 2.11). Let $\rho : \tilde{X} \to X$ be the*

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blow-up of X in Z, and set $\tilde{\mathcal{X}} := (X, \rho^* D)$. There is a canonical isomorphism in $D(X_{\mathrm{Nis}})$:

$$R\rho_*\Omega^q_{\tilde{X}}(\log \rho^* D)(\rho^* D - |\rho^* D|) \cong \Omega^q_X(\log D)(D - |D|) \oplus \bigoplus_{r=1}^{c-1} i_*\Omega^{q-r}_Z(\log i^* D)(i^* D - |i^* D|)[-r].$$

Same with Ω^q replaced by $\Omega^q_{/k}$.

Proof. This follows from Corollary 7.3, Corollary 11.2, and (11.1.1), respectively, (11.1.2). □

Corollary 11.4. *Let the assumption be as in Corollary 11.3. There is a distinguished triangle in $D(X_{\mathrm{Nis}})$:*

$$i_*\Omega^{q-c}_Z(\log i^* D)(i^* D - |i^* D|)[-c] \xrightarrow{g_{Z/\mathcal{X}}} \Omega^q_X(\log D)(D - |D|) \xrightarrow{\rho^*} R\rho_*\Omega^q(\log \rho^* D + E)(\rho^* D - |\rho^* D|) \xrightarrow{\partial},$$

where $E = \rho^{-1}(Z)$. Same with Ω^q replaced by $\Omega^q_{/k}$.

Proof. This follows from Theorem 7.16, Corollary 11.2 and (11.1.1), respectively, (11.1.2). □

Remark 11.5.

(1) We can extend the statements from the Corollaries 11.3 and 11.4 to complexes as in 9.11 to obtain similar formulas for $\Omega^\bullet$, $\Omega^{\geq n}$, $\tau_{\leq n}\Omega^\bullet$; same with $\Omega^\bullet_{/k}$.
(2) One can check that in the case $c = 1$, the distinguished triangle in Corollary 11.4 is up to shift and sign induced by the exact sequence:

$$0 \to \Omega^q_X(\log D)(D - |D|) \to \Omega^q_X(\log D + Z)(D - |D|) \xrightarrow{\mathrm{Res}_Z} \Omega^{q-1}_Z(\log i^* D)(i^* D - |i^* D|) \to 0.$$

Corollary 11.6. *Let k be a perfect field, $(X, D) \in \mathbf{MCor}_{ls}$, and let $i : Z \hookrightarrow X$ be a smooth closed subscheme of codimension c intersecting D transversally. Denote by $\rho : \tilde{X} \to X$ the blow-up of X in Z.*

(1) *Assume that $\mathrm{char}(k) = 0$. Denote by Conn^1 the reciprocity sheaf, whose sections over X are rank 1 connections on X. Recall from [RS21, Theorem 6.11] that the group $\widetilde{\mathrm{Conn}^1}(X, D)$ consists of the rank 1 connections on $X \setminus |D|$, whose nonlog-irregularity is bounded by D. If $c = 1$, then there is an exact sequence:*

$$0 \to \widetilde{\mathrm{Conn}^1}(X, D) \to \widetilde{\mathrm{Conn}^1}(X, D + Z) \to H^0(Z, \mathcal{O}_Z(i^* D - |i^* D|))/\mathbb{Z} \to H^1\left(X, \frac{\Omega^1_{X/k}(\log D)(D-|D|)}{\mathrm{dlog}\, j_*\mathcal{O}^\times_{X\setminus|D|}}\right) \to H^1\left(X, \frac{\Omega^1_{X/k}(\log D+Z)(D-|D|)}{\mathrm{dlog}\, j_*\mathcal{O}^\times_{X\setminus|D+Z|}}\right).$$

If $c \geq 2$, then:

$$\widetilde{\mathrm{Conn}^1}(X, D) \cong \widetilde{\mathrm{Conn}^1}(\tilde{X}, \rho^* D + E).$$

(2) *Assume that $\mathrm{char}(k) = p > 0$, and fix a prime $\ell \neq p$. Denote by Lisse^1 the presheaf whose sections over X are the lisse $\bar{\mathbb{Q}}_\ell$ sheaves of rank 1. By [RS21, Corollary 8.10 and Theorem 8.8], we have $\mathrm{Lisse}^1 \in \mathbf{RSC}_{\mathrm{Nis}}$, and $\widetilde{\mathrm{Lisse}^1}(X, D)$ is the group of lisse $\bar{\mathbb{Q}}_\ell$-sheaves of rank 1 on $X \setminus |D|$ whose Artin conductor is bounded by D. If $c \geq 2$, then:*

$$\widetilde{\mathrm{Lisse}^1}(X, D) \cong \widetilde{\mathrm{Lisse}^1}(\tilde{X}, \rho^* D + E).$$

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Proof. For $c \geq 2$, both (1) and (2) follow directly from the Gysin sequence, Theorem 7.16. We consider the case $c = 1$ in (1). We have an isomorphism of reciprocity sheaves $(\Omega^1_{/k}/\mathrm{dlog}\,\mathbf{G}_m)_{\mathrm{Nis}} \cong \mathrm{Conn}^1$ (cf. [RS21, Section 6.10]), whence an isomorphism in $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$:

$$\underline{\omega}^{\mathbf{CI}}(\Omega^1_{/k}/\mathrm{dlog}\,\mathbf{G}_m)_{\mathrm{Nis}} \xrightarrow{\simeq} \widetilde{\mathrm{Conn}^1}. \tag{11.6.1}$$

We claim that the induced composite map:

$$\widetilde{\Omega^1_{/k}} \to \underline{\omega}^{\mathbf{CI}}(\Omega^1_{/k}/\mathrm{dlog}\,\mathbf{G}_m)_{\mathrm{Nis}} \xrightarrow{\simeq} \widetilde{\mathrm{Conn}^1} \tag{11.6.2}$$

is surjective. Indeed, by Lemma 1.3 and resolution of singularities, it suffices to show that its restriction to any $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ is surjective. The latter is a local question. Let $A = \mathcal{O}^h_{X,x}$ be the local ring at some $x \in X$ and $f \in A$ and equation for D at x. Since $X \in \mathbf{Sm}$, the local ring A is regular, and, hence, the localisation A_f is factorial. By the exact sequence:

$$\Omega^1_{A_f/k} \to \mathrm{Conn}^1(A_f) \to \mathrm{Pic}(A_f) = 0,$$

(induced by taking cohomology of the complex $[\mathbb{G}_m \xrightarrow{\mathrm{dlog}} \Omega^1_{/k}]$), we can lift any rank 1 connection E to a differential $\omega_E \in \Omega^1_{A_f}$. It is direct to check from [RS21, Theorem 6.11] that $E \in \widetilde{\mathrm{Conn}^1}(A, f)$ if and only if $\omega_E \in \widetilde{\Omega^1}(A, f)$ (with the obvious abuse of notation), proving the claim.

By (11.6.1), the surjectivity of (11.6.2) and the left exactness of $\underline{\omega}^{\mathbf{CI}}$, we have an exact sequence:

$$0 \to \underline{\omega}^*(\mathbf{G}_m/(k^{\mathrm{alg}})^\times) \xrightarrow{\mathrm{dlog}} \widetilde{\Omega^1_{/k}} \to \widetilde{\mathrm{Conn}^1} \to 0,$$

where $(k^{\mathrm{alg}})^\times = \mathrm{Ker}(\mathrm{dlog} : \mathbf{G}_m \to \Omega^1)$. Since $\gamma(\underline{\omega}^*(k^{\mathrm{alg}})^\times) = 0$ (e.g. by the projective bundle formula) and $\gamma(\underline{\omega}^*\mathbf{G}_m) = \gamma(\mathbb{Z}(1)) = \mathbb{Z}$ by the weak cancellation theorem, Corollary 11.2 and Lemma 9.2 yield:

$$\gamma(\widetilde{\mathrm{Conn}^1}) = \widetilde{\mathbf{G}_a}/\mathbb{Z}.$$

The statement follows from this and the Gysin sequence. □

11.3. Results without modulus

11.7. Assume $\mathrm{char}(k) = p > 0$. For $F \in \mathbf{RSC}_{\mathrm{Nis}}$, denote by $h^0_{\mathbf{A}^1}(F)$ the maximal $\mathbf{A}^1$-invariant subsheaf of F. In particular, $\mathrm{dlog} : K^M_r \to W_n\Omega^r$ factors via the inclusion $h^0_{\mathbf{A}^1}(W_n\Omega^r) \hookrightarrow W_n\Omega^r$. We obtain an induced map in $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$:

$$\mathrm{dlog} : \underline{\omega}^* K^M_r \to \underline{\omega}^* h^0_{\mathbf{A}^1}(W_n\Omega^r) = \underline{\omega}^{\mathbf{CI}} h^0_{\mathbf{A}^1}(W_n\Omega^r) \hookrightarrow \widetilde{W_n\Omega^r}.$$

Thus, for $q \geq r \geq 0$, $n \geq 1$, $X \in \mathbf{Sm}$ and $\mathcal{Y} \in \underline{\mathbf{M}}\mathbf{Cor}$, we can define:

$$\phi^r_{X,\mathcal{Y}} : W_n\Omega^{q-r}(X) \to \mathrm{Hom}(\underline{\omega}^* K^M_r(\mathcal{Y}), \widetilde{W_n\Omega^q}(X \otimes \mathcal{Y}))$$

by:

$$\phi^r_{X,\mathcal{Y}}(\alpha)(\beta) := p_X^*\alpha \cdot \mathrm{dlog}\, p_{\mathcal{Y}}^*\beta,$$

where $\alpha \in W_n\Omega^{q-r}(X)$, $\beta \in \underline{\omega}^* K^M_r(\mathcal{Y})$ and $p_X : X \otimes \mathcal{Y} \to X$ and $p_{\mathcal{Y}} : X \otimes \mathcal{Y} \to Y$ denote the projections.

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The following result is essentially a corollary of the projective bundle formula for reciprocity sheaves and the computation of the cohomology of the Kähler differentials of the projective line. The case $\mathrm{char}(k) = 0$ was proved in [MS20, Theorem 6.1] by a slightly different method. Recall from Proposition 9.3 that we have $\underline{\mathrm{Hom}}_{\mathbf{PST}}(K^M_n, F) = \gamma^n F$, $F \in \mathbf{RSC}_{\mathrm{Nis}}$.

Theorem 11.8.

(1) *Assume* $\mathrm{char}(k) = p > 0$. *For* $q, r \geq 0$, $n \geq 1$, *the collection* $\{\phi^r_{X,Y}\}_{X,\mathcal{Y}}$ *from 11.7 induces an isomorphism:*

$$\phi^r : W_n\Omega^{q-r} \xrightarrow{\simeq} \gamma^r W_n\Omega^q \quad \text{in } \mathbf{RSC}_{\mathrm{Nis}},$$

where we set $W_n\Omega^{q-r} := 0$, *if* $q < r$. *We have:*

$$\phi^{r+s} = \gamma^s(\phi^r) \circ \phi^s, \quad \text{for } r, s \geq 0. \tag{11.8.1}$$

Furthermore, ϕ^r *commutes with R, F, V, d (see 11.1(4) for notation), that is,*

$$\phi^r \circ f = \gamma^r(f) \circ \phi^r, \quad \text{for } f \in \{R, F, V, d\}. \tag{11.8.2}$$

(2) *Assume* $\mathrm{char}(k) = 0$. *Then similarly as in (1), we have an isomorphism* $\phi^r : \Omega^{q-r} \xrightarrow{\simeq} \gamma^r\Omega^q$, $q, r \geq 0$, *which satisfies (11.8.1); also for relative differentials* $\Omega^\bullet_{/k}$.

Proof. We will consider the situation in (1) and make a remark on (2) later. First, assume $q \geq r$. Fix $\alpha \in W_n\Omega^{q-r}(X)$. We claim that $\phi^r_X(\alpha) := \{\phi^r_{X,\mathcal{Y}}(\alpha)\}$ with varying $\mathcal{Y}$ defines a morphism $\underline{\omega}^* K^M_r \to \widetilde{W_n\Omega^q}(X \otimes -)$ in $\underline{\mathbf{M}}\mathbf{PST}$, that is, we have to show for $\Gamma \in \underline{\mathbf{M}}\mathbf{Cor}(\mathcal{Y}', \mathcal{Y})$:

$$(\mathrm{id}_X \otimes \Gamma)^* \phi^r_{X,\mathcal{Y}}(\alpha)(\beta) = \phi^r_{X,\mathcal{Y}'}(\alpha)(\Gamma^*\beta) \quad \text{in } \widetilde{W_n\Omega^q}(X \otimes \mathcal{Y}').$$

Since restriction to open subsets is injective on $W_n\Omega^q$, it suffices to check this for X affine and $\mathcal{Y} = (Y, \emptyset)$, $\mathcal{Y}' = (Y', \emptyset)$. In this case, we can lift α to $\tilde{\alpha} \in W\Omega^{q-r}(X)$ and we can use this to lift $\phi^r_X(\alpha)$ to $\phi^r_X(\tilde{\alpha}) : K^M_r \to W\Omega^q(X \times -)$ via $\beta \mapsto p_X^*\tilde{\alpha} \cdot \mathrm{dlog}\, p_{\mathcal{Y}}^*\beta$. Since $\phi^r_X(\tilde{\alpha})$ is obviously a map of sheaves (without transfers), and since $W\Omega^q(X \times -)$ is p-torsion free (see [Ill79, Chapter I, Corollary 3.5]), it follows from [MS20, Lemma 1.1] that $\phi_X(\tilde{\alpha})$ is compatible with transfers, and, hence, so is $\phi^r_X(\alpha)$. Thus,

$$\begin{aligned}\phi^r_X(\alpha) &\in \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(\underline{\omega}^* K^M_r, \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), \widetilde{W_n\Omega^q})) \\ &= (\underline{\omega}_! \gamma^r \widetilde{W_n\Omega^q})(X) = (\gamma^r W_n\Omega^q)(X),\end{aligned}$$

where the last equality holds by (9.1.4). Next we claim that:

$$\phi^r : W_n\Omega^{q-r} \to \gamma^r W_n\Omega^q, \quad \alpha \text{ on } X \mapsto \phi^r_X(\alpha)$$

is a morphism in $\mathbf{PST}$ (hence, also in $\mathbf{RSC}_{\mathrm{Nis}}$). Indeed, this can be checked similarly as above. We show (11.8.1). Let $\alpha \in W_n\Omega^{q-r-s}(X)$. Then $\phi^s(\alpha)$ is determined by: $\phi^s_{X,\mathcal{Y}}(\alpha)(\beta_s)$ for $\beta_s \in \underline{\omega}^* K^M_s(\mathcal{Y})$ and $\gamma^s(\phi^r)(\phi^s(\alpha))$ is determined by

$$\phi^r_{X \otimes \mathcal{Y}, \mathcal{Z}}(\phi^s_{X,\mathcal{Y}}(\alpha)(\beta_s))(\beta_r) = p_X^*\alpha \cdot \mathrm{dlog}(p_{\mathcal{Y}}^*\beta_s \cdot p_{\mathcal{Z}}^*\beta_r)$$

for $\beta_r \in \underline{\omega}^* K^M_r(\mathcal{Z})$, $\mathcal{Z} \in \underline{\mathbf{M}}\mathbf{Cor}$. Here, $p_X, p_{\mathcal{Y}}$ and $p_{\mathcal{Z}}$ denote the obvious projections.

For $\alpha \in W_n\Omega^{q-r-s}(X)$, the map $(\beta_s, \beta_r) \mapsto p_X^*\alpha \cdot \mathrm{dlog}(p_{\mathcal{Y}}^*\beta_s \cdot p_{\mathcal{Z}}^*\beta_r)$ also induces an element in:

$$\underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\underline{\omega}^* K^M_s \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \underline{\omega}^* K^M_r, \widetilde{W_n\Omega^q})(X) \cong (\gamma^{s+r} W_n\Omega^q)(X),$$

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where the isomorphism follows from (4.5.1) and (9.1.4). This yields (11.8.1). Furthermore, (11.8.2) follows from the following general formulas in $W_\bullet\Omega^*$:

$$R(\alpha \cdot \operatorname{dlog}\beta) = R(\alpha)\cdot \operatorname{dlog}\beta, \quad d(\alpha \cdot \operatorname{dlog}\beta) = d(\alpha)\cdot \operatorname{dlog}\beta,$$

$$F(\alpha \cdot \operatorname{dlog}\beta) = F(\alpha)\cdot \operatorname{dlog}\beta, \quad V(\alpha \cdot \operatorname{dlog}\beta) = V(\alpha)\cdot \operatorname{dlog}\beta.$$

Next we prove that:

$$\phi^1 : \Omega^{q-1} \to \gamma^1\Omega^q \tag{11.8.3}$$

is an isomorphism. Indeed, by Lemma 6.2, the map:

$$H^1(\lambda^1_{\mathcal{O}^{\oplus 2}}) : (\gamma^1\Omega^q)_X \xrightarrow{\simeq} R^1\pi_*(\Omega^q_{\mathbf{P}^1_X}) \tag{11.8.4}$$

is an isomorphism, where $\pi : \mathbf{P}^1_X \to X$ is the projection. On the other hand, by definition of $\lambda^1_{\mathcal{O}^{\oplus 2}}$ and ϕ^1, the precomposition of (11.8.4) with ϕ^1_X is exactly the cup product with:

$$-\cup c_1(\mathcal{O}_{\mathbf{P}^1_X}(1)) : \Omega^{q-1}_X \to R^1\pi_*\Omega^q_{\mathbf{P}^1_X},$$

which is well known to be an isomorphism. Hence, (11.8.3) is an isomorphism as well. Iterating and (11.8.1) yield the isomorphism:

$$\phi^r : \Omega^{q-r} \xrightarrow{\simeq} \gamma^r\Omega^q, \quad \text{for } q \geq r.$$

Note that a similar argument also works in characteristic zero for $\Omega^\bullet$ and $\Omega^\bullet_{/k}$; hence, (2) holds. Since γ^r is exact (see Lemma 9.2) and ϕ^r is compatible with d, we obtain isomorphisms (which we will denote by ϕ^r again):

$$Z^{q-r} \xrightarrow{\simeq} \gamma^r Z^r, \quad B^{q-r} \xrightarrow{\simeq} \gamma^r B^q, \quad Z^{q-r}/B^{q-r} \xrightarrow{\simeq} \gamma^r(Z^q/B^q),$$

where $Z^q = \operatorname{Ker}(d : \Omega^q \to \Omega^{q+1})$ and $B^q = d\Omega^{q-1}$. Denote by $C^{-1} : \Omega^q \xrightarrow{\simeq} Z^q/B^q$ the inverse Cartier operator. One easily checks that the following diagram commutes:

$$\begin{array}{ccc} \Omega^{q-r} & \xrightarrow{\phi^r} & \gamma^r\Omega^q \\ \downarrow{\scriptstyle C^{-1}} & & \downarrow{\scriptstyle \gamma^r(C^{-1})} \\ Z^{q-r}/B^{q-r} & \xrightarrow{\phi^r} & \gamma^r(Z^q/B^q), \end{array}$$

(in fact, this follows from the compatibility of ϕ^r with F and [Ill79, I, Proposition 3.3]). It is direct from this to check that we also have isomorphisms:

$$B^{q-r}_n \xrightarrow{\simeq} \gamma^r B^q_n, \quad Z^{q-r}_n \xrightarrow{\simeq} \gamma^r Z^q_n,$$

where B^q_n and Z^q_n are defined as in [Ill79, Chapter 0, Eq. (2.2.2)]. Set $\operatorname{gr}^{q,n} := \operatorname{Ker}(W_{n+1}\Omega^q \xrightarrow{R} W_n\Omega^q)$. By compatibility of ϕ^r with R, we see that ϕ^r on $W_{n+1}\Omega^{q-r}$ restricts to:

$$\phi^r : \operatorname{gr}^{q-r,n} \to \gamma^r \operatorname{gr}^{q,n}. \tag{11.8.5}$$

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By [Ill79, Chapter I, Corollary 3.9], we have an exact sequence:

$$0 \to \Omega^q/B_n^q \xrightarrow{V^n} \mathrm{gr}^{q,n} \xrightarrow{\beta} \Omega^{q-1}/Z_n^{q-1} \to 0,$$

where β is determined by $\beta(V^n(a) + dV^n(b)) = b$. It follows from this and the above that (11.8.5) is an isomorphism. Hence, $\phi^r : W_n\Omega^{q-r} \to \gamma^r W_n\Omega^q$ is an isomorphism by induction on n.

It remains to show that $\gamma^r W_n\Omega^q = 0$, for $q < r$. By the above, it suffices to show $\gamma^1 W_n = 0$. By the exactness of γ^1, it suffices to show $\gamma^1 \mathbf{G}_a = 0$. By Lemma 6.2, we have $(\gamma^1 \mathbf{G}_a)_X \cong R^1\pi_*\mathcal{O}_{\mathbf{P}^1_X} = 0$. This completes the proof of the theorem. □

11.9. Given the fact that the de Rham-Witt sheaves are reciprocity sheaves, we obtain, as a corollary of the above and the abstract, results for reciprocity sheaves new (motivic) proofs of the following results of Gros: projective bundle formula (by Theorem 6.3 with empty modulus, cf. [Gro85, Chapter I, Theorem 4.1.11]), blow-up formula (by (7.3.1) with empty modulus, cf. [Gro85, Chapter IV, Corollary 1.1.11]) and a proper pushforward and Gysin morphisms for smooth quasi-projective schemes (for $r \geq 0$ in 9.5, take $a = 0$ and $b = r$ and precompose f_* with the map induced by $W_n\Omega_Y^q \to \gamma^r(W_n\Omega^q\langle r\rangle)_Y$; for $r < 0$, take in 9.5 $a = -r$ and $b = -r$ and post compose with the isomorphism from the weak Cancellation theorem, cf. [Gro85, Chapter II]). We also obtain the (opposed) action of properly supported Chow correspondences constructed in [CR12] (by 9.9 and 9.10). However, let us remind the reader that, at the moment, the finite transfers on the de Rham-Witt complex are defined by restricting the action of properly supported Chow correspondences from [CR12] (the construction of which uses all the results above) to the case of finite correspondences. It is therefore an interesting problem to have a more direct construction of the transfers structure for the de Rham-Witt complex (cf. the discussion in the characteristic zero case in 11.1(3)). The Gysin sequence, however, is to our knowledge new:

Corollary 11.10. *Let k be a perfect field. Let $X \in \mathbf{Sm}$, and let $i : Z \hookrightarrow X$ be a smooth closed subscheme of codimension c. Denote by $\rho : \tilde{X} \to X$ the blow-up of X in Z, and set $E := \rho^{-1}(Z)$.*

(1) *There is a distinguished triangle in $D(X_{\mathrm{Nis}})$:*

$$i_*\Omega_Z^{q-c}[-c] \xrightarrow{g_{Z/X}} \Omega_X^q \to R\rho_*\widetilde{\Omega^q}_{(\tilde{X},E)} \xrightarrow{\partial} i_*\Omega_Z^{q-c}[-c+1].$$

Similarly with $\Omega^\bullet_{/k}$.

(2) *Assume $\mathrm{char}(k) = p > 0$. Then we have a distinguished triangle (for all n):*

$$i_*W_n\Omega_Z^{q-c}[-c] \xrightarrow{g_{Z/X}} W_n\Omega_X^q \to R\rho_*\widetilde{W_n\Omega^q}_{(\tilde{X},E)} \xrightarrow{\partial} i_*W_n\Omega_Z^{q-c}[-c+1].$$

(3) *Let $(X/W_n)_{\mathrm{crys}}$ be the Nisnevich-crystalline site of X relative to* Spec $W_n(k)$ *and $u_X : (X/W_n)_{\mathrm{crys}} \to X_{\mathrm{Nis}}$ the map of sites. There is distinguished triangle in $D(X_{\mathrm{Nis}})$:*

$$i_*Ru_{Z*}\mathcal{O}_{Z/W_n}[-2c] \xrightarrow{g_{Z/X}} Ru_{X*}\mathcal{O}_{X/W_n} \to R\rho_*\widetilde{W_n\Omega^\bullet}_{(\tilde{X},E)} \xrightarrow{\partial} \dots$$

Proof. (1) and (2) follow directly from Theorems 7.16 and 11.8. (3) follows from Illusie's isomorphism $Ru_{X*}\mathcal{O}_{X/W_n} \cong W_n\Omega_X^\bullet$ and the above (cf. 9.11). □

Remark 11.11.

(1) Note that in characteristic zero, we have $\widetilde{\Omega^q}_{(\tilde{X},E)} = \Omega^q_{\tilde{X}}(\log E)$. In positive characteristic, this is expected to hold but is not yet known (also for the de Rham-Witt sheaves).

(2) If $\mathrm{char}(k) = 0$, then $R\rho_*\Omega^\bullet_{\tilde{X}/k}(\log E) \cong Rj_*\Omega^\bullet_{U/k}$, with $j : U := X \setminus Z \hookrightarrow X$ the open immersion, and the Gysin sequence becomes the classical one for de Rham cohomology.

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Corollary 11.12. *Assume $char(k) = p > 0$. Let $f : Y \to X$ be a morphism of relative dimension $r \geq 0$ between smooth projective k-schemes. Assume that X is ordinary in the sense of [BK86, Definition (7.2)]. Then the Ekedahl-Grothendieck pushfoward (see [Gro85, II, 1.]) factors via:*

$$R\Gamma(Y, W_n\Omega^q_Y)[r] \to R\Gamma(Y, W_n\Omega^q_Y/B^q_{n,\infty})[r] \xrightarrow{f_*} R\Gamma(X, W_n\Omega^{q-r}_X), \tag{11.12.1}$$

where $B^q_{n,\infty} = \bigcup_s F^{s-1}dW_{n+s-1}\Omega^{q-1}$ (see [IR83, IV, (4.11.2)]) and f_* is induced by the pushforward from 9.5.

Proof. The exactness of γ and Theorem 11.8 imply $\gamma^r B^q_{n,\infty} = B^{q-r}_{n,\infty}$. Thus, by functoriality, the pushforward induce a morphism of triangles:

$$\begin{array}{ccccccc} R\Gamma(Y, B^q_{n,\infty})[r] & \longrightarrow & R\Gamma(Y, W_n\Omega^q)[r] & \longrightarrow & R\Gamma(Y, W_n\Omega^q/B^q_{n,\infty})[r] & \longrightarrow & \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\ R\Gamma(X, B^{q-r}_{n,\infty}) & \longrightarrow & R\Gamma(X, W_n\Omega^{q-r}) & \longrightarrow & R\Gamma(X, W_n\Omega^{q-r}/B^{q-r}_{n,\infty}) & \longrightarrow & . \end{array} \tag{11.12.2}$$

Since X is ordinary, we have $R\Gamma(X, B^{q-r}_{n,\infty}) = 0$, by [IR83, Theorem IV.4.13]. This yields the factorisation. That the pushforward coincides (up to sign) with the Ekedahl-Grothendieck pushforward, follows from the construction of the pushforward in 9.5 and the explicit description of the projective trace (see Definition 8.2) and the Gysin map (see Theorem 7.12), as well as the corresponding description for the Ekedahl-Grothendieck pushforward (see [Gro85, Chapter II, Sections 2.6 and 3.3], also [CR12, Proposition 2.4.1 and Corollary 2.4.3]). □

Remark 11.13. Let $f : Y \to X$ be as in Corollary 11.12, but without assuming that X is ordinary. Then the above proof shows that there is a factorisation:

$$R\Gamma(Y, W_n\Omega^r_Y)[r] \to R\Gamma(Y, W_n\Omega^r_Y/B^r_{n,\infty})[r] \xrightarrow{f_*} R\Gamma(X, W_n),$$

which simply follows from the diagram (11.12.2) and the fact that $B^0_{n,\infty} = 0$.

Corollary 11.14. *Assume $char(k) = p > 0$. Let $f : X \to S$ be a surjective morphism between smooth projective connected k-schemes. Assume that the generic fibre has index prime to p. Then:*

$$X \text{ is ordinary} \implies S \text{ is ordinary.}$$

Proof. Let $B^q = d\Omega^{q-1}$. This is a reciprocity sheaf. By [BK86, Definition (7.2)], X is ordinary if and only if $H^i(X, B^q) = 0$, for $i, q \geq 0$. Thus, the statement follows from Corollary 10.2. □

Remark 11.15. Let $f : X \to S$ be as in Corollary 11.14, and assume, moreover, that the generic fibre has a zero cycle of degree prime to p. Then it is possible, with a similar argument, to prove the implication:

$$X \text{ is Hodge-Witt} \implies S \text{ is Hodge-Witt,}$$

(see [IR83, Chapter IV, Section 4.12]). Similarly, if the crystalline cohomology $H^*(X/W)$ of X is torsion free, the existence of the splitting in Corollary 10.2 implies that the crystalline cohomology of S is torsion free as well.

Corollary 11.16. *Let S be a separated k-scheme of finite type, X and Y be integral smooth quasi-projective k-schemes both of dimension N and $f : X \to S$, $g : Y \to S$ be morphism of k-schemes. Assume that X and Y are properly birational over S (see 10.2). Then any proper birational correspondence over S between X and Y induces an isomorphism:*

$$Rf_*F_X \xrightarrow{\simeq} Rg_*F_Y, \tag{11.16.1}$$

where F is one of the following sheaves (respectively, complexes):

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(1) *any complex of reciprocity sheaves whose terms are subquotients of* $\Omega^N_{/k}$*, for example,*

$$\Omega^N_{/k}, \quad \Omega^N_{/k}/\mathrm{dlog}\, K^M_N, \quad \Omega^N_{/k}/h^0_{\mathbf{A}^1}(\Omega^N_{/k}),$$

where $h^0_{\mathbf{A}^1}(\Omega^N_{/k})$ *is the maximal* $\mathbf{A}^1$*-invariant subsheaf of* $\Omega^N_{/k}$*;*

and in the case of $\mathrm{char}(k) = p > 0$ *:*

(2) *any complex of reciprocity sheaves whose terms are subquotient of* $W_n\Omega^N$ $(n \geq 1)$*, for example,*

$$W_n\Omega^N_{\log}, \quad W_n\Omega^N/F^{r-1}dW_{n+r-1}\Omega^{N-1}, \quad F^{r-1}dW_{n+r-1}\Omega^{N-1}, \quad (r \geq 1),$$
$$R\varepsilon_*(\mathbb{Z}/p^n(N)), \quad R^i\varepsilon_*(\mathbb{Z}/p^n(N)),\ \textit{all } i \quad (\textit{see } 11.1(5));$$

(3)

$$G\langle N\rangle,$$

where G *is a smooth commutative unipotent* k*-group;*

(4)

$$H^1(G)\langle N\rangle,$$

where G *is a finite commutative* p*-group scheme over* k *and* $H^1(G)(X) = H^1(X_{\mathrm{fppf}}, G)$*.*

Proof. First note that all the listed examples are (respecively, complexes of) reciprocity sheaves. This follows directly from the examples discussed in 11.1. Therefore, the assertions follow directly from Theorem 10.10 and Lemma 10.12. □

Remark 11.17.

(1) The birational invariance of the cohomology of $\Omega^N_{/k}$ in characteristic zero is classical (using resolution of singularities), in positive characteristic, this was proven in [CR11] by a similar method as here. However, the statement in *loc. cit.* was only for the cohomology sheaves, not the complexes in the derived category. The whole statement can, in this case, be also deduced from [Kov17]. To our knowledge, the other statements in Corollary 11.16 are new.

(2) We get an isomorphism (11.16.1) up to bounded torsion for all $F \in \mathbf{RSC}_{\mathrm{Nis}}$ which are successive extensions of subquotients of $\mathbf{G}_a$. This follows from Theorem 10.9, the vanishing $\gamma(\mathbf{G}_a) = 0$ (see Theorem 11.8) and the fact that γ is exact. In this case, the statement can, however, be also deduced without the 'up-to-bounded-torsion' assumption from [Kov17, Theorem 1.4], the existence of Macaulayfication, and the fact that smooth schemes have pseudo-rational singularities (theorem of Lipman-Teissier).

11.18. Let K be a field of characteristic p. For $\ell \neq p$, the cohomological ℓ-dimesnion $\dim_\ell(K)$ is defined, for example, in [Ser94]. For $\ell = p$, the cohomological p-dimension of K is defined in [KK86, Definition 1] by:

$$\dim_p(K) := \inf\{i \in \mathbb{N} \mid \Omega^{i+1}_K = 0 \text{ and } H^1_{\text{ét}}(K', \Omega^i_{\log}) = 0, \text{ for all } K'/K \text{ finite}\},$$

where K'/K ranges over all finite field extensions.

The corollary below generalises results of Pirutka [Pir12] and Colliot-Thélène-Voisin [CTV12] (see Remark 11.20). We thank Colliot-Thélène for pointing to these results.

Corollary 11.19. *Let* S *be a separated* k*-scheme of finite type,* X *and* Y *be integral smooth quasi-projective* k*-schemes both of dimension* d *and* $f : X \to S$*,* $g : Y \to S$ *be morphism of* k*-schemes. Assume that* X *and* Y *are properly birational over* S *(see 10.2). Let* ℓ *be a prime, and assume* $e := \dim_\ell(k) < \infty$

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(note $e = 0$, if $\ell = p$). Then any proper birational correspondence over S between X and Y induces an isomorphism:

$$Rf_*(R^{d+e}\varepsilon_*\mathbb{Z}/\ell^r(j))_X \xrightarrow{\simeq} Rg_*(R^{d+e}\varepsilon_*\mathbb{Z}/\ell^r(j))_Y, \quad \text{for all } j \geq 0, r \geq 1. \tag{11.19.1}$$

Proof. Set $F := R^{d+e}\varepsilon_*\mathbb{Z}/\ell^r(j)$. We know $F \in \mathbf{RSC}_{\mathrm{Nis}}$. By Theorem 10.10, it suffices to show $F(K) = 0$, for any finitely generated field K/k with $\mathrm{trdeg}(K/k) < d$. Assume $\ell \neq p$. By [GL01, Theorem 1.5], we have $F(K) = H^{d+e}_{\text{ét}}(K, \mu_{\ell^r}^{\otimes j})$, which vanishes since $\dim_\ell(K) \leq d - 1 + e$, by [Ser94, Chapter II, §4, Proposition 11]. Now assume $\ell = p$ (hence, $e = 0$). By [GL00] (see 11.1(5)), we have:

$$F(K) = \begin{cases} W_n\Omega^d_K, & \text{if } j = d \\ H^1_{\text{ét}}(K, W_n\Omega^{d-1}_{\log}), & \text{if } j = d-1 \\ 0, & \text{else.} \end{cases}$$

Thus, in this case, $F(K) = 0$, since $\dim_p(K) \leq d - 1$ by [KK86, 3., Corollary 2]. □

Remark 11.20. If we assume $\ell \neq p, f$ and g projective and generically smooth and we take in (11.19.1) the stalks in the generic point of S and cohomology, we get back the first statement of [Pir12, Theorem 3.3] (which generalises [CTV12, Proposition 3.4]), at least in the case where the base field is a finitely generated field over a perfect field.

To our knowledge, the global statement for S of arbitrary dimension is new even for $\ell \neq p$. The case $\ell = p$ and $j = d - 1$ seems to be completely new.

The following corollary holds for all reciprocity sheaves. We just spell it out for one special example.

Corollary 11.21. *Let S be the henselisation of a smooth k-scheme in a 1-codimensional point or a regular connected affine scheme of dimension ≤ 1 and of finite type over a function field K over k. Let $f : X \to S$ be a smooth projective morphism of relative dimension d, and let $\eta \in S$ be the generic point. Assume the diagonal of the generic fibre X_η decomposes as:*

$$[\Delta_{X_\eta}] = p_2^*\xi + (i \times \mathrm{id})_*\beta \quad \text{in } \mathrm{CH}_0(X_\eta \times_\eta X_\eta), \tag{11.21.1}$$

where $\xi \in \mathrm{CH}_0(X_\eta)$ and $\beta \in \mathrm{CH}_d(Z \times_\eta X_\eta)$ with $i : Z \hookrightarrow X_\eta$ a closed immersion of codimension ≥ 1.

Then for all $n, i \geq 0$:

$$H^0(X, R^i\varepsilon_*\mathbb{Q}/\mathbb{Z}(n)) = H^0(S, R^i\varepsilon_*\mathbb{Q}/\mathbb{Z}(n))$$

(see 11.1, (6) for notation). In particular, taking $n = 1$ and $i = 2$, we obtain $\mathrm{Br}(X) = \mathrm{Br}(S)$.

Proof. This follows from Theorem 10.13 (note, in the case S is a regular affine curve over K, it follows, for example, from [RS21, Lemma 2.4] that S satisfies the conditions of that theorem). □

Corollary 11.22. *Let $f : X \to S$ be a flat projective morphism of relative dimension d between smooth integral and quasi-projective k-schemes. Let $\dim X = N$. Assume the diagonal of the generic fibre of f decomposes as in (11.21.1). For $F = F^N$, as in Corollary 11.16(1)–((4)) (in (1) and ((2)), we only consider the explicitly listed examples with the exception of $F = \Omega^N_{/k}/h^0_{\mathbf{A}^1}(\Omega^N_{/k})$), the pushforward:*

$$Rf_*F^N_X \xrightarrow{f_*} F^{N-d}_S[-d]$$

is an isomorphism.

Proof. The proof is similar to the proof of Corollary 11.16 except that we have to replace the reference to Theorem 10.10 by a reference to Theorem 10.16. Furthermore, we have to observe that in the cases considered, we have $\gamma^d F^N \cong F^{N-d}$, as follows directly from the exactness of γ (see Lemma 9.2), Theorem 11.8 and the weak Cancellation theorem from [MS20] (see (9.1.5)). □

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Remark 11.23.

(1) If the diagonal only decomposes rationally, then we obtain a similar statement as in Corollary 11.22, up to bounded torsion.
(2) If F is a successive extensions of subquotients of $\mathbf{G}_a$, then $\gamma F = 0$ (as follows from Theorem 11.8 and the exactness of γ), and we can similarly apply Theorem 10.15.

The following statement seems to be new if S is not an algebraically closed field.

Corollary 11.24. *Let $f : X \to S$ and $g : Y \to S$ be two flat, geometrically integral and projective morphisms between smooth connected k-schemes. We furthermore assume that the generic fibre of f and the generic fibre of g have index 1 over the function field $k(S)$. Denote by $\mathrm{Pic}_{X/S}$ the relative Picard functor (which is representable, e.g. [BLR90, Section 8.2, Theorem 1]) and by $\mathrm{Pic}_{X/S}[n]$, its n-th torsion subfunctor.*

If X and Y are stably properly birational over S (see 10.2(2)), then any proper birational correspondence between projective bundles over X and Y induces an isomorphism of sheaves on S_{Nis}:

$$\mathrm{Pic}_{X/S}[n] \cong \mathrm{Pic}_{Y/S}[n], \quad \textit{for all } n.$$

Proof. By definition, we have that for any morphism of schemes $T \to S$, $\mathrm{Pic}_{X/S}(T) = H^0(T, R^1 f_* \mathcal{O}^\times_{X_{\mathrm{fppf}}})$. The assumptions on f ensure that for any such T, we have $f_{T*}\mathcal{O}_{X_T} = \mathcal{O}_T$, where $f_T : X_T \to T$ denotes the base change of f, (and similarly with g). Hence, $f_* \mathcal{O}^\times_{X_{\mathrm{fppf}}} = \mathcal{O}^\times_{S_{\mathrm{fppf}}}$, and, therefore, also:

$$f_* \mu_{n, X_{\mathrm{fppf}}} = \mu_{n, S_{\mathrm{fppf}}}, \tag{11.24.1}$$

where $\mu_{n, X_{\mathrm{fppf}}}$ denotes the sheaf of n-th roots of unity in the fppf-topology on X. This yields an exact sequence on S_{fppf}:

$$0 \to R^1 f_* \mu_{n, X_{fppf}} \to R^1 f_* \mathcal{O}^\times_{X_{\mathrm{fppf}}} \xrightarrow{n\cdot} R^1 f_* \mathcal{O}^\times_{X_{\mathrm{fppf}}}.$$

Thus, on $S_{\text{ét}}$:

$$\mathrm{Pic}_{X/S}[n] = v_*(R^1 f_* \mu_{n, X_{\mathrm{fppf}}}), \tag{11.24.2}$$

where $v : S_{\mathrm{fppf}} \to S_{\text{ét}}$ denotes the morphisms of sites (we denote the corresponding morphism on X by the same letter). The spectral sequence for the composition $Rv_* \circ Rf_*$ yields an exact sequence:

$$\begin{aligned} 0 \to R^1 v_*(f_* \mu_{n, X_{\mathrm{fppf}}}) \to R^1 (v \circ f)_* \mu_{n, X_{\mathrm{fppf}}} \\ \to v_* R^1 f_* \mu_{n, X_{\mathrm{fppf}}} \to R^2 v_*(f_* \mu_{n, X_{\mathrm{fppf}}}). \end{aligned} \tag{11.24.3}$$

Write $n = mp^r$, where p is the exponential characteristic of k, $(m, p) = 1$ and $r \geq 0$. By [Gro68, Theorem 11.7], we have:

$$Rv_* \mu_{m, X_{\mathrm{fppf}}} \cong \mu_{m, X_{\text{ét}}}, \quad Rv_* \mathbf{G}_{m, X_{\mathrm{fppf}}} \cong \mathbf{G}_{m, X_{\text{ét}}}, \tag{11.24.4}$$

in particular, the second isomorphism of (11.24.4) gives:

$$Rv_* \mu_{p^r, X_{\mathrm{fppf}}} \cong \mathcal{O}^\times_{X_{\text{ét}}} / (\mathcal{O}^\times_{X_{\text{ét}}})^{p^r}[-1] \cong W_r \Omega^1_{X_{\text{ét}}, \log}[-1], \tag{11.24.5}$$

where the second isomorphism holds by [Ill79, Chapter I, Proposition 3.23.2], and we use that X is smooth (also for the first isomorphism). Putting (11.24.1) – (11.24.5) together and using $R(v \circ f)_* = Rf_* Rv_*$ and $R^2 v_*(f_* \mu_{n, X_{\mathrm{fppf}}}) = R^2 v_*(\mu_{n, S_{\mathrm{fppf}}}) = 0$ (note that S is smooth too), we obtain an exact

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sequence on $S_{\text{ét}}$:

$$0 \to W_r\Omega^1_{S_{\text{ét}},\log} \to R^1 f_* \mu_{m,X_{\text{ét}}} \oplus f_* W_r\Omega^1_{X_{\text{ét}},\log} \to \mathrm{Pic}_{X/S}[n] \to 0. \tag{11.24.6}$$

Let $\varepsilon : \mathbf{Sm}_{\text{ét}} \to \mathbf{Sm}_{\text{Nis}}$ be the morphism of sites. We saw in (11.0.2) that $F^i := R^i\varepsilon_* W_r\Omega^1_{\log}$ is a reciprocity sheaf, for all i. The assumption on the index of the general fibre of f is equivalent to the existence of a zero cycle of degree 1 on the generic fibre $X_K \to K = k(S)$, so that we can apply Corollary 10.2 to find that:

$$f^* : F^1_S \to f_* F^1_X$$

is split injective. We can factor this map as the following composition:

$$F^1_S \xrightarrow{R^1\varepsilon_*(f^*)} R^1\varepsilon_*(f_* W_r\Omega^1_{X_{\text{ét}},\log}) \xrightarrow{d^1_{\varepsilon,f}} R^1(\varepsilon\circ f)_* W_r\Omega^1_{X_{\text{ét}},\log} \xrightarrow{e^1_{f,\varepsilon}} f_* F^1_X, \tag{11.24.7}$$

where $d^1_{\varepsilon,f}$ (respectively, $e^1_{f,\varepsilon}$) is an edge map of the spectral sequence associated to $R\varepsilon_* Rf_*$ (respectively, to $Rf_* R\varepsilon_*$). Since, by the above, the composition (11.24.7) is injective, so is the first map in that composition; it follows that applying ε_* to (11.24.6) yields an exact sequence on S_{Nis}:

$$0 \to F^0_S \to \varepsilon_* R^1 f_* \mu_{m,X_{\text{ét}}} \oplus f_* F^0_X \to \varepsilon_* \mathrm{Pic}_{X/S}[n] \to 0. \tag{11.24.8}$$

Furthermore, the spectral sequence for $R\varepsilon_* \circ Rf_*$ and the restriction of (11.24.1) to $S_{\text{ét}}$ yield an exact sequence:

$$\begin{aligned} 0 \to R^1\varepsilon_*\mu_{m,S_{\text{ét}}} \to R^1(\varepsilon\circ f)_*\mu_{m,X_{\text{ét}}} \to \varepsilon_* R^1 f_*\mu_{m,X_{\text{ét}}} \\ \to R^2\varepsilon_*\mu_{m,S_{\text{ét}}} \xrightarrow{d^2_{\varepsilon,f}} R^2(\varepsilon\circ f)_*\mu_{m,X_{\text{ét}}}. \end{aligned} \tag{11.24.9}$$

We claim that $d^2_{\varepsilon,f}$ is injective. Indeed, set $M^j := R^j\varepsilon_*\mu_m \in \mathbf{HI}_{\text{Nis}}$. We can factor $f^* : M^2_S \to f_* M^2_X$ as:

$$f^* : M^2_S \xrightarrow{d^2_{\varepsilon,f}} R^2(\varepsilon\circ f)_*\mu_{m,X_{\text{ét}}} \xrightarrow{e^2_{f,\varepsilon}} f_* M^2_X,$$

where the maps are the edge morphisms of the sprectral sequence to $R\varepsilon_* Rf_*$ (respectively, $Rf_* R\varepsilon_*$). By the assumption on the index of the general fibre of f and Corollary 10.2, we find that $f^* : M^2_S \to f_* M^2_X$ is injective; hence, so is $d^2_{\varepsilon,f}$. Furthermore, since the Nisnevich cohomology of a constant sheaf is trivial, we have $R^j f_*(\varepsilon_*\mu_{m,X_{\text{ét}}}) = 0$, for all $j \geq 1$. This yields:

$$R^1(f\circ\varepsilon)_*\mu_{m,X_{\text{ét}}} \cong f_* R^1\varepsilon_*\mu_{m,X_{\text{ét}}} = f_* M^1_X.$$

Thus, (11.24.9) yields an exact sequence:

$$0 \to M^1_S \to f_* M^1_X \to \varepsilon_* R^1 f_*\mu_{m,X_{\text{ét}}} \to 0.$$

Together with (11.24.8), we obtain:

$$\varepsilon_* \mathrm{Pic}_{X/S}[n] = \mathrm{Coker}(F^0_S \oplus M^1_S \xrightarrow{f^*} f_*(F^0_X \oplus M^1_X)).$$

Since we get a similar description for $g : Y \to S$, the statement follows from Theorem 10.7. □

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Conflicts of Interest. None

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